Biconic aggregation functions with a given diagonal or opposite diagonal section

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Abstract

A new method to construct aggregation functions is introduced. These aggregation functions are called biconic aggregation functions with a given diagonal (resp. opposite diagonal) section and their construction method is based on linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square and the points $(0,1)$ and $(1,0)$ (resp. $(0,0)$ and $(1,1)$). Special classes of biconic aggregation functions such as biconic semi-copulas, quasi-copulas and copulas are studied in detail.

Keywords: Aggregation function, Quasi-copula, Copula, Diagonal section, Opposite diagonal section, Linear interpolation

1. Introduction

A binary aggregation function A is a $[0,1]^2 \to [0,1]$ function satisfying the following conditions:

(i) $A(0,0) = 0$ and $A(1,1) = 1$;
(ii) for any $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $A(x,y) \leq A(x',y')$.

Aggregation functions are of great importance in many fields of application. Their most prominent uses is as logical connectives in fuzzy set theory [2]. To increase modelling flexibility, new methods to construct aggregation functions are being proposed continuously in the literature [6, 18].

Special classes of aggregation functions are of particular interest, such as semi-copulas [15, 16], triangular norms [1, 22], quasi-copulas [17, 24] and copulas [1, 25]. They are all conjunctors, in the sense that they extend the classical Boolean conjunction.

Recall that an aggregation function is a semi-copula if it has 1 as neutral element, i.e. $A(x,1) = A(1,x) = x$ for any $x \in [0,1]$. Evidently, any semi-copula S has 0 as annihilator, i.e. $S(0,x) = S(x,0) = 0$ for any $x \in [0,1]$. A semi-copula S is a triangular norm (t-norm for short) if it is commutative and associative. The aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ and $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are examples of t-norms. Moreover, for any semi-copula S the inequality $T_{\mathbf{D}} \leq S \leq T_{\mathbf{M}}$ holds. A semi-copula S is a quasi-copula if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$|S(x',y') - S(x,y)| \leq |x'-x| + |y'-y| \,.$$

A semi-copula S is a copula if it is 2-increasing, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $V_S([x,x'] \times [y,y']) :=$

$$S(x',y') + S(x,y) - S(x',y) - S(x,y') \geq 0 \,.$$

V_S is called the volume of the rectangle $[x,x'] \times [y,y']$. Any copula is a quasi-copula since the 2-increasingness of a semi-copula implies its 1-Lipschitz continuity. The (quasi-)copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ with $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$, are respectively the greatest and the smallest (quasi-)copulas, i.e. for any (quasi-)copula C, it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$.

The diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\delta_F : [0,1] \to [0,1]$ defined by $\delta_F(x) = F(x,x)$. A diagonal function [13] is a function $\delta : [0,1] \to [0,1]$ satisfying the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;

(D2) δ is increasing;

(D3) for all $x \in [0,1]$, it holds that $\delta(x) \leq x$;

(D4) δ is 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$ it holds that

$$|\delta(x') - \delta(x)| \leq 2|x'-x| \,.$$

The set of all diagonal functions will be denoted by $\mathcal{D}$. The set of all $[0,1] \to [0,1]$ functions that satisfy conditions **D1** and **D2** (resp. **D1**, **D2** and **D3**) will be denoted by $\mathcal{D}_A$ (resp. $\mathcal{D}_S$). The diagonal section of a copula C is a diagonal function. Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$. For instance, the copula

$$C_\delta(x,y) = \min\left(x, y, \frac{\delta(x)+\delta(y)}{2}\right)$$

is the greatest symmetric copula with diagonal section δ [12, 14, 26]. Similarly, the opposite diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\omega_F : [0,1] \to [0,1]$ defined by $\omega_F(x) = F(x, 1-x)$. An opposite diagonal function [7] is a function $\omega : [0,1] \to [0,1]$ satisfying the following conditions:

(OD1) for all $x \in [0,1]$, it holds that $\omega(x) \leq \min(x, 1-x)$;

(OD2) ω is 1-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$|\omega(x') - \omega(x)| \leq |x' - x| \,.$$

The set of all opposite diagonal functions will be denoted by $\mathcal{O}$. The set of all $[0,1] \to [0,1]$ functions that satisfy condition **OD1** will be denoted by $\mathcal{O}_S$. The opposite diagonal section ω_C of a copula C is an opposite diagonal function. Conversely, for any opposite diagonal function ω, there exists at least one copula C with opposite diagonal section $\omega_C = \omega$. For instance, the copula F_ω defined by $F_\omega(x,y) = T_{\mathbf{L}}(x,y) + \min\{\omega(t) \mid t \in [\min(x,1-y), \max(x,1-y)]\}$ is the greatest copula with opposite diagonal section ω [7, 23]. Note that F_ω is opposite symmetric [7], i.e. $F_\omega(x,y) - F_\omega(1-y, 1-x) = x + y - 1$, for any $(x,y) \in [0,1]^2$. Diagonal and opposite diagonal functions have been used recently to construct several subclasses of aggregation functions such as quasi-copulas and copulas [5, 7, 11, 12, 13, 14, 20].

Characteristic for the aggregation function $T_{\mathbf{M}}$ is that its surface is constituted from linear segments connecting its diagonal section to the points $(0,1,0)$ and $(1,0,0)$. Also characteristic for $T_{\mathbf{M}}$ is that its surface is constituted from linear segments connecting its opposite diagonal section to the points $(0,0,0)$ and $(1,1,1)$. Inspired by the above interpretation, we introduce a new method to construct aggregation functions. These aggregation functions are constructed by linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points $(0,1)$ and $(1,0)$ (resp. $(0,0)$ and $(1,1)$).

This paper is organized as follows. In the next section we introduce the definition of a biconic function with a given diagonal section and characterize the class of biconic aggregation functions, as well as the classes of biconic semi-copulas, biconic quasi-copulas, biconic copulas and singular biconic copulas. The class of biconic functions with a given opposite diagonal section is introduced in Section 3. Finally, some conclusions are given.

2. Biconic functions with a given diagonal section

2.1. Biconic aggregation functions with a given diagonal section

In this subsection we introduce biconic functions with a given diagonal section. Their construction is based on linear interpolation on segments connecting the diagonal of the unit square and the points $(0,1)$ and $(1,0)$. Throughout this paper the convention $\frac{0}{0} = 0$ is assumed. For any $(x,y) \in [0,1]^2$, we introduce in this subsection the following notations

$$u = \frac{x}{1+x-y}\,, \quad v = \frac{y}{1+y-x}\,.$$

Let $\delta \in \mathcal{D}_A$ and $\alpha, \beta \in [0,1]$. The function $A_\delta^{\alpha,\beta} : [0,1]^2 \to [0,1]$ defined by

$$A_\delta^{\alpha,\beta}(x,y) = \begin{cases} \alpha(x-y) + y\,\dfrac{\delta(v)}{v} & \text{, if } y \leq x\,, \\[2ex] \beta(y-x) + x\,\dfrac{\delta(u)}{u} & \text{, otherwise} \end{cases} \tag{1}$$

is well defined. This function is called *a biconic function with a given diagonal section.*

Proposition 1 *Let $\delta \in \mathcal{D}_A$. The function $A_\delta^{\alpha,\beta}$ defined in (1) is an aggregation function if and only if*

(i) *the functions $\mu_{\delta,\alpha}, \mu_{\delta,\beta} :]0,1] \to \mathbb{R}$, defined by*

$$\mu_{\delta,\alpha}(x) = \frac{\delta(x)-\alpha}{x}\,, \quad \mu_{\delta,\beta}(x) = \frac{\delta(x)-\beta}{x}\,,$$

are increasing;

(ii) *the functions $\lambda_{\delta,\alpha}, \lambda_{\delta,\beta} : [0,1[\to \mathbb{R}$, defined by*

$$\lambda_{\delta,\alpha}(x) = \frac{\delta(x)-\alpha}{1-x}\,, \quad \lambda_{\delta,\beta}(x) = \frac{\delta(x)-\beta}{1-x}\,,$$

are increasing.

Inspired by the above proposition, the biconic function $A_\delta^{\alpha,\beta}$ is called *a biconic aggregation function with a given diagonal section.*

Example 1 *Consider the diagonal section $\delta_{\mathbf{M}}$ of $T_{\mathbf{M}}$, i.e. $\delta_{\mathbf{M}}(x) = x$ for any $x \in [0,1]$. Obviously, conditions (i) and (ii) of Proposition 1 are satisfied. The resulting biconic aggregation function is a Choquet integral* [4, 10]*, i.e.*

$$A_{\delta_{\mathbf{M}}}^{\alpha,\beta}(x,y) = \begin{cases} \alpha x + (1-\alpha)y & \text{, if } y \leq x\,, \\[2ex] (1-\beta)x + \beta y & \text{, otherwise}\,. \end{cases}$$

Taking $\beta = 1-\alpha$, the resulting biconic aggregation function is the weighed arithmetic mean, i.e.

$$A_{\delta_{\mathbf{M}}}^{\alpha,1-\alpha}(x,y) = \alpha x + (1-\alpha)y\,.$$

Proposition 2 *Let $A_\delta^{\alpha,\beta}$ be a biconic aggregation function. Then the inequality*

$$\max(\alpha x, \beta x) \leq \delta(x) \leq \min(\alpha + (1-\alpha)x, \beta + (1-\beta)x)\,, \tag{2}$$

holds for any $x \in [0,1]$.

Now we identify the functions in $\mathcal{D}_A$ which characterize the extreme biconic aggregation functions with fixed α and β. Let $\alpha, \beta \in [0,1]$ and consider the functions $\underline{\delta}^{\alpha,\beta}, \overline{\delta}^{\alpha,\beta} : [0,1] \to [0,1]$, defined by

$$\underline{\delta}^{\alpha,\beta}(x) = \begin{cases} \max(\alpha x, \beta x) & \text{, if } x < 1\,, \\ 1 & \text{, if } x = 1\,, \end{cases}$$

$$\overline{\delta}^{\alpha,\beta}(x) = \begin{cases} \min(\alpha + (1-\alpha)x, \beta + (1-\beta)x) & \text{, if } x > 0\,, \\ 0 & \text{, if } x = 0\,. \end{cases}$$

Obviously, $\underline{\delta}^{\alpha,\beta}, \overline{\delta}^{\alpha,\beta} \in \mathcal{D}_A$ and the conditions of Proposition 1 are satisfied. Note also that for any two biconic aggregation functions $A^{\alpha,\beta}_{\delta_1}$ and $A^{\alpha,\beta}_{\delta_2}$, it holds that $A^{\alpha,\beta}_{\delta_1} \leq A^{\alpha,\beta}_{\delta_2}$ if and only if $\delta_1 \leq \delta_2$. The following proposition is then obvious.

Proposition 3 *Let $A^{\alpha,\beta}_{\delta}$ be a biconic aggregation function. Then it holds that*

$$A^{\alpha,\beta}_{\underline{\delta}^{\alpha,\beta}} \leq A^{\alpha,\beta}_{\delta} \leq A^{\alpha,\beta}_{\overline{\delta}^{\alpha,\beta}}\,.$$

Example 2 *The functions $\underline{\delta}^{0,0}$ and $\overline{\delta}^{0,0}$ are given by*

$$\underline{\delta}^{0,0}(x) = \begin{cases} 0 & \text{, if } x < 1\,, \\ 1 & \text{, if } x = 1\,, \end{cases}$$

and $\overline{\delta}^{0,0}(x) = x$ for any $x \in [0,1]$. The corresponding biconic aggregation functions are respectively the smallest t-norm, i.e. $A^{0,0}_{\underline{\delta}^{0,0}} = T_{\mathbf{D}}$, and the greatest t-norm, i.e. $A^{0,0}_{\overline{\delta}^{0,0}} = T_{\mathbf{M}}$.

Example 3 *The functions $\underline{\delta}^{1,1}$ and $\overline{\delta}^{1,1}$ are given by $\underline{\delta}^{1,1}(x) = x$ for any $x \in [0,1]$ and*

$$\overline{\delta}^{1,1}(x) = \begin{cases} 1 & \text{, if } x > 0\,, \\ 0 & \text{, if } x = 0\,. \end{cases}$$

The corresponding biconic aggregation functions are respectively the smallest t-conorm, i.e. $A^{1,1}_{\underline{\delta}^{1,1}}(x,y) = \max(x,y)$, and the greatest t-conorm, i.e. $A^{1,1}_{\overline{\delta}^{1,1}}(x,y) = \max(x,y)$ whenever $\min(x,y) = 0$, and $A^{1,1}_{\overline{\delta}^{1,1}}(x,y) = 1$ elsewhere.

Example 4 *The functions $\underline{\delta}^{1,0}$, $\overline{\delta}^{1,0}$, $\underline{\delta}^{0,1}$ and $\overline{\delta}^{0,1}$ all coincide with $\delta_{\mathbf{M}}$. The corresponding biconic aggregation functions coincide with the projection to the first and second coordinate* [2, 3], i.e. $A^{1,0}_{\underline{\delta}^{1,0}}(x,y) = A^{1,0}_{\overline{\delta}^{1,0}}(x,y) = x$ and $A^{0,1}_{\underline{\delta}^{0,1}}(x,y) = A^{0,1}_{\overline{\delta}^{0,1}}(x,y) = y$.

Evidently, any biconic function $A^{\alpha,\beta}_{\delta}$ is continuous if and only if δ is continuous. The functions $\underline{\delta}^{\alpha,\beta}$ and $\overline{\delta}^{\alpha,\beta}$ need not to be continuous in general. In fact, the only case in which they are both continuous is when

$$\max(\alpha,\beta) = 1 \text{ and } \min(\alpha,\beta) = 0\,.$$

However, as Example 4 shows, it then holds that

$$\underline{\delta}^{\alpha,\beta} = \delta = \overline{\delta}^{\alpha,\beta} = \delta_{\mathbf{M}}\,,$$

and $A^{\alpha,\beta}_{\delta}$ coincides with one of the projections.

Proposition 4 *Let $\delta \in \mathcal{D}_A$. The function $A^{\alpha,\beta}_{\delta}$ defined in (1)*

(i) *is symmetric, i.e. $A^{\alpha,\beta}_{\delta}(x,y) = A^{\alpha,\beta}_{\delta}(y,x)$ for any $(x,y) \in [0,1]^2$ if and only if $\alpha = \beta$;*
(ii) *has 0 as annihilator if and only if $\alpha = \beta = 0$;*
(iii) *has 1 as neutral element if and only if $\alpha = \beta = 0$.*

From here on, we will only consider biconic functions with a given diagonal section that have 1 as neutral element, i.e. $\alpha = \beta = 0$. We then abbreviate $A^{0,0}_{\delta}$ as A_δ. In this case, A_δ is symmetric and is given by

$$A_\delta(x,y) = \begin{cases} y\,\dfrac{\delta(v)}{v} & \text{, if } y \leq x\,, \\ x\,\dfrac{\delta(u)}{u} & \text{, otherwise}\,. \end{cases} \tag{3}$$

Suppose that the diagonal section of a biconic aggregation function A_δ contains a (linear) segment with endpoints $(x_1, \delta(x_1))$ and $(x_2, \delta(x_2))$. From the definition of a biconic function with a given diagonal section, it follows that A_δ is linear on the triangle $T_1 := \Delta_{\{(x_1,x_1),(x_2,x_2),(1,0)\}}$ as well as on the triangle $T_2 := \Delta_{\{(x_1,x_1),(x_2,x_2),(0,1)\}}$. This situation is depicted in Figure 1. For any $(x,y) \in T_1$, it

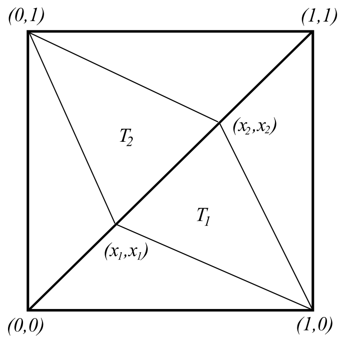

Figure 1: Illustration for the triangles T_1 and T_2.

holds that

$$A_\delta(x,y) = ax + by + c\,. \tag{4}$$

Furthermore,

$$\begin{aligned} ax_1 + bx_1 + c &= \delta(x_1) \\ ax_2 + bx_2 + c &= \delta(x_2) \\ a + c &= 0\,. \end{aligned}$$

Solving this system of equations and using the symmetry of A_δ, we obtain

$$A_\delta(x,y) = \begin{cases} \dfrac{rx + sy - r}{t} & \text{, if } (x,y) \in T_1\,, \\[2ex] \dfrac{sx + ry - r}{t} & \text{, if } (x,y) \in T_2\,, \end{cases} \tag{5}$$

where

$$\begin{aligned} r &= x_1\delta(x_2) - x_2\delta(x_1) \\ s &= (1-x_1)\delta(x_2) - (1-x_2)\delta(x_1) \\ t &= x_2 - x_1\,. \end{aligned}$$

2.2. Biconic semi-copulas with a given diagonal section

In this subsection we characterize the set of functions in $\mathcal{D}_S$ for which the corresponding biconic function is a semi-copula.

Lemma 1 *Let $\delta \in \mathcal{D}_S$. The function $\lambda_\delta : [0,1[\, \to [0,\infty[\,$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{1-x}$, is increasing;*

Proposition 5 *Let $\delta \in \mathcal{D}_S$. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3) is a semi-copula if and only if the function $\mu_\delta :]0,1] \to [0,\infty[\,$, defined by $\mu_\delta(x) = \frac{\delta(x)}{x}$, is increasing.*

Example 5 *Consider the diagonal functions $\delta_{\mathbf{M}}$ and $\delta_{\mathbf{L}}$ with $\delta_{\mathbf{L}}$ being the diagonal section of $T_{\mathbf{L}}$, i.e. $\delta_{\mathbf{L}}(x) = \max(2x-1, 0)$ for any $x \in [0,1]$. Clearly, the functions $\mu_{\delta_{\mathbf{M}}}$ and $\mu_{\delta_{\mathbf{L}}}$, defined in Proposition 5, are increasing. The corresponding biconic semi-copulas are respectively $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.*

Example 6 *Consider the diagonal function $\delta(x) = x^{1+\theta}$ with $\theta \in [0,1]$. Clearly, the function μ_δ, defined in Proposition 5, is increasing for any $\theta \in [0,1]$. The corresponding family of biconic semi-copulas is given by*

$$A_\theta(x,y) = \begin{cases} \dfrac{y^{1+\theta}}{(1+y-x)^\theta} & \text{, if } y \le x\,, \\[2ex] \dfrac{x^{1+\theta}}{(1+x-y)^\theta} & \text{, otherwise}\,. \end{cases}$$

Proposition 6 *Let A_δ be a biconic semi-copula and suppose that $\delta(x_0) = x_0$ for some $x_0 \in]0,1[\,$. Then it holds that $\delta(x) = x$ for any $x \in [x_0, 1]$.*

2.3. Biconic quasi-copulas with a given diagonal section

An interesting class of aggregation functions is the class of quasi-copulas. Quasi-copulas are of increasing importance in various studies in fuzzy set theory, such as preference modelling and similarity measurement [8, 9].

Next we characterize the diagonal functions for which the corresponding biconic function is a quasi-copula.

Lemma 2 *Let $\delta \in \mathcal{D}$. Then it holds that*

(i) *the function $\nu_\delta :]0,1] \to [2,\infty[\,$, defined by $\nu_\delta(x) = \frac{1+\delta(x)}{x}$, is decreasing;*
(ii) *the function $\phi_\delta : [0,1/2[\cup]1/2,1] \to \mathbb{R}$, defined by $\phi_\delta(x) = \frac{\delta(x)}{1-2x}$, is increasing on the interval $[0,1/2[$ and on the interval $]1/2,1]$.*

Proposition 7 *Let $\delta \in \mathcal{D}$. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3) is a quasi-copula if and only if*

(i) *the function μ_δ, defined in Proposition 5, is increasing;*
(ii) *the function $\xi_\delta : [0,1[\, \to [0,1[\,$, defined by $\xi_\delta(x) = \frac{x-\delta(x)}{1-x}$, is increasing.*

Example 7 *Consider the diagonal function in Example 6. Clearly, the functions μ_δ and ξ_δ, defined in Propositions 5 and 7, are increasing. The corresponding family of biconic semi-copulas is a family of biconic quasi-copulas.*

Example 8 *Consider the diagonal function δ defined by*

$$\delta(x) = \begin{cases} 0 & \text{, if } x \le \frac{1}{6}\,, \\[1ex] 2x - \dfrac{1}{3} & \text{, if } \frac{1}{6} \le x \le \frac{1}{4}\,, \\[1ex] \dfrac{2}{3}x & \text{, if } \frac{1}{4} \le x \le \frac{3}{4}\,, \\[1ex] 2x - 1 & \text{, otherwise}\,. \end{cases}$$

Clearly, the function μ_δ, defined in Proposition 5, is increasing. Note also that the function ξ_δ, defined in Proposition 7, is not increasing. Hence, the corresponding biconic semi-copula is a proper semi-copula. Consequently, the class of biconic quasi-copulas with a given diagonal section is a proper subclass of the class of biconic semi-copulas with a given diagonal section.

Proposition 8 *Let A_δ be a biconic quasi-copula. Then it holds that*

(i) *if $\delta(x_0) = x_0$ for some $x_0 \in]0,1[\,$, then $A_\delta = T_{\mathbf{M}}$;*
(ii) *if $\delta(x_0) = 2x_0 - 1$ for some $x_0 \in [1/2, 1[\,$, then $\delta(x) = 2x - 1$ for any $x \in [x_0, 1]$.*

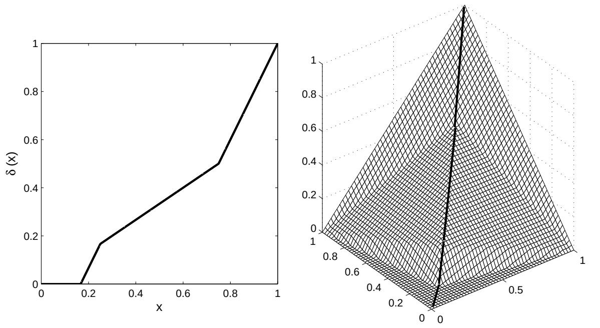

Figure 2: The diagonal function and the corresponding biconic semi-copula of Example 8.

2.4. Biconic copulas with a given diagonal section

Another relevant class of aggregation functions is the class of copulas. Due to Sklar's theorem [27], copulas have received increasing attention from researchers in statistics and probability theory [19]. We denote the *(linear) segment* with endpoints $\mathbf{x}, \mathbf{y} \in [0,1]^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\theta \mathbf{x} + (1-\theta)\mathbf{y} \mid \theta \in [0,1]\}\,.$$

Proposition 9 *Let δ be a piecewise linear diagonal function. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3) is a copula if and only if δ is convex.*

Lemma 3 *Let C_δ be a biconic copula and $m_1, m_2 \in\,]-\infty, 0[$ such that $m_1 > m_2$. Consider three points $\mathbf{b}_1 := (x_1, x_1)$, $\mathbf{b}_2 := (x_2, x_2)$ and $\mathbf{b}_3 := (x_3, x_3)$ such that $0 \le x_1 < x_2 < x_3 \le 1$ and the segments $\langle \mathbf{b}_1, (1,0)\rangle$, $\langle \mathbf{b}_2, (1,0)\rangle$ and $\langle \mathbf{b}_3, (1,0)\rangle$ have slopes m_1, $-\sqrt{m_1 m_2}$ and m_2, respectively. Then it holds that*

(i) *there exists a rectangle $[x, x'] \times [y, y']$ such that the segment connecting the points (x, y') and (x', y) is a subset of the segment $\langle \mathbf{b}_2, (1,0)\rangle$ and the points (x, y) and (x', y') are located on the segments $\langle \mathbf{b}_1, (1,0)\rangle$ and $\langle \mathbf{b}_3, (1,0)\rangle$ respectively.*

(ii) *the point $(x_2, \delta(x_2))$ lies below or on the segment $\langle (x_1, \delta(x_1)), (x_3, \delta(x_3))\rangle$.*

Lemma 3 and Proposition 9 are used to show that for any convex diagonal function, the function A_δ defined in (3) is a copula.

Proposition 10 *Let $\delta \in \mathcal{D}$. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3) is a copula if and only if δ is convex.*

Example 9 *Consider the diagonal function in Example 6. Clearly, δ is convex for any $\theta \in [0,1]$. The corresponding family of biconic semi-copulas is a family of biconic copulas.*

Example 10 *Consider the diagonal function given by $\delta(x) = \frac{x^2}{1-\theta(1-x)^2}$ with $\theta \in [-1,1]$. Clearly, δ is convex for any $\theta \in [-1,1]$. The corresponding family of biconic copulas is given by*

$$C_\theta(x,y) = \begin{cases} \dfrac{y^2(1+y-x)}{(1+y-x)^2 - \theta(1-x)^2} & \text{, } \textit{if } y \le x\,, \\[2ex] \dfrac{x^2(1+x-y)}{(1+x-y)^2 - \theta(1-y)^2} & \text{, } \textit{otherwise}\,. \end{cases}$$

Example 11 *Consider the diagonal function δ given by*

$$\delta(x) = \begin{cases} 0 & \text{, } \textit{if } x \le \frac{1}{4}\,, \\[1ex] \dfrac{1}{3}(4x-1) & \text{, } \textit{if } \frac{1}{4} \le x \le \frac{2}{5}\,, \\[1ex] \dfrac{1}{2}x & \text{, } \textit{if } \frac{2}{5} \le x \le \frac{2}{3}\,, \\[1ex] 2x-1 & \text{, } \textit{otherwise}\,. \end{cases}$$

Clearly, the functions μ_δ and ξ_δ, defined in Propositions 5 and 7, are increasing. Note also that δ is not convex. Hence, A_δ is a proper biconic quasi-copula. Consequently, the class of biconic copulas with a given diagonal section is a proper subclass of the class of biconic quasi-copulas with a given diagonal section.

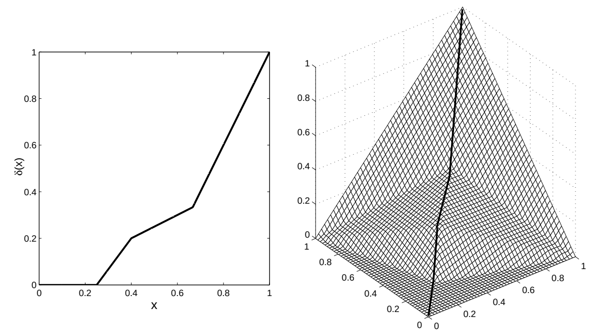

Figure 3: The diagonal function and the corresponding biconic quasi-copula of Example 11.

In the following lemma the opposite symmetry property of a biconic copula with a given diagonal section is studied.

Proposition 11 *A biconic copula C_δ is opposite symmetric if and only if the function $f(x) = x - \delta(x)$ is symmetric with respect to the point $(1/2, 1/2)$, i.e. $\delta(x) - \delta(1-x) = 2x - 1$ for any $x \in [0, 1/2]$.*

Next we characterize the class of singular biconic copulas with a given diagonal section. The support of a copula C is the complement of the union of all non-degenerated open rectangles of the unit square such that the C-volume of the closed rectangle is equal to zero. A copula C is called singular if its support has Lebesgue measure zero.

Proposition 12 *Let C_δ be a biconic copula. Then it holds that C_δ is singular if and only if δ is piecewise linear.*

Example 12 *The family of biconic copulas given in (7) is a family of singular biconic copulas.*

We focus now on associative biconic copulas with a given diagonal section and conclude that the only two associative biconic copulas with a given diagonal section are $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$. Every 1-Lipschitz t-norm is a copula, while every associative copula is a t-norm (the commutativity can be obtained from the continuity of a copula [22]).

Proposition 13 *$T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are the only biconic associative copulas (1-Lipschitz t-norms) with a given diagonal section.*

We conclude this subsection by finding the intersection between the set of biconic copulas with a given diagonal section and the set of conic copulas. Conic copulas were introduced in [21] and their construction was based on linear interpolation on segments connecting the upper boundary curve of the zero-set and the point $(1,1)$. In other words, the surface of any conic copula is constituted from its zero-set and segments connecting the upper boundary curve of its zero-set to the point $(1,1,1)$. The *zero-set* Z_C of a copula C is the inverse image of the value 0, i.e.

$$Z_C := C^{-1}(\{0\}) = \left\{(x,y) \in [0,1]^2 \mid C(x,y) = 0\right\} .$$

Lemma 4 *Let $\delta \in \mathcal{D}$ and suppose that $\theta = 1 - 2^{-\frac{1}{\theta_1}}$, with $\theta_1 \in [1,\infty[$, is the maximum value such that $\delta(\theta) = 0$. Then the biconic copula C_δ has the zero set Z_{C_δ} given by*

$$Z_{C_\delta} = \{(x,y) \in [0,1]^2 \mid y \le f_{\theta_1}(x)\}$$

where the function $f_{\theta_1} : [0,1] \to [0,1]$ is given by

$$f_{\theta_1}(x) = \begin{cases} (1-2^{\frac{1}{\theta_1}})^{-1}x + 1 & , \textit{if } x \le 1 - 2^{-\frac{1}{\theta_1}} , \\ (1-2^{\frac{1}{\theta_1}})(x-1) & , \textit{if } x \ge 1 - 2^{-\frac{1}{\theta_1}} . \end{cases} \tag{6}$$

Due to the above lemma and the definition of a conic copula, the following proposition is obvious.

Proposition 14 *Let C_δ be a biconic copula and suppose further that C_δ is a conic copula. Then it holds that*

$$C_\delta(x,y) = \begin{cases} \max(y + (1-x)(1-2^{\frac{1}{\theta_1}}), 0) & , \textit{if } y \le x , \\ \max(x + (1-y)(1-2^{\frac{1}{\theta_1}}), 0) & , \textit{otherwise} , \end{cases} \tag{7}$$

with $\theta_1 \in [1,\infty[$. This family of copulas was introduced in [21].

3. Biconic functions with a given opposite diagonal section

In this section we introduce biconic functions with a given opposite diagonal section. Their construction is based on linear interpolation on segments connecting the opposite diagonal of the unit square and the points $(0,0)$ and $(1,1)$. For any $(x,y) \in [0,1]^2$, we introduce in this section the following notations

$$u' = \frac{x}{x+y} , \quad v' = \frac{1-y}{2-x-y} .$$

Let $\omega : [0,1] \to [0,1]$ and $\alpha, \beta \in [0,1]$. The function $A_\omega^{\alpha,\beta} : [0,1]^2 \to [0,1]$ defined by $A_\omega^{\alpha,\beta}(x,y) =$

$$\begin{cases} \alpha(1-x-y) + x\,\dfrac{\omega(u')}{u'} & , \text{if } x+y \le 1 , \\ \beta(x+y-1) + (1-y)\,\dfrac{\omega(v')}{v'} & , \text{otherwise} . \end{cases} \tag{8}$$

is well defined. This function is called *a biconic function with a given opposite diagonal section.* Evidently, the boundary conditions of an aggregation function imply that $\alpha = 0$ and $\beta = 1$. We then abbreviate $A_\omega^{0,1}$ as A_ω with $A_\omega(x,y) =$

$$\begin{cases} x\,\dfrac{\omega(u')}{u'} & , \text{if } x+y \le 1 , \\ x+y-1 + (1-y)\,\dfrac{\omega(v')}{v'} & , \text{otherwise} . \end{cases} \tag{9}$$

Clearly, the function A_ω defined in (9) has 1 as neutral element. Therefore, if A_ω is an aggregation function then it is also a semi-copula.

In the next proposition, we characterize the functions in $\mathcal{O}_S$ for which the corresponding biconic function is a biconic aggregation function.

Proposition 15 *Let $\omega \in \mathcal{O}_S$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (9) is an aggregation function if and only if*

(i) *the functions $\mu_\omega, \rho_\omega :]0,1] \to [0,1]$, defined by $\mu_\omega(x) = \frac{\omega(x)}{x}$, $\rho_\omega(x) = \frac{1-\omega(x)}{x}$, are decreasing;*
(ii) *the functions $\lambda_\omega, \xi_\omega : [0,1[\to [0,1]$, defined by $\lambda_\omega(x) = \frac{\omega(x)}{1-x}$ and $\xi_\omega = \frac{x-\omega(x)}{1-x}$, are increasing.*

Let C be a quasi-copula (resp. copula) with opposite diagonal section ω. The function C', defined by $C'(x,y) = x - C(x,1-y)$, is again a quasi-copula (resp. copula) whose diagonal section $\delta_{C'}$ is given by $\delta_{C'}(x) = x - \omega(x)$. This transformation permits to derive in a straightforward manner the conditions that have to be satisfied by an opposite diagonal function to obtain a biconic quasi-copula (resp. copula), which has that opposite diagonal function as opposite diagonal section.

Proposition 16 *Let $\omega \in \mathcal{O}$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (9) is a quasi-copula if and only if the functions μ_ω and λ_ω, defined in Proposition 15, are respectively decreasing and increasing.*

Proposition 17 *Let $\omega \in \mathcal{O}$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (9) is a copula if and only if ω is concave.*

Example 13 *Consider the opposite diagonal functions $\omega_{\mathbf{M}}(x) = \min(x, 1-x)$ and $\omega_{\mathbf{L}}(x) = 0$. Obviously, $\omega_{\mathbf{M}}$ and $\omega_{\mathbf{L}}$ are concave functions. The corresponding biconic copulas are respectively $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.*

Example 14 *Consider the opposite diagonal function $\omega_\Pi(x) = x(1-x)$. Obviously, ω_Π is concave. The corresponding biconic copula is given by*

$$C_{\omega_\Pi}(x,y) = \begin{cases} x(1-u') & , \text{ if } x+y \le 1\,, \\ \\ x-(1-y)v' & , \text{ otherwise}\,. \end{cases}$$

We focus now on the symmetry and opposite symmetry properties of biconic copulas with a given opposite diagonal section.

Proposition 18 *Let C_ω be a biconic copula. Then it holds that*

(i) *C_ω is opposite symmetric;*
(ii) *C_ω is symmetric if and only if ω is symmetric with respect to the point $(1/2, 1/2)$, i.e. $\omega(x) = \omega(1-x)$ for any $x \in [0, 1/2]$.*

We conclude this section by finding the intersection between the class of biconic copulas with a given opposite diagonal section and the class of biconic copulas with a given diagonal section and the class of conic copulas.

Proposition 19 *Let C be a biconic copula with a given opposite diagonal section and suppose further that C is a biconic copula with a given diagonal section. Then it holds that C is a member of the following family*

$$\theta T_{\mathbf{M}} + (1-\theta) T_{\mathbf{L}} \qquad \text{with } \theta \in [0,1]\,.$$

Let C_ω be a biconic copula. Due to the definition of C_ω, the only possible zero-sets are

$$Z_{C_\omega} = Z_{T_{\mathbf{M}}} = [0,1]^2 \setminus]0,1]^2$$

and

$$Z_{C_\omega} = Z_{T_{\mathbf{L}}} = \{(x,y) \in [0,1]^2 \mid x+y \le 1\}\,.$$

Recalling that every conic copula is uniquely determined by its zero-set [21], the following proposition is clear.

Proposition 20 *Let C_ω be a biconic copula with given opposite diagonal section ω and suppose further that C_ω is a conic copula. Then it holds that $C_\omega = T_{\mathbf{M}}$ or $C_\omega = T_{\mathbf{L}}$.*

4. Conclusion

We have introduced biconic aggregation functions with a given diagonal (resp. opposite diagonal) section. We have also characterized the classes of biconic semi-copulas, quasi-copulas and copulas with a given diagonal (resp. opposite diagonal) section. The t-norms $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ turn out to be the only 1-Lipschitz biconic t-norms with a given diagonal section. Moreover, a copula that is a biconic copula with a given diagonal section as well as with a given opposite diagonal section turns out to be a convex combination of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.

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