Riesz MV-algebras and their logic

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Abstract

We develop the general theory of RMV-algebras, which are essentially unit intervals in Riesz spaces with strong unit. Since the variety of RMV-algebras is generated by $[0,1]$, we get an equational characterization of the real product on [0,1] interpreted as scalar multiplication.

Keywords: RMV-algebra, MV-algebra, Riesz space, Łukasiewicz logic.

1. Introduction

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is an abelian monoid and the following identities hold for all $x, y \in A$: $(x^*)^* = x$, $0^* \oplus x = 0^*$ and $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. Note that every MV-algebra A is a bounded distributive lattice, where $x \vee y = x \oplus (x \oplus y^*)^*$ and $x \wedge y = (x^* \vee y^*)^*$ for any $x, y \in A$. If we set $x \odot y = (x^* \oplus y^*)^*$ then $\odot$ is the Łukasiewicz t-norm on $[0,1]$. The residuum is defined by $x \to y := x^* \oplus y$.

MV-algebras are the algebraic structures of Łukasiewicz ∞-valued logic. The real unit interval $[0,1]$ equipped with the operations $x^* = 1 - x$ and $x \oplus y = \min(1, x+y)$ is the *standard* MV-algebra, i.e. an equation holds in any MV-algebra if and only if it holds in $[0,1]$. In [19] Mundici proved that MV-algebras are categorically equivalent with abelian lattice-ordered groups with strong unit. Consequently, any MV-algebra is, up to isomorphism, the unit interval of an abelian lattice-ordered group with strong unit. We refer to [2] for all the unexplained notions concerning MV-algebras.

If we consider Riesz spaces [3, 16] with strong unit instead of lattice-ordered groups, then the unit interval is closed to the scalar multiplication with scalars from $[0,1]$.

The idea of considering these structures is also related to the problem of axiomatizing the real product on $[0,1]$. These investigations led to the definition of *PMV-algebras* (*product MV-algebras*) [4]. The analogue of Mundici's theorem for PMV-algebras was obtained by Di Nola and Dvurečenskij [4]: there exists a categorical equivalence between PMV-algebras and lattice-ordered rings with strong unit (ℓu-rings). Due to a result of Isbell [12], the class of PMV-algebras is larger than intended. In [17], Montagna axiomatized the quasi-variety of PMV-algebras generated by $[0,1]$.

Interpreting the product as scalar multiplication with scalars from $[0,1]$, the standard algebra $[0,1]$ generates the variety of *Riesz MV-algebras* (RMV-algebras, shortly). These structures are, up to isomorphism, unit intervals in Riesz spaces with strong unit. Our goal is to develop a theory for these structures and to investigate their relevance within MV-algebras.

Note that RMV-algebras are particular MV-modules, structures defined in [6]. Consequently, some results presented in this paper are obtained from general results proved in [6, 14].

The second section contains basic facts on RMV-algebras. We specialize Mundici's equivalence to RMV-algebras and Riesz spaces with strong unit and we establish an adjunction between MV-algebras and RMV-algebras. Consequently, any MV-algebra has an RMV-algebra cover. We get a particular characterization for semisimple RMV-algebra and we recall a construction from [5], where the Riesz spaces are proved to be categorical equivalent with a particular class of RMV-algebras.

In the third section we characterize the variety of RMV-algebras and we prove that it is generated by the standard RMV-algebra $[0,1]$. We also characterize the free RMV-algebras with n generators as algebras of Mc Naughton functions with real coefficients.

The last section present $\mathcal{L}_R$, a propositional calculus which has RMV-algebras as models. This calculus has standard completeness with respect to $[0,1]$. In the end we prove a theorem of approximation: any continuous function $h : [0,1]^n \to [0,1]$ is uniform limit of functions corresponding to formulas of $\mathcal{L}_R$.

2. RMV-algebras

In [6] the authors defined the structure of *MV-module over a PMV-algebra*. The RMV-algebras are the unital MV-modules over $[0,1]$, where the PMV-algebra structure of $[0,1]$ is given by the real product. Hence the general theory of MV-modules [6, 14] can be applied to RMV-algebras.

Definition 2.1 A *Riesz MV-algebra* (*RMV-algebra*) is a structure $(R, \cdot)$, where R is an MV-algebra and $\cdot : [0,1] \times R \longrightarrow R$ is such that the following properties hold for any $x, y \in R$ and $r, q \in [0,1]$:
(RMV1)$(r\cdot x) \odot (r\cdot y) = 0$ and $r\cdot(x \oplus y) = (r\cdot x) \oplus (r\cdot y)$ whenever $x \odot y = 0$,

(RMV2) $(r\cdot x)\odot(q\cdot x)=0$ and $(r\oplus q)\cdot x=(r\cdot x)\oplus(q\cdot x)$ whenever $r\odot q=0$,
(RMV3) $(r\cdot q)\cdot x=r\cdot(q\cdot x)$,
(RMV4) $1\cdot x=x$.

We shall frequently denote an RMV-algebra $(R,\cdot)$ by its MV-algebra support R and we shall simply write rx instead of $r\cdot x$ for $r\in[0,1]$ and $x\in R$.

Example 2.2 (1) $([0,1],\cdot)$ is an RMV-algebra, where $\cdot$ is the real product. Moreover, due to a result of Hion [9][Chapter IV, Proposition 2], one can prove that this is the only structure of RMV-algebra on $[0,1]$ and it will be called the *standard RMV-algebra*.
(2) If X is a compact Hausdorff space, then $C(X)=\{f\colon X\to[0,1]|f \text{ continuous}\}$ is an RMV-algebra, with the scalar multiplications defined by $(rf)(x):=rf(x)$ for any $x\in X$.

The following example is the motivation of our theory.

Example 2.3 *The unit interval of a Riesz space.* Let (V,u) be a Riesz space with strong unit [16, 3]. Hence the unit interval $\Gamma(V,u)=([0,u]_V,\oplus,^*,0)$ is an MV-algebra by Mundici's categorical equivalence [18]. Moreover, $rx\in[0,u]$ whenever $r\in[0,1]$ and $x\in[0,u]$. It is straightforward that $\Gamma(V,u)$ is an RMV-algebra.

If R is an RMV-algebra and $I\subseteq R$ is an MV-ideal, then $rx\in I$ for any $r\in[0,1]$ and $x\in I$ [6][Remark 3.15]. Hence, the MV-ideals and the RMV-ideals coincide, i.e. the MV-algebra congruences are compatible with the scalar multiplication. Consequently, if $f:R_1\to R_2$ is an MV-algebra homomorphism then $f(rx)=rf(x)$ $r\in[0,1]$ and $x\in R_1$, so RMV-algebra homomorphisms are MV-algebra homomorphisms between RMV-algebras, so we specialize Mundici's categorical equivalence as follows.

Theorem 2.4 [6] The category of RMV-algebras with MV-algebra homomorphisms is equivalent to the category of Riesz spaces with strong unit with unit preserving Riesz homomorphisms. As a consequence, for any RMV-algebra R there exists a Riesz space with strong unit (V,u) such that R is isomorphic with $\Gamma(V,u)$.

Chang's representation theorem for MV-algebras [2] immediately yields a similar representation for RMV-algebras.

Theorem 2.5 Any RMV-algebra R is isomorphic with a subdirect product of linearly ordered RMV-algebras.

Proof. There is an MV-algebra embedding $h: R\to\prod_{P\in Spec(R)}R/P$, where $Spec(R)$ is the prime ideal space of R. But any ideal P is an RMV-algebra ideal, so R/P is an RMV-algebra. Hence h is an RMV-algebra embedding.

The relation between MV-algebras and RMV-algebras can be expressed using the tensor product of MV-algebras $\otimes$ defined in [20].

Proposition 2.6 For any MV-algebra A, the tensor product $[0,1]\otimes A$ has an RMV-algebra structure such that the following properties hold:
(a) $r(q\otimes x)=(rq)\otimes x$ for any r, $q\in[0,1]$, $x\in A$,
(b) the function $\iota_A: A\to[0,1]\otimes A$ defined by $\iota_A(x):=1\otimes x$ for any $x\in A$ is an MV-algebra embedding.
Moreover, for any RMV-algebra R and any MV-algebra homomorphism $f: A\to R$ there exists a unique RMV-algebra homomorphism $f_\otimes:[0,1]\otimes A\to R$ such that $f_\otimes\circ\iota_A=f$.

Proof. (a) is proved in [7][Theorem 4.1].
(b) By [20][Proposition 2.3], ι_A is an MV-algebra homomorphism. The fact that ι_A is an embedding was proved by F. Montagna and T. Flaminio (private communication).

The above results asserts that any MV-algebra has an RMV-algebra hull. This construction yields an adjunction between the category $\mathcal{MV}$ of MV-algebras and the category $\mathcal{RMV}$ of RMV-algebras. Let us define the functors

$$\mathcal{U}:\mathcal{RMV}\to\mathcal{MV} \text{ and } \mathcal{T}_\otimes:\mathcal{MV}\to\mathcal{RMV}$$

as follows: $\mathcal{U}$ is the forgetful functor forgets the scalar multiplication and $\mathcal{T}_\otimes(A):=[0,1]\otimes A$ for any MV-algebra A. If $h:A\to B$ is an MV-algebra homomorphism then $\iota_B\circ h:A\to[0,1]\otimes B$, using Proposition 2.6, we get an RMV-algebra homomorphism $(\iota_B\circ h)_\otimes:[0,1]\otimes B\to[0,1]\otimes A$. Hence we define $\mathcal{T}_\otimes(h):=(\iota_B\circ h)_\otimes$ whenever $h:A\to B$ is an MV-algebra homomorphism.

Theorem 2.7 $(\mathcal{T}_\otimes,\mathcal{U})$ is an adjoint pair.

Proof. For a detailed proof in the general setting of MV-modules see [14][Proposition 7.29]. It is obvious that $\mathcal{T}_\otimes$ is a functor. Let A be an MV-algebra and R an RMV-algebra. By Proposition 2.6, for any MV-algebra homomorphism $f: A\to\mathcal{U}(R)$ there exists a unique RMV-algebra homomorphism $f_\otimes:\mathcal{T}_\otimes(A)\to R$ such that $\mathcal{U}(f_\otimes)\circ\iota_A=f$. This proves that $\mathcal{T}_\otimes$ is left adjoint to $\mathcal{U}$.

Proposition 2.8 An MV-algebra A admits an RMV-algebra structure if and only if $A\simeq[0,1]\otimes A$.

Proof. If A is an RMV-algebra then, by Proposition 2.6, there exists a unique RMV-algebra homomorphism $(I_A)_\otimes$ such that $(I_A)_\otimes\circ\iota_A=I_A$, where $I_A:A\to A$ is the identity function. We only have to prove that $\iota_A\circ(I_A)_\otimes=I_{[0,1]\otimes A}$, but this is true since the two functions coincide on the generators of

$[0,1] \otimes A$, i.e. $r \otimes x = r(1 \otimes x) = r\iota_A(x) = \iota_A(rx) = \iota_A((I_A)_\otimes(r \otimes x))$ for any $r \in [0,1]$ and $x \in A$.

We further emphasize some properties of RMV-algebras.

Lemma 2.9 For any RMV-algebra $(R, \cdot)$ the following properties hold.
(a) The function $r \mapsto r \cdot 1_R$ is an embedding of $[0,1]$ in R.
(b) For any maximal ideal $M \subseteq R$, $R/M \simeq [0,1]$.

Proof. (a) By Theorem 2.4, we can take $R = \Gamma(V, u)$ for some Riesz space with strong unit u and the intended result follows from the properties of Riesz spaces.
(b) For any maximal ideal $M \subseteq R$, the MV-algebra R/M is simple, so it is a subalgebra of $[0,1]$. Using (a), we get $R/M \simeq [0,1]$.

As a consequence of the previous lemma, the only simple RMV-algebra is $[0,1]$.

Recall that an MV-algebra is *archimedean* if the corresponding lattice-ordered group is archimedean. Archimedean MV-algebras are equivalent with the semisimple ones, i.e. those with the property that $Rad(A) = \bigcap\{M \subseteq A | M \in Max(A)\} = \{0\}$, where $Max(A)$ is the maximal ideal space of A. Consequently, semisimple and archimedean RMV-algebras will coincide. Since the unique simple RMV-algebra is $[0,1]$, any semisimple RMV-algebra is isomorphic with a subdirect product of copies of $[0,1]$.

For any MV-algebra A, $Max(A)$ is a compact Hausdorff space with respect to the spectral topology. For a nonempty set X, an MV-subalgebra S of $[0,1]^X$ is *separating* if whenever $x \neq y \in X$ there exists $f \in S$ such that $f(x) = 0$ and $f(y) > 0$. It is known that any archimedean MV-algebra A is isomorphic with a separating MV-subalgebra of $C(Max(A))$ [2].

We can further specialize the characterization of semisimple RMV-algebras.

Let us firstly recall the MV-algebraic version of Stone-Weierstrass theorem.

Theorem 2.10 (*Stone-Weierstrass for RMV-algebras*)[15] Assume X is a compact Hausdorff space. Every separating RMV-subalgebra A of $C(X)$ is dense in $C(X)$ with respect to the *sup*-norm.

Theorem 2.11 (*Characterization of semisimple RMV-algebra*) Any semisimple RMV-algebra R is isomorphic with a dense (w.r.t. to the *sup*-norm) subalgebra of $C(Max(R))$.

Proof. By Stone-Weierstrass theorem, R is dense in $C(X)$.

In the end of this section we recall an important construction defined in [5]

Remark 2.12 Let V be a Riesz space and $\mathbb{R} \times_{lex} V$ be the lexicographic product. Hence $(1,0)$ is a strong unit, so $R = \Gamma(\mathbb{R} \times_{lex} V, (1,0))$ is an RMV-algebra. Denote $\mathcal{RMV}_{lex}$ the class of RMV-algebras R with the property that $R \simeq \Gamma(\mathbb{R} \times_{lex} V, (1,0))$ for some Riesz space V. This class is axiomatized in [5].

Theorem 2.13 [5][Theorem 4.6] $\mathcal{RMV}_{lex}$ is equivalent with the category of Riesz spaces.

If T is an MV-algebra and $a \neq 0$ in T then the interval $([0,a], \oplus_a, {}^{*a}, 0)$ is an MV-algebra with the operations defined by: $x \oplus_a y := (x \oplus y) \wedge a$, $x^{*a} := x^* \odot a$. Note that, whenever T is an RMV-algebra, the interval $[0,a]$ is closed to scalar multiplication, so $[0,a]$ is an RMV-algebra.

The following result allows us to assert that the class $\mathcal{RMV}_{lex}$ stands to RMV-algebras as perfect MV-algebras stand to MV-algebras.

Lemma 2.14 For any RMV-algebra R there exists an RMV-algebra T in $\mathcal{RMV}_{lex}$ and an element $a \neq 0$ in T such that $R \simeq [0,a]$.

Proof. If R is an RMV-algebra then $R \simeq \Gamma(V,u)$ for some Riesz space with strong unit (V,u). Set $T := \Gamma(\mathbb{R} \times_{lex} V, (1,0))$ and $a := (0,u)$. Then the intended conclusion is straightforward.

3. Equational characterization. Free RMV-algebras.

We show that the class of RMV-algebras is the variety generated by $[0,1]$.

Theorem 3.1 If R is an MV-algebra and $\cdot : [0,1] \times R \to R$ then $(R, \cdot)$ is an RMV-algebra if and only if the following identities are satisfied for any r, $q \in [0,1]$ and x,$y \in R$:
(R1) $(rx) \odot ((r \vee q)x)^* = 0$,
(R2) $(r \odot q^*)x = (rx) \odot ((r \wedge q)x)^*$,
(R3) $r(qx) = (r \cdot q)x$,
(R4) $r(x \odot y^*) = (rx) \odot (ry)^*$,
(R5) $1x = x$.

Proof. It follows by [8][Corollary 3.13]. For a detailed proof one can see [14][Corollary 6.45].

The free objects in a variety always exists. In the category of RMV-algebras we get the following particular characterization.

Proposition 3.2 For any set X, the free RMV-algebra generated by X is $[0,1] \otimes Free_{MV}(X)$, where $Free_{MV}(X)$ is the free MV-algebra generated by X.

Proof. It is straightforward by Proposition 2.6.

We prove that an identity holds in the theory of RMV-algebras if and only if it holds in the standard RMV-algebra $[0,1]$. Our approach follows closely the proof of Chang's completeness theorem for Łukasiewicz logic [1]. To any sentence in the first-order theory of RMV-algebras we associate a sentence in the first-order theory of Riesz spaces such that the satisfiability is preserved by the Γ functor. The first-order theory of RMV-algebras, as well as the theory of Riesz spaces, are obtained considering for each scalar r an unary function ρ_r which denotes in a particular model the scalar multiplication by r, i.e. $x \overset{\rho_r}{\mapsto} rx$. In the following, the language of RMV-algebras is $\mathcal{L}_{RMV} = \{\oplus, ^*, 0, \{\rho_r\}_{r\in[0,1]}\}$ and the language of Riesz spaces is $\mathcal{L}_{Riesz} = \{\leq, +, -, \vee, \wedge, 0, \{\rho_r\}_{r\in\mathbb{R}}\}$.

Let $t(v_1, \ldots, v_k)$ be a term of $\mathcal{L}_{RMV}$ and v a propositional variable different from $v_1, \ldots, v_k$. We define $\widetilde{t}$ as follows:
- if $t = 0$ then $\widetilde{0}$ is 0,
- if $t = v$ then $\widetilde{t}$ is v
- if $t = t_1^*$ then $\widetilde{t}$ is $v - \widetilde{t_1}$,
- if $t = t_1 \oplus t_2$ then $\widetilde{t}$ is $(\widetilde{t_1} + \widetilde{t_2}) \wedge v$,
- if $t = \rho_r(t_1)$ then $\widetilde{t}$ is $\rho_r(\widetilde{t_1})$.

Let $\varphi(v_1, \ldots, v_k)$ be a formula of $\mathcal{L}_{RMV}$ such that all the free and bound variables of φ are in $\{v_1, \ldots, v_k\}$ and v a propositional variable different from $v_1, \ldots, v_k$. We define $\widetilde{\varphi}$ as follows:
- if φ is $t_1 = t_2$ then $\widetilde{\varphi}$ is $\widetilde{t_1} = \widetilde{t_2}$,
- if φ is $\neg\psi$ then $\widetilde{\varphi}$ is $\neg\widetilde{\psi}$,
- if φ is $\psi \vee \chi$ then $\widetilde{\varphi}$ is $\widetilde{\psi} \vee \widetilde{\chi}$ and similarly for $\wedge$, $\rightarrow$, $\leftrightarrow$,
- if φ is $(\forall v_i)\psi$ then $\widetilde{\varphi}$ is $\forall v_i((0 \leq v_i) \wedge (v_i \leq v) \rightarrow \widetilde{\psi})$,
- if φ is $\exists v_i \psi$ then $\widetilde{\varphi}$ is $\exists v_i((0 \leq v_i) \wedge (v_i \leq v) \rightarrow \widetilde{\psi})$.

Thus to any formula $\varphi(v_1, \ldots, v_k)$ of $\mathcal{L}_{RMV}$ we associate a formula $\widetilde{\varphi}(v_1, \ldots, v_k, v)$ of $\mathcal{L}_{Riesz}$. As a consequence, to any sentence σ of $\mathcal{L}_{RMV}$ corresponds a formula with only one free variable $\widetilde{\sigma}(v)$ of $\mathcal{L}_{Riesz}$.

Proposition 3.3 Let (V, u) be a Riesz space with strong unit and $R = \Gamma(V, u)$. If σ is a sentence in the first-order theory of RMV-algebras then

$$R \models \sigma \text{ if and only if } V \models \widetilde{\sigma}[u].$$

Proof. By structural induction on terms it follows that $t[a_1, \ldots, a_n] = \widetilde{t}[a_1, \ldots, a_n, u]$ whenever $t(v_1, \ldots, v_n)$ is a term of $\mathcal{L}_{RMV}$ and $a_1, \ldots, a_n \in R$. The rest of the proof is straightforward.

Theorem 3.4 An equation σ in the theory of RMV-algebras holds in all RMV-algebras if and only if it holds in the standard RMV-algebra $[0,1]$.

Proof. One implication is obvious. To prove the other one, let R be an RMV-algebra such that $R \not\models \sigma$. Since $R \simeq \Gamma(V, u)$ for some Riesz space with strong unit (V, u), we have that $\Gamma(V, u) \not\models \sigma$. Using Proposition 3.3, we infer that $V \not\models \widetilde{\sigma}[u]$ in the theory of Riesz spaces. Since the order relation in any lattice can be expressed equationally, we note that $\widetilde{\sigma}(v)$ is a quasi-identity. By [13][Corollary 2.6] a quasi-identity is satisfied by all Riesz spaces if and only if it is satisfied by $\mathbb{R}$. Hence there exists a real number $c \geq 0$ such that $\mathbb{R} \not\models \widetilde{\sigma}[c]$. Since $\mathbb{R} \models \widetilde{\sigma}[0]$, we get $c > 0$. If follows that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) \mapsto x/c$ is an automorphism of Riesz spaces. We infer that $\mathbb{R} \not\models \widetilde{\sigma}[1]$, so $[0,1] \not\models \sigma$.

Corollary 3.5 $[0,1]$ generates the variety of RMV-algebras.

Given $t(v_1, \cdots, v_n)$ a term in $\mathcal{L}_{RMV}$ we define the term function $f_t : [0,1]^n \to [0,1]$ as usual (see [**?**] for the general theory). In the theory of MV-algebras, the term functions are Mc Naughton functions [2], i.e. continuous piecewise affine functions with integer coefficients. We immediately obtain a similar description for the term functions in $\mathcal{L}_{RMV}$.

Definition 3.6 Let $n > 1$ be a natural number. A *Mc Naughton function with real coefficients* is a continuous function $f : [0,1]^n \longrightarrow [0,1]$ which satisfies the following condition:

there exists a finite number of affine functions (with real coefficients) $q_1, \ldots, q_k : \Re^n \longrightarrow \Re$ such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there is $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = q_i(a_1, \ldots, a_n)$.

Theorem 3.7 If $t(v_1, \cdots, v_n)$ is a term in $\mathcal{L}_{RMV}$ then the term function $f_t : [0,1]^n \to [0,1]$ is a Mc Naughton function with real coefficients.

Proof. We prove the conclusion by structural induction on terms:
- if t is v_i for some $i \in \{1, \ldots, n\}$ then $f_t = \pi_i$ (the i-th projection);
- if t is $t_1 \oplus t_2$ and let $q_1, \ldots, q_m$ be the polynomials of f_{t_1} and $p_1, \ldots, p_k$ be the polynomials of f_{t_2}; then f_t is defined by the polynomials $\{1\} \cup \{s_{ij}\}_{i,j}$, where $s_{ij} = q_i + p_j$ for any $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$;
- t is t_1^* and $q_1, \ldots, q_s$ are the corresponding polynomials of f_{t_1}, then i $1 - q_1, \ldots, 1 - q_s$ are the polynomials of f_t;
- t is $\rho_r(t_1)$ for some $r \in [0,1]$ and $q_1, \ldots, q_s$ are the corresponding polynomials of f_{t_1}, then i $rq_1, \ldots, rq_s$ are the polynomials of f_t.

Remark 3.8 Term functions and Mc Naughton functions with integer coefficients coincide in the case of MV-algebras. It is an open question if this holds for RMV-algebras too:
($\star$) *given $f : [0,1]^n \to [0,1]$ a Mc Naughton function with real coefficients, can we find a term t in $\mathcal{L}_R$ such that $f = f_t$?*
Note that, for $f : [0,1]^n \to [0,1]$ a Mc Naughton function with real coefficients, there are finite sets I and J such that

$$f = \bigvee_{i\in I} \bigwedge_{j\in J} f_{ij},$$

where $f_{ij} : [0,1]^n \to \mathbb{R}$ are affine functions with real coefficients [21, Theorem 2.1]. It follows that it would be enough to answer ($\star$) for affine functions with real coefficents.

In Section 4 we develop a propositional calculus for RMV-algebras. Since the primary connectives of Łukasiewicz logic are $\to$ and $\neg$, we have to provide an equational characterization of the scalar multiplication using implication and negation.

Remark 3.9 [14, Section 6.4] Let R be an MV-algebra and $\circ : [0,1] \times R \to R$ such that the following properties hold for any $x, y \in R$ and $r, q \in [0,1]$:
(1°) $r \circ (x \to y) = (r \circ x) \to (r \circ y)$,
(2°) $(r \odot q^*) \circ x = ((r \wedge q) \circ x) \to (r \circ x)$,
(3°) $r \circ (q \circ x) = (r \cdot q) \circ x$,
(4°) $((r \vee q) \circ x) \to (r \circ x) = 1$,
(5°) $1 \circ x = x$.
We call *dual RMV-algebra* a structure $(R, \circ)$ as above. If R is an MV-algebra and $\circ : [0,1] \times R \to R$ we define
$r \cdot x := (r \circ (x^*))^*$ for any $x \in R$, $r \in [0,1]$.
Hence $(R, \cdot)$ is an RMV-algebra if and only if $(R, \circ)$ is a dual RMV-algebra [14].

4. A propositional calculus for RMV-algebra

We develop in a classical way a propositional calculus $\mathcal{L}_R$ that have RMV-algebra as models. One can see [14] for detailed proof in the general setting of MV-modules. Note that Theorem 4.5 and Theorem 4.6 are not proved in general.

The *language* of the propositional calculus $\mathcal{L}_R$ consists of:
- denumerable many propositional variables: v_1, $\ldots$,v_n, $\cdots$
(the set of all the propositional variables will be denoted by Var),
- the logical connectives of $\mathcal{L}$: $\to$ (binary) and $\neg$ (unary),
- unary logical connectives : $\Diamond_r$ for any $r \in [0,1]$, - parentheses: (and).
We denote by $Form(\mathcal{L}_R)$ the set of formulas, which are defined inductively as usual.

Definition 4.1 The *axioms* of $\mathcal{L}_R$ are defined as follows:
a formula which has one of the following forms is an axiom (where φ, ψ and χ are arbitrary formulas and r, q are arbitrary elements of $[0,1]$):
(L1) $\varphi \to (\psi \to \varphi)$,
(L2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(L3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,
(L4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$
(F1) $\Diamond_r(\varphi \to \psi) \leftrightarrow (\Diamond_r\varphi \to \Diamond_r\psi)$,
(F2) $\Diamond_{(r\odot q^*)}\varphi \leftrightarrow (\Diamond_{r\wedge q}\varphi \to \Diamond_r\varphi)$,
(F3) $\Diamond_r\Diamond_q\varphi \leftrightarrow \Diamond_{r\cdot q}\varphi$,
(F4) $\Diamond_{r\vee q}\varphi \to \Diamond_r\varphi$,
(F5) $\Diamond_1\varphi \leftrightarrow \varphi$.

Note that (L1)-(L4) are the axiom of Łukasiewicz logic. Note that the axioms (F1)-(F5) are the duals of (R1)-(R5) if we consider that the term function associated to $\Diamond_r\varphi$ is $(rf_\varphi^*)^*$, where f_φ is the term function of φ.

The *deduction rule* of $\mathcal{L}_R$ is modus ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$. Proofs are defined as usual.

Proposition 4.2 (*Deduction theorem*)

$$\Theta \cup \{\varphi\} \vdash \psi \text{ iff } \Theta \vdash \varphi^n \to \psi \text{ for some } n \geq 1,$$

where φ^n denotes $\underbrace{\varphi \odot \cdots \odot \varphi}_{n\ times}$.

We define the Lindenbaum-Tarski algebra. In the sequel $\Theta \subseteq Form(\mathcal{L}_R)$ is a fixed set of formulas. For any two formulas φ and ψ we define

$$\varphi \equiv_\Theta \psi \text{ iff } \Theta \vdash \varphi \to \psi \text{ and } \Theta \vdash \psi \to \varphi.$$

Since $\equiv_\Theta$ is an equivalence relation on $Form(\mathcal{L}_R)$, we denote $[\varphi]_\Theta$ the equivalence class of φ with respect to $\equiv_\Theta$.

On $Form(\mathcal{L}_R)/\equiv_\Theta$ we define the following operations:
$[\varphi]_\Theta^* : = [\neg\varphi]_\Theta$,
$[\varphi]_\Theta \to [\psi]_\Theta : = [\varphi \to \psi]_\Theta$,
$[\varphi]_\Theta \oplus [\psi]_\Theta : = [(\neg\varphi) \to \psi]_\Theta$,
$1_\Theta : = [\varphi]_\Theta$, where $\Theta \vdash \varphi$,
$0_\Theta : = 1_\Theta^*$,
$r[\varphi]_\Theta : = [\neg(\Diamond_r(\neg\varphi))]_\Theta$

With the above operations, $Form(\mathcal{L}_R)/\equiv_\Theta$ is an RMV-algebra by Remark 3.9. When $\Theta = \emptyset$, we denote $Linda_R := Form(\mathcal{L}_R)/\equiv_\emptyset$ and this is the Lindenbaum-Tarski algebra of RMV-logic.

Since the Lindenbaum-Tarski algebra is freely generated by the set of variables, by Proposition 3.2, we get the following.

Corollary 4.3 $Linda_R \simeq [0,1] \otimes Lind_{Luk}$, where $Linda_{Luk}$ is the Lindenbaum-Tarski algebra of Łukasiewicz logic.

Let R be an RMV-algebra. Following Remark 3.9, an *R-evaluation* is a function $e : Form(\mathcal{L}_R) \to R$ which satisfies the following conditions:
(e1) $e(\varphi \to \psi) = e(\varphi)^* \oplus e(\psi)$,
(e2) $e(\neg\varphi) = e(\varphi)^*$,
(e3) $e(\Diamond_a\varphi) = (ae(\varphi)^*)^*$,
for any $\varphi \in Form(\mathcal{L}_R)$ and $r \in [0,1]$.

The notions of satisfaction and semantic consequence are defined as usual.

According to [11], the system $\mathcal{L}_R$ is a *core fuzzy logic*, hence the strong completeness with respect to linearly ordered structures follows by [11, Theorem 2.11].

For $\mathcal{L}_R$ we also prove completeness with respect to the standard model.

Theorem 4.4 (*Strong completeness theorem*) Assume Θ be a set of formulas and φ a formula of $\mathcal{L}_R$. The following are equivalent:
(a) $\Theta \vdash \varphi$,
(b) $\Theta \models_R \varphi$ for any RMV-algebra R,
(c) $\Theta \models_R \varphi$ for any linearly-ordered RMV-algebra R.

Proof. The equivalence of (a) and (b) is straightforward. The equivalence with (c) follows by Theorem 2.5. See also [11, Theorem 2.11].

Theorem 4.5 (*Standard completeness*) For a formula φ of $\mathcal{L}_R$, the following are equivalent:
(a) $\vdash \varphi$,
(b) $\models_{[0,1]} \varphi$.

Proof. It follows by Theorem 3.4.

As a direct consequence of the standard completeness it follows that the logic of RMV-algebras is a conservative extension of Łukasiewicz logic.

Finally, we prove an approximation result.

Theorem 4.6 (*Approximation of continuous functions*) Let $n \geq 1$ be a natural number. For any continuous function $h : [0,1]^n \to [0,1]$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathcal{L}_R$ such that h is the uniform limit of $(f_{\varphi_n})_n$.

Proof. If $Form_n$ is the set of the formulas which contain only the variables $v_1, \ldots, v_n$, then $R_n = Form_n/{\equiv_\emptyset}$ is the free RMV-algebra with n-generators. By Theorem 3.7, R_n is a semisimple RMV-algebra. By Theorem 2.11, R_n is dense in $C(X)$ in the *sup*-norm which proves our result.

Remark 4.7 The logical system briefly presented in this chapter is strongly related with Rational Łukasiewicz Logic developed in [10], where only multiplication by rationals is considered. The algebraic structures of Rational Łukasiewicz Logic are the divisible MV-algebras. Our system is also a conservative extension of Rational Łukasiewicz Logic.

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References

[1] C. C. Chang, A new proof of the completeness of the Łukasiewicz axioms, Transactions of the American Mathematical Society 93, pages 74-80, 1959.
[2] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic Foundations of many-valued Reasoning, Kluwer, Dordrecht, 2000.
[3] E. de Jonge, A.C.M. van Rooij, Introduction to Riesz spaces. Mathematical Centre Tracts 78, Amsterdam, 1977.
[4] A. Di Nola , A. Dvurečenskij, Product MV-algebras, Multiple -Valued Logics 6, pages 193-215, 2001.
[5] A. Di Nola, A. Lettieri, Coproduct MV-algebras, nonstandard reals, and Riesz spaces, J. Algebra 185, No.3, pages 605-620, 1966.
[6] A. Di Nola, I. Leuştean, P. Flondor, MV-modules, Journal of Algebra 267, pages 21-40, 2003.
[7] P. Flondor, I. Leuştean, Tensor products of MV-algebras, Soft Computing 7, pages 446-457, 2003.
[8] P. Flondor, I. Leuştean, MV-algebras with operators (the commutative and the non-commutative case), Discrete Mathematics 274, pages 41-76, 2004.
[9] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, 1963.
[10] B. Gerla, Rational Łukasiewicz logic and DMV-algebras, Neural Networks World, vol 11, pages 579-584, 2001.
[11] P. Hájek, P. Cintula, On theories and models in fuzzy predicate logics, The Journal of Symbolic Logic 71, pages 863888, 2006.
[12] J.R. Isbell, Notes on ordered rings, Algebra Universalis, vol. 1, pages 393399, 1971.
[13] C.C.A. Labuschagne, C.J. van Alten, On the variety of Riesz spaces, Indagationes Mathematicae, N.S., 18 (1), pages 61-68, 2007.
[14] I. Leuştean, Contributions to the theory of MV-algebras: MV-modules, Editura Universitara, 2009. PhD Thesis, University of Bucharest, 2004.
[15] I. Leustean, V. Marra, Kakutani duality for MV-algebras, draft.
[16] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North-Holland, Amsterdam, 1971.
[17] F. Montagna, Subreducts of MV-algebras with product and product residuation, Algebra Universalis 53, pages 109-137, 2005.
[18] D. Mundici, Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, J. Functional Analysis 65, pages 15-63, 1986.
[19] D. Mundici, Averaging the truth value Łukasiewicz logic, Studia Logica 55, pages 113-127, 1995.
[20] D. Mundici, Tensor products and the Loomis-Sikorski theorem for MV-algebras, Advances in Applied Mathematics 22, pages 227-248, 1999.
[21] S. Ovchinnikov, Max-Min Representation of Piecewise Linear Functions, Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry Volume 43, No. 1, pages 297-302, 2002.