$1/f$ Fractal Signals Denoising with Dual-Tree Complex Wavelet Transform

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Abstract

In the paper, an algorithm based on Dual-Tree Complex Wavelet Transform is proposed for $1/f$ process denoising. Use the variance of the wavelet coefficients at different scales to estimate the parameters of $1/f$ process. Adopting Maximum a Posteriori estimator estimates the wavelet coefficients of $1/f$ process. The simulation results show that the method is effective. And comparing with other methods this method doesn't need to know the statistical characteristic of the added white noise and the parameters of the $1/f$ process.

Keywords: Fractal signals; Dual-Tree Complex Wavelet Transform; Bayesian Estimation;

1. Introduction

There are many fractal signals in natural, such as the ocean waves, the turbulent flows, and in the pattern of errors on communication channels. An important class of fractal signals is $1/f$ process. $1/f$ process exhibits Long-Term Correlation structure and nonstationarity. It can not be captured with traditional method for processing signals.

The emergence of powerful wavelet basis representations is important to signal processing. This theory arise highly natural and useful representations for fractal phenomena. Researchers have proposed some methods based on wavelet transform to solve this problem. Wornell [1] applied the method of Maximum Likelihood in wavelet domain to estimate $1/f$ process; Chen [2] proposed using Multiscale Wiener filter for the restoration of fractal signals; He Kai [3] use DWT denoising the $1/f$ process.

This paper, according to the wavelet-based features of $1/f$ process, exploits a new method to estimate the parameter of $1/f$ process. Consider Dual-Tree Complex Wavelet Transform as a filter for $1/f$ processes which are embedded in white background noise. Then wavelet coefficients of the original signal are estimated by Bayesian framework with Maximum a Posterior density. Contract with other method, this algorithm does not need to know the parameter of the fractal signal and the statistical characteristic of added white noise. It does well in estimating the $1/f$ process from the added white noise.

2. Bayesian Denoising

The term $1/f$ process [4] has been described as the form:

$$s_x(w) \sim \frac{\sigma_x^2}{|w|} \qquad 1 \le \gamma \le 2 \tag{1}$$

γ is spectral parameter.

It is supposed that we have observed $r(t)$ of a $1/f$ process $x(t)$ embedded in zero-mean added white noise $w(t)$, so:

$$r(t) = x(t) + w(t) \tag{2}$$

The aim of denoising is to attain $\hat{x}(t)$, an estimation of $x(t)$ from the receiving signal $r(t)$, and let the mean squared error is minimum.

Because the Wavelet Transform is linear, in wavelet domain wavelet coefficients can be described as:

$$r_n^m = x_n^m + w_n^m \tag{3}$$

r_n^m is the wavelet coefficients vector of the receiving signal $r(t)$; x_n^m is the wavelet coefficients vector of $x(t)$ that is going to estimate; w_n^m is the wavelet coefficients vector of the noise $w(t)$.

There are many methods to estimate x_n^m. The Maximum Posteriori estimation (MAP) is the most classical.

$$\begin{aligned}\hat{x}(r) &= \arg\max_x p(x \mid r) \\ &= \arg\max_x [p(r \mid x)p(x)] \\ &= \arg\max_x [p_w(r-x)p(x)]\end{aligned} \tag{4}$$

To be estimated $x(t)$, noise distribution and distribution of $x(t)$ should be known. In general,

$p(x)$ is unknown. Supposed $p(x) \sim N(\mu,\sigma^2)$, the MAP estimation for x_n^m is:

$$\hat{x} = (\sigma_r^2 + \sigma_w^2 \mu) \big/ (\sigma + \sigma_w^2) \tag{5}$$

3. Wavelet Based features for $1/f$ Process[4]

It has been proved that the wavelet coefficients of $1/f$ process in the same scale m are approximately stationary random process. The variance of the wavelet coefficients x_n^m is $\sigma^2 \cdot 2^{-\gamma m}$ [4].

Let $w(t)$ be the added white noise. The different scales in wavelet transform, the wavelet coefficients of $w(t)$ are respectively w_k^j , w_k^{j+1} . It exhibits that the wavelet coefficients of white noise are uncorrelated, and the variance of the white noise is still σ_w^2 [5]. $r_n^m = x_n^m + w_n^m$, $\operatorname{var} x_n^m = \sigma^2 \cdot 2^{-\gamma m}$, then the variance of r_n^m can be described as:

$$\begin{aligned} \operatorname{var} r_n^m &= \sigma_r^2 \\ &= \sigma^2 \cdot 2^{-\gamma m} + \sigma_w^2 \end{aligned} \tag{6}$$

4. Dual-Tree Complex Wavelet Transform

Approximate shift invariance and good directionality of DT CWT can remove “ringing” of the edge. The Discrete Wavelet Transform (DWT) is commonly used in signal processing. This work is good for compression but its use for other signal analysis and reconstruction tasks has been hampered by two main disadvantages [6]:

- Lack of shift invariance, which means that small shifts in the input signal can cause major variations in the distribution of energy between DWT coefficients at different scales.
- Poor direction selectivity for diagonal features, because the wavelet filters are separable and real.

The DT-CWT ensures filtering results with no distortion and good ability for feature localization. N Kingsbury [6] observed that it can be achieved approximate shift invariance with a real DWT by doubling the sampling rate at each level of the tree. The samples must be evenly spaced, such as Tree a in fig.1. There are two parallel fully-decimated trees, a and b in fig.1, supposed that the delays of filters H_{0b} and H_{1b} are one sample offset from the delays of H_{0a} and H_{1a}, which ensures that the level 1 down samplers in tree b pick the opposite samples to those in tree a [5]. We then find that, to get uniform intervals between samples from the two trees below level 1, the filters in one tree must provide delays that are half a sample different from those in the opposite tree. For linear phase, it is required that the odd-length filters should be in one tree and even-length filters in another. We can interpret the outputs of each tree as the real part and imaginary part through the complex wavelet transform. So in the domain of DT CWT domain, then $r_n^m = x_n^m + w_n^m$ can express as:

$$r_{m,n}^c = x_{m,n}^c + w_{m,n}^c \tag{7}$$

$r_{m,n}^c$, $x_{m,n}^c$ and $w_{m,n}^c$ can also express as:

$$\begin{aligned} r_{m,n}^c &= r_{real,m} + j r_{imag,m} \\ x_{m,n}^c &= x_{real,m} + j x_{imag,m} \\ w_{m,n}^c &= w_{real,m} + j w_{imag,m} \end{aligned} \tag{8}$$

$r_{real,m}$ is the real part wavelet coefficients of $r_{m,n}^c$, and $r_{imag,m}$ is the imaginary part wavelet coefficients of $r_{m,n}^c$.

With Formula 7, 8, we can attain Formula 9:

$$\begin{aligned} r_{real,m} &= x_{real,m} + j w_{real,m} \\ r_{imag,m} &= x_{imag,m} + j w_{imag,m} \end{aligned} \tag{9}$$

Using Formula 9 and Formula 5, the estimation for real part wavelet coefficients and imaginary part wavelet coefficients of $x(t)$ can be attained:

$$\begin{aligned} \hat{x}_{real} &= (\sigma^2_{real,m,y_{real,m}} + \sigma_w^2 \mu_{real,m}) \big/ (\sigma^2_{real,m} + \sigma_w^2) \\ \hat{x}_{imag} &= (\sigma^2_{imag,m,y_{imag,m}} + \sigma_w^2 \mu_{imag,m}) \big/ (\sigma^2_{imag,m} + \sigma_w^2) \end{aligned} \tag{10}$$

Complex wavelet coefficients that can express as:

$$\hat{x} = \hat{x}_{real} + j\hat{x}_{imag} \tag{11}$$

Then reconstruct $\hat{x}$ and attain the signal after denoising the added white noise.

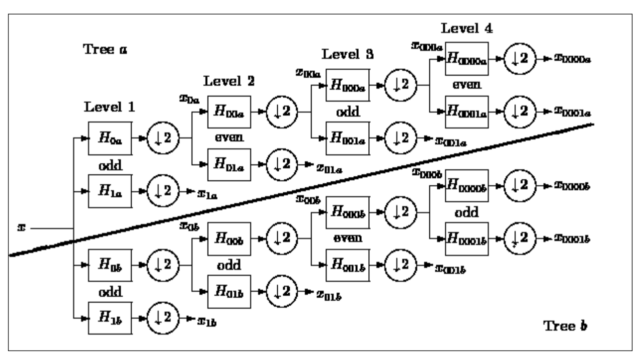


Fig. 1: DT CWT of 1-D Signal [6]

5. Estimate σ_w, γ and $1/f$ Process

The standard deviation σ_w has been estimated by a robust median estimator.

$$\sigma_w = median(|y_{diag}|)/0.675 \tag{12}$$

y_{diag} is the coefficient of the direction of $75°$

r_n^m is the wavelet coefficients vector of the receiving signal $r(t)$ at scale m, r_n^{m+1} is the wavelet coefficients vector of $r(t)$ at scale $m+1$. We can use r_n^m and r_n^{m+1} to compute parameter γ.

$$\begin{aligned}\mathrm{var}[r_n^{m+1}]/\mathrm{var}[r_n^m] &= (\sigma^2 \cdot 2^{-\gamma(m+1)} + \sigma_w^2)/(\sigma^2 \cdot 2^{-\gamma m} + \sigma_w^2) \\ &= [2^{-\gamma}(\sigma^2 \cdot 2^{-\gamma m} + \sigma_w^2) - \sigma_w^2 \cdot 2^{-\gamma} + \sigma_w^2]/(\sigma^2 \cdot 2^{-\gamma m} + \sigma_w^2) \\ &= 2^{-\gamma} + \sigma_w^2(1 - 2^{-\gamma})/\mathrm{var}[r_n^m] \end{aligned} \tag{13}$$

For Formula 13 use logarithm, then parameter γ can get though Formula 13.

$$\gamma = \log_2[(\mathrm{var}[r_n^{m+1}] - \sigma_w^2)/(\mathrm{var}[r_n^m] - \sigma_w^2)] \tag{14}$$

There are m scales, so it can be estimated $m-1$ γ. We use the average to reflect γ.

$$\hat{\gamma} = (1/M-1)\sum_{i=1}^{M-1}\gamma_i \tag{15}$$

Because $\mathrm{var}\, r_n^m = \sigma_r^2 = \sigma^2 \cdot 2^{-\gamma m} + \sigma_w^2$ (Formula 6), then we can get variance σ^2 of original signal as follow:

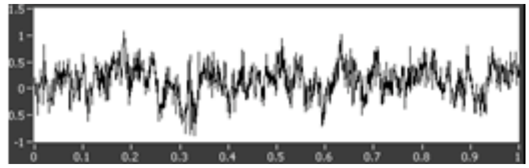
$$\sigma^2 = (\mathrm{var}[r_n^m] - \sigma_w^2)/2^{-\gamma m} \tag{16}$$

As mentioned above, the algorithm to estimate $1/f$ process can be described as follow:

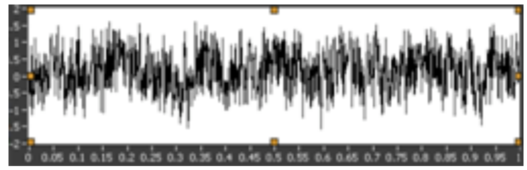
- Step1: processing $r(t)$ with DT CWT, gain the wavelet coefficients of $r(t)$;
- Step2: using Formula 12, 13, 14 and 15 to estimate the variance of noise σ_w^2, γ.
- Step3: using Formula 9, 10, 11 and 16 to estimate the $1/f$ process $x(t)$
- Step4: reconstruct $\hat{x}$.

6. conclusions

The algorithm carried out by Matlab and Labview. The $1/f$ process and added white noise come into being with Labview by stochastic. Fig.2 is one sample in a set of $1/f$ process experiment. The algorithm programmed with Matlab. a) $1/f$ fractal single brought with Labview, b) receiving signal that has been added white noise, c) $1/f$ signal that we estimated by the program.



a)



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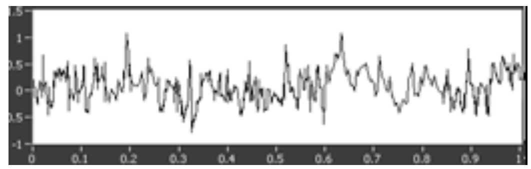
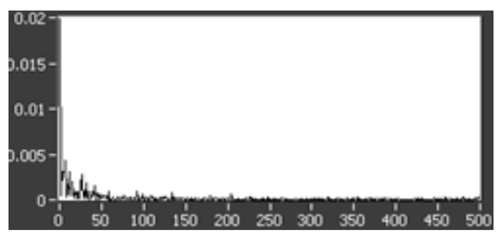


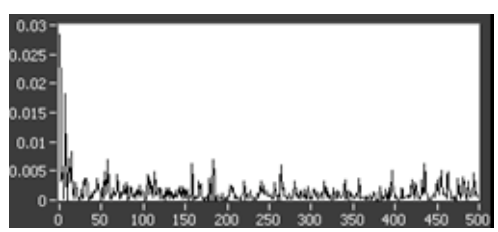
c)

Fig.2: The Result of Experiment

Fig.3 is the frequency spectrum of the receiving signal, the original signal, and the estimation signal in Fig.2. It illustrates the frequency spectrum of the original signal and the estimation signal is very similar. Experimental results show that the method can effectively estimate the $1/f$ signal.



a)



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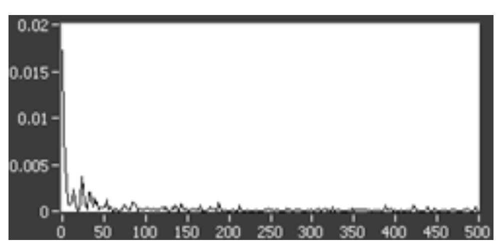

c)

Fig.3: Frequency Spectrum

Contract with other method, the method does not need to know the parameter of the fractal signal and the statistical characteristic of assed white noise. The simulation results show that the method is effective and simple.

This work only considers the correlation between the scales, and do not consider the correlation in each scale. So in future, the work will be in this aspect. For the good directionality of DT CWT, the method can be used in 2-D image denoising; it will be another key work in future.

7. References

[1] Gregory W. Wornell, "Estimation of Fractal Signals from Noisy Measurements Using Wavelet," *IEEE Trans. Signal Processing,* 40(3), pp.611-623, 1992

[2] B. S. Chen, G. W. Lin, "Multiscale Wiener Filter for the Restoration of Fractal Signal: Wavelet Filter Bank Approach," *IEEE Trans. Signal Processing,* 42(11), pp.972-982, 1994.

[3] K. He, S.X. Wang, Y. S. Dai, "Study on the Wavelet-based Denoising for $1/f$ Fractional Signals," Journal of Jilin University (Engineering and Technology Edition), 33(1), pp.77-81, 2003.

[4] Gregory W. Wornell, "Wavelet-Based Representations for the $1/f$ Family of Fractal Process," *Proc. IEEE,* 81, pp.1428-1450, 1993.

[5] S. Mallet, W. L. Wen, "Singularity Detection and Processing with Wavelets," *IEEE Trans. Information Theory,* 38(2), pp.617-643, 1992.

[6] N. Kingsbury, "Complex Wavelet for Shift Invariant Analysis and Filtering of Signals," *Journal of Applied and Computational Harmonic Analysis,* 10(3), pp.234-253, 2001.

[7] Z. Y. Xiao, S. L. Peng, H. Han, "Image Denoising Based on Local Mixture Gaussian Model of Dual-Tree Complex Wavelet Domain," *Journal of Computer Aided Design and Computer Graphics,* 17(7), pp.1536-1543, 2005.