# At-least At-most Modifications in a Space with Fuzzy Preoreder

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**Abstract**

In this paper, we utilize the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder and propose a justification of solvability of systems that are modified with “at least” (“at most”) quantifiers. We show that the respectively modified fuzzy sets are upper (lower) sets of a fuzzy preorder on the space of reals. On the basis of this, we show that the systems with the sup $*$ composition and with the same type of modifications on both sides are solvable. Moreover, we explain why the solvability of the similarly modified systems with the inf $\rightarrow$composition cannot be established. Last but not least we show that only opposite modifications on the left and right-hand sides of the modified system with the inf $\rightarrow$composition guarantee its solvability.

## 1. Introduction

The aim of this contribution is to show that the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder can be successfully used in analyzing monotonous fuzzy rule based systems. The following two problems are connected with this analysis: interpolativity and monotonicity of the corresponding function. This paper is devoted to the first problem.

A monotonously increasing system of fuzzy IF-THEN rules is called *interpolative* if at least one of corresponding to it systems (1) or (2) of fuzzy relation equations is solvable. It turned out that not all monotonously increasing (decreasing) systems are interpolative, see e.g., [18, 19]. With this respect, it has been proposed to replace fuzzy sets $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in (3) by their “at least” (or “at most”) modifications (see [2]). The interpolativity of modified systems has been investigated in [19] where a number of sufficient conditions has been proved. In the current contribution, we show that the proposed modifications transform fuzzy sets into upper and lower sets of a certain fuzzy preorder (see [7, 10]). On the basis of this, we utilize the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder which has been started in [10] and propose a justification of solvability of the modified systems with the sup $*$ composition and with the same type of modifications on both sides. Moreover, we explain why the solvability of the similarly modified systems with the inf $\rightarrow$composition cannot be established (see [19]).

Because the problem of interpolativity leads to the problem of solvability of systems of fuzzy relation equations that differ in a type of composition, we will be concentrating on the latter. By types of composition, we mean either sup $*$ composition which is usually denoted by $\circ$, or inf $\rightarrow$composition which is denoted by $\lhd$. The first one of composition was introduced by L. Zadeh [23] and the second one by W. Bandler and L. Kohout [1].

Both systems were extensively investigated in the literature; see e.g., [3–6, 9, 12, 14–17, 21, 22] for various results about the system with the sup $*$ composition, and [3, 8, 14, 20] for respective results about the system with the inf $\rightarrow$composition. An overview of the contemporary progress in this topic can be found in [4] and (a short one) in [14]. Irrespective of the used composition, the obtained results can be divided into two groups: the first containing solvability criteria (see [3–6, 9, 12, 14–16]), and the second containing various characterizations of the solution sets (see [17, 21, 22]). A finer classification within each group is based on the specification of the $*$-operation that is used in the sup $*$ composition.

The main purpose of this investigation is to embed the problem of interpolativity of modified systems into the general problem of solvability of systems of fuzzy relation equations in a space with fuzzy preorder and by this, to simplify the analysis of the former problem. We will characterize fuzzy sets on both sides of fuzzy relation equations that guarantee solvability of corresponding systems.

The paper is organized as follows. In Section 2, we give preliminary information on residuated lattices, fuzzy sets, fuzzy relations and systems of fuzzy relation equations. In Section 3, we explain the notion of monotonous fuzzy rule based systems and formulate the problem of our research. We propose another than in [19] proof to the solvability of the modified systems and show how the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder helps to analyze the problem of solvability of modified systems with the inf $\rightarrow$composition.

## 2. Preliminaries

Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a fixed, complete, integral, residuated, commutative l-monoid (*a complete residuated lattice*) that is extended by the binary operation $\leftrightarrow$ (bi-implication):

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

Let $X$ and $Y$ be two non-empty sets (universes) that are not necessary different, and $L^X$, $L^Y$ and $L^{X\times Y}$ be respective classes of *fuzzy sets* on $X$, $Y$ and *fuzzy relations* on $X \times Y$, so that fuzzy sets and fuzzy relations are identified with their membership functions. A fuzzy set $A$ is *normal* if there exists $x_A \in X$ such that $A(x_A) = 1$. The (ordinary) set $Core(A) = \{x \in X \mid A(x) = 1\}$ is the *core* of the normal fuzzy set $A$. Fuzzy sets $A \in L^X$ and $B \in L^X$ are *equal* $(A = B)$ if for all $x \in X$, $A(x) = B(x)$.

If a universe is the set $\mathbb{R}$ of real numbers, then we say that a fuzzy set $A$ is *convex* if for all $\lambda \in [0, 1]$ and for all $x_1, x_2 \in \mathbb{R}$, $A(\lambda x_1 + \lambda x_2) \geq \min(A(x_1), A(x_2)$. The set of normal convex fuzzy sets on $\mathbb{R}$ will be denoted by CNV. We say that $A_1 \in$ CNV is *less than or equal* to $A_2 \in$ CNV, if for all $\alpha \in (0, 1]$, $A_1(\alpha) \leq A_2(\alpha)$ where $A_1(\alpha), A_2(\alpha)$ are respective $\alpha$-cuts of $A_1, A_2$. We will use the standard denotation $\leq$ for this relation.

The lattice operations $\vee$ and $\wedge$ induce the union and intersection of fuzzy sets, respectively. Two other binary operations, $*$ and $\rightarrow$ of $\mathcal{L}$, are used below for set-relation compositions. We consider two of them: the sup $*$ composition, which is usually denoted by $\circ$, and the inf $\rightarrow$composition, which is denoted by $\lhd$. Let $A \in L^X$ and $R \in L^{X\times Y}$. Then

$$(A \circ R)(y) = \bigvee_{x\in X} (A(x) * R(x, y)),$$
$$(A \lhd R)(y) = \bigwedge_{x\in X} (A(x) \rightarrow R(x, y)).$$

The first composition was introduced by L. Zadeh [23] and the second one - by W. Bandler and L. Kohout [1].

The following system of equations

$$A_i \circ R = B_i, \quad i = 1, \dots, n, \tag{1}$$

that is considered with respect to the unknown fuzzy relation $R \in L^{X\times Y}$, is called a *system of fuzzy relation equations with* sup $*$ *composition.* Its counterpart is a *system of fuzzy relation equations with* inf $\rightarrow$*composition*

$$A_i \lhd R = B_i, \quad i = 1, \dots, n. \tag{2}$$

Both systems of fuzzy relation equations arise when a system of fuzzy IF-THEN rules

$$\text{IF} X \text{ is } A_i \text{ THEN } Y \text{ is } B_i, \quad i = 1, \dots, n, \tag{3}$$

is modeled by a fuzzy relation, say $R$, and the model is requested to be *correct* and *continuous* [13]. To explain this request, we recall that a relation model determined by $R$ establishes a correspondence between a dependent value $B \in L^Y$ and an independent value $A \in L^X$ via either sup $*$ composition,

$$B = A \circ R, \tag{4}$$

or inf $\rightarrow$composition,

$$B = A \lhd R. \tag{5}$$

The relational model is *continuous* if the dependent values that correspond to close (in a specific sense) independent values are close as well. In [13], we proved that this is possible if and only if $R$ solves either system (1) or (2) of fuzzy relation equations. This fact places further importance on the solvability of systems of fuzzy relation equations.

In general, solutions of (1) or (2) may not exist. Therefore, an investigation of necessary and sufficient conditions for solvability (or at least sufficient conditions) is needed. This problem has been widely studied in the literature, and many theoretical results have been obtained in the papers cited above. The most known criterion [16] states that system (1) or (2) is solvable if and only if the fuzzy relation

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \rightarrow B_i(y)), \tag{6}$$

is its solution. Similarly, system (2) is solvable if and only if the fuzzy relation

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)), \tag{7}$$

is its solution. $\hat{R}$ is the greatest solution of (1) and $\check{R}$ is the least solution of (2).

## 3. Rule bases modified by "at least", "at most"

Recently, a certain interest has been shown to monotonous fuzzy rule based systems. We will explain this notion and formulate the problem of our research using the denotation and results of [19]. Assume that fuzzy sets $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in the system (3) belong to CNV. Then the system (3) is monotonously increasing (decreasing) if for all $i, j = 1, \dots, n$, $A_i \leq A_j$ implies that $B_i \leq B_j$ $(B_j \leq B_i)$.

A monotonously increasing system (3) is called *interpolative* if at least one of systems (1) or (2) is solvable. It turned out that not all monotonously increasing (decreasing) systems are interpolative, see e.g., [18, 19]. With this respect, it has been proposed to replace fuzzy sets $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in (3) by their "at least" (or "at most") modifications (see [2]). The interpolativity of modified systems has been investigated in [19] where a number of sufficient conditions has been proved. In this contribution, we will show that the proposed modifications transform fuzzy sets into upper and lower sets of a certain fuzzy preorder (see [7, 10]). On the basis of this, we will utilize the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder which has been started in [10] and propose a justification of the solvability of the modified systems (10) and (11) with the sup $*$ composition. Moreover, we will explain why the solvability of the similarly modified systems with the inf $\rightarrow$composition was not established in [19].

### 3.1. Systems of fuzzy relation equations with "at least", "at most" modifications

Let us remind that modifications "at least" and "at most" have been proposed for those fuzzy sets that map $\mathbb{R}$ onto $[0,1]$ (see [2]). In details, if $A \in [0,1]^{\mathbb{R}}$, then "at least" $A$ is the fuzzy set $A^{\uparrow}$ such that for all $y \in \mathbb{R}$,

$$A^{\uparrow}(y) = \bigvee_{x \leq y} A(x), \tag{8}$$

and "at most" $A$ is the fuzzy set $A^{\downarrow}$ such that for all $y \in \mathbb{R}$,

$$A^{\downarrow}(y) = \bigvee_{x \geq y} A(x). \tag{9}$$

The problem of solvability of the two systems of types (1) and (2) with modified fuzzy sets has been investigated in e.g., [19]. If we take into account that there are two modifications and two sides of each system, then we easily come to four new systems of each type (1) and (2) with modified left and right sides. However, only two new systems with the same modifications on each side have been discussed in [19] for each type of composition. Below, we reproduce only those two systems of the type (1), i.e. with the sup $*$ composition.

$$A_i^{\uparrow} \circ R = B_i^{\uparrow}, \quad i = 1, \ldots, n, \tag{10}$$

and

$$A_i^{\downarrow} \circ R = B_i^{\downarrow}, \quad i = 1, \ldots, n, \tag{11}$$

The two principal results of [19] are formulated for a monotonously increasing system (3) where antecedents $A_i$, $i = 1, \ldots, n$ are convex and fulfil the Ruspini condition. It is shown that both systems (10) and (11) are solvable. Moreover, it has been remarked that the solvability of similarly modified systems with the inf $\rightarrow$composition (type (2)) is an open problem. Below, we will give an answer to this problem.

Our approach consists in utilizing the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder which has been started in [10] and then elaborated in [11]. We will show that in order to guarantee solvability of the modified system of type (2) the fuzzy sets in the left and the right sides of (2) should be modified oppositely.

### 3.2. "At least", "at most" in view of upper and lower fuzzy sets

Let us show that expressions (8) and (9) define upper and lower sets of the ordinary order relation $\leq$ on $\mathbb{R}$ with the characteristic function

$$Q_{\leq}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 1** *Let $\mathcal{L}_{[0,1]} = \langle [0,1], \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ be a residuated lattice on $[0,1]$. Then for all $y \in \mathbb{R}$,*

$$A^{\uparrow}(y) = \bigvee_{x \in \mathbb{R}} A(x) * Q_{\leq}(x,y), \tag{12}$$

*and*

$$A^{\downarrow}(y) = \bigvee_{x \in \mathbb{R}} A(x) * Q_{\leq}(y,x). \tag{13}$$

Let us remind that a fuzzy relation $Q : X \times X \rightarrow [0,1]$ is called a $(*)$-fuzzy preorder (see e.g., [7] if it is reflexive and $*$-transitive. In [10], we proposed to extend this notion to fuzzy relations whose values are in $L$, where $L$ is a support of a complete residuated lattice. In the sequel, we will be using this extension.

A fuzzy set $A : X \rightarrow L$ is an *upper set* (*lower set*) [7] with respect to a fuzzy preorder $Q$ on $X$ if for all $x, y \in X$,

$$A(x) * Q(x,y) \leq A(y), \quad (A(y) * Q(x,y) \leq A(x)).$$

Let us show that the left (right) $Q$-image of any fuzzy set on $X$ is a lower (upper) set with respect to the fuzzy preorder $Q$ on $X$.

**Proposition 2** *Let*

- *$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ be a complete residuated lattice on $L$,*
- *$Q : X \times X \rightarrow L$ is a fuzzy preorder on $X$,*
- *$A : X \rightarrow L$ is a fuzzy set on $X$.*

*Then the right $Q$-image*

$$A_Q^{\uparrow}(y) = \bigvee_{x \in \mathbb{R}} A(x) * Q(x,y) \tag{14}$$

*and respectively, the left $Q$-image*

$$A_Q^{\downarrow}(y) = \bigvee_{x \in \mathbb{R}} Q(y,x) * A(x) \tag{15}$$

*is an upper (lower) set of $A$ with respect to $Q$.*

PROOF: We will give the proof for the first claim.

$$A_Q^{\uparrow}(y) * Q(y,z) = Q(y,z) * \bigvee_{x \in \mathbb{R}} A(x) * Q(x,y) =$$

$$\bigvee_{x \in \mathbb{R}} (A(x) * Q(x,y) * Q(y,z)) \leq \bigvee_{x \in \mathbb{R}} A(x) * Q(x,z) =$$

$$A_Q^{\uparrow}(z).$$

□

**Remark 1** *By Propositions 1, 2, the "at least" ("at most") modification of a fuzzy set $A$ on $\mathbb{R}$ is the upper (lower) set of $A$ with respect to $Q_{\leq}$.*

### 3.3. Interpolativity of modified fuzzy rule bases with $\sup *$ composition

The purpose of this Section is to utilize the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder which has been started in [10] and then elaborated in [11]. With the help of this theory we propose other than in [19] proof to the solvability of the modified systems (10) and (11). We will make use of the fact that "at least" ("at most") modification is an upper (lower) set with respect to several fuzzy preorders.

In order to simplify the denotation, all systems of types (1) or (2) will be considered for a fixed value of $y \in Y$, i.e. in the following forms:

$$A_i \circ R = b_i, \quad i = 1, \dots, n,$$

or

$$A_i \lhd R = b_i, \quad i = 1, \dots, n,$$

where $R \in L^X$ is the unknown fuzzy set, $b_1, \dots, b_n \in L$.

The following theorem is an adaptation of the result that has been proved in [10, 11].

**Theorem 1** *Let fuzzy sets $A_1, \dots, A_n \in L^X$ be normal and $x_1, \dots, x_n \in X$ be pairwise different elements such that for all $i = 1, \dots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \dots, n$,*

$$A_i(x_j) \leq \bigwedge_{x \in X} (A_j(x) \rightarrow A_i(x)) \tag{16}$$

*holds true. Then there exists a fuzzy preorder $Q$ such that for all $i = 1, \dots, n$,*

$$A_i = A^{\uparrow}_{Q,i},$$

*where $A^{\uparrow}_{Q,i}$ is the upper set of $A_i$ with respect to $Q$. Moreover, both systems*

$$A_i \circ R = b_i, \quad i = 1, \dots, n,$$

*and*

$$A^{\uparrow}_{Q,i} \circ R = b_i, \quad i = 1, \dots, n,$$

*are solvable if and only if*

$$A_i(x_j) \leq (b_j \rightarrow b_i). \tag{17}$$

Let us show that under the same conditions as those in [19] solvability of systems (10) and (11) follows from Theorem 1.

**Theorem 2** *Let fuzzy sets $A_1, \dots, A_n \in [0,1]^X$ and $B_1, \dots, B_n \in [0,1]^Y$, where $X, Y \subseteq \mathbb{R}$, be normal, convex and linearly ordered in the sense that if $i \leq j$, then $A_i \leq A_j$ and $B_i \leq B_j$.* [*)] *Let moreover, fuzzy sets $A_1, \dots, A_n$ fulfil the Ruspini condition, i.e.*

$$\sum_{i=1}^{n} A_i(x) = 1, \; x \in X.$$

*Then the system (10) is solvable.*

[*)] We say that normal, convex fuzzy sets $A \in [0,1]^{\mathbb{R}}$ and $B \in [0,1]^{\mathbb{R}}$ are such that $A \leq B$ if for all $\alpha \in (0,1]$, $A_\alpha \leq B_\alpha$.

PROOF: Let us choose arbitrary $y \in Y$ and denote $b_i = B^{\uparrow}_i(y)$, $i = 1, \dots, n$. We will prove that the system

$$A^{\uparrow}_i \circ R = b_i, \quad i = 1, \dots, n,$$

where the "at least" modification $A^{\uparrow}_i$ is computed in accordance with (8), $R \in [0,1]^X$ is unknown, is solvable. For this purpose we will verify the conditions of Theorem 1. At first, we will show that for all $i, j = 1, \dots, n$, the inequality (16) is valid for the sets $A^{\uparrow}_i$ and $A^{\uparrow}_j$. Assume that $i < j$ and on the basis of the assumption about linear ordering and the Ruspini condition, choose core points $x_i, x_j$ of $A_i, A_j$, such that $x_i < x_j$. It is not difficult to show that $A^{\uparrow}_i(x_i) = 1$, $A^{\uparrow}_i(x_j) = 1$, $A^{\uparrow}_j(x_i) = 0$ and $A^{\uparrow}_j(x_j) = 1$. Moreover, for all $x \in X$, $A^{\uparrow}_j(x) \leq A^{\uparrow}_i(x)$. Therefore, the condition (16) in the forms

$$A^{\uparrow}_i(x_j) \leq \bigwedge_{x \in X} (A^{\uparrow}_j(x) \rightarrow A^{\uparrow}_i(x)),$$

and

$$A^{\uparrow}_j(x_i) \leq \bigwedge_{x \in X} (A^{\uparrow}_i(x) \rightarrow A^{\uparrow}_j(x)),$$

is valid.

Let us verify the condition (17). Similarly to above, for all $y \in Y$, $B^{\uparrow}_j(y) \leq B^{\uparrow}_i(y)$ and therefore, $b_j \leq b_i$. It follows that $b_j \rightarrow b_i = 1$ so that

$$A^{\uparrow}_i(x_j) \leq b_j \rightarrow b_i.$$

Because $A^{\uparrow}_j(x_i) = 0$, it also holds that

$$A^{\uparrow}_j(x_i) \leq b_i \rightarrow b_j.$$

Thus, the conditions of Theorem 1 are verified and the system (10) is solvable. □

**Remark 2** *The proofs of Theorem 1 and Theorem 2 reveal that "at least" modifications of normal convex fuzzy sets with the Ruspini condition are upper sets of several fuzzy preorders, the coarsest is the one given by*

$$\bigwedge_{i=1}^{n} (A^{\uparrow}_i(x) \rightarrow A^{\uparrow}_i(y)).$$

**Remark 3** *Theorems 1 and 2 have been formulated and proved for "at least" modifications of fuzzy sets. Similar results (about solvability of the system (11)) can be easily obtained for "at most" modifications if we notice that lower sets are at the same time upper sets of the opposite fuzzy relation that is a fuzzy preorder too.*

### 3.4. Interpolativity of modified fuzzy rule bases with $\inf \rightarrow$composition

In this Section, we show how the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder helps to analyze the problem of solvability of modified systems with the inf $\rightarrow$composition. Let us remark that in [19], this problem was not solved and been proclaimed as open. Our explanation is as follows.

On the basis of our theory, the system with the inf $\rightarrow$composition admits only opposite modifications on the left and right sides, i.e. if fuzzy sets $A_i$ in (2) are modified by "at least", then the corresponding fuzzy sets $B_i$ should be modified by "at most" and vice versa. In [19], the systems with the inf $\rightarrow$composition have been considered with the same types of modifications, and therefore, they could not be solved. Below we give the justification of this conclusion.

The following theorem is again an adaptation of the result about solvability of the system (2) that has been proved in [10, 11].

**Theorem 3** *Let fuzzy sets $A_1, \ldots, A_n \in L^X$ be normal and $x_1, \ldots, x_n \in X$ be pairwise different elements such that for all $i = 1, \ldots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \ldots, n$, (16) holds true. Then there exists a fuzzy preorder $Q$ such that for all $i = 1, \ldots, n$,*

$$A_i = A^{\uparrow}_{Q,i},$$

*where $A^{\uparrow}_{Q,i}$ is the upper set of $A_i$ with respect to $Q$. Moreover, both systems*

$$A_i \lhd R = b_i, \quad i = 1, \ldots, n,$$

*and*

$$A^{\uparrow}_{Q,i} \lhd R = b_i, \quad i = 1, \ldots, n,$$

*are solvable if and only if*

$$A_i(x_j) \leq (b_i \rightarrow b_j). \tag{18}$$

If we compare conditions (17) and (18) in Theorems 1 and 3, then we see that elements $b_i, b_j$ in their right-hand sizes are oppositely related. This fact together with the proof of Theorem 2 leads us to the conjecture that solvability of the modified system of fuzzy relation equations with the inf $\rightarrow$composition can be established if modifications of left- and right-hand sizes are different (opposite).

Let us show that under the same conditions as those in [19] solvability of the following systems

$$A^{\uparrow}_i \lhd R = B^{\downarrow}_i, \quad i = 1, \ldots, n, \tag{19}$$

and

$$A^{\downarrow}_i \lhd R = B^{\uparrow}_i, \quad i = 1, \ldots, n, \tag{20}$$

follows from Theorem 3. We will consider the case (19) only, because the system (20) can be analyzed similarly.

**Theorem 4** *Let fuzzy sets $A_1, \ldots, A_n \in [0, 1]^X$ and $B_1, \ldots, B_n \in [0, 1]^Y$, where $X, Y \subseteq \mathbb{R}$, be normal, convex and linearly ordered in the sense that if $i \leq j$, then $A_i \leq A_j$ and $B_i \leq B_j$ (the meaning of $\leq$ is the same as in Theorem 2). Let moreover, fuzzy sets $A_1, \ldots, A_n$ fulfil the Ruspini condition, i.e.*

$$\sum_{i=1}^{n} A_i(x) = 1, \, x \in X.$$

*Then the system (19) is solvable.*

PROOF: Let us choose arbitrary $y \in Y$ and denote $b_i = B^{\uparrow}_i(y)$, $i = 1, \ldots, n$. We will prove that the system

$$A^{\uparrow}_i \lhd R = b_i, \quad i = 1, \ldots, n,$$

where the "at least" modification $A^{\uparrow}_i$ is computed in accordance with (8), $R \in [0, 1]^X$ is unknown, is solvable. For this purpose we will verify the conditions of Theorem 3. Similarly to the proof of Theorem 2, we can show that for all $i, j = 1, \ldots, n$, the inequality (16) is valid for the sets $A^{\uparrow}_i$ and $A^{\uparrow}_j$. Therefore, it remains to proof (18).

Assume that $i < j$. It is not difficult to prove that under the assumptions on $B_1, \ldots, B_n$, for all $y \in Y$, $B^{\downarrow}_i(y) \leq B^{\downarrow}_j(y)$ and therefore, $b_i \leq b_j$. It follows that $b_i \rightarrow b_j = 1$ and then

$$A^{\uparrow}_i(x_j) \leq b_i \rightarrow b_j.$$

Because $A^{\uparrow}_j(x_i) = 0$, it holds that

$$A^{\uparrow}_j(x_i) \leq b_j \rightarrow b_i.$$

Thus, the conditions of Theorem 2 are verified and the system (19) is solvable. $\square$

## 4. Conclusion

We investigated monotonously increasing (decreasing) systems and their interpolativity. With this respect, fuzzy sets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in (3) are replaced by their "at least" (or "at most") modifications. In this contribution, we proved that the proposed modifications transform fuzzy sets into upper and lower sets of a certain fuzzy preorder. On the basis of this, we utilized the theory of solvability of systems of fuzzy relation equations in a space with fuzzy preorder and proposed a justification of the solvability of the modified systems (10) and (11) with the sup $*$ composition. We explained why the solvability of the similarly modified systems with the inf $\rightarrow$composition cannot be established. Moreover, we showed that only opposite modifications on the left and right-hand sides of the system (2) guarantee its solvability.


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