R-implications and the Exchange Principle: A Complete Characterization

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Abstract

It is well-known that the residual I_T of a left-continuous t-norm T satisfies the exchange principle (EP), viz., $I_T(x, I_T(y,z)) = I_T(y, I_T(x,z))$ for all $x,y,z \in [0,1]$. However, the left-continuity of T is only sufficient and not necessary, as many examples in the literature illustrate. In this work we study the necessary conditions on a t-norm T for its residual to satisfy (EP). The work presents a complete characterization of the class of t-norms whose residuals satisfy (EP).

Keywords: R-implication, t-norm, exchange principle, fuzzy implication.

1. Introduction

The family of R-implications is one of the most established classes of fuzzy implications. In fact, one of the earliest methods for obtaining implications was from conjunctions as their residuals, when no additional logical connectives are given. In this way Gödel extended the three-valued implication of Heyting, while discussing the possible relationships between many-valued logic on the one hand, and intuitionistic logic on the other. Residuals of conjunctions on a lattice $\mathcal{L}$, be it from t-norms, uninorms, t-subnorms, copulas, etc., have attracted the most attention from researchers, since they can transform the underlying lattice $\mathcal{L}$ into a residuated lattice. In this article we will consider only R-implications generated from t-norms.

Definition 1.1. A function $I\colon [0,1]^2 \to [0,1]$ is called an *R-implication*, if there exists a t-norm T (see Definition 2.3) such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\},$$

for all $x,y \in [0,1]$. If an R-implication is generated from a t-norm T, then we will often denote it by I_T.

R-implications also have a parallel origin other than its logical foundations. They were also obtained from the study of solutions of systems of fuzzy relational equations and have been known under different names, for example, as a Φ-operator in Pedrycz [9], as T-relative pseudocomplement and α_T-operator in [8].

1.1. A first characterization of R-implications generated from left-continuous t-norms

Sanchez [10] showed that the greatest solution of $\sup - \min$ composition of fuzzy relations is the relation obtained from the residual of min. In fact, Miyakoshi and Shimbo [8] generalized this result to any left-continuous t-norm. They also showed that their α_T-operator is equivalent to the Φ-operator of Pedrycz. Most importantly, they gave the first characterization of R-implications obtained from left-continuous t-norms (for the proof see also [1, Theorem 2.5.17]).

Theorem 1.2. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an R-implication generated from a left-continuous t-norm.

(ii) I is non-decreasing with respect to the second variable, it satisfies the exchange principle, i.e., for all $x,y,z \in [0,1]$

$$I(x, I(y,z)) = I(y, I(x,z)), \tag{EP}$$

it satisfies the ordering property, i.e., for all $x,y \in [0,1]$

$$x \le y \iff I(x,y) = 1, \tag{OP}$$

and I is right continuous with respect to the second variable.

As we see, there are two important axioms of multivalued implications above: (EP) and (OP). The characterization of t-norms, which residuals satisfy the ordering property (OP) have been obtained by Baczyński and Jayaram [2].

Definition 1.3. A function $T\colon [0,1]^2 \to [0,1]$ is said to be *border-continuous*, if it is continuous on the boundary of the unit square $[0,1]^2$, i.e., on the set $[0,1]^2 \setminus]0,1[^2$.

Proposition 1.4 ([2, Proposition 5.8], [1, Proposition 2.5.9])**.** *For a t-norm T the following statements are equivalent:*

(i) T is border-continuous.

(ii) I_T satisfies the ordering property (OP)*.*

Our main goal in this article is to obtain a similar characterization but for the exchange principle, i.e., we want to characterize those t-norms whose residuals satisfy (EP). To see that this condition is different from (OP), let us analyze the following examples.

Example 1.5. (i) Consider the least t-norm, also called the drastic product, given as follows

$$T_{\mathbf{D}}(x, y) = \begin{cases} 0, & \text{if } x, y \in [0, 1[, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Observe that it is a non-left-continuous t-norm. Then the R-implication generated from $T_{\mathbf{D}}$ is given by

$$I_{\mathbf{TD}}(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases}$$

It satisfies (EP), but does not satisfy (OP).

(ii) Consider the non-left-continuous t-norm given in [5, Example 1.24 (i)] as follows

$$T_{\mathbf{B}^*}(x, y) = \begin{cases} 0, & \text{if } (x, y) \in]0, 0.5[^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{B}^*}$ is

$$I_{\mathbf{TB}^*}(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ 0.5, & \text{if } x > y \text{ and } x \in [0, 0.5[, \\ y, & \text{otherwise.} \end{cases}$$

Obviously, $I_{\mathbf{TB}^*}$ satisfies (OP) but not (EP), since

$$I_{\mathbf{TB}^*}(0.4, I_{\mathbf{TB}^*}(0.5, 0.3)) = 0.5,$$

while

$$I_{\mathbf{TB}^*}(0.5, I_{\mathbf{TB}^*}(0.4, 0.3)) = 1.$$

(iii) Consider now the non-left-continuous t-norm T given in [5, Example 1.24 (ii)] as follows:

$$T_{\mathbf{B}}(x,y) = \begin{cases} 0, & \text{if } (x, y) \in]0, 1[^2 \setminus [0.5, 1[^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{B}}$ is

$$I_{\mathbf{TB}}(x,y) = \begin{cases} 1, & \text{if } x \le y \text{ or } x, y \in [0, 0.5[, \\ 0.5, & \text{if } x \in [0.5, 1[\text{ and } y \in [0, 0.5[, \\ y, & \text{otherwise.} \end{cases}$$

It is obvious that $I_{\mathbf{TB}}$ does not satisfy (OP). $I_{\mathbf{TB}}$ also does not satisfy (EP) since

$$I_{\mathbf{TB}}(0.8, I_{\mathbf{TB}}(0.5, 0.3)) = I_{\mathbf{TB}}(0.8, 0.5) = 0.5,$$

while

$$I_{\mathbf{TB}}(0.5, I_{\mathbf{TB}}(0.8, 0.3)) = I_{\mathbf{TB}}(0.5, 0.5) = 1.$$

(iv) Finally, consider the largest t-norm, $T_{\mathbf{M}}(x, y) = \min(x, y)$ whose residual is the Gödel implication

$$I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{if } x > y, \end{cases}$$

which satisfies both (EP) and (OP).

1.2. Left-continuity of T for (EP) of I_T: Sufficient but necessary?

Left-continuity of T is sufficient for I_T to satisfy (EP), but is not necessary. As a counterexample consider the non-left-continuous nilpotent minimum t-norm (see [6, p. 851]):

$$T_{\mathbf{nM}^*}(x, y) = \begin{cases} 0, & \text{if } x + y < 1, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{nM}^*}$ is the following Fodor implication

$$I_{\mathbf{FD}}(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ \max(1 - x, y), & \text{if } x > y, \end{cases}$$

which satisfies both (EP) and (OP). This leads us to the following natural question:

> *What is(are) the most general condition(s) on T to ensure that I_T has* (EP)?

In this work, we take up this study and present a complete characterization of the class of t-norms whose residuals satisfy (EP). Towards this end, we firstly partition the class of t-norms into those that are border-continuous and those that are not and deal with each of them separately.

2. Preliminaries

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we introduce some notations used in the text and we briefly mention some of the concepts and results employed in the rest of the work.

Definition 2.1. A function $I\colon [0, 1]^2 \to [0, 1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable}, \tag{I1}$$

$$I \text{ is increasing in the second variable}, \tag{I2}$$

$$I(0, 0) = 1, \quad I(1, 1) = 1, \quad I(1, 0) = 0. \tag{I3}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Remark 2.2 (see [3, Theorem 7.6])**.** If a function $T\colon [0, 1]^2 \to [0, 1]$ is border-continuous, commutative, monotonic increasing with neutral element 1, then the residual $I_T \in \mathcal{FI}$ and it satisfies (OP).

Definition 2.3. (i) A function $M\colon [0,1]^2 \to [0,1]$ is called a *t-subnorm*, if it is increasing in both variables, commutative, associative and $M(x,y) \le \min(x,y)$ for all $x,y \in [0,1]$.

(ii) A t-norm T is a t-subnorm that has 1 as the neutral element.

Definition 2.4. Two functions $F,G\colon [0,1]^2 \to [0,1]$ form an adjoint pair if they satisfy the residuation property, i.e., for all $x,y,z \in [0,1]$,

$$F(x,z) \le y \iff G(x,y) \ge z. \tag{RP}$$

Theorem 2.5 (cf. [1, Proposition 2.5.2 & Theorem 2.5.7])**.** *If M is a left-continuous t-subnorm, then*

(i) $I_M(x,y) = \max\{t \in [0,1] \mid M(x,t) \le y\}$,
(ii) M *and* I_M *form an adjoint pair,*
(iii) I_M *satisfies* (EP).

Theorem 2.6 ([1, Theorem 2.5.14])**.** *If a function $I\colon [0,1]^2 \to [0,1]$ satisfies* (EP), (OP) *and is both monotonic non-decreasing and right-continuous with respect to the second variable, then T_I defined as below*

$$T_I(x,y) = \min\{t \in [0,1] \mid I(x,t) \ge y\}$$

is a left-continuous t-norm, where the right side exists for all $x,y \in [0,1]$.

Lemma 2.7. *If $T\colon [0,1]^2 \to [0,1]$ is monotonic non-decreasing, commutative and associative, then the function T^* defined as below*

$$T^*(x,y)=\begin{cases}\sup\{T(u,v)|u<x, v<y\}, & \textit{if } x,y \in]0,1[\\ T(x,y), & \textit{otherwise,}\end{cases} \tag{1}$$

for all $x,y \in [0,1]$, is monotonic non-decreasing and commutative. Moreover, T^* is called the conditionally left-continuous completion of T.

Observe firstly that in general T^* may not be left-continuous. For example when $T = T_{\mathbf{D}}$, the drastic t-norm, then $T^* = T$, but $T_{\mathbf{D}}$ is not left-continuous. This explains the word 'conditionally'.

In next example we show that T^* may not satisfy the associativity.

Example 2.8. Consider the following non-left continuous Viceník t-norm given by the formula

$$T_{\mathbf{VC}}(x,y) = \begin{cases} 0.5, & \text{if } \min(x,y) \ge 0.5 \\ & \text{and } x+y \le 1.5, \\ \max(x+y-1,0), & \text{otherwise.}\end{cases}$$

Then the conditionally left-continuous completion of $T_{\mathbf{VC}}$ is given by

$$T^*_{\mathbf{VC}}(x,y) = \begin{cases} 0.5, & \text{if } \min(x,y) > 0.5 \\ & \text{and } x+y < 1.5, \\ \max(x+y-1,0), & \text{otherwise.}\end{cases}$$

One can easily check that $T^*_{\mathbf{VC}}$ is not a t-norm since it is not associative. Indeed, we have

$$T^*_{\mathbf{VC}}(0.55, T^*_{\mathbf{VC}}(0.95,0.95)) = 0.5,$$

while

$$T^*_{\mathbf{VC}}(T^*_{\mathbf{VC}}(0.55,0.95),0.95) = 0.45.$$

Definition 2.9 (cf. [4, Definition 5.7.2])**.** A monotonic non-decreasing, commutative and associative function $T\colon [0,1]^2 \to [0,1]$ is said to satisfy the *(CLCC-A)-property*, if its conditionally left-continuous completion T^*, as defined by (1), is associative.

Remark 2.10. Let T be a t-norm.

(i) By the monotonicity of T we have

$$T^*(x,y) = \begin{cases} T(x^-,y^-), & \text{if } x,y \in]0,1[, \\ T(x,y), & \text{otherwise,}\end{cases}$$

for any $x,y \in [0,1]$, where the value $T(x^-,y^-)$ denotes the left-hand limit.

(ii) T^* has 1 as its neutral element.

(iii) If T is border-continuous, then T^* is left-continuous (in particular it is also border-continuous).

(iv) One can easily check that I_{T^*} is a fuzzy implication.

(v) By the monotonicity of T we have $T^* \le T$ and hence $I_{T^*} \ge I_T$.

(vi) If $x \le y$ then $I_{T^*}(x,y) = I_T(x,y) = 1$.

(vii) Also, if $x = 1$, then by neutrality $I_{T^*}(x,y) = I_T(x,y)$.

3. Border-continuous t-norms

In this section, we consider the class of border-continuous t-norms and determine its sub-class whose residuals satisfy (EP). Note that the t-norm $T_{\mathbf{B}}$ in Example 1.5(iii) is a border-continuous but non-left-continuous t-norm whose residual does not satisfy (EP).

Lemma 3.1. *Let T be a border-continuous t-norm and let I_T satisfy* (EP). *Then $I_T = I_{T^*}$.*

Proof. From formula for T^* and Remark 2.10 we know that $I_T(x,y) = I_{T^*}(x,y)$ when $x \le y$ or $(x,y) \in [0,1]^2 \setminus]0,1[^2$. Therefore assume that there exist $x_0, y_0 \in]0,1[$ such that $x_0 > y_0$ and

$$\beta = I_{T^*}(x_0,y_0) > I_T(x_0,y_0) = \alpha.$$

Since T^* is left-continuous we have that $\beta = I_{T^*}(x_0,y_0) \implies T^*(x_0,\beta) \le y_0$. Thus, $\beta < 1$ and for every $\delta \in (\alpha,\beta)$ we have

$$\begin{aligned} y_0 \ge T^*(x_0,\beta) &= T(x_0^-,\beta^-) \\ &\ge T(x_0^-,\delta) \ge T(x_0^-,\alpha). \end{aligned} \tag{5}$$

Fix arbitrarily $\delta \in (\alpha,\beta)$. Now, we have 2 cases:

1. $\alpha \in \{t | T(x_0, t) \le y_0\}$, in which case

$$T(x_0, \alpha) \le y_0 < T(x_0, \delta).$$

2. $\alpha \notin \{t | T(x_0, t) \le y_0\}$, in which case

$$T(x_0, \alpha^-) \le y_0 < T(x_0, \alpha) \le T(x_0, \delta).$$

From (5) and any of the above 2 cases we have

$$\begin{aligned} &T(x_0^-, \delta) \le y_0 < T(x_0, \delta) \\ &\quad \Longrightarrow I_T(\delta, y_0) = \sup\{t | T(\delta, t) \le y_0\} = x_0. \end{aligned}$$

Now, since I_T satisfies (EP) and (OP) we get

$$\begin{aligned} I_T(x_0, I_T(\delta, y_0)) &= I_T(x_0, x_0) = 1 \\ &= I_T(\delta, I_T(x_0, y_0)) \\ &= I_T(\delta, \alpha), \end{aligned}$$

thus $\delta \le \alpha$, by (OP); a contradiction. Hence $\beta = I_{T^*}(x_0, y_0) = I_T(x_0, y_0) = \alpha$. □

Lemma 3.2. *Let T be a border-continuous t-norm and let I_T satisfy* (EP). *Then T satisfies the (CLCC-A)-property, i.e., its conditionally left-continuous completion T^* is associative.*

Proof. To prove the associativity of T^* we show that T^* is equal to the t-norm $T_{I_{T^*}}$ obtained from its residual I_{T^*}. We prove this in a series of claims.
• The pair (T^*, I_{T^*}) form an adjoint pair, i.e.,

$$T^*(x, z) \le y \iff I_{T^*}(x, y) \ge z.$$

for all $x, y, z \in [0, 1]$.
Since T is border-continuous, T^* is a left-continuous function and assume, that $T^*(x, z) \le y$ for some $x, y, z \in [0, 1]$. This implies, that

$$z \in \{t \in [0, 1] \mid T^*(x, t) \le y\},$$

and hence $I_{T^*}(x, y) \ge z$. On the other side assume, that $z \le I_{T^*}(x, y)$ for some $x, y, z \in [0, 1]$. We consider two cases now. If $z < I_{T^*}(x, y)$, then there exists some $t' > z$ such that $T^*(x, t') \le y$, so by monotonicity $T^*(x, z) \le y$. If $z = I_{T^*}(x, y)$, then either $z \in \{t \in [0, 1] \mid T^*(x, t) \le y\}$ and therefore $T^*(x, z) \le y$, or $z \notin \{t \in [0, 1] \mid T^*(x, t) \le y\}$. Thus there exists an increasing sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_i < z$, $T^*(x, t_i) \le y$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} t_i = z$. By the left-continuity of T^* we get

$$T^*(x, z) = T^*(x, \lim_{i\to\infty} t_i) = \lim_{i\to\infty} T^*(x, t_i) \le y.$$

• I_{T^*} is right-continuous in the second variable.
Let us assume that I_{T^*} is not right-continuous with respect to the second variable in some point $(x_0, y_0) \in [0, 1] \times [0, 1[$. Since I_{T^*} is monotonic in the second variable, there exist $a, b \in [0, 1]$, such that $a > b$ and

$$\begin{aligned} I_{T^*}(x_0, y) &\ge a, \qquad \text{for all } y > y_0, \\ I_{T^*}(x_0, y_0) &= b. \end{aligned}$$

By just showed (RP) for the pair (T^*, I_{T^*}) we get

$$T^*(x_0, a) \le y, \qquad \text{for all } y > y_0.$$

In the limit $y \to y_0$ we have $T^*(x_0, a) \le y_0$. Again from (RP) for the pair (T^*, I_{T^*}) we obtain $b = I_{T^*}(x_0, y_0) \ge a$, a contradiction to $a > b$. Therefore I_{T^*} is a right-continuous function with respect to the second variable.
• The pair $(T_{I_{T^*}}, I_{T^*})$ form an adjoint pair.
This fact follows from [1, Proposition 2.5.13].
• $T_{I_{T^*}}$ is a left-continuous t-norm.
Since T^* is border-continuous, by Remark 2.2 we see that I_{T^*} satisfies (OP). By Lemma 3.1, we obtain that $I_T = I_{T^*}$ and hence I_{T^*} satisfies (EP). Thus, by Theorem 2.6, we get the claim.
• $T^* = T_{I_{T^*}}$.
Fix arbitrarily $x, y \in [0, 1]$. Then

$$\begin{aligned} &I_{T^*}(x, T^*(x, y)) = \\ &\quad = \max\{t \in [0, 1] \mid T^*(x, t) \le T^*(x, y)\} \ge y, \end{aligned}$$

so $T^*(x, y) \in \{t \in [0, 1] \mid I_{T^*}(x, t) \ge y\}$, thus

$$T^*(x, y) \ge T_{I_{T^*}}(x, y). \tag{2}$$

On the other side, since obviously $I_{T^*}(x, z) \ge I_{T^*}(x, z)$, from (RP) for the pair (T^*, I_{T^*}) we get $T^*(x, I_{T^*}(x, z)) \le z$ for all $x, z \in [0, 1]$. Now, if we put $z = T_{I_{T^*}}(x, y)$, then

$$T_{I_{T^*}}(x, y) \ge T^*(x, I_{T^*}(x, T_{I_{T^*}}(x, y))). \tag{3}$$

Further, from (RP) for the pair $(T_{I_{T^*}}, I_{T^*})$ we get also $I_{T^*}(x, T_{I_{T^*}}(x, y)) \ge y$. Using this inequality in (3) and by monotonicity of T^* we have

$$T_{I_{T^*}}(x, y) \ge T^*(x, y). \tag{4}$$

From (2) and (4) we get our claim. □

Theorem 3.3. *For a border-continuous t-norm T the following statements are equivalent:*

(i) I_T *satisfies* (EP).
(ii) T *satisfies the (CLCC-A)-property (i.e., T^* is a associative), and $I_T = I_{T^*}$.*

Proof. (i) $\Longrightarrow$ (ii): Follows from Lemmas 3.2 and 3.1.
(ii) $\Longrightarrow$ (i): If T satisfies the (CLCC-A)-property, then T^* is a left-continuous t-norm. Therefore I_{T^*} satisfies (EP). But $I_T = I_{T^*}$, so I_T also satisfies (EP). □

Using obtained result we able to present the characterization of t-norms, whose residuals satisfy both exchange principle and ordering property.

Corollary 3.4. *For a t-norm T the following statements are equivalent:*

(i) I_T *satisfies* (EP) *and* (OP).
(ii) T *is border-continuous, satisfies the (CLCC-A)-property and $I_T = I_{T^*}$.*

4. Non border-continuous t-norms

In this section, we consider the class of non-border-continuous t-norms and determine its sub-class whose residuals satisfy (EP). Note that the t-norm $T_{\mathbf{B}^*}$ in Example 1.5(ii) is neither border-continuous nor left-continuous and its residual does not satisfy (EP).

Let M be a t-subnorm and T_M the t-norm obtained from M as follows (see [5, Corollary 1.8]):

$$T_M(x,y) = \begin{cases} M(x,y), & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{5}$$

Then, as it can be verified, the corresponding residual is related as follows:

$$I_{T_M}(x,y) = \begin{cases} I_M(x,y), & \text{if } x \neq 1, \\ y, & \text{if } x = 1. \end{cases} \tag{6}$$

The following definition discusses the reverse process of the above, i.e., obtaining a t-subnorm from a t-norm.

Definition 4.1. Let T be a t-norm. The *border-continuous projection* (BCP) of T is the operation $M_T\colon [0,1]^2 \to [0,1]$ obtained from T as follows:

$$M_T(x,y) = \begin{cases} T(x,y), & \text{if } (x,y) \in [0,1[^2, \\ T(x^-,y^-), & \text{otherwise,} \end{cases} \tag{BCP}$$

for all $x, y \in [0,1]$.

Remark 4.2. Let T be a t-norm.

(i) Basically, (BCP) of a T redraws the boundary of the t-norm so that the resulting operation M_T is border-continuous. Note that this is the reverse of lifting a t-subnorm to a t-norm by suitably redefining the boundary.

(ii) Note that the following is true: T is border-continuous if and only if $T = M_T$.

(iii) The conditionally left-continuous completion of the M_T obtained from a T denoted by M_T^* is given by $M_T^*(x,y) = M_T(x^-,y^-)$.

(iv) It can be easily shown that (BCP) of a T is commutative, monotonic and $M_T(1,x) \leq x$. However, note that while M_T is always associative, i.e., M_T is always a t-subnorm, it may not always have the (CLCC-A)-property, i.e., M_T^* is not always associative (see Example 4.3).

Example 4.3. Consider the non-border-continuous t-norm given by (cf. [5, Proposition 3.66])

$$T_{0.5}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.5[^2, \\ 0.5, & \text{if } (x,y) \in [0.5,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

The (BCP) $M_{T_{\mathbf{0.5}}}$ and its conditionally left-continuous completion $M^*_{T_{\mathbf{0.5}}}$ are defined as the following:

$$M_{T_{0.5}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.5[^2, \\ 0.5, & \text{if } (x,y) \in [0.5,1]^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

$$M^*_{T_{0.5}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.5]^2, \\ 0.5, & \text{if } (x,y) \in]0.5,1]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Observe that $M^*_{T_{\mathbf{0.5}}}$ is not associative, since

$$M^*_{T_{0.5}}(0.7, M^*_{T_{0.5}}(0.7,0.5)) = M^*_{T_{0.5}}(0.7,0.5) = 0.5,$$

while

$$M^*_{T_{0.5}}(M^*_{T_{0.5}}(0.7,0.7),0.5) = M^*_{T_{0.5}}(0.5,0.5) = 0.$$

Example 4.4. Consider the non-border-continuous t-norm given by (cf. [5, Example 3.19])

$$T_{\mathbf{Z}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in ([0,0.2[\times[0,1[) \\ & \quad \cup([0,1[\times[0,0.2[), \\ 0.4 + \frac{5}{3}(y-0.4)(x-0.4), & \\ & \text{if } (x,y) \in [0.4,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Figure 1 gives the plots of the t-norm $T_{\mathbf{Z}}$ along with its (BC)-projected $M_{T_{\mathbf{Z}}}$ and its conditionally left-continuous completion $M^*_{T_{\mathbf{Z}}}$ defined as the following:

$$M_{T_{\mathbf{Z}}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in ([0,0.2[\times[0,1]) \\ & \quad \cup([0,1]\times[0,0.2[), \\ 0.4 + \frac{5}{3}(y-0.4)(x-0.4), & \\ & \text{if } (x,y) \in [0.4,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

$$M^*_{T_{\mathbf{Z}}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in ([0,0.2]\times[0,1]) \\ & \quad \cup([0,1]\times[0,0.2]), \\ 0.4 + \frac{5}{3}(y-0.4)(x-0.4), & \\ & \text{if } (x,y) \in]0.4,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

One can easily check that $M^*_{T_{\mathbf{Z}}}$ is associative.

Theorem 4.5. *For a t-norm T the following statements are equivalent:*

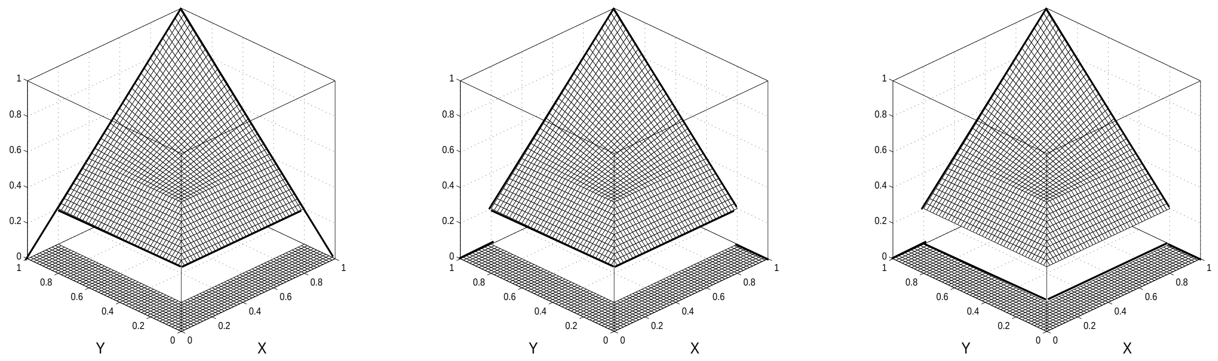
(i) I_T satisfies (EP).

(ii) M_T satisfies (CLCC-A)-property (i.e., M_T^* is associative) and $I_{M_T} = I_{M_T^*}$.

By Remark 4.2(ii), the above result subsumes Theorem 3.3.

Example 4.6. Consider the non-border-continuous t-norm $T_{\mathbf{Z}}$ given in Example 4.4. It can be verified that $I_{M_{T_{\mathbf{Z}}}} = I_{M^*_{T_{\mathbf{Z}}}}$, hence its residual $I_{T_{\mathbf{Z}}}$ does satisfy (EP). In fact, $I_{M_{T_{\mathbf{Z}}}}$ is given by

$$I_{M_{T_{\mathbf{Z}}}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \text{ or } x \in [0,0.2[, \\ 0.2, & \text{if } x > y \ \&\ y \leq 0.2, \\ y, & \text{if } x > y \ \&\ y \in [0.2,0.4[, \\ 0.4 + \frac{3(y-0.4)}{5(x-0.4)}, & \text{if } x > y \ \&\ y \geq 0.4, \end{cases}$$



(a) $T_{\mathbf{Z}}$ (b) $M_{T_{\mathbf{Z}}}$ (c) $M^*_{T_{\mathbf{Z}}}$

Figure 1: The t-norm $T_{\mathbf{Z}}$ along with its (BC)-projected $M_{T_{\mathbf{Z}}}$ and its conditionally left-continuous completion $M^*_{T_{\mathbf{Z}}}$ (see Example 4.4)

and satisfies (EP). Further, we have that

$$I_{T_{\mathbf{Z}}}(x,y) = \begin{cases} I_{M_{T_{\mathbf{Z}}}}(x,y), & \text{if } x \neq 1, \\ y, & \text{if } x = 1. \end{cases}$$

The plots of both functions are given on Figure 2.

5. Ordinal sums of t-norms

Just as there exists a complete representation of continuous t-norms in terms of an ordinal sum representation, see [5, Theorem 5.11], the following representation of left-continuous t-norms as the ordinal sum of t-subnorms can be given.

Theorem 5.1 ([7, Theorem 1]). *A function $T\colon [0,1]^2 \to [0,1]$ is a left-continuous t-norm if and only if there exist a family of pairwise disjoint open sub-intervals $\{]\alpha_k, \beta_k[\}_{k\in\mathcal{K}}$ of $[0,1]$ and a family of left-continuous t-subnorms $(M_k)_{k\in\mathcal{K}}$ such that if either $\beta_k = 1$ for some $k \in \mathcal{K}$ or $\beta_k = \alpha_{k^*}$ for some $k, k^* \in \mathcal{K}$ and M_{k^*} has zero-divisors, then M_k is a t-norm, so that*

$$T(x,y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)\cdot M_k\left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}\right), \\ \qquad\qquad \textit{if } x, y \in]\alpha_k, \beta_k], \\ \min(x,y), \qquad \textit{otherwise.} \end{cases}$$

Theorem 5.2 ([7, Theorem 5]). *If T is a left-continuous t-norm with the ordinal sum structure as given in Theorem 5.1, then*

$$\begin{aligned} I_T(x,y) &= \begin{cases} 1, & \textit{if } x \leq y, \\ \alpha_k + (\beta_k - \alpha_k)\cdot I_{M_k}\left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}\right), \\ & \textit{if } \alpha_k < y < x \leq \beta_k, \\ y, & \textit{otherwise.} \end{cases} \\ &= \begin{cases} \alpha_k + (\beta_k - \alpha_k)\cdot I_{M_k}\left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}\right), \\ \qquad\qquad \textit{if } \alpha_k < y < x \leq \beta_k, \\ I_{\mathbf{GD}}(x,y), \qquad \textit{otherwise,} \end{cases} \end{aligned}$$

where $I_{\mathbf{GD}}$ is the Gödel implication (see Example 1.5(iv)).

Obviously, I_T given in Theorem 5.2 satisfies (EP) and thus the formula in Theorem 5.1 can be used as a construction method for t-norms yielding residual implications possessing (EP). This method of construction (based on left-continuous triangular subnorms) of t-norms for which the residual implication satisfies (EP) can be further generalized, not requiring the left-continuity of single summands in the ordinal sum. We show such a generalization considering t-norms summands only (i.e., we will deal with ordinal sums of t-norms only). Firstly, we consider t-norms obtained as an ordinal sum with a single summand.

Theorem 5.3. *Let T_1 be a t-norm and let $T = (\langle \alpha_1, \beta_1, T_1\rangle)$. Then the following statements are equivalent:*

(i) *I_T satisfies* (EP).
(ii) *I_{T_1} satisfies* (EP) *and if $\beta_1 < 1$ then T_1 is border-continuous.*

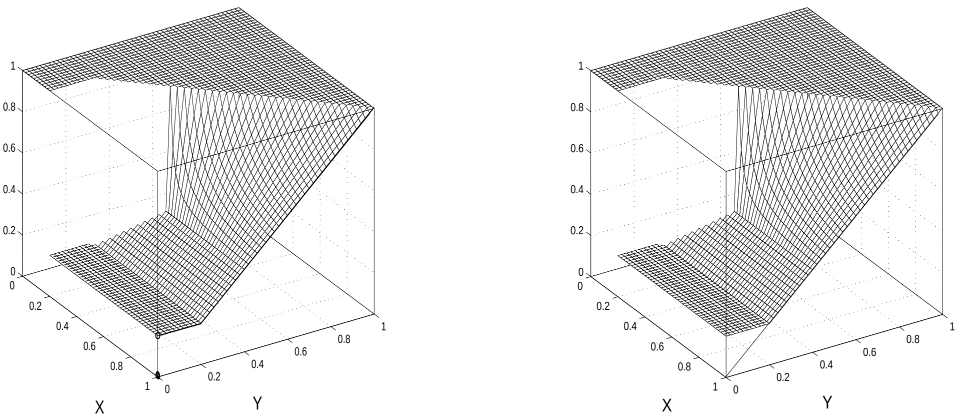
A generalization of the above result to t-norms with countable ordinal summands is straightforward.

Corollary 5.4. *Let $T = (\langle \alpha_k, \beta_k, T_k\rangle)_{k\in\mathcal{K}}$ be an ordinal sum t-norm. Then the following statements are equivalent:*

(i) *I_T satisfies* (EP).
(ii) *For every $k \in \mathcal{K}$, I_{T_k} satisfies* (EP) *and either T_k is border-continuous or $\beta_k = 1$.*

6. Concluding Remarks

In this work we have given a complete characterization of the class of t-norms whose residuals satisfy the exchange principle. The study reveals that the concept of conditionally left-continuous completion of a t-norm plays an important role. In fact, it can



(a) $I_{M_{T_{\mathbf{Z}}}}$ (b) $I_{T_{\mathbf{Z}}}$

Figure 2: The residuals of the t-subnorm $M_{T_{\mathbf{Z}}}$ and $T_{\mathbf{Z}}$ (see Example 4.6)

be seen that unless a t-norm can be embedded into a left-continuous t-norm, in some rather precise manner as presented in the work, its residual does not satisfy the exchange principle.

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References

[1] M. Baczyński and B. Jayaram, *Fuzzy implications*, (Studies in Fuzziness and Soft Computing, Vol. 231), Springer, Berlin, 2008.

[2] M. Baczyński and B. Jayaram, (S,N)- and R-implications: a state-of-the-art survey, *Fuzzy Sets and Systems* 159:1836–1859, 2008.

[3] B. Jayaram and R. Mesiar, I-Fuzzy equivalences and I-fuzzy partitions, *Info. Sci.* 179:1278–1297, 2009.

[4] S. Jenei, A survey on left-continuous t-norms and pseudo t-norms. In: E.P. Klement and R. Mesiar (ed.) *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*, pp. 113–142, Elsevier, Amsterdam, 2005.

[5] E.P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Kluwer, Dordrecht, 2000.

[6] K.C. Maes and B. De Baets, On the structure of left-continuous t-norms that have a continuous contour line. *Fuzzy Sets and Systems* 158:843–860, 2007.

[7] R. Mesiar and A. Mesiarová, Residual implications and left-continuous t-norms which are ordinal sums of semigroups, *Fuzzy Sets and Systems* 143:47–57, 2004.

[8] M. Miyakoshi and M. Shimbo, Solutions of composite fuzzy relational equations with triangular norms, *Fuzzy Sets and Systems* 16:53–63, 1985.

[9] W. Pedrycz, Fuzzy relational equations with triangular norms and their resolutions, *BUSEFAL* 11:24–32, 1982.

[10] E. Sanchez, Resolution of composite fuzzy relation equations. *Info. Control* 30:38–48, 1976.