Qualitative integrals and desintegrals as lower and upper possibilistic expectations

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Abstract

Any capacity (i.e., an increasing set function) has been proved to be a lower possibility measure and an upper necessity measure. Similarly, it is shown that any anti-capacity (i.e., a decreasing set function) can be viewed both as an upper guaranteed possibility measure and as a lower weak necessity measure. These results are the basis for establishing that qualitative integrals (including Sugeno integrals) are lower and /or upper possibilistic expectations wrt a possibility measure, while qualitative desintegrals are upper or lower possibilistic expectations wrt a guaranteed possibility measure. The results are presented in a finite qualitative setting, and apply to multiple criteria aggregation or decision under uncertainty.

Keywords: capacity; Heyting algebras; Sugeno integral; possibility theory.

1. Introduction

Monotonic set functions are a basic representation tool that can be encountered in many areas, in particular in uncertainty modeling, multiple criteria aggregation, group decision and game theory. While quantitative settings rely on weighted sums, qualitative ones use weighted maximum or minimum operators. In a qualitative perspective, it is remarkable that in a finite setting, a capacity (i.e., an increasing set function) can be represented by a conjunction of possibility measures or by a disjunction of necessity measures [1, 2]. Dually, decreasing set functions can also be represented in terms of decreasing max or min decomposable measures. These observations are the starting point in this paper looking for the expression of general qualitative integrals (resp. desintgrals) in terms of the maximum or minimum of finite families of integrals (resp. desintegrals) of the same type with respect to possibility or necessity (resp. guaranteed possibility and weak necessity) measures.

In multi-criteria decision making, Sugeno integrals are commonly used as qualitative aggregation functions [11]. They are counterparts to Choquet integrals that apply to a quantitative setting. The definition of Sugeno integrals depends on a capacity which represent the importance of the subsets of criteria. Importance levels may affect aggregation operators in different ways due to several variants of the Sugeno integrals, named qualitative integrals, when the evaluation scale is a Heyting algebra and the residuum is the Gödel implication [4, 5].

Qualitative integrals are such that the resulting global evaluation increases with the partial evaluations. In such a case the criteria are said to be *positive* (the higher their values, the better the corresponding evaluations). If the global evaluation increases when the partial evaluations decrease the criteria are said to be *negative*. In such a case other variants of the Sugeno integrals, named qualitative desintegrals, can be defined. The capacities are then replaced by fuzzy anti-measures which are decreasing set functions. Besides, it has been recently shown that a Sugeno integral is a lower prioritized maximum [2]. This paper extends this result to qualitative integrals and qualitative desintegrals. It continues a systematic investigation of the qualitative setting for the representation of uncertainty and preferences, which has been started by the authors two years ago.

The paper is structured as follows. The next section is devoted to the representation of qualitative capacities and anti-capacities in terms of possibility and necessity measures, and in terms of guaranteed possibility measures and their duals respectively. The definitions of the qualitative integrals and desintegrals are then recalled in Section 3. For each of them, Section 4 presents equivalent expressions that require the comparison of only n cases (the number of criteria). In Section 5, qualitative integrals are shown to be upper and lower possibility integrals, and before concluding, Section 6 presents similar results for the qualitative desintegrals.

2. Qualitative capacities and anti-capacities

We first recall the qualitative setting of Heyting algebras in which evaluations take place, as well as the finite framework of multiple criteria aggregation. We then restate how a fuzzy measure, or capacity, can be naturally associated to a possibilistic core, thus pointing out a parallel with cooperative game theory. In particular, the minimal elements of this core enable us to obtain a representation of a capacity as a lower possibility measure or as an upper necessity measure. Similar results are established for the support associated to anti-capacities, which are decreasing set functions, with respect to guaranteed possibility measures and their duals. In this paper, these results are then used in multiple criteria aggregation. It should be clear that these results would be also meaningful for decision under uncertainty [3].

2.1. Algebraic framework

We consider a finite set of criteria $S = \{s_1, \cdots, s_n\}$. Objects are evaluated using these criteria. The evaluation scale, L, associated to each criterion is assumed to be totally ordered. It may be finite or be the interval $[0, 1]$. Then an object is represented by its evaluation on the different criteria, i.e., by $f = (f_1, \cdots, f_n) \in L^n$ where f_i is the evaluation of f according to the criterion s_i. In the following the evaluation of f according to the criterion s can also be denoted $f(s)$.

Moreover we consider L as a Heyting algebra i.e., as a complete residuated lattice with a greatest element denoted by 1 and a least element denoted by 0. More precisely, $\langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a complete lattice: $\langle L, \wedge, 1\rangle$ is a commutative monoid (i.e., $\wedge$ is associative, commutative and for all $a \in L$, $a \wedge 1 = a$). The associated residuated implication, denoted by $\rightarrow$ is then the Gödel implication defined by $a \rightarrow b = \bigvee_{a\wedge x\leq b}\ x = 1$ if $a \leq b$ and b otherwise.

In the following, we will consider positive criteria and negative criteria. The criteria are positive when the global evaluation of objects increases with the partial evaluation. When we consider negative criteria, the global evaluation increases when the partial evaluations decrease. Partial evaluations model degrees of defect. In this latter case, 0 will be a good local evaluation, 1 will be a bad evaluation and the scale will be said to be decreasing (the scale is increasing in the case of positive criteria). To play with the directionality of the scale, we also need an operation that reverses it. This operation (a decreasing involution) defined on L is denoted by η, and $\langle L, \wedge, \vee, \eta\rangle$ is a De Morgan algebra.

2.2. Fuzzy measures. Their possibilistic core

A capacity (or fuzzy measure) is set increasing. Formally, it is a mapping $\gamma : 2^S \rightarrow L$ such that $\gamma(\emptyset) = 0$, $\gamma(S) = 1$, and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$. When we consider positive criteria, $\gamma(A)$ represents the importance of the subset A of criteria.

The conjugate $\gamma^c(A)$ of capacity γ is a capacity defined by $\gamma^c(A) = \eta(\gamma(A^c)), \forall A \subseteq S$, where A^c is the complement of subset A.

A special case of capacity is a possibility measure [16, 7] which is a maxitive capacity, i.e., a capacity Π such that $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$. Since S is finite, the possibility distribution π : $\pi(s) = \Pi(\{s\})$ is enough to recover the set-function: $\Pi(A) = \vee_{s\in A}\ \pi(s)$. When modeling uncertainty, the value $\pi(s)$ is understood as the possibility that s be the actual state of the world: $\exists s \in S : \pi(s) = 1$. When modeling priorities, $\pi(s)$ is the importance of criterion s, and normalization means that there is at least one criterion that is fully important. A possibility measure Π_1 is said to be more specific than another possibility measure Π_2 if $\forall A \subset S, \Pi_1(A) \leq \Pi_2(A)$ (equivalently $\forall s \in S, \pi_1(s) \leq \pi_2(s)$). If Π_1 and Π_2 are possibility measures, then $\Pi_1 \vee \Pi_2$ is a possibility measure too, which is less specific than both Π_1 and Π_2 [8].

The conjugate of a possibility measure Π is a necessity measure $N(A) = \eta(\Pi(A^c))$, and then N is a minitive capacity, i.e., $N(A \cap B) = N(A) \wedge\ N(B)$. Moreover, $N(A) = \wedge_{s\notin A}\ \iota(s)$ where $\iota(s) = N(S \setminus \{s\})$ (this is the degree of impossibility of s when dealing with uncertainty), and $\iota(s) = \eta(\pi(s))$, where π defines the conjugate possibility measure $\Pi = N^c$.

There is always at least one possibility measure that dominates any capacity: the vacuous possibility measure, based on the distribution $\pi^?$ expressing ignorance, since then $\forall A \neq \emptyset \subset S, \Pi^?(A) = 1 \geq \gamma(A), \forall$ capacity γ, and $\Pi^?(\emptyset) = \gamma(\emptyset) = 0$. Let

$$S(\gamma) = \{\pi : \Pi(A) \geq \gamma(A), \forall A \subseteq S\}$$

be the set of possibility distributions whose corresponding set-functions Π dominate γ. We call $S(\gamma)$ the *possibilistic core* of the capacity γ.

By analogy with game theory, it may be also viewed as the possibilistic *core* of the capacity γ. Indeed, in game theory, the core, which is the set of feasible allocations p that cannot be improved upon by a coalition, can be defined by the conditions $\sum_{s_i\in A} p(s_i) \geq \gamma(A)$, and $\sum_{s_i\in S} p(s_i) = \gamma(S)$, where γ is now the characteristic function of the game. The possibilistic *core* is thus just a qualitative maxitive counterpart to the additive definition of the core in game theory.

Let us recall some results on the structure of this set of possibility distributions. Let σ be a permutation of the $n = |S|$ elements in S. The ith element of the permutation is denoted by $s_{\sigma(i)}$. Moreover let $S^i_\sigma = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$. Define the possibility distribution π^γ_σ as follows:

$$\forall i = 1 \ldots, n, \pi^\gamma_\sigma(s_{\sigma(i)}) = \gamma(S^i_\sigma) \qquad (1)$$

There are at most $n!$ (number of permutations) such possibility distributions. It can be checked that the possibility measure Π^γ_σ induced by π^γ_σ lies in $S(\gamma)$ and that the $n!$ such possibility distributions enable γ to be reconstructed (as already pointed out by Banon [1]). More precisely, for each permutation σ: $\forall A \subseteq S, \Pi^\gamma_\sigma(A) \geq \gamma(A)$. Moreover, $\forall A \subseteq S, \gamma(A) = \bigwedge_\sigma\ \Pi^\gamma_\sigma(A)$. As a consequence, $\forall \pi \in S(\gamma), \pi(s) \geq \pi^\gamma_\sigma(s), \forall s \in S$ for some permutation σ of S.

This result says that the possibility distributions π^γ_σ (we call them the *marginals* of γ) include the least elements of $S(\gamma)$ in the sense of fuzzy set inclusion, i.e., the most specific possibility distributions dominating γ. In other terms, $S(\gamma) = \{\pi, \exists\sigma, \pi \geq$

$\pi_\sigma^\gamma\}$. Of course the maximal element of $S(\gamma)$ is the vacuous possibility distribution $\pi^?$.

In the qualitative case, $S(\gamma)$ is closed under the qualitative counterpart of a convex combination or mixture: namely, if $\pi_1, \pi_2 \in S(\gamma)$ then $\forall a, b \in L$, such that $a \vee b = 1$, it holds that $(a \wedge \pi_1) \vee (b \wedge \pi_2) \in S(\gamma)$. In fact, $S(\gamma)$ is an upper semi-lattice. Let $\mathcal{C}_*(\gamma) = \min S(\gamma)$ be the set of minimal elements in $S(\gamma)$.

It is interesting to look at the properties of the set of qualitative mixtures that can be built from $\mathcal{C}_*(\gamma)$. Let us denote this set by

$$\mathcal{QM}_{\mathcal{C}_*}(\gamma) = \{ \bigvee_{\pi_i \in \mathcal{C}_*(\gamma)} a_i \wedge \pi_i | \vee_i a_i = 1\}.$$

This is the qualitative counterpart of a credal set (convex set of probabilities). It can be checked that

- $\mathcal{QM}_{\mathcal{C}_*}(\gamma) \subset S(\gamma)$, since in general $\Pi_? \notin \mathcal{QM}_{\mathcal{C}_*}$;
- $\mathcal{QM}_{\mathcal{C}_*}(\Pi) = \{\Pi\}$ for any possibility measure Π;
- $\min \mathcal{QM}_{\mathcal{C}_*}(\gamma) = \mathcal{C}_*(\gamma)$;
- $\max \mathcal{QM}_{\mathcal{C}_*}(\gamma) = \{\pi_\gamma\}$ with $\pi_\gamma = \bigvee_{\pi_i \in \mathcal{C}_*(\gamma)} \pi_i$.

In order to better characterize π_γ, we need to first introduce the (inner) qualitative Mœbius transform $\gamma_\#$ of the capacity γ. It is a mapping $\gamma_\# : 2^S \to L$ defined by [13, 10]

$$\gamma_\#(E) = \gamma(E) \text{ if } \gamma(E) > \bigvee_{B \subsetneq E} \gamma(B)$$

and $\gamma_\#(E) = 0$ otherwise. The qualitative Mœbius transform contains the minimal information needed to reconstruct the capacity γ, and the qualitative counterpart of a belief function [6, 14] based on the basic possibilistic assignment $\gamma_\#$ (note that $\bigvee_E \gamma_\#(E) = 1$ and $\gamma_\#(\emptyset)$=0) is nothing but γ itself, namely

$$\gamma(A) = \bigvee_{E \subseteq A} \gamma_\#(E).$$

Moreover, it can be shown [2] that $\pi_\gamma(s) = \bigvee_{s \in A} \gamma_\#(A)$. Then we have

$$\Pi_\gamma(A) = \bigvee_{E : E \cap A \neq \emptyset} \gamma_\#(E),$$

which expresses that Π_γ is the qualitative counterpart [6, 14] of a Shafer plausibility function [15]. Thus, the maximal element of $\mathcal{QM}_{\mathcal{C}_*}$, i.e., the possibility distribution π_γ, is the contour function of γ.

Besides, it follows from the definition of the possibilistic core that $\gamma(A) = \bigwedge_{\Pi \in S(\gamma)} \Pi(A)$, and thus the following proposition holds [2].

Proposition 1 *Any capacity can be viewed either as a lower possibility measure or as an upper necessity measure:*

$$\gamma(A) = \bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \Pi(A)$$

$$\gamma(A) = \bigvee_{\pi \in \mathcal{C}_*(\gamma^c)} N(A)$$

where $\mathcal{C}_*(\gamma)$ the set of minimal elements in the possibilistic core of γ.

The second result can be obtained by applying the first one to γ^c, and using $\gamma^c(A) = \eta(\gamma(A^c)), \forall A \subseteq S$, the involution of η, together with $N(A^c) = \eta(\Pi(A))$. Moreover, the inner qualitative Mœbius transform $\gamma^c_\#$ of the conjugate defines the outer qualitative Mœbius transforms $\gamma^\#$ of a capacity γ

$$\gamma^\#(A) = \eta(\gamma^c_\#(A^c)),$$

and

$$\gamma^\#(A) = \gamma(A) \text{ if } \gamma(A) < \bigwedge_{A \subset E} \gamma(E)$$

and $\gamma^\#(A) = 1$ otherwise.

2.3. Fuzzy anti-measures and their support

A fuzzy anti-measure (or anti-capacity) is set decreasing, and is formally defined as a mapping $\nu : 2^S \to L$ such that $\nu(\emptyset) = 1$, $\nu(S) = 0$, and if $A \subseteq B$ then $\nu(B) \leq \nu(A)$.

When we consider negative criteria, $\nu(A)$ is the level of tolerance of the subset A of criteria: the greater $\nu(A)$, the less important is A. It makes sense for negative scales: the degrees of defect in the criteria forming set A have to be higher than $\nu(A)$ in order to be considered significant.

The conjugate ν^c of an anti-capacity ν is defined by $\nu^c(A) = \eta(\nu(A^c))$. This is also a fuzzy anti-measure.

A special case of anti-capacity is the guaranteed possibility measure [9] defined by $\Delta(A) = \wedge_{s \in A} \delta(s)$, where δ is a possibility distribution such that $\wedge_s \delta(s) = 0$. In a multiple criteria perspective, $\delta(s)$ is the tolerance level of criterion s. The conjugate ∇ of Δ is the weak necessity measure $\nabla(A) = \eta(\Delta(A^c))$.

Just as capacities have possibilistic cores, an anti-capacity ν has a possibilistic support $\mathcal{S}(\nu)$, defined by

$$\mathcal{S}(\nu) = \{\delta : \Delta(A) \leq \nu(A), \forall A \subseteq S\}.$$

The set $\mathcal{S}(\nu)$ is not empty since there is always at least one guaranteed possibility under any anti-measure based on the following tolerance function t expressing complete intolerance, since then $\forall A \neq \emptyset \subset S, t(A) = 0 \leq \nu(A), \forall$ anti-measure ν, and $t(\emptyset) = \nu(\emptyset) = 1$.

A result dual of Proposition 1 can then be established. It is clear that $\mathcal{S}(\nu)$ is a lower semi-lattice, and that one can restrict the $\bigvee$ and $\bigwedge$ to the maximal elements of $\mathcal{S}(\nu)$ in the following proposition.

Proposition 2

$$\nu(A) = \bigvee_{\delta \in \mathcal{S}(\nu)} \Delta(A).$$

$$\nu(A) = \bigwedge_{\delta \in \mathcal{S}(\nu^c)} \nabla(A).$$

Proof:

For all A, we have $\bigvee_{\delta\in\mathcal{S}(\nu)}\Delta(A)\leq\nu(A)$.

Let σ be a permutation of the $n=|S|$ elements in S. We denote the i-th element of the permutation by $s_{\sigma(i)}$, and let $S^i_\sigma=\{s_{\sigma(i)},\cdots,s_{\sigma(n)}\}$.

Let us define the tolerance $t^\sigma_\nu(i)=\nu(S^i_\sigma)$. The associated guaranteed possibility Δ^σ_ν belongs to $\mathcal{S}(\nu)$.

Let $\mathcal{C}^{i_\sigma}_\sigma$ be the smallest set in the sequence $\{\mathcal{C}^i_\sigma\}_i$ such that $A\subseteq\mathcal{C}^{i_\sigma}_\sigma$. Note that we have $s_{\sigma(i_\sigma)}\in A$. $A\subseteq\mathcal{C}_{i_\sigma}$ entails $\nu(A)\geq\nu(\mathcal{C}^{i_\sigma}_\sigma)=t^\sigma_\nu(i_\sigma)$. Moreover $\Delta^\sigma_\nu(A)=\wedge_{j\in A}t^\sigma_\nu(j)\leq t^\sigma_\nu(i_\sigma)\leq\nu(A)$.

When we consider a set of criteria A, there exists a permutation σ_0 such that $A=S^i_{\sigma_0}$. Hence, $\nu(A)=\nu(S^i_{\sigma_0})=t^{\sigma_0}_\nu(i)$.

Moreover we have $\Delta^{\sigma_0}_\nu(A)=t^{\sigma_0}_\nu(i)\wedge\cdots\wedge t^{\sigma_0}_\nu(n)=\nu(S^i_{\sigma_0})\wedge\cdots\wedge\nu(S^n_{\sigma_0})=\nu(S^i_{\sigma_0})=t^{\sigma_0}_\nu(i)$. So $\nu(A)=\Delta^{\sigma_0}_\nu(A)$ and $\nu(A)\leq\max_{\Delta\in\mathcal{S}(\nu)}\Delta(A)$.

Applying the first result to ν^c, and applying the relations linking ν^c to ν and Δ to ∇ yields the second expression.

3. Qualitative integrals and desintegrals

Last year, we have introduced different variants of a Sugeno integral based on various interpretations of a qualitative weighting system. We have distinguished between integrals which increase when the criteria evaluations increase (positive criteria) and desintegrals which decrease when the criteria evaluations increase (negative criteria). In this section, we restate these different integrals and desintegrals, starting with Sugeno integral.

3.1. Qualitative integrals

When the criteria are assumed to be positive the global evaluation of objects can be calculated using the Sugeno integral or one of its variants. Let us present a brief reminder.

Let $f:S\to L$ be a function that describes a vector of utility values for some object according to several attributes (features, criteria) $s\in S$. Sugeno integral is often defined as follows:

$$\mathcal{S}_\gamma(f)=\bigvee_{\lambda\in L}\lambda\wedge\gamma(F_\lambda)\qquad(2)$$

where $F_\lambda=\{s:f(s)\geq\lambda\}$ is the set of attributes having best ratings for some object, above threshold λ, and $\gamma(A)$ is the degree of importance of feature set A. An equivalent expression is [12]:

$$\mathcal{S}_\gamma(f)=\bigvee_{A\subseteq S}(\gamma(A)\wedge\bigwedge_{s\in A}f(s))$$

In this disjunctive form, the set-function γ can be replaced without loss of information by the inner qualitative Moebius transform $\gamma_\#$ defined earlier.

$$\mathcal{S}_\gamma(f)=\max_{A\in\mathcal{F}^\gamma}\min(\gamma_\#(A),f_A)\qquad(3)$$

where $f_A=\min_{s\in A}f(s)$. The above expression of Sugeno integral has the standard maxmin form, viewing $\gamma_\#$ as a possibility distribution over 2^S.

When γ is a possibility measure Π, Sugeno integral in form (3) simplifies into:

$$\mathcal{S}_\Pi(f)=\max_{s\in S}\min(\pi(s),f(s))\qquad(4)$$

which is the prioritized max, since $\Pi_\#$ and π coincide.

There are two variants of Sugeno integrals that use implication to model the effect of priority weights on utility values $f(s)$.

The first variant interprets a weight $\pi(s)$ as a softening threshold that makes local evaluations $f(s)$ less demanding:

- $f(i)\geq\pi(i)$ is enough to reach full satisfaction:
- $f(i)$ is kept otherwise

this is clearly modeled by Gödel implication, replacing $f(s)$ by $\pi(s)\to f(s)$. The corresponding variant of the Sugeno integral is:

$$\oint^{\uparrow}_\gamma(f)=\bigwedge_{A\subseteq S}\gamma(A)\to\vee_{s\in A}f(s)\qquad(5)$$

If γ is a possibility measure, then $\oint^{\uparrow}_\gamma(f)=\wedge_{s\in S}\pi(s)\to f(s)$ which is a form of prioritized minimum. This is because it can be checked that again $\gamma(A)$ can be changed into $\gamma_\#(A)$ in (5). Note that $\oint^{\uparrow}_\gamma(f)=1$ as soon as for all subsets $A,\exists s\in A,f(s)\geq\gamma(A)$, which means that $\oint^{\uparrow}_\gamma(f)<1$ as soon as $f(s)<1,\forall s\in S$. Likewise $\oint^{\uparrow}_\gamma(f)=0$ as soon as for some subset $A,\forall s\in A,f(s)=0$, that is $f(s)=0$ for some feature s for which $\gamma(\{s\})>0$.

In second variant, when the original rating $f(s)$ is higher than a threshold, the criterion s is considered fully satisfied, but the modified rating is severely decreased otherwise, if the criterion is important, and increased, if not. In other words: if $f(s)\geq\pi(s)$, the rating becomes maximal, i.e. 1, otherwise it is always turned into $\eta(\pi(s))$. The corresponding aggregation scheme is based on contraposed Gödel implication ($a\Rightarrow b=\eta(b)\to\eta(a)$), since the final rating is either 1 or $\eta(\pi(s))$.

The corresponding aggregation scheme when priority weights bear on subsets of S is:

$$\oint^{\Uparrow}_\gamma(f)=\bigwedge_{A\subseteq S}\wedge_{s\in A}\eta(f(s))\to\gamma(\overline{A})\qquad(6)$$

If γ is a necessity measure based on π, then $\oint^{\Uparrow}_\gamma(f)=\wedge_{s\in S}(\eta(f(s))\to\eta(\pi(s)))$, which is another form of prioritized minimum. Indeed, as $\gamma(\overline{A})=\eta(\gamma^c(A))$, in (4) one can replace $\gamma(\overline{A})$ by $\eta(\gamma^c_\#(A))$ $(=\eta(\Pi_\#(A))$ if γ is a necessity measure). If there is no criterion such that $f(s)=1$ then $\oint^{\Uparrow}_\gamma(f)=0$. Moreover, $\oint^{\Uparrow}_\gamma(f)=1$ if and only if $\oint^{\uparrow}_\gamma(f)=1$.

Example 1 *We consider two criteria $S = \{s_1, s_2\}$, $L = [0, 1]$, $\eta = 1-$ and γ defined by $\gamma(\{s_1\}) = 0.4$ and $\gamma(\{s_2\}) = 0.6$. Then:*
$\mathcal{S}_\gamma(0.2, 0.8) = \vee(0.2, 0.6) = 0.6$
$\oint_\gamma^{\uparrow}(0.2, 0.8) = \wedge(0.4 \to 0.2, 0.6 \to 0.8, 1 \to 0.8) = 0.2$
$\oint_\gamma^{\Uparrow}(0.2, 0.8) = 0.$

3.2. Qualitative desintegrals

In this part, the evaluation scale for each criterion is decreasing, i.e., 0 is better than 1, but the scale for the global evaluation is increasing. In this case the aggregation functions must be decreasing and the capacities are replaced by the anti-measures.

A first desintegral is obtained by a saturation effect on a reversed scale:

$$\oint_\nu^{\natural}(f) = \bigvee_{A \subseteq S} \nu(\overline{A}) \wedge (\wedge_{s \in A} \eta(f(s))) \qquad (7)$$

Note that $\oint_\nu^{\natural}(f) = \mathcal{S}_{\nu(\bar{\cdot})}(\eta(f))$.

Another viewpoint is to consider that if $f(s) > t(s)$ then the local evaluation is bad and $f(s)$ becomes $t(s)$. Otherwise the local evaluation is good and $f(s)$ becomes 1. Note that tolerance thresholds turn local degrees of defect into local degrees of merit. This corresponds to the use of the Gödel implication and the global evaluation $\bigwedge_{i=1,\dots,n} f(s) \to t(s)$ which is generalized by the following desintegral:

$$\oint_\nu^{\Downarrow}(f) = \bigwedge_{A \subseteq S} (\wedge_{s \in A} f(s)) \to \nu(A)) \qquad (8)$$

If there is no defect-free criterion s (such that $f(s) = 0$) then $\oint_\nu^{\Downarrow}(f) = 0$ (the alternative is globally bad). Note that $\oint_\nu^{\Downarrow}(f) = \oint_{\nu(\bar{\cdot})}^{\Uparrow}(\eta(f))$.

In the last viewpoint t_i is viewed as a tolerance threshold such that it is enough to have $f(s) \leq t(s)$ (i.e. the degree of defect remains smaller than the threshold) for the requirement to be totally fulfilled. Recall that now the purpose is to avoid defects (a strictly positive evaluation of a negative criterion means a defect). If the object possesses the defect to an extent higher than t_i, then the rating value is reversed, leading to a poor positive local rating. This weighting scheme is captured by the formula $\eta(t(s)) \to \eta(f(s))$ where $\to$ is Gödel implication.

$$\oint_\nu^{\downarrow}(f) = \bigwedge_{A \subseteq S} \eta(\nu(A)) \to \vee_{s \in A} \eta(f(s)). \qquad (9)$$

Note that $\oint_\nu^{\downarrow}(f) = \oint_{\eta(\nu)}^{\uparrow}(\eta(f))$.

Example 2 *We consider two criteria $S = \{s_1, s_2\}$, $L = [0, 1]$, $\eta = 1-$ and ν an fuzzy anti-measure defined by $\nu(\{s_1\}) = 0.2$ and $\nu(\{s_2\}) = 0.6$.*
$\oint_\nu^{\natural}(0.3, 0.5) = \vee(0.6 \wedge 0.7, 0.2 \wedge 0.5, 1 \wedge 0.5) = 0.6$
$\oint_\nu^{\downarrow}(0.3, 0.5) = \wedge(0.8 \to 0.7, 0.4 \to 0.5, 1 \to 0.7) = 0.7$
$\oint_\nu^{\Downarrow}(0.3, 0.5) = 0.$

4. Equivalent expressions

This part presents equivalent expressions for the qualitative integrals and desintegrals which will be instrumental when expressing them as lower or upper possibility integrals. More precisely, we are going to prove that the qualitative integrals and desintegrals require the comparison of only n cases. This result is already known for the Sugeno integrals.

Without loss of generality we can suppose that $f(s_1) = f_1 \leq \cdots \leq f(s_n) = f_n$. We define the sets $A_i = \{s_i, \cdots, s_n\}$ with the convention $A_{n+1} = \emptyset$. It is well-known that for Sugeno integrals:

$$\mathcal{S}_\gamma(f) = \bigvee_{i=1}^{n} (f_i \wedge \gamma(A_i)) = \bigwedge_{i=1}^{n} (f_i \vee \gamma(A_{i+1}))$$

See [12] for more details. We establish similar expressions for the other integrals and desintegrals.

Proposition 3

$$\oint_\gamma^{\uparrow}(f) = \bigwedge_{i=1}^{n} (\gamma(\overline{A_{i+1}}) \to f_i)$$

where $\overline{A_{i+1}}$ denotes the complement of A_{i+1}.

Proof $\oint_\gamma^{\uparrow}(f) = \bigwedge_{i=1}^{n} (\gamma(\overline{A_{i+1}}) \to f_i) \wedge \bigwedge_{A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}} (\gamma(A) \to \vee_{i \in A} f_i)$.

We consider $A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}$ and let f_{i_A} be a shorthand for $\vee_{s_i \in A} f_i$.
Now, $A \subseteq \overline{A_{i_A+1}}$, for some index i_A. Then clearly $\gamma(A) \leq \gamma(\overline{A_{i_A+1}})$.

- If $\gamma(A) \leq f_{i_A}$ then $\gamma(A) \to f_{i_A} = 1 \geq \gamma(\overline{A_{i_A+1}}) \to f_{i_A}$.
- If $f_{i_A} < \gamma(A)$ then $f_{i_A} < \gamma(\overline{A_{i_A+1}})$ and $\gamma(A) \to f_{i_A} = f_{i_A} = \gamma(\overline{A_{i_A+1}}) \to f_{i_A}$.

So $\gamma(A) \to \vee_{i \in A} f_i \geq \gamma(\overline{A_{i_A+1}}) \to f_{i_A} \geq \wedge_{i=1}^{n} (\gamma(\overline{A_{i+1}}) \to f_i)$ which concludes the proof.

Proposition 4

$$\oint_\gamma^{\Uparrow}(f) = \wedge_{i=1}^{n} (\eta(f_i) \to \gamma(A_{i+1})).$$

Proof: We have $\eta(f_n) \leq \cdots \leq \eta(f_1)$.
$\oint_\gamma^{\Uparrow}(f) = [\bigwedge_{i=1}^{n} (\eta(f_i) \to \gamma(A_{i+1}))] \wedge \bigwedge_{A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}} [\wedge_{i \in A} \eta(f_i) \to \gamma(\overline{A})]$ so $\oint_\gamma^{\Uparrow}(f) \leq \wedge_{i=1}^{n} (\eta(f_i) \to \gamma(A_{i+1}))$. Let us prove the converse.

We consider $A \notin \{\overline{A_2}, \cdots, \overline{A_{n+1}}\}$ and note that $\eta(f_{i_A}) = \wedge_{s_i \in A} \eta(f_i)$. As f is supposed to be well ordered, $A \subseteq \overline{A_{i_A+1}}$ for some index i_A, i.e., $A_{i_A+1} \subseteq \overline{A}$ which entails $\gamma(A_{i_A+1}) \leq \gamma(\overline{A})$.

- If $\eta(f_{i_A}) \leq \gamma(\overline{A})$ then $\eta(f_{i_A}) \to \gamma(\overline{A}) = 1 \geq \eta(f_{i_A}) \to \gamma(A_{i_A+1})$.
- If $\eta(f_{i_A}) > \gamma(\overline{A})$ then $\eta(f_{i_A}) > \gamma(A_{i_A+1})$ and $\eta(f_{i_A}) \to \gamma(\overline{A}) = \gamma(\overline{A}) \geq \gamma(A_{i_A+1}) = \eta(f_{i_A}) \to \gamma(A_{i_A+1})$.

So $\eta(f_{i_A}) \to \gamma(\overline{A}) \geq \eta(f_{i_A}) \to \gamma(A_{i_A+1}) \geq \wedge_{i=1}^{n}(\eta(f_i) \to \gamma(A_{i+1}))$
and $\oint_{\gamma}^{\Uparrow}(f) = \bigwedge_{i=1}^{n}(\eta(f_i) \to \gamma(A_{i+1}))$.

When we use the relations between the desintegrals and the qualitative integrals we obtain the following expressions.

Proposition 5

$$\oint_{\nu}^{\lightning}(f) = \bigvee_{i=1}^{n} \eta(f_i) \wedge \nu(A_{i+1}) = \bigwedge_{i=1}^{n} \eta(f_i) \vee \nu(A_i).$$

$$\oint_{\nu}^{\downarrow}(f) = \bigwedge_{i=1}^{n}(\eta(\nu(A_i))) \to \eta(f_i)).$$

$$\oint_{\nu}^{\Downarrow}(f) = \bigwedge_{i=1}^{n}(f_i \to \nu(A_i)).$$

Proof We define g by $g_i = \eta(f_{n-i+1})$. We have $g_1 = \eta(f_n) \leq \cdots \leq g_n = \eta(f_1)$.

- $\oint_{\nu(\cdot)}(\eta(f)) = \bigvee_{i=1}^{n} g_i \wedge \nu(\overline{A_i^g})$ where $A_i^g = \{1, \cdots, n-i+1\} = \overline{A_{n-i+2}}$. When we define the index $j = n-i+1$ we have $\oint_{\nu(\cdot)}(\eta(f)) = \bigvee_{j=1}^{n} g_{n-j+1} \wedge \nu(\overline{A_{n-j+1}^g}) = \bigvee_{j=1}^{n} \eta(f_j) \wedge \nu(\overline{A_{n-j+1}^g}) = \bigvee_{j=1}^{n} \eta(f_j) \wedge \nu(A_{j+1})$.
  $\oint_{\nu(\cdot)}(\eta(f)) = \bigwedge_{i=1}^{n} g_i \vee \nu(\overline{A_{i+1}^g})$ where $A_{i+1}^g = \{1, \cdots, n-i\} = \overline{A_{n-i+1}}$. When we define the index $j = n-i+1$ we have $\oint_{\nu(\cdot)}(\eta(f)) = \bigwedge_{j=1}^{n} g_{n-j+1} \vee \nu(\overline{A_{n-j+2}^g}) = \bigwedge_{j=1}^{n}(\eta(f_j)) \vee \nu(\overline{A_{n-j+2}^g}) = \bigwedge_{j=1}^{n}(\eta(f_j)) \vee \nu(A_j)$.
- $\oint_{\eta(\nu)}^{\uparrow}(\eta(f)) = \bigwedge_{i=1}^{n} \eta(\nu(\overline{A_{i+1}^g})) \to g_i$. When we consider the index $j = n-i+1$ we have $\oint_{\eta(\nu)}^{\uparrow}(\eta(f)) = \bigwedge_{j=1}^{n} \eta(\nu(\overline{A_{n-j+2}^g})) \to g_{n-j+1} = \bigwedge_{j=1}^{n} \eta(\nu(A_j)) \to \eta(f_j)$.
- $\oint_{\nu(\cdot)}^{\Uparrow}(\eta(f)) = \bigwedge_{i=1}^{n} \eta(g_i) \to \nu(\overline{A_{i+1}^g}))$
  $= \bigwedge_{j=1}^{n} \eta(g_{n-j+1}) \to \nu(\overline{A_{n-j+2}^g}))$
  $= \bigwedge_{j=1}^{n} f_j \to \nu(A_j)$.

5. Qualitative integrals as upper and lower possibility integrals

It was proved in [2] that Sugeno integral is a lower prioritized maximum:

Proposition 6 $\mathcal{S}_{\gamma}(f) = \bigwedge_{\pi \in \mathcal{C}*(\gamma)} \mathcal{S}_{\Pi}(f)$.

Proof: Viewing γ as a lower possibility, it comes (with $f_A = \wedge_{s \in A} f(s)$):
$\mathcal{S}_{\gamma}(f) = \bigvee_{A \subseteq S}(\wedge_{\pi \in \mathcal{C}_*(\gamma)} \Pi(A)) \wedge f_A = \bigvee_{A \subseteq S} \bigwedge_{\pi \in \mathcal{C}*(\gamma)}(\Pi(A) \wedge f_A)$
$\leq \bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \bigvee_{A \subseteq S}(\Pi(A) \wedge f_A)$, hence $\mathcal{S}_{\gamma}(f) \leq \bigwedge_{\pi \in \mathcal{C}*(\gamma)} \mathcal{S}_{\Pi}(f)$.
Conversely, let π_f be the marginal of γ obtained from the nested sequence of sets F_λ induced by function f, then it is clear that $\Pi_f(F_\lambda) = \gamma(F_\lambda)$, and thus $\mathcal{S}_{\gamma}(f) = \mathcal{S}_{\Pi_f}(f)$. As $\exists \pi \in \mathcal{C}_*(\gamma), \pi_f \geq \pi$, by definition, $\mathcal{S}_{\Pi_f}(f) \geq \mathcal{S}_{\Pi}(f) \geq \bigwedge_{\pi \in \mathcal{C}_*(\gamma)} \mathcal{S}_{\Pi}(f)$.

The obtained equality may sound surprizing. If $\gamma = \gamma_1 \wedge \gamma_2$, it only holds in general that $\mathcal{S}_{\gamma_1 \wedge \gamma_2}(f) \leq \mathcal{S}_{\gamma_1}(f) \wedge \mathcal{S}_{\gamma_2}(f)$. Nevertheless, equality holds if $\gamma = \Pi_1 \wedge \Pi_2$, for two possibility measures, that is $\mathcal{C}_*(\gamma) = \{\pi_1, \pi_2\}$. This is because any possibility measure $\Pi \geq \gamma$ is such that $\pi \geq \pi_1$ or $\pi \geq \pi_2$. To see it directly, suppose first that $\pi(s_1) < \pi_1(s_1)$ and $\pi(s_2) < \pi_2(s_2)$ for some $s_1, s_2 \in S$. Then $\Pi(\{s_1, s_2\}) < \gamma(\{s_1, s_2\})$, which violates the assumption $\Pi \geq \gamma$. So, $\forall s \in S, \Pi \geq \gamma$ implies either $\pi(s) \geq \pi_1(s)$ or $\pi(s) \geq \pi_2(s)$. Now suppose $\pi_2(s_1) > \pi(s_1) \geq \pi_1(s_1)$. Then if for some other $s \neq s_1, \pi(s) < \pi_1(s)$, again $\Pi(\{s_1, s\}) < \gamma(\{s_1, s\})$. Hence, $\pi \geq \pi_1$. So, $\mathcal{S}_{\gamma}(f) = \mathcal{S}_{\Pi_f}(f) \geq \mathcal{S}_{\Pi_1}(f) \wedge \mathcal{S}_{\Pi_2}(f)$ since $\Pi_f \geq \gamma$.

Note that in the numerical case, the same feature occurs, namely, lower expectations with respect to a convex probability set are sometimes Choquet integrals with respect to the capacity equal to the lower probability constructed from this probability set (for instance convex capacities, and belief functions). However, this is not true for any capacity and any convex probability set.

Finally, using conjugacy properties, one can prove that

Proposition 7 $\mathcal{S}_{\gamma}(f) = \bigvee_{\pi \in \mathcal{C}_*(\gamma^c)} \mathcal{S}_N(f)$, *where* $\mathcal{S}_N(f) = \bigwedge_{s \in S} \eta(\pi(s)) \vee f(s)$.

In the following we show that these results hold for residuated variants of Sugeno integrals.

Proposition 8

$$\oint_{\gamma}^{\uparrow}(f) = \bigvee_{\pi \in \mathcal{C}_*(\gamma)} \oint_{\Pi}^{\uparrow}(f).$$

The fact that we have a $\bigvee$ in place of a $\bigwedge$ in Proposition 6 should not come as a surprise, since when we are approaching γ from above by Π, we are approaching $\oint_{\gamma}^{\uparrow}$ from below by $\oint_{\Pi}^{\uparrow}$.

Proof: Viewing γ as a lower possibility, it comes (with $f^A = \vee_{s \in A} f(s)$):
$\oint_{\gamma}^{\uparrow}(f) = \bigwedge_{A \subseteq S}(\wedge_{\pi \in \mathcal{C}_*(\gamma)} \Pi(A)) \to f^A = \bigwedge_{A \subseteq S} \bigvee_{\pi \in \mathcal{C}*(\gamma)}(\Pi(A) \to f^A)$
$\geq \bigvee_{\pi \in \mathcal{C}_*(\gamma)} \bigwedge_{A \subseteq S} \Pi(A) \to f^A$, hence $\oint_{\gamma}^{\uparrow}(f) \geq \bigvee_{\pi \in \mathcal{C}*(\gamma)} \oint_{\Pi}^{\uparrow}(f)$.

Conversely, let f be such that $f_1 \leq \cdots \leq f_n$. Hence we can define the possibility measure $\overline{\Pi}_f$, with distribution $\overline{\pi}_f^{\gamma}$, dominating γ, such that $\overline{\Pi}_f^{\gamma}(\overline{A_{i+1}}) = \gamma(\overline{A_{i+1}}), \forall i = 0, \ldots, n-1$, and by Proposition 3 $\oint_{\gamma}^{\uparrow}(f) = \bigwedge_{i=1}^{n}(\gamma(\overline{A_{i+1}}) \to f_i)$. As $\exists \pi \in \mathcal{C}_*(\gamma), \overline{\pi}_f^{\gamma} \geq \pi$, by definition, $\oint_{\gamma}^{\uparrow}(f) = \oint_{\overline{\Pi}_f^{\gamma}}^{\uparrow}(f) \leq \bigvee_{\pi \in \mathcal{C}_*(\gamma)} \oint_{\Pi}^{\uparrow}(f)$.

Proof 2: The result can be also obtained by a direct proof. We suppose f well ordered: $f_1 \leq \cdots \leq f_n$. Hence we can define the possibility distribution $\pi_f^\gamma(s_i) = \gamma(\overline{A_{i+1}})$.
Let us prove that $\forall A \subseteq C$, $\Pi_f^\gamma(A) \geq \gamma(A)$:

Let $\overline{A_{i_A+1}}$ be the smallest set in the sequence $\{\overline{A_i}\}_i$ such that $A \subseteq \overline{A_i}$. Note that we have $i_A \in A$.
$A \subseteq \overline{A_{i_A+1}}$ entails $\gamma(A) \leq \gamma(\overline{A_{i_A+1}}) = \pi_f^\gamma(s_{i_A})$.
$\Pi_f^\gamma(A) = \vee_{i\in A}\pi_f^\gamma(s_i) \geq \pi_f^\gamma(s_{i_A})$.
So we have $\gamma(A) \leq \Pi_f^\gamma(A)$.

Let us prove that for all $\pi \in S(\gamma)$ $\oint_\Pi^\uparrow(f) \leq \oint_\gamma^\uparrow(f)$:
For all i, $\Pi(\overline{A_{i+1}}) \to f_i \leq \gamma(\overline{A_{i+1}}) \to f_i$ which entails $\wedge_j \Pi(\overline{A_{j+1}}) \to f_j \leq \gamma(\overline{A_{i+1}}) \to f_i$ for all i. So we have $\oint_\Pi^\uparrow(f) \leq \oint_\gamma^\uparrow(f)$.

Hence we have $\oint_\gamma^\uparrow(f) \geq \bigvee_{\pi\in S(\gamma)} \oint_\Pi^\uparrow(f)$.

Moreover we have $\Pi_f^\gamma(\overline{A_{i+1}}) = \pi_f^\gamma(1) \vee \cdots \vee \pi_f^\gamma(s_i) = \gamma(\overline{A_2}) \vee \cdots \vee \gamma(\overline{A_{i+1}}) = \gamma(\overline{A_{i+1}})$ so $\oint_\gamma^\uparrow(f) = \oint_{\Pi_f^\gamma}^\uparrow(f) \leq \bigvee_{\pi\in S(\gamma)} \oint_\Pi^\uparrow(f)$.

Proposition 9

$$\oint_\gamma^{\Uparrow}(f) = \bigwedge_{\pi\in S(\gamma)} \oint_\Pi^{\Uparrow}(f).$$

Proof: We suppose f well ordered $f_1 \leq \cdots \leq f_n$. Hence we can use the possibility distribution defined in [2]: $\pi_f^\gamma(s_i) = \gamma(A_i)$.

$\forall A \subseteq C$ we have $\Pi_f^\gamma(A) \geq \gamma(A)$ and $\gamma(A_i) = \Pi_f^\gamma(A_i)$ so $\oint_\gamma^{\Uparrow}(f) = \oint_{\Pi_f^\gamma}^{\Uparrow}(f) \geq \bigwedge_{\pi\in S(\gamma)} \oint_\Pi^{\Uparrow}(f)$.

For all $\Pi \in S(\gamma)$ we have
$\eta(f_i) \to \gamma(A_i) \leq \eta(f_i) \to \Pi^\gamma(A_i)$
so $\wedge_i \eta(f_i) \to \gamma(A_i) \leq \eta(f_i) \to \Pi^\gamma(A_i)$
and $\oint_\gamma^{\Uparrow}(f) \leq \oint_\Pi^{\Uparrow}(f)$ for all $\Pi \in S(\gamma)$. Hence we have $\oint_\gamma^{\Uparrow}(f) \leq \bigwedge_{\pi\in S(\gamma)} \oint_\Pi^{\Uparrow}(f)$.

6. Qualitative desintegrals as upper and lower possibility desintegrals

When we consider guaranteed anti- possibilities instead of possibilities the desintegrals appear to be the upper bound of possibility desintegrals.

Proposition 10

$$\oint_\nu^{\downarrow}(f) = \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\downarrow}(f).$$

Proof: We suppose f well ordered: $f_1 \leq \cdots \leq f_n$. Hence we can define the guaranteed possibility distribution $t_f^\nu(i) = \nu(A_i)$.
Let us prove that $\forall A \subseteq C$, $\Delta_f^\mu(A) \leq \nu(A)$:

Let A_{i_A} be the smallest set in the sequence $\{A_i\}_i$ such that $A \subseteq A_i$. Note that we have $i_A \in A$.
$A \subseteq A_{i_A}$ entails $\nu(A) \geq \nu(A_{i_A}) = t_f^\nu(s_{i_A})$.
$\Delta_f^\nu(A) = \wedge_{i\in A} t_f^\nu(i) \leq t_f^\nu(s_{i_A})$.
So we have $\nu(A) \geq \Delta_f^\nu(A)$.

Let us prove that for all $\Delta \in \mathcal{S}(\nu)$ $\oint_\Delta^\downarrow(f) \leq \oint_\nu^\downarrow(f)$:
For all i, $\eta(\Delta(A_i)) \to \eta(f_i) \leq \eta(\nu(A_i)) \to \eta(f_i)$ which entails $\wedge_j \eta(\Delta(A_j)) \to \eta(f_j) \leq \eta(\nu(A_i)) \to \eta(f_i)$ for all i. So we have $\oint_\Delta^\downarrow(f) \leq \oint_\nu^\downarrow(f)$.

Hence we have $\oint_\nu^\downarrow(f) \geq \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^\downarrow(f)$.

Moreover we have $\Delta_f^\nu(A_i) = t_f^\nu(i) \wedge \cdots \wedge t_f^\nu(n) = \nu(A_i) \wedge \cdots \wedge \nu(A_n) = \nu(A_i)$ so $\oint_\nu^\downarrow(f) = \oint_{\Delta_f^\nu}^\downarrow(f) \leq \bigvee_{\Delta\in\mathcal{S}(\mu)} \oint_\Delta^\downarrow(f)$.

Proposition 11

$$\oint_\nu^{\Downarrow}(f) = \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\Downarrow}(f).$$

Proof: We suppose f well ordered $f_1 \leq \cdots \leq f_n$. Hence we can use the guaranteed possibility distribution defined for the previous proof: $t_f^\mu(i) = \nu(A_i)$.

$\forall A \subseteq C$ we have $\Delta_f^\nu(A) \leq \nu(A)$ and $\nu(A_i) = \Delta_f^\mu(A_i)$ which entails $\oint_\mu^{\Downarrow}(f) = \oint_{\Delta_f^\mu}^{\Downarrow}(f) \leq \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\Downarrow}(f)$.

For all $\Delta \in \mathcal{S}(\nu)$ we have
$f_i \to \Delta(A_i) \leq f_i \to \nu(A_i)$
so $\wedge_i f_i \to \Delta(A_i) \leq f_i \to \nu(A_i)$
and $\oint_\Delta^{\Downarrow}(f) \leq \oint_\nu^{\Downarrow}(f)$ for all $\Delta \in \mathcal{S}(\nu)$. Hence we have $\oint_\nu^{\Downarrow}(f) \geq \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\Downarrow}(f)$.

Proposition 12

$$\oint_\nu^{\nmid}(f) = \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\nmid}(f).$$

Proof: We suppose f well ordered $f_1 \leq \cdots \leq f_n$. Hence we can use the guaranted possibility distribution $t_f^\mu(i) = \nu(A_i)$.

$\forall A \subseteq C$ we have $\Delta_f^\nu(A) \leq \nu(A)$ and $\nu(A_i) = \Delta_f^\mu(A_i)$ which entails $\oint_\nu^{\nmid}(f) = \oint_{\Delta_f^\mu}^{\nmid}(f) \leq \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\nmid}(f)$.

For all $\Delta \in \mathcal{S}(\nu)$ we have $\Delta(A_i) \leq \nu(A_i)$ so $\eta(f_i) \wedge \Delta(A_{i+1}) \leq \eta(f_i) \wedge \nu(A_{i+1})$ and $\oint_\Delta^{\nmid}(f) \leq \oint_\nu^{\nmid}(f)$ for all $\Delta \in \mathcal{S}(\nu)$.
Hence we have $\oint_\nu^{\nmid}(f) \geq \bigvee_{\Delta\in\mathcal{S}(\nu)} \oint_\Delta^{\nmid}(f)$.

Proposition 13

$$\oint_\nu^{\nmid}(f) = \bigwedge_{\delta\in\mathcal{S}(\nu^c)} \oint_\nabla^{\nmid}(f)$$

Proof: We consider a permutation
$f_1 \leq \cdots \leq f_n$
$\delta(i) = \nu^c(A_i) = \eta(\nu(\overline{A_i}))$.
A_{i_0} the smallest set such that $A \subseteq A_{i_0}$.
We have $i_0 \in A$
$\delta(i_0) = \nu^c(A_{i_0}) \leq \nu^c(A)$.
$\Delta(A) = \wedge_{i\in A}\delta(i) \leq \delta(i_0) \leq \nu^c(A)$.

When we consider A we can find a permutation such that $A = A_{i_0}$ hence $\Delta(A) = \nu^c(A_{i_0}) = \nu^c(A)$.
$\eta(\Delta(\overline{A})) = \nabla(A) = \eta(\nu^c(\overline{A})) = \nu(A)$

$$\oint_{\nu}^{\lightning}(f) = \oint_{\nabla}^{\lightning}(f) \geq \bigwedge_{\delta \in \mathcal{S}(\nu^c)} \oint_{\nabla}^{\lightning}(f). \qquad (10)$$

If $\delta \in \mathcal{S}(\nu^c)$ then $\eta(\nu^c(\overline{A})) \leq \eta(\Delta(\overline{A}))$ i.e $\nu(A) \leq \nabla(A)$. So $\oint_{\nu}^{\lightning}(f) \leq \oint_{\nabla}^{\lightning}(f)$ for all $\delta \in \mathcal{S}(\nu^c)$. So $\oint_{\nu}^{\lightning}(f) \leq \bigwedge_{\delta \in \mathcal{S}(\nu^c)} \oint_{\nabla}^{\lightning}(f)$.

The qualitative desintegrals can also be expressed as lower or upper bounds using the relations between the qualitative integrals and desintegrals. This may be used for obtaining the results for desintegrals from the ones regarding integrals.

$\oint_{\nu}^{\lightning}(f) = \mathcal{S}_{\nu(\bar{\cdot})}(\eta(f)) = \bigwedge_{\pi \in S(\nu(\bar{\cdot}))} \mathcal{S}_{\Pi}(\eta(f))$.
$\oint_{\nu}^{\downarrow}(f) = \oint_{\eta(\nu)}^{\uparrow}(\eta(f)) = \bigvee_{\pi \in S(\eta(\nu))} \oint_{\Pi}^{\uparrow}(\eta(f))$.
$\oint_{\nu}^{\Downarrow}(f) = \oint_{\nu(\bar{\cdot})}^{\Uparrow}(\eta(f)) = \bigwedge_{\pi \in S(\nu(\bar{\cdot}))} \oint_{\Pi}^{\Uparrow}(\eta(f))$.

For instance, Proposition 13 can be obtained by noticing that $\oint_{\nu}^{\lightning}(f) = \mathcal{S}_{\nu(\bar{\cdot})}(\eta(f)) = \bigwedge_{\pi \in S(\nu(\bar{\cdot}))} \mathcal{S}_{\Pi}(\eta(f)) = \bigwedge_{\eta(\delta) \in S(\nu(\bar{\cdot}))} \mathcal{S}_{\nabla}(f)$
since $\nabla_{\delta}(A) = \Pi_{\eta(\delta)}(\overline{A})$. It can be checked that $\pi \in S(\eta(\nu)) \Leftrightarrow \eta(\pi) \in \mathcal{S}(\nu)$, and thus
$\eta(\delta) \in S(\nu(\bar{\cdot})) \Leftrightarrow \delta \in \mathcal{S}(\eta(\nu(\bar{\cdot}))) \Leftrightarrow \delta \in \mathcal{S}(\nu^c)$, which completes the checking of Proposition 13.

7. Conclusion

Retrospectively, one may wonder why one has only considered Sugeno integrals for a long time, since the other qualitative integrals are as simple. The results obtained in this paper makes Sugeno integrals, and other integrals or desintegrals easier to compute as a simple combination of integrals or desintegrals with respect to possibility, necessity, guaranteed possibility measures (i.e. basically weighted max and min). Besides, the representation of capacities as a finite conjunction of possibility measures, or as a finite disjunction of necessity measures has strong links with k-maxitivity and k-minitivity axioms, and the representation of imprecise possibilities by means of possibilistic focal elements of limited size. This will motivate further investigation.

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