Functionally Expressible Multidistances

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Abstract

In this paper we deal with the problem of aggregating pairwise distance values in order to obtain a multi-argument distance function. After introducing the concept of functionally expressible multidistance, several essential types of multidimensional aggregation functions are considered to construct such kind of multidistances. An example of non functionally expressible multidistance is exhibited.

Keywords: Distance, multidistance, multidimensional aggregation functions, smallest enclosing ball.

1. Introduction

Given a non-empty set X, a distance (metric) d on X is a function that distinguish between every two different points x and y of X by assigning to the ordered pair (x, y), in a symmetric manner, a single positive real number, $d(x, y)$, in such a way that, given any point z of X, $d(x, y)$ does not exceed the sum of $d(x, z)$ and $d(z, y)$, and we always set $d(x, x) = 0$. These axioms arise when one examines the fundamental properties of the point-to-point-along-a straight-line-segment prototype with a view to developing a theory around those properties, a theory that will then be applicable in many situations, some very different from that of the prototype. The conventional definition of distance over a space specifies properties that must be obeyed by any measure of "how separated" two points in that space are.

However often one wants to measure *how separated* the members of a collection of more than two elements are. The usual way to do this is to combine the pairwise distance values for all pairs of elements in the collection, into an aggregate measure. Thus, given a Euclidean triangle (A, B, C) we can combine the distances AB, AC, BC using, for instance, a 3-dimensional OWA operator, $F(x_1, x_2, x_3) = w_1x_{(1)} + w_2x_{(2)} + w_3x_{(3)}$. Then, we measure "how separated" (A, B, C) are by means of the formula $D(A, B, C) = F(AB, AC, BC)$. It is clear that we have to choose the weighting vector (w_1, w_2, w_3) such that the multi-argument distance function D satisfies a group of axioms that extend to some degree those for ordinary distance functions. We can consider other procedures to measure how separated the vertices (A, B, C) are: $D(A, B, C) = FA + FB + FC$ where F is the Fermat point of a triangle (A, B, C), also called Torricelli point (F is the point for which the sum of the distances from it to the vertices is as small as possible).

In [5, 6] the formal definition of a distance function is extended to apply to collections of more than two elements. The measure presented there applies to n-dimensional ordered lists of elements, and it can be directly incorporated into many domains where ad hoc combinations of pairwise distance values are currently used.

In other previous papers we have introduced and studied some aspects of these multi-argument distance functions, thus in [4] we proposed an extension of the concept "degree of similarity between two elements" in order to be used to measure the similarity between all the element of a finite list of elements. This extension to multiple arguments was called multi-indistinguishability, and we dealt also with its counterpart in the field of metric spaces, namely multidistance.

In 2004, D. H. Wolpert [10] presented the definition of a multi-argument metric (multimetric, for short) in a rather different manner. The measure introduced by Wolpert applies even to collections with "fractional" numbers of elements, however its axiomatic, an extension of the usual conditions defining a metric, is stronger than the one we present here. To be more precise, Wolpert's multimetrics can be viewed as similar to our strong multidistances. In [2, 3, 5], terms like n-distances and multimetrics are introduced in certain contexts, but with a very different meaning with respect to those defined by Wolpert and ourselves.

The present paper is devoted to introduce the class of functionally expressible multidistances. Roughly speaking, a multidistance is functionally expressible if it can be obtained from an ordinary distance function by aggregation of all pairwise distance values. In this way, our main concern is the construction of such multidistances by means of appropriate multidimensional aggregation functions (Section 3).

2. Multidistances

We recall here some definitions, properties and examples related to multidistances. For more details see [5, 6].

Definition 1 *We say that a function* $D : \bigcup_{n\geqslant 1} \mathbb{X}^n \to [0, \infty)$ *is a* multidistance *on a non empty set* $\mathbb{X}$ *when the following properties hold, for all* n *and* $x_1, \dots, x_n, y \in \mathbb{X}$*:*

(m1) $D(x_1,\ldots,x_n)=0$ *if and only if* $x_i=x_j$ *for all* $i,j=1,\ldots n$.

(m2) $D(x_1,\ldots,x_n) = D(x_{\pi(1)},\ldots,x_{\pi(n)})$ *for any permutation* π *of* $1,\ldots,n$,

(m3) $D(x_1,\ldots,x_n)\leqslant D(x_1,y)+\ldots+D(x_n,y)$,

We say that D is a strong *multidistance if it fulfills (m1), (m2) and a third condition, stronger than (m3):*

(m3') $D(\mathbf{x_1},\ldots\mathbf{x_k})\leqslant D(\mathbf{x_1},\mathbf{y})+\ldots+D(\mathbf{x_k},\mathbf{y})$ *for all* $\mathbf{x_1},\ldots,\mathbf{x_k},\mathbf{y}\in\bigcup_{n\geqslant 1}\mathbb{X}^n$.

Here, expressions like $D(\mathbf{x},\mathbf{y})$, that is, the function D applied to two lists $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{X}^n$ and $\mathbf{y}=(y_1,\ldots,y_m)\in\mathbb{X}^m$, have the following meaning:

$$D(\mathbf{x},\mathbf{y})=D(x_1,\ldots,x_n,y_1,\ldots,y_m).$$

Remark 1

i) If D *is a multidistance on* $\mathbb{X}$*, then the restriction of* D *to* $\mathbb{X}^2$, $D|_{\mathbb{X}^2}$*, is an ordinary distance on* $\mathbb{X}$.

ii) An ordinary distance d *on* $\mathbb{X}$ *can be* extended *in order to obtain a multidistance. For example, we can define* $D(x_1,\ldots,x_n)$ *in this way:*

$$D(x_1,\ldots,x_n)=\max\{d(x_i,x_j);\ i<j\}.$$

Then, D *is a multidistance on* $\mathbb{X}$ *such that* $D|_{\mathbb{X}^2}=d$*. This multidistance,* D_M *in the sequel, is strong.*

As in the case of ordinary distances we can state the following.

Proposition 1 *Let* D *and* D' *be multidistances on a set* $\mathbb{X}$.

i) $D+D'$ *is a multidistance on* $\mathbb{X}$.

ii) If $k>0$*, then* kD *is a multidistance on* $\mathbb{X}$.

iii) $\frac{D}{1+D}$ *and* $\min\{1,D\}$ *are also multidistances on* $\mathbb{X}$*, with values in* $[0,1]$.

The following are relevant examples of multidistances. Note that most of them come from combining in some way all pairwise ordinary distance values.

Example 1 *Let* $(\mathbb{X},d)$ *be a metric space.*

- *The* Fermat *multidistance is the function* $D_F\colon\bigcup_{n\geqslant 1}\mathbb{X}^n\to[0,\infty)$ *defined by:*

$$D_F(x_1,\ldots,x_n)=\inf_{x\in\mathbb{X}}\{\sum_{i=1}^n d(x_i,x)\}. \tag{1}$$

- *The* sum–based *multidistances are the functions* $D_\lambda\colon\bigcup_{n\geqslant 1}\mathbb{X}^n\to[0,\infty)$ *defined by*

$$D_\lambda(\mathbf{x})=\begin{cases}0 & \text{if } n=1,\\ \lambda(n)\sum_{i<j}d(x_i,x_j), & \text{if } n\geqslant 2,\end{cases} \tag{2}$$

where:

(i) $\lambda(2)=1$,

(ii) $0<\lambda(n)\leqslant\frac{1}{n-1}$ *for any* $n>2$.

- *Let us consider a triangle of weights* W *as the following:*

$$\begin{array}{ccccccc} & & & \omega_1^1 & & & \\ & & \omega_1^2 & & \omega_2^2 & & \\ & \omega_1^3 & & \omega_2^3 & & \omega_3^3 & \\ \cdots & & \cdots & & \cdots & & \cdots \end{array}$$

with $w_i^j\geqslant 0$ *and* $\sum_{i=1}^j w_i^j=1$.
A function $D_W\colon\bigcup_{n\geqslant 1}\mathbb{X}^n\to[0,\infty)$ *can be defined from this triangle in this way: for all* $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{X}^n$,

$$D_W(\mathbf{x})=\begin{cases}0 & \text{if } n=1,\\ W_n(\overbrace{d(x_1,x_2),\ldots,d(x_{n-1},x_n)}^{\binom{n}{2}}) & \text{if } n\geqslant 2,\end{cases} \tag{3}$$

where W_n *is the OWA operator whose weights are those of the triangle's* $\binom{n}{2}$*–row.*
An OWA–based function like this is a multidistance if and only if $\omega_1^{\binom{n}{2}}<1$*, for all* $n\geqslant 3$.

Three multidistances on $\mathbb{R}^2$

Given a distance d on $\mathbb{R}^2$, the formula:

$$D(P_1,\ldots,P_n)=2\min_{P\in\mathbb{R}}\{\max_{i=1,\ldots,n}\{d(P_i,P)\}\}, \tag{4}$$

provides remarkable examples of multidistances on $\mathbb{R}^2$. Note that $D(P_1,\ldots,P_n)$ is the diameter of the smallest ball containing the points $P_1,\ldots,P_n$.

The smallest circle problem (SCP) is a mathematical problem of computing the smallest circle that contains all the points of a given list in the Euclidean plane (see Fig. 1). This problem was initially proposed by J.J. Sylvester in 1857 [8]. The SCP in the plane is an example of a facility location problem in which the location of a new facility must be chosen to provide service to a number of customers, minimizing the farthest distance that any customer must travel to reach the new facility. Generalization to higher dimensions and more details on this topic can be found in [9].

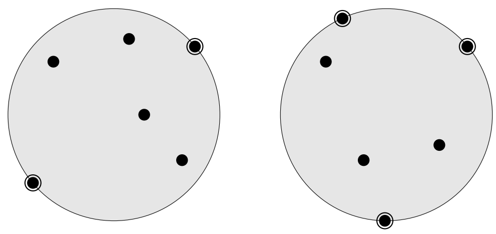

Figure 1: Smallest enclosing balls in the Euclidean plane; two cases.

Now we focus our attention on the Minkowski distance, defined for any points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ in this way:

$$d^p(P_1, P_2) = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{\frac{1}{p}},$$

with $p > 0$. Minkowski distance is typically used with p being 1 or 2. The latter is the Euclidean distance d^2:

$$d^2(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

while the former is sometimes known as the Manhattan distance d^1:

$$d^1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$

In the limiting case of p reaching infinity we obtain the Chebyshev (maximum) distance d^∞:

$$d^\infty(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

The shape of the balls for these three distances is shown in Fig. 2.

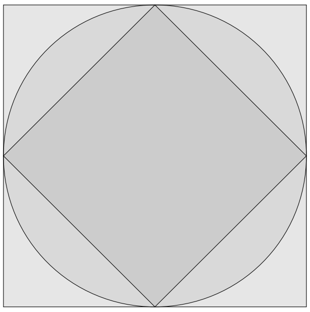

Figure 2: Balls of d^1 (romb), d^2 (circle) and d^∞ (square).

Proposition 2 *The corresponding multidimensional functions D^1, D^2 and D^∞, obtained from the distances d^1, d^2 and d^∞ via expression (4) are multidistances.*

Proof. Let us prove only condition m3 for D^2. There are two different cases, reflected in Fig. 1. The first one is that of the left–ball: there are two diametrically opposed points in the frontier, say P_1, P_2.

$$\begin{aligned} D^2(P_1, \ldots, P_n) &= d^2(P_1, P_2) \\ &\leqslant d^2(P_1, Q) + d^2(P_2, Q) \\ &\leqslant \textstyle\sum_{i=1}^n d^2(P_i, Q), \end{aligned}$$

for all $Q \in \mathbb{R}^2$, and so condition m3 is fulfilled.

The other case is when there are three points in the frontier, P_1, P_2, P_3 for example, such that the triangle $P_1P_2P_3$ is acute. Let R be the radius of its circumscribed circle. We have $D(P_1, \ldots, P_n) = 2R$ and $\sum_{i=1}^n d^2(P_i, Q) \geqslant d^2(P_1, F) + d^2(P_2, F) + d^2(P_3, F)$, where F is the Fermat point of the triangle $P_1P_2P_3$. And so, condition m3 reduces to this inequality:

$$2R \geqslant d^2(P_1, F) + d^2(P_2, F) + d^2(P_3, F),$$

which is a result of the Euclidean Geometry. ■

Note that the above multidistances are extensions of their associated ordinary distances. This is also true for any multidistance obtained from (4). Multidimensional functions D^1, D^2 and D^∞ will be revisited in Section 4.

3. Functionally expressible multidistances

Definition 2 *Let D be a multidistance on a set $\mathbb{X}$ and d an ordinary distance on the same set. We will say that D is functionally expressible from d (or d–functionally expressible) if there exist a function $F\colon \bigcup_{m\geqslant 1}(\mathbb{R}^+)^m \to \mathbb{R}^+$ such that for all $n \geqslant 2$,*

$$\begin{aligned} &D(x_1, \ldots, x_n) \\ &\quad = F(x_1x_2, \ldots, x_ix_j, \ldots, x_{n-1}x_n), \end{aligned} \tag{5}$$

where x_ix_j stands for the distance $d(x_i, x_j)$, for all $1 \leqslant i < j \leqslant n$.

Observe that if D is a d–functionally expressible multidistance then, in particular, $D(x_1, x_2) = F(d(x_1, x_2))$ for all $x_1, x_2 \in \mathbb{X}$ and so, due to the fact that the restriction D_2 of D to $\mathbb{X}^2$ is an ordinary distance (see Remark 1), function F must transform the distance d into D_2. For more details on functions transforming distances into distances, see [1]. From now on, we assume to be $d = D_2$ (in this case we will take $F(a) = a$ for all $a \in \mathbb{R}^+$).

The main problem treated here is to find multidimensional functions F, for a given ordinary distance d on $\mathbb{X}$, which allow to obtain multidistances on $\mathbb{X}$ with the expression (5).

Proposition 3 *Let $(\mathbb{X}, d)$ be an ordinary metric space. If $F\colon \bigcup_{m\geqslant 1}(\mathbb{R}^+)^m \to \mathbb{R}^+$ is a function such that $F(a) = a$ for all $a \in \mathbb{R}^+$, fulfilling for all $m \geqslant 2$ the following conditions:*

(i) $F(a_1, \ldots, a_m) = 0$ *if and only if* $a_1 = \ldots = a_m = 0$,
(ii) F *is symmetric,*
(iii) *if* $(a_{12}, \ldots, a_{ij}, \ldots, a_{n-1n})$ *and* $(b_1, \ldots, b_n)$ *are such that* $a_{ij} \leqslant b_i + b_j$ *for all* i, j, $1 \leqslant i < j \leqslant n$, *then* $F(a_{12}, \ldots, a_{ij}, \ldots, a_{n-1n}) \leqslant b_1 + \ldots + b_n$,

then

$$D(x_1, \ldots, x_n) = F(x_1x_2, \ldots, x_ix_j, \ldots, x_{n-1}x_n),$$

where x_ix_j represents $d(x_i, x_j)$ for all $1 \leqslant i < j \leqslant n$, is a multidistance on $\mathbb{X}$ extending the distance d, functionally expressible by means of F.

Proof. First of all, note that $D(x_1, x_2) = F(d(x_1, x_2)) = d(x_1, x_2)$. Thus, expression (5) extends the distance d. Taking into account axioms of d and under hypothesis i to iii, let us prove conditions m1, m2 and m3 in Definition 1.

(m1) Considering i, we have $D(x_1, \ldots, x_n) = F(x_1x_2, \ldots, x_ix_j, \ldots, x_{n-1}x_n) = 0$ if and only if $x_ix_j = d(x_i, x_j) = 0$ for all $1 \leqslant i < j \leqslant 1$, that is, $x_1 = \ldots = x_n$.

(m2) The symmetry of D follows from condition ii.

(m3) To prove the extended triangle inequality $D(x_1,\dots,x_n) \leqslant \sum D(x_k,y)$, let us denote $a_{ij} = d(x_i,x_j)$, $1 \leqslant i < j \leqslant 1$, and $b_k = d(x_k,y)$, $k = 1,\dots,n$. We have

$$a_{ij} = d(x_i,x_j) \leqslant d(x_i,y) + d(x_j,y) = b_i + b_j;$$

then, according to condition iii we can write

$$\begin{aligned} &D(x_1,\dots,x_n) \\ &\quad \leqslant F(a_{12},\dots,a_{ij},\dots,a_{n-1n}) \\ &\quad \leqslant b_1 + \dots + b_n \\ &\quad = \sum d(x_k,y) \\ &\quad = \sum D(x_k,y). \end{aligned}$$

■

Remark 2

i) The function $F = \max$ fulfills the three conditions of Proposition 3. On the other hand, $F = \min$ does not fulfill condition i, but satisfies ii and iii.

ii) Obviously, conditions in Proposition 3 are not necessary in order to get a multidistance from expression (5). Consider de drastic multidistance:

$$D(x_1,\dots,x_n) = \begin{cases} 0 & \text{if } x_1 = \dots = x_n, \\ 1 & \text{otherwise.} \end{cases}$$

This multidistance is functionally expressible by using the following multidimensional function:

$$F(a_1,\dots,a_m) = \begin{cases} 0 & \text{if } a_1 = \dots = a_m, \\ 1 & \text{otherwise.} \end{cases}$$

Let us see that this function does not fulfill condition iii. Consider for example $(a_{12},a_{13},a_{23}) = (b_1,b_2,b_3) = (0,\frac{1}{3},\frac{1}{3})$. Note that $a_{ij} \leqslant b_i + b_j$ but:

$$F(0,\frac{1}{3},\frac{1}{3}) = 1 \nleqslant 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

iii) If (a,b,c) is a triangle triplet, that is, $a \leqslant b+c$, $b \leqslant a+c$, $c \leqslant a+b$, and F is a symmetric function satisfying condition iii, then $F(a,b,c) \leqslant a+b+c$.

iv) If F satisfies iii, the following must be fulfilled, for all $(b_1,\dots,b_n)$:

$$\begin{aligned} &F(b_1+b_2,\dots,b_i+b_j,\dots,b_{n-1}+b_n) \\ &\quad \leqslant b_1 + \dots + b_n. \end{aligned} \tag{6}$$

In particular, $F(k,\dots,k) \leqslant \frac{nk}{2}$ for all $k \geqslant 0$ (F is applied to k $\binom{n}{2}$ times).

v) Let us observe also that if F is increasing, then condition 6 implies iii.

Proposition 4 *Let $(\mathbb{X},d)$ be an ordinary metric space. If $F\colon \bigcup_{m\geqslant 1}(\mathbb{R}^+)^m \to \mathbb{R}^+$ is a function such that $F(a) = a$ for all $a \in \mathbb{R}^+$, fulfilling for all $m \geqslant 2$ the following conditions:*

(i) $F(a_1,\dots,a_m) = 0$ if and only if $a_1 = \dots = a_m = 0$,

(ii) F is symmetric,

(iii) $F(a_1,\dots,a_m) \leqslant \frac{2}{m+1}(a_1+\dots+a_m)$ if $m \geqslant 3$,

then

$$D(x_1,\dots,x_n) = F(x_1x_2,\dots,x_ix_j,\dots,x_{n-1}x_n),$$

where x_ix_j represents $d(x_i,x_j)$ for all $1 \leqslant i < j \leqslant n$, is a multidistance on $\mathbb{X}$ extending the distance d, functionally expressible by means of F.

Proof. Let us see that this condition iii implies condition iii in Proposition 3.

Consider lists $(a_{12},\dots,a_{ij},\dots,a_{n-1n})$ and $(b_1,\dots,b_n)$ such that $a_{ij} \leqslant b_i + b_j$ for all $1 \leqslant i < j \leqslant n$. Then we have, for all $n \geqslant 3$:

$$\begin{aligned} F(a_{12},\dots,a_{ij},\dots,a_{n-1n}) &\leqslant \tfrac{2}{\binom{n}{2}+1}\textstyle\sum_{i<j} a_{ij} \\ &\leqslant \tfrac{2(n-1)}{\binom{n}{2}+1}\textstyle\sum_{i=1}^k b_i \\ &\leqslant \textstyle\sum_{i=1}^k b_i. \end{aligned}$$

■

Remark 3

i) The function $F = \max$ does not fulfill condition iii in Proposition 4. And $F = \min$ does not fulfill i, but satisfies the other three conditions.

ii) Note that the arithmetic mean $M(a_1,\dots,a_m) = \frac{1}{m}(a_1+\dots+a_m)$ satisfies all of the conditions. Therefore, the function D defined by

$$D(x_1,\dots,x_n) = \tfrac{x_1x_2+\dots+x_ix_j+\dots+x_{n-1}x_n}{\binom{n}{2}}$$

is a multidistance.

A generalization of the arithmetic mean is the family of the so–called power means defined by

$$M_{[r]}(a_1,\dots,a_m) = (\frac{1}{m}\sum_{i=1}^m a_i^r)^{\frac{1}{r}}, \quad r > 0.$$

Note that they are symmetric and take the value 0 only at $(0,\dots,0)$. A further generalization of power means is the family of quasi–arithmetic means:

$$M_f(a_1,\dots,a_m) = f^{-1}(\frac{1}{m}\sum_{i=1}^m f(a_i)),$$

where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and strictly increasing function with $f(0) = 0$. They also satisfy conditions i and ii in Proposition 4 and, under some hypothesis on the generator f, the condition iii holds, as the following shows.

Proposition 5 *Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and strictly increasing function with $f(0) = 0$. If f is concave then the quasi–arithmetic mean*

$$M_f(a_1,\dots,a_m) = f^{-1}(\frac{1}{m}\sum_{i=1}^m f(a_i)),$$

with generator f satisfies condition iii in Proposition 4.

Proof. We know that f is concave if and only if it satisfies the inequality

$$f((1-t)a+tb) \geqslant (1-t)f(a)+tf(b) \qquad (7)$$

for all $a, b \geqslant 0$ and $t \in [0,1]$. It can be extended by induction to more than two summands as follows:

$$\begin{aligned}
&f(\textstyle\sum_{i=1}^{m} \frac{1}{m} a_i) \\
&\quad = f((1-\frac{1}{m}) \sum_{i=1}^{m-1} \frac{\frac{1}{m}}{1-\frac{1}{m}} a_i + \frac{1}{m} a_m) \\
&\quad \geqslant (1-\frac{1}{m}) f(\sum_{i=1}^{m-1} \frac{\frac{1}{m}}{1-\frac{1}{m}} a_i) + \frac{1}{m} f(a_m) \\
&\quad = (1-\frac{1}{m}) f(\sum_{i=1}^{m-1} \frac{1}{m-1} a_i) + \frac{1}{m} f(a_m) \\
&\quad \geqslant (1-\frac{1}{m}) \frac{1}{m-1} \sum_{i=1}^{m-1} f(a_i) + \frac{1}{m} f(a_m) \\
&\quad = \sum_{i=1}^{m} \frac{1}{m} f(a_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_f(a_1, \ldots, a_m) &= f^{-1}(\tfrac{1}{m} \textstyle\sum_{i=1}^{m} f(a_i)) \\
&\leqslant \textstyle\sum_{i=1}^{m} \frac{1}{m} a_i \\
&\leqslant \tfrac{2}{m+1}(a_1 + \ldots + a_m).
\end{aligned}$$

■

Remark 4

i) *Due to the fact that $f(0) = 0$, if f is concave then f is subadditive:*

$$f(a+b) \leqslant f(a) + f(b).$$

Let us prove it. Putting $a = 0$ in (7) we have, for any $b \geqslant 0$,

$$\begin{aligned}
f(b) &= f((1-t) \cdot 0 + tb) \\
&\geqslant (1-t) f(0) + t f(b) \\
&= t f(b).
\end{aligned}$$

Therefore, for any $a, b \geqslant 0$,

$$\begin{aligned}
f(a) + f(b) &= f((a+b)\tfrac{a}{a+b} + (a+b)\tfrac{b}{a+b}) \\
&\geqslant \tfrac{a}{a+b} f(a+b) + \tfrac{b}{a+b} f(a+b) \\
&= f(a+b).
\end{aligned}$$

ii) *Note that if*

$$f(\sum_{i=1}^{m} \frac{1}{m} a_i) \geqslant \sum_{i=1}^{m} \frac{1}{m} f(a_i),$$

then

$$f^{-1}(\frac{1}{m} \sum_{i=1}^{m} b_i) \leqslant \sum_{i=1}^{m} \frac{1}{m} f^{-1}(b_i).$$

This implies that f^{-1} is convex, and thus f is concave.
Therefore, under the above hypothesis on f, $M_f \leqslant M_{[1]}$ if and only if f is concave.

iii) *Basic examples of concave functions are:*
   – $f(t) = t^k$, $0 < k < 1$,
   – $f(t) = \log_k(t+1)$, $k > 1$,
   – $f(t) = \arctan t$, $t \geqslant 0$.

4. Existence of non functionally expressible multidistances

We recall in this section the three multidistances on $\mathbb{R}^2$ defined at the end of Section 2: D^2 ,D^1 and D^∞, as examples of functionally expressible muldidistances.

Also, the existence of multidistances whose values do not depend only on the pairwise distances between the points of the list will be shown.

The multidistance D^2, applied to a list of points $P_1, \ldots, P_n \in \mathbb{R}^2$, give as a result the diameter of the smallest circle containing them. As the relative position of the points is determined by the pairwise distances

$$d^2(P_1, P_2), \ldots, d^2(P_i, P_j), \ldots, d^2(P_{n-1}, P_n),$$

up to isometries, the diameter of the circumcircle also is, and so D^2 is functionally expressible from d^2.

The Chebyshev multidistance D^∞ can be expressed as follows:

$$D^\infty(P_1, \ldots, P_n) = \max\{d^\infty(P_i, P_j), 1 \leqslant i < j \leqslant n\}.$$

Now the balls are squares with sides parallel to the axes and the smallest ball containing the points is not unique. So, $D^\infty(P_1, \ldots, P_n)$ is the diameter of one of the smallest squares containing the points $P_1, \ldots, P_n$. See Fig. 3.

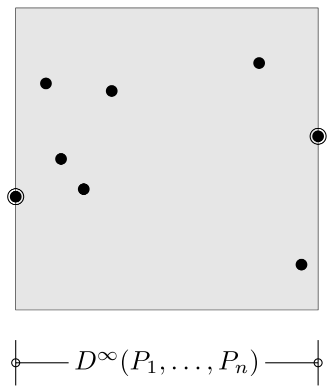

Figure 3: A smallest enclosing ball in the d^∞-plane.

The formula for the Manhattan multidistance D^1 is similar:

$$D^1(P_1, \ldots, P_n) = \max\{d^1(P_i, P_j), 1 \leqslant i < j \leqslant n\}.$$

Also in this case the smallest ball is not unique. The shape is as Fig. 4 shows.

So, D^1 is d^1–functionally expressible, also with $F = \max$.

That is, D^1, D^2 and D^∞ are functionally expressible.

But there exist non–functionally expressible multidistances on $\mathbb{R}^2$. Let us see an example.

Consider the Euclidean plane $(\mathbb{R}^2, d^2)$ and the function $D\colon \bigcup_{n \geqslant 1} (\mathbb{R}^2)^n \to \mathbb{R}^2$ defined in this way:

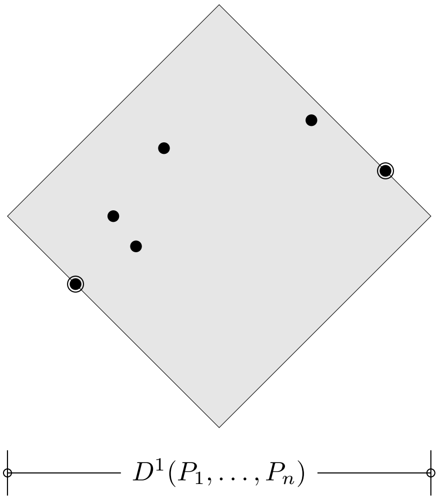

Figure 4: A smallest enclosing ball in the d^1-plane.

$D(P_1, \ldots, P_n)$ is the length of the diagonal of the smallest rectangle, with sides parallel to the axes, containing the points $P_1, \ldots, P_n$. Note that the restriction of D to $(\mathbb{R}^2)^2$ is d^2.

It can be proved that D is a multidistance. But it is not d^2–functionally expressible: if we take, for example, the points $P_1 = (0, 0)$, $P_2 = (0, 1)$ and $P_3 = (1, 0)$, their pairwise distances are $d^2(P_1, P_2) = d^2(P_1, P_3) = 1$, $d^2(P_2, P_3) = \sqrt{2}$, and their multidistance is

$$D(P_1, P_2, P_3) = \sqrt{2}.$$

But if we change the last two ones to $P_2' = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $P_3' = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, the pairwise distances are the same but the multidistance changes:

$$D(P_1, P_2', P_3') = \sqrt{\frac{5}{2}}.$$

So, the value taken by the multidistance is not determined by the pairwise distances, hence D is not d^2–functionally expressible.

5. Conclusions

- The concept of functionally expressible multidistance has been introduced. Some procedures to generate such multidistances has been studied.
- We have dealt with this notion on $\mathbb{R}^2$, equipped with the basic Minkowski distances.
- An example of non–functionally expressible multidistance has been shown.

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