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Uniform observability analysis for structured bilinear systems: A graph-theoretic approach

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Abstract

This paper deals with the analysis of generic uniform observability for structured bilinear systems. The proposed method is based on a graph-theoretic approach and assumes only the knowledge of the system's structure. We express, in graphical terms two conditions, the first one is necessary and the second one is sufficient, to check whether or not they are generically uniformly observable. These conditions are quite easy to verify because they are based on comparisons of integers and on finding edges and subgraphs in a digraph. Therefore, our approach is very well suited to study large scale and/or uncertain systems.

Keywords: Structured bilinear systems, uniform observability, graph theory, generic properties.

1 Introduction

For nonlinear systems, the observability notion is based on the concept of indistinguishability of two initial states for every input. In fact, it results from [3, 13, 15, 21, 24] that a nonlinear system is observable if it can be found an input such that any pair of initial states are distinguishable by observation of the corresponding output. The distinguishability of two initial states is thus depending on the input applied to the system. In this respect, for nonlinear systems, the notion of universal input is defined [13, 21] as an input for which every pair of initial states can be distinguished by observation of the output. A system, for which every input is universal is said uniformly observable knowing that we are interested in the inputs which take only real values. We call these

inputs admissible inputs. In other words, a system is uniformly observable when we can express its whole state components in function of the measured variables and their derivatives for all the admissible input values. The analysis of the uniform observability is important since it informs us on the possible alteration of the state distinguishability for some input values. This information is useful obviously for the design of nonlinear observers and also for control, fault detection and isolation and reconfiguration schemes which both use the estimated state variables.

Many studies are dedicated to the uniform observability [10, 11, 24]. In most cases, either the authors are interested only in the single output case or they provide geometric trivial conditions to achieve uniform observability. These conditions are mainly difficult to apply to large scale systems because they are based on the computation of the rank of the observability matrix. Note that some studies assume that the system can be transformed into some triangular form which ensures the uniform observability and which is used in the design of nonlinear observers [2, 7, 14, 19]. Finally, in the single output case, a triangular canonical form for uniformly observable systems is provided [9].

In this context, this paper is dedicated to the characterization of the uniform observability of a particular class of nonlinear systems. According to their applicative interest and their intrinsic simplicity, we focus on multi-output bilinear systems. This class of systems, whose dynamics are jointly linear in the state and the input was introduced in control theory in the 1960's. Industrial process control, economics and biology (switched circuits, mechanical brakes, controlled suspension systems, immunological systems, population growth, enzyme kinetics, ...) provide examples of bilinear systems (BLS). Furthermore, this kind of systems are simpler and better understood than most other nonlinear systems. Finally, the usual linearization of a nonlinear control system near an equilibrium point can be improved by using a bilinear approximation. For these reasons many works deal with BLS.

Generally, BLS are studied in connection with the theories of time-variant linear systems and matrix Lie groups [4, 16, 20]. However, the use of such tools assumes the exact knowledge of the state space matrices characterizing the system's model. In many modeling problems, these matrices have a number of fixed zero entries determined by the physical laws while the remaining entries are not known precisely. This is also true during the system conception stage. To study the properties of these systems in spite of the poor knowledge we have on them, the idea is that we only keep the zero/nonzero entries in the state space matrices. Thus, we consider models where the fixed zeros are conserved while the nonzero entries are replaced by free parameters. There is a huge amount of interesting works in the literature using this kind of models called structured

models. These models are useful to describe the class of systems having the same structure and they capture most of the structural available information from physical laws. Moreover, their study requires a low computational burden which allows to deal with large scale systems. Because of these features, we think that structured systems are suited to study the uniform observability and subsequently this paper deals with this kind of systems.

Many studies on structured systems are related to the graph-theoretic approach. Until now, this approach is principally dedicated to linear systems for which structural properties, such as controllability, observability, solvability of several classical control problems including disturbance rejection, input-output decoupling, fault detection and isolation, are studied. Survey paper [8] reviews the most significant results in this area. From these works, it results that the graph-theoretic approach provides simple and elegant solutions and so is very well suited to analyse large scale or/and uncertain systems. Unfortunately, few works based on graph-theoretic methods deal with nonlinear systems. Among them, [22] gives sufficient conditions to fulfill the observability of bilinear systems and more recently, [1] provides sufficient conditions for checking the uniform observability of nonlinear systems which are preliminarily put in a canonical form and where the nonlinearities satisfy Lipschitz conditions.

The aim of this paper is to provide a necessary and a sufficient conditions to achieve the uniform observability of multi-output structured bilinear systems (SBLS). The study of this property is based on a graphic representation, which does not necessitate any preliminary computation. Comparatively with the results of [9] which enounce necessary and sufficient conditions to the uniform observability of single output nonlinear systems, our results concern only a particular class of nonlinear system, but for multi-output and structured systems. In the single output case, our conditions are also necessary and sufficient. Moreover, in [24], the author give a geometric necessary and sufficient criterion to the uniform observability of a multi-output bilinear system. This criterion is not a structural one *i.e.* depends on the parameters of the system, is quite trivial as it assesses that a system is uniformly observable if and only if the rank of the observability matrix is equal to the dimension of the state for every input. This criterion is difficult to verify especially in the case of large scale or uncertain systems. Starting from an equivalent criterion to the one given in [24] and using original mathematical tools, we give two conditions (one necessary and the other sufficient) to the uniform observability of structured bilinear systems which are more "computable" and which depends only on the structure of the systems and not on its parameters. The main interest of the proposed conditions are their easy computability and the fact that they allows to handle system's structure.

The paper is organised as follows: after section 2, which is devoted to the problem formulation, a digraph representation of SBLS is given in section 3. The main result of the paper is provided in section 4 and is illustrated with an example in section 5. Finally, some concluding remarks are made.

2 Problem statement

In this paper, we are interested in the uniform observability of SBLS:

$$(\Sigma_\Lambda) : \begin{cases} \dot{x} &= A^\lambda x + \sum_{i=1}^m u_i B_i^\lambda x \\ y &= C^\lambda x \\ z &\stackrel{def}{=} \dot{x} - \sum_{i=1}^m u_i B_i^\lambda x \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state vector, the input vector and the output vector. New variable $z \in \mathbb{R}^n$ called dynamic variable, is introduced in this paper. $A^\lambda \in \mathbb{R}^{n \times n}$, $B_i^\lambda \in \mathbb{R}^{n \times n}$ for $i = 1, \dots, m$ and $C^\lambda \in \mathbb{R}^{p \times n}$ represent matrices whose elements are either fixed to zero or assumed to be free non-zero parameters. The vector of these parameters is $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)^T$. If all the non-zero parameters λ_i are fixed, we obtain an admissible realization of structured system (Σ_Λ) . Theoretic properties of each realization can be studied according to the values of λ_i .

We say that a property is true generically if it is true for almost all realizations of the system or equivalently for almost all parameter values [6]. Here “for almost all parameter values” is to be understood [8] as “for all parameter values ($\Lambda \in \mathbb{R}^h$) except for those in some proper algebraic variety in the parameter space”. The proper algebraic variety for which the property is not true is the zero set of some nontrivial polynomial with real coefficients in the h parameters of the system, which can be written down explicitly. Recall that a proper algebraic variety is an algebraic variety which has Lebesgue measure zero. A property which is true for almost all parameters values is also said to be true generically or to be a generic property for the considered structured system.

In order to precise the notion of generic uniform observability, we adapt hereafter the definition of the uniform observability [9, 24] to the SBLS:

Definition 1 *Structured bilinear system (Σ_Λ) is generically uniformly observable if, for almost all its realizations and for any input value, any pair of initial states $x^0(0)$ and $x^1(0)$ are distinguishable by observation of the corresponding outputs $y^0(t)$ and $y^1(t)$ for $t \geq 0$.*

From the latter definition, we deduce that a structured system is generically uniformly observable iff for almost all its realizations and for all the input values we can express $x(t)$ in function of $y(t)$,

$u(t)$ and their derivatives *i.e.* there always exists a nonsingular function f such that:

$$x(t) = f\left(y(t), \dot{y}(t), y^{(2)}(t), \dots, y^{(\kappa_y)}(t), u(t), \dot{u}(t), u^{(2)}(t), \dots, u^{(\kappa_u)}(t)\right) \quad (2)$$

Indeed, according to Definition 1, if a structured system is generically uniformly observable then we can always (for all the input values) express its initial conditions in function of the output and its derivatives. In this case, using the system's dynamics (Σ_Λ) , we can write an expression of the form (2). Reciprocally, if we can write an expression like (2), then we can determine in a unique way, using the system's dynamics (Σ_Λ) , the initial conditions $x(0)$ in function of the corresponding output $y(t)$, whatever the value of the known input $u(t)$.

In the case of bilinear systems, since $y(t)$ is a linear function of $x(t)$ and since the dynamics are jointly linear in the state and the input, if we have an expression of $x(t)$ in function of the measured variables, then it is necessarily in the form:

$$x(t) = \Gamma\left(u(t), \dot{u}(t), u^{(2)}(t), \dots, u^{(\kappa_u)}(t)\right) \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\kappa_y)}(t) \end{pmatrix} \quad (3)$$

$\Gamma\left(u(t), \dot{u}(t), u^{(2)}(t), \dots, u^{(\kappa_u)}(t)\right)$ is a matrix whose elements are polynomials in the input components and their derivatives, $\kappa_y < n$ and $\kappa_u < n$.

From the previous settings, a system is uniformly observable if and only if we can deduce from the equations (Σ_Λ) , according to the differentiation of output $y(t)$, a nonsingular system parameterized by the input and its derivatives and where the unknown variables are the state components.

Hereafter, we enounce a first proposition which allows to define a triangular form ensuring the generic uniform observability of a bilinear system:

Proposition 1 *A structured bilinear system is generically uniformly observable if there exist matrices $\Gamma_0^\lambda, \Gamma_1^\lambda, \dots, \Gamma_\kappa^\lambda$ such that we can generically deduce from equations (Σ_Λ) relations of the form:*

$$\begin{array}{l} 1^{st} \text{ derivation order of the output} \\ 2^{nd} \text{ derivation order of the output} \\ \vdots \\ k^{th} \text{ derivation order of the output} \end{array} \left| \begin{array}{l} Y_0 \stackrel{def}{=} y = C^\lambda x \\ \Gamma_0^\lambda \dot{Y}_0 = C_1^\lambda x + \sum_{i=1}^m u_i H_{0,i}^\lambda Y_0 \\ Y_1 \stackrel{def}{=} \begin{pmatrix} C^\lambda \\ C_1^\lambda \end{pmatrix} x \\ \Gamma_1^\lambda \dot{Y}_1 = C_2^\lambda x + \sum_{i=1}^m u_i H_{1,i}^\lambda Y_1 \\ \vdots \\ Y_{k-1} \stackrel{def}{=} \begin{pmatrix} C^\lambda \\ C_{k-1}^\lambda \end{pmatrix} x \\ \Gamma_{k-1}^\lambda \dot{Y}_{k-1} = C_k^\lambda x + \sum_{i=1}^m u_i H_{k-1,i}^\lambda Y_{k-1} \\ Y_k \stackrel{def}{=} \begin{pmatrix} C^\lambda \\ C_k^\lambda \end{pmatrix} x \end{array} \right. \quad (4)$$

with $g_rank \begin{pmatrix} C^\lambda \\ C_1^\lambda \\ \vdots \\ C_k^\lambda \end{pmatrix} = n$. Note that, by construction of form (4), $g_rank \begin{pmatrix} C^\lambda \\ C_1^\lambda \\ \vdots \\ C_k^\lambda \end{pmatrix} = g_rank \begin{pmatrix} C^\lambda \\ C_k^\lambda \end{pmatrix}$.

Here $g_rank(M^\lambda)$ denotes the generic rank [8, 12, 17, 18] of structured matrix M^λ . Roughly speaking the generic rank of a structured matrix M^λ with h free parameters $\lambda_1, \dots, \lambda_h$ is defined as $\max_{(\lambda_1, \dots, \lambda_h)^T \in \mathbb{R}^h} rank(M^\lambda)$.

In form (4), to be coherent with (Σ_Λ) 's equations, $C_1^\lambda \stackrel{def}{=} \Gamma_0^\lambda C^\lambda A^\lambda$, $H_{0,i}^\lambda$ is such that $H_{0,i}^\lambda C^\lambda = \Gamma_0^\lambda C^\lambda B_i^\lambda$ and for $j = 1, \dots, k-1$, $C_{j+1}^\lambda \stackrel{def}{=} \Gamma_j^\lambda \begin{pmatrix} C^\lambda \\ C_j^\lambda \end{pmatrix} A^\lambda$ and $H_{j+1,i}^\lambda$ is such that $H_{j+1,i}^\lambda \begin{pmatrix} C^\lambda \\ C_j^\lambda \end{pmatrix} = \Gamma_j^\lambda \begin{pmatrix} C^\lambda \\ C_j^\lambda \end{pmatrix} B_i^\lambda$.

Proof:

Assume that we can put the system into the form (4), then we can write

$$\begin{pmatrix} C^\lambda \\ C_1^\lambda \\ C_2^\lambda \\ \vdots \\ C_k^\lambda \end{pmatrix} x(t) = \begin{pmatrix} y(t) \\ \Gamma_0^\lambda \dot{y}(t) \\ \Gamma_1^\lambda \dot{Y}_1(t) \\ \vdots \\ \Gamma_{k-1}^\lambda \dot{Y}_{k-1}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ \sum_{i=1}^m u_i(t) H_{0,i}^\lambda Y_0(t) \\ \sum_{i=1}^m u_i(t) H_{1,i}^\lambda Y_1(t) \\ \vdots \\ \sum_{i=1}^m u_i(t) H_{k-1,i}^\lambda Y_{k-1}(t) \end{pmatrix} \quad (5)$$

Since we have $Y_0(t) = y(t)$ and for $j \geq 1$, $Y_j(t) = \begin{pmatrix} y(t) \\ \Gamma_{j-1}^\lambda \dot{Y}_{j-1}(t) - \sum_{i=1}^m u_i(t) H_{j-1,i}^\lambda Y_{j-1}(t) \end{pmatrix}$, then all the Y_j can be expressed in function of the output, the input and their derivatives i.e.

$$Y_j(t) = \phi_j(u(t), \dot{u}(t), \dots, u^{(j)}(t)) \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(j)}(t) \end{pmatrix}.$$

Therefore, we can write $\begin{pmatrix} C^\lambda \\ C_1^\lambda \\ \vdots \\ C_k^\lambda \end{pmatrix} x(t) = \Phi(u(t), \dot{u}, u^{(2)}(t), \dots, u^{(\kappa_u)}(t)) \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\kappa_y)}(t) \end{pmatrix}$ as the generic rank of $\Gamma_x \stackrel{def}{=} \begin{pmatrix} C^\lambda \\ C_1^\lambda \\ \vdots \\ C_k^\lambda \end{pmatrix}$

is equal to $n = \dim(x)$, we can write $x(t) = (\Gamma_x^T \Gamma_x)^{-1} \Phi(u(t), \dot{u}, u^{(2)}(t), \dots, u^{(\kappa_u)}(t)) \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\kappa_y)}(t) \end{pmatrix}$. Con-

sequently, starting from equations (4) we obtain, for any input value, an expression on the form (3). This concludes to the generic uniform observability of structured system (Σ_Λ) . $\triangle$

Remark 1 For single output systems as it was proved in [9], the condition of Proposition 1 is necessary and sufficient to ensure the generic observability of a bilinear system. Nevertheless, in the case of multi-output systems, the following counter-example shows that this condition is not

necessary:

$$\begin{aligned}
\dot{x}_1 &= \lambda_1 x_3 + \lambda_2 u_1 x_3 + \lambda_3 x_4 \\
\dot{x}_2 &= \lambda_4 x_3 + \lambda_5 u_1 x_4 \\
\dot{x}_3 &= \lambda_7 x_4 \\
\dot{x}_4 &= \lambda_8 x_3 \\
y_1 &= \lambda_9 x_1 \\
y_2 &= \lambda_{10} x_2
\end{aligned} \tag{6}$$

This system cannot be put on the form 4 of the paper. The observability matrix (by differentiating twice each output) is equal to

$$O(u, \dot{u}) = \begin{pmatrix} \lambda_9 & 0 & 0 & 0 \\ 0 & \lambda_{10} & 0 & 0 \\ 0 & 0 & \lambda_9(\lambda_1 + \lambda_2 u_1) & \lambda_9 \lambda_3 \\ 0 & 0 & \lambda_{10} \lambda_4 & \lambda_{10} \lambda_5 u_1 \\ 0 & 0 & \lambda_9(\lambda_2 \dot{u}_1 + \lambda_3 \lambda_8) & \lambda_9 \lambda_7(\lambda_1 + \lambda_2 u_1) \\ 0 & 0 & \lambda_{10} \lambda_5 \lambda_8 u_1 & \lambda_{10}(\lambda_4 \lambda_7 + \lambda_5 \dot{u}_1) \end{pmatrix}$$

To establish the non-uniform observability of system (6) we must have simultaneously:

$$\begin{aligned}
&\det \begin{pmatrix} \lambda_1 + \lambda_2 u_1 & \lambda_3 \\ \lambda_4 & \lambda_5 u_1 \end{pmatrix} = 0, \det \begin{pmatrix} \lambda_1 + \lambda_2 u_1 & \lambda_3 \\ \lambda_2 \dot{u}_1 + \lambda_3 \lambda_8 & \lambda_7(\lambda_1 + \lambda_2 u_1) \end{pmatrix} = 0, \det \begin{pmatrix} \lambda_1 + \lambda_2 u_1 & \lambda_3 \\ \lambda_5 \lambda_8 u_1 & \lambda_4 \lambda_7 + \lambda_5 \dot{u}_1 \end{pmatrix} = \\
&0, \det \begin{pmatrix} \lambda_4 & \lambda_5 u_1 \\ \lambda_2 \dot{u}_1 + \lambda_3 \lambda_8 & \lambda_7(\lambda_1 + \lambda_2 u_1) \end{pmatrix} = 0, \det \begin{pmatrix} \lambda_4 & \lambda_5 u_1 \\ \lambda_5 \lambda_8 u_1 & \lambda_4 \lambda_7 + \lambda_5 \dot{u}_1 \end{pmatrix} = 0 \quad \text{and} \\
&\det \begin{pmatrix} \lambda_2 \dot{u}_1 + \lambda_3 \lambda_8 & \lambda_7(\lambda_1 + \lambda_2 u_1) \\ \lambda_5 \lambda_8 u_1 & \lambda_4 \lambda_7 + \lambda_5 \dot{u}_1 \end{pmatrix} = 0.
\end{aligned}$$

Generically, u and $\dot{u}$ cannot verify simultaneously all these equations. So, System (6) is generically uniformly observable.

In the present work, we are interested in the question whether or not SBLS (Σ_Λ) can be generically transformed into the form (4) using a graph-theoretic approach. It turns out that this question can be answered by checking some simple graphic conditions. These conditions are sufficient to the generic uniform observability of SBLS (Σ_Λ). Note that we provide also a necessary condition to the generic uniform observability of SBLS (Σ_Λ).

Before presenting the main results of the paper, we give, in the next section, some definitions related to a graphical representation of SBLS.

3 Graphical representation of a structured bilinear system

This section is devoted to the definition of the digraph which is used to represent structured system (Σ_Λ) . This digraph is noted $\mathcal{G}(\Sigma_\Lambda)$ and is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. More precisely, the vertex set is defined by: $\mathcal{V} = \mathbf{X} \cup \mathbf{XU} \cup \mathbf{Y} \cup \mathbf{Z}$, where $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is the set of state vertices, $\mathbf{XU} = \{\mathbf{x}_j \mathbf{u}_i | \exists k \in \{1, \dots, n\} \text{ such that } B_i^\lambda(k, j) \neq 0\}$ is the set of X - U mixed vertices, $\mathbf{Y} = \{\mathbf{y}_1, \dots, \mathbf{y}_p\}$ is the set of output vertices, $\mathbf{Z} = \{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ is the set of dynamic variable vertices. The edge set is $\mathcal{E} = I\text{-edges} \cup A\text{-edges} \cup C\text{-edges} \cup B_i\text{-edges}$ with $I\text{-edges} = \{(\mathbf{x}_i, \mathbf{z}_i), i = 1, \dots, n\}$, $A\text{-edges} = \{(\mathbf{z}_i, \mathbf{x}_j) | A^\lambda(i, j) \neq 0\}$, $C\text{-edges} = \{(\mathbf{y}_i, \mathbf{x}_j) | C^\lambda(i, j) \neq 0\}$, for $i = 1, \dots, m$, $B_i\text{-edges} = \{(\mathbf{x}_j \mathbf{u}_i, \mathbf{z}_k) | B_i^\lambda(k, j) \neq 0\}$.

Here $M^\lambda(i, j)$ is the (i, j) th element of matrix M^λ and $(\mathbf{s}_1, \mathbf{s}_2)$ denotes a directed edge from vertex $\mathbf{s}_1 \in \mathcal{V}$ to vertex $\mathbf{s}_2 \in \mathcal{V}$. Note that to differentiate the state, the output and the dynamic variables from their corresponding vertices, the latter are written in bold fonts.

Hereafter, we illustrate our proposed digraph representation for SBLS with a simple example:

Example 1 Consider the SBLS defined by:

$A^\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & \lambda_4 & \lambda_5 \end{pmatrix}$, $B_1^\lambda = \begin{pmatrix} 0 & \lambda_6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B_2^\lambda = \begin{pmatrix} 0 & \lambda_7 & 0 \\ 0 & 0 & \lambda_8 \\ 0 & 0 & 0 \end{pmatrix}$ and $C^\lambda = \begin{pmatrix} \lambda_9 & 0 & 0 \end{pmatrix}$. To such a model, we associate the digraph in figure 1.

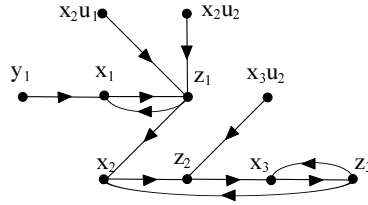


Figure 1: Digraph associated to Example 1

We give now some useful definitions and notations:

- For each edge $e = (\mathbf{s}_0, \mathbf{s}_1) \in \mathcal{E}$, $\mathbf{s}_0$ is the begin vertex and $\mathbf{s}_1$ the end vertex of e .
- Two edges $e_1 = (\mathbf{s}_1, \mathbf{s}'_1)$ and $e_2 = (\mathbf{s}_2, \mathbf{s}'_2)$ elements of $\mathcal{E}$ are v -disjoint if $\mathbf{s}_1 \neq \mathbf{s}_2$ and $\mathbf{s}'_1 \neq \mathbf{s}'_2$.

Note that e_1 and e_2 can be v -disjoint even if $\mathbf{s}'_1 = \mathbf{s}_2$ or $\mathbf{s}_1 = \mathbf{s}'_2$. In Example 1, $(\mathbf{x}_1, \mathbf{z}_1)$ and $(\mathbf{x}_2, \mathbf{z}_2)$ are v -disjoint. It is also the case for $(\mathbf{y}_1, \mathbf{x}_1)$ and $(\mathbf{x}_1, \mathbf{z}_1)$. Nevertheless, neither $(\mathbf{z}_1, \mathbf{x}_1)$ and $(\mathbf{z}_1, \mathbf{x}_2)$ nor $(\mathbf{x}_2 \mathbf{u}_1, \mathbf{z}_1)$ and $(\mathbf{x}_2 \mathbf{u}_2, \mathbf{z}_1)$ are v -disjoint.

- Some edges are v -disjoint if they are mutually v -disjoint.
- A subgraph $\mathcal{S}_\mathcal{G}$ of $\mathcal{G}(\Sigma_\Lambda)$ is defined by an edge subset $\mathcal{S}_\mathcal{E} \subseteq \mathcal{E}$ and a vertex subset $\mathcal{S}_\mathcal{V} \subseteq \mathcal{V}$ such

that $\mathcal{S}_{\mathcal{V}}$ is constituted by the begin and the end vertices of all the elements of $\mathcal{S}_{\mathcal{E}}$.

- An edge e belongs to a subgraph $\mathcal{S}_{\mathcal{G}} = (\mathcal{S}_{\mathcal{E}}, \mathcal{S}_{\mathcal{V}})$ if $e \in \mathcal{S}_{\mathcal{E}}$. A subgraph $\mathcal{S}_{\mathcal{G}}$ of $\mathcal{G}(\Sigma_{\Lambda})$ is said to cover a vertex $\mathbf{s}$ if there exists an edge e belonging to $\mathcal{S}_{\mathcal{G}}$ such that $\mathbf{s}$ is the end vertex of e .
- A subgraph $\mathcal{S}_{\mathcal{G}}$ of $\mathcal{G}(\Sigma_{\Lambda})$ is a ν -disjoint subgraph if all its edges are ν -disjoint.
- For any $\mathcal{V}_0 \subseteq \mathcal{V}$, we denote by $2^{\mathcal{V}_0}$ the power set of $\mathcal{V}_0$ composed of all the subsets of $\mathcal{V}_0$.

Function φ is defined by:

$$\begin{aligned} \varphi: 2^{\mathcal{V}} &\rightarrow 2^{\mathcal{V}} \\ \mathbf{V}_0 &\mapsto \mathbf{V}_1 = \varphi(\mathbf{V}_0) = \{\mathbf{s} \in \mathcal{V} / \exists \mathbf{s}_0 \in \mathbf{V}_0 \text{ where } (\mathbf{s}_0, \mathbf{s}) \in \mathcal{E}\} \end{aligned} \quad (7)$$

Roughly speaking, $\varphi(\mathbf{V}_0)$ represents the reachable set from $\mathbf{V}_0$ using any edge of $\mathcal{E}$ beginning at $\mathbf{V}_0$. In Example 1, $\varphi(\{\mathbf{y}_1, \mathbf{z}_3\}) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. We define also function $\bar{\varphi}$ such that:

$$\begin{aligned} \bar{\varphi}: 2^{\mathcal{Z}} &\rightarrow 2^{\mathbf{X} \cup \mathbf{X}\mathbf{U}} \\ \mathbf{V}_1 &\mapsto \bar{\mathbf{V}}_0 = \bar{\varphi}(\mathbf{V}_1) = \{\mathbf{s} \in \mathbf{X} \cup \mathbf{X}\mathbf{U} / \exists \mathbf{s}_1 \in \mathbf{V}_1 \text{ where } (\mathbf{s}, \mathbf{s}_1) \in \mathcal{E}\} \end{aligned} \quad (8)$$

In Example 1, $\bar{\varphi}(\{\mathbf{z}_2, \mathbf{z}_3\}) = \{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_3\mathbf{u}_2\}$.

- α is the function defined by:

$$\begin{aligned} \alpha: 2^{\mathbf{X}\mathbf{U}} &\rightarrow 2^{\mathbf{X}} \\ \mathbf{V}_{\mathbf{X}\mathbf{U}} &\mapsto \mathbf{V}_{\mathbf{X}} = \alpha(\mathbf{V}_{\mathbf{X}\mathbf{U}}) = \{\mathbf{x}_i \in \mathbf{X} / \exists j \in \{1, \dots, m\} \text{ such that } \mathbf{x}_i\mathbf{u}_j \in \mathbf{V}_{\mathbf{X}\mathbf{U}}\} \end{aligned} \quad (9)$$

In Example 1, $\alpha(\{\mathbf{x}_2\mathbf{u}_1, \mathbf{x}_2\mathbf{u}_2\}) = \{\mathbf{x}_2\}$.

4 Main results

In order to graphically analyse the generic uniform observability of SBLS, we follow an iterative approach where the i^{th} stage corresponds to the i^{th} differentiation order of the output.

Hereafter, we detail the first step of our approach. As it is shown in the form (4), only combinations $\Gamma_0^\lambda y$ such that we can deduce from (Σ_{Λ}) an equation of the form:

$$\Gamma_0^\lambda \dot{y} = C_1^\lambda x + \sum_{i=1}^m u_i H_{0,i}^\lambda Y_0, \quad Y_0 \stackrel{def}{=} y = C^\lambda x \quad (10)$$

are useful. In equation (10), $C_1^\lambda = \Gamma_0^\lambda C^\lambda A^\lambda$ and $H_{0,i}^\lambda C^\lambda = \Gamma_0^\lambda C^\lambda B_i^\lambda$. Determining these combinations amounts to find the combined vertices associated to some combinations $P^\lambda z$ which satisfy $P^\lambda z = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$. These vertices are called **useful vertices**. The first part of this section is dedicated to the determination of these useful vertices.

Let us first precise the notion of combined vertices corresponding to combinations $P^\lambda z$:

- We denote by $\mathbf{z}_{i_1, i_2, \dots, i_k}$ the vertex associated to any combination of dynamic variables

$z_{i_1}, z_{i_2}, \dots, z_{i_k}$ whatever $i_1 < i_2 < \dots < i_k$. $\mathbf{z}_{i_1, i_2, \dots, i_k}$, which we call combined vertex, can be defined by means of bijective function π associating a combined vertex to each nonempty subset of $\mathbf{Z}$: $\{\mathbf{z}_{i_1}, \mathbf{z}_{i_2}, \dots, \mathbf{z}_{i_k}\} \mapsto \pi(\{\mathbf{z}_{i_1}, \mathbf{z}_{i_2}, \dots, \mathbf{z}_{i_k}\}) \stackrel{\text{def}}{=} \mathbf{z}_{i_1, i_2, \dots, i_k}$. We denote by $\bar{\pi}$ the inverse function of π . The set of combined vertices is noted $\mathbf{Z}_C$. Obviously, $\mathbf{Z} \subseteq \mathbf{Z}_C$.

- For each vertex subset $\mathbf{V}_0 \subseteq \mathcal{V} \cup \mathbf{Z}_C$, $\text{card}(\mathbf{V}_0)$ represents the cardinality of $\mathbf{V}_0$ *i.e.* the number of vertices included in $\mathbf{V}_0$.
- A nonempty vertex subset $\mathbf{V}_C \subseteq \mathbf{Z}_C$ is said “structurally minimal” if

$$\forall \bar{\mathbf{V}} \subseteq \mathbf{V}_C, \quad \text{card}\left(\bigcup_{\mathbf{v} \in \bar{\mathbf{V}}} \bar{\pi}(\mathbf{v})\right) - \text{card}(\bar{\mathbf{V}}) \geq \max_{\mathbf{v} \in \bar{\mathbf{V}}} [\text{card}(\bar{\pi}(\mathbf{v}))] - 1 \quad (11)$$

For example, $\mathbf{V}_1 = \{\mathbf{z}_1, \mathbf{z}_{2,3}, \mathbf{z}_{2,4}\}$ is structurally minimal while $\mathbf{V}_2 = \{\mathbf{z}_{1,2,3}, \mathbf{z}_{2,3}, \mathbf{z}_{2,4}\}$ is not structurally minimal because for $\bar{\mathbf{V}} = \{\mathbf{z}_{1,2,3}, \mathbf{z}_{2,3}\}$, $\text{card}\left(\bigcup_{\mathbf{v} \in \bar{\mathbf{V}}} \bar{\pi}(\mathbf{v})\right) - \text{card}(\bar{\mathbf{V}}) = 3 - 2$ and $\max_{\mathbf{v} \in \bar{\mathbf{V}}} (\text{card}(\bar{\pi}(\mathbf{v}))) - 1 = 3 - 1$. Similarly, $\mathbf{V}_3 = \{\mathbf{z}_{1,2}, \mathbf{z}_{1,3}, \mathbf{z}_{2,3}\}$ is not structurally minimal because for $\bar{\mathbf{V}} = \mathbf{V}_3$, condition (11) is not satisfied.

The notion of structural minimality is related to the notion of vector linear independence. Indeed, consider a structurally minimal vertex subset $\mathbf{V}_Z = \{\mathbf{z}_{\Omega_1}, \mathbf{z}_{\Omega_2}, \dots, \mathbf{z}_{\Omega_k}\} \subseteq \mathbf{Z}_C$ and its associated vector $\tilde{\mathbf{z}} = (z_{\Omega_1}, z_{\Omega_2}, \dots, z_{\Omega_k})^T = (P_1^\lambda z, P_2^\lambda z, \dots, P_k^\lambda z)^T$, where $P_i^\lambda \in \mathbf{R}^{1 \times n}$. We can easily prove that the family of vectors $(P_1^\lambda)^T, \dots, (P_k^\lambda)^T$ is generically linearly independent because of the structural minimality of $\mathbf{V}_Z$.

- To each $\mathbf{V}_C \subseteq \mathbf{Z}_C$, we can associate a vertex subset $\varphi(\mathbf{V}_C)$ constituted by all the elements $\mathbf{s} \in \mathbf{Z}_C$ for which there exists $\mathbf{V}_0 \subseteq \mathbf{V}_C$ such that:

$$\begin{cases} \text{card}\left(\bigcup_{\mathbf{v} \in \mathbf{V}_0} \bar{\pi}(\mathbf{v})\right) - \text{card}(\bar{\pi}(\mathbf{s})) = \text{card}(\mathbf{V}_0) - 1, \\ \forall \bar{\mathbf{V}}_0 \subset \mathbf{V}_0 \mid \text{card}(\bar{\mathbf{V}}_0) = \text{card}(\mathbf{V}_0) - 1, \quad \bigcup_{\mathbf{v} \in \bar{\mathbf{V}}_0} \bar{\pi}(\mathbf{v}) \cup \bar{\pi}(\mathbf{s}) = \bigcup_{\mathbf{v} \in \mathbf{V}_0} \bar{\pi}(\mathbf{v}) \end{cases} \quad (12)$$

For example, for $\mathbf{V}_C = \{\mathbf{z}_{1,2}, \mathbf{z}_{1,3}, \mathbf{z}_4\}$, we have $\varphi(\mathbf{V}_C) = \{\mathbf{z}_{1,2}, \mathbf{z}_{1,3}, \mathbf{z}_{2,3}, \mathbf{z}_4\}$. Note that $\forall \mathbf{V}_Z \subseteq \mathbf{Z}$ we have $\varphi(\mathbf{V}_Z) = \mathbf{V}_Z$. Obviously, we have $\forall \mathbf{V}_C \subseteq \mathbf{Z}_C, \mathbf{V}_C \subseteq \varphi(\mathbf{V}_C)$.

- The definition of φ in section 3 is adapted for taking into account $\mathbf{Z}_C$. In the sequel, φ_e is such that:

$$\begin{aligned} \varphi_e: 2^{\mathcal{V} \cup \mathbf{Z}_C} &\rightarrow 2^{\mathcal{V}} \\ \mathbf{V}_0 &\mapsto \mathbf{V}_1 = \varphi_e(\mathbf{V}_0) = \left(\bigcup_{\mathbf{s} \in \mathbf{V}_0 \cap \mathbf{Z}_C} \varphi(\bar{\pi}(\mathbf{s})) \right) \cup \varphi(\mathbf{V}_0 \cap \mathcal{V}) \end{aligned} \quad (13)$$

In Example 1, $\varphi_e(\{\mathbf{y}_1, \mathbf{z}_{2,3}\}) = \varphi(\{\mathbf{y}_1\}) \cup \varphi(\{\mathbf{z}_2\}) \cup \varphi(\{\mathbf{z}_3\}) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

We define, finally, function θ as:

$$\begin{aligned} \theta: 2^{\mathbf{Y} \cup \mathbf{Z}_C} &\rightarrow \mathbb{N} \\ \mathbf{V}_0 &\mapsto \theta(\mathbf{V}_0) = \text{maximal number of } \nu\text{-disjoint edges starting from } \mathbf{V}_0 \end{aligned} \quad (14)$$

knowing that, to take into account $\mathbf{Z}_C$, we add virtually (there is no need to redraw the digraph associated to the system) to the digraph the following edge subset A_C -edges = $\{(\mathbf{z}_{i_1, i_2, \dots, i_k}, \mathbf{x}_j) \text{ if } \exists r \in \{i_1, i_2, \dots, i_k\} \mid A^\lambda(r, j) \neq 0\}$.

In Example 1, $\theta(\mathbf{Y} \cup \mathbf{Z}) = 3$ $((\mathbf{y}_1, \mathbf{x}_1), (\mathbf{z}_1, \mathbf{x}_2), (\mathbf{z}_2, \mathbf{x}_3))$ are 3 ν -disjoint edges starting from $\mathbf{Y} \cup \mathbf{Z}$ and $\theta(\{\mathbf{z}_1, \mathbf{z}_{2,3}\}) = 2$ $((\mathbf{z}_1, \mathbf{x}_2), (\mathbf{z}_{2,3}, \mathbf{x}_3))$ are 2 ν -disjoint edges starting from $\{\mathbf{z}_1, \mathbf{z}_{2,3}\}$.

Function θ has the following property:

P1. Consider $\mathbf{V}_Y = \{\mathbf{y}_{i_1}, \mathbf{y}_{i_2}, \dots, \mathbf{y}_{i_k}\} \subseteq \mathbf{Y}$ and its associated vector $\tilde{\mathbf{y}} = (y_{i_1}, y_{i_2}, \dots, y_{i_k})^T$. Let us denote by $\tilde{C}^\lambda$ the matrix such that $\tilde{\mathbf{y}} = \tilde{C}^\lambda \mathbf{x}$, then from the results on the matrices generic rank [18,23], $\theta(\mathbf{V}_Y) = g_rank(\tilde{C}^\lambda)$.

Similarly, consider a structurally minimal vertex subset $\mathbf{V}_Z = \{\mathbf{z}_{\Omega_1}, \mathbf{z}_{\Omega_2}, \dots, \mathbf{z}_{\Omega_k}\} \subseteq \mathbf{Z}_C$ and its asso-

ciated vector $\tilde{\mathbf{z}} = (z_{\Omega_1}, z_{\Omega_2}, \dots, z_{\Omega_k})^T = (P_1^\lambda z, P_2^\lambda z, \dots, P_k^\lambda z)^T$, where $P_i^\lambda \in \mathbf{R}^{1 \times n}$. Let $\tilde{E}^\lambda = \begin{pmatrix} P_1^\lambda \\ P_2^\lambda \\ \vdots \\ P_k^\lambda \end{pmatrix} A^\lambda$.

We have then $\tilde{\mathbf{z}} = \tilde{E}^\lambda \mathbf{x}$ and so $\theta(\mathbf{V}_Z) = g_rank(\tilde{E}^\lambda)$.

This property can be directly deduced from the results of [23] or from the results related to the maximum matching notion [17].

Using function θ , we define a particular vertex set $\mathbf{V}_{\mathbf{M}^0}^*$:

Definition 2 Let $\mathbf{V}_{\mathbf{M}^0}^*$ be a subset of $\mathbf{Y}$ such that

$$\begin{cases} \theta(\mathbf{V}_{\mathbf{M}^0}^*) = \text{card}(\varphi_e(\mathbf{V}_{\mathbf{M}^0}^*)) \\ \text{card}(\mathbf{V}_{\mathbf{M}^0}^*) \text{ is maximal} \end{cases} \quad (15)$$

Moreover, we take the following notations $\mathbf{V}_{\mathbf{X}^0}^* = \varphi_e(\mathbf{V}_{\mathbf{M}^0}^*)$ and $\mathbf{V}_{\mathbf{XU}^0}^* = \{\mathbf{s} \in \mathbf{XU} \text{ such that } \alpha(\{\mathbf{s}\}) \subseteq \mathbf{V}_{\mathbf{X}^0}^*\}$.

If $\mathbf{V}_{\mathbf{M}^0}^*$ exists, then it is unique because if relation $\theta(\mathbf{V}_M) = \text{card}(\varphi_e(\mathbf{V}_M))$ is satisfied for $\mathbf{V}_M = \mathbf{V}_{M_1}$ and for $\mathbf{V}_M = \mathbf{V}_{M_2}$, then it is also satisfied for $\mathbf{V}_M = \mathbf{V}_{M_1} \cup \mathbf{V}_{M_2}$.

We give now, in the following lemma, the necessary and sufficient conditions to be satisfied in order to express state component x_i by means of output y :

Lemma 1 Consider structured bilinear system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. State component x_i can be generically expressed in function of the output i.e. there exists a matrix T_i^λ such that

$$x_i = T_i^\lambda y, \text{ iff in } \mathcal{G}(\Sigma_\Lambda)$$

$$\exists \mathbf{V}_Y \subseteq \mathbf{Y} \text{ such that } \begin{cases} (i) & \mathbf{x}_i \in \varphi_e(\mathbf{V}_Y) \\ (ii) & \theta(\mathbf{V}_Y) = \text{card}(\varphi_e(\mathbf{V}_Y)) \end{cases} \quad (16)$$

or equivalently $\mathbf{x}_i \in \mathbf{V}_{\mathbf{X}^0}^*$

Proof:

Sufficiency: Assume that (16) is satisfied. Let x_Ω be the part of the state vector associated to vertex set $\varphi_e(\mathbf{V}_Y)$ and let y_Ω be the output vector associated to $\mathbf{V}_Y$. According to the graph construction, there exists necessarily a matrix C_Ω such that $y_\Omega = C_\Omega x_\Omega$. If $\theta(\mathbf{V}_Y) = \text{card}(\varphi_e(\mathbf{V}_Y))$ then using Property **P1** of function θ , the generic rank of C_Ω equals $\text{card}(\varphi_e(\mathbf{V}_Y)) = \dim(x_\Omega)$. Thus, there exists generically a matrix T_Ω of full row rank such that $x_\Omega = T_\Omega y_\Omega$ where T_Ω is the pseudo-inverse of C_Ω . Since $\mathbf{x}_i \in \varphi_e(\mathbf{V}_Y)$, we have $x_i = T_i^\lambda y$ where T_i^λ is the row of T_Ω corresponding to the state component x_i in the equality $x_\Omega = T_\Omega y_\Omega$.

Necessity: Suppose that there exists a matrix T_i^λ such that $x_i = T_i^\lambda y$, then there exists necessarily a square nonsingular linear system of independent equations linking an output subvector y_Ω to a state subvector x_Ω which includes x_i *i.e.* there exists a square nonsingular matrix T_Ω such that $x_\Omega = T_\Omega y_\Omega$. So, $y_\Omega = T_\Omega^{-1} x_\Omega$. Let $\mathbf{V}_Y$ the vertex subset associated to the output components included in y_Ω , and $\mathbf{V}_x$ the vertex subset associated to the state components included in x_Ω . On the one hand, as $y_\Omega = T_\Omega^{-1} x_\Omega$, we have that $\mathbf{V}_x = \varphi_e(\mathbf{V}_Y)$ and since state subvector x_Ω includes x_i , we have that $\mathbf{x}_i \in \mathbf{V}_x$. On the other hand, as T_Ω is of full rank, we have that $\text{g_rank}(T_\Omega) = \text{g_rank}(T_\Omega^{-1}) = \theta(\mathbf{V}_Y) = \text{card}(\mathbf{V}_x) = \text{card}(\varphi_e(\mathbf{V}_Y))$. $\triangle$

4.1 Determination of the useful vertices

If $\theta(\mathbf{Y}) = n$, or equivalently $\mathbf{V}_{\mathbf{X}^0}^* = \mathbf{X}$, then $\text{g_rank}(C^\lambda) = n$ and so the system is generically uniformly observable. Otherwise, the first derivative of the output is necessary to achieve the state observation. Hereafter, we study the conditions ensuring that a combination of dynamical variables z_i can be expressed using y and u only *i.e.* there exists generically matrix P^λ such that $P^\lambda z = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$. This amounts to find matrix Γ_0 of the form (4). To do so, our first objective is to construct a subset of $\mathbf{Z}_C$ which represents all the output combinations having derivatives independent from $x_i u_j$ if x_i cannot be expressed using only y . In this respect, the following lemma aims to provide the conditions which guarantee that a combined vertex is useful.

Lemma 2 Consider system (Σ_Λ) , represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. Let $\mathbf{V}_0 = \{\mathbf{z}_{i_1}, \mathbf{z}_{i_2}, \dots, \mathbf{z}_{i_k}\}$ be a subset of $\mathbf{Z}$ and note by $\pi(\mathbf{V}_0) = \mathbf{z}_{i_1, i_2, \dots, i_k}$ its corresponding combined vertex. Assume that

$\nexists \mathbf{V}'_0 \subset \mathbf{V}_0$ such that $\pi(\mathbf{V}'_0)$ is useful.

Combined vertex $\mathbf{z}_{i_1, i_2, \dots, i_k}$ is generically useful and there exists vector $P_k^\lambda \in \mathbb{R}^{1 \times k}$, such that $P_k^\lambda(z_{i_1}, z_{i_2}, \dots, z_{i_k})^T$ can be expressed in function of $\dot{y}$ and $u_i y$ only i.e. $P_k^\lambda(z_{i_1}, z_{i_2}, \dots, z_{i_k})^T = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$ iff $\exists \mathbf{V}_Y \subseteq \mathbf{Y}$ satisfying:

$$\theta(\mathbf{V}_Y) + k + \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = \text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_0)) + 1 \quad (17)$$

Proof:

Sufficiency: Assume that $\exists \mathbf{V}_Y \subseteq \mathbf{Y}$ such that (17) is satisfied and let us take the following notations:

- $\overline{\mathbf{V}_{\mathbf{XU}^0}^*}$ is the complementary set of $\mathbf{V}_{\mathbf{XU}^0}^*$ in $\mathbf{XU}$ i.e. $\overline{\mathbf{V}_{\mathbf{XU}^0}^*} = \mathbf{XU} \setminus \mathbf{V}_{\mathbf{XU}^0}^*$;
- x_Ω is the vector whose components correspond to the elements of subset $\varphi_e(\mathbf{V}_Y) \cup (\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{X})$.
- xu_Ω is the vector whose components correspond to the elements of vertex set $\bar{\varphi}(\mathbf{V}_0) \cap \overline{\mathbf{V}_{\mathbf{XU}^0}^*}$;
- $z_\Omega = (z_{i_1}, z_{i_2}, \dots, z_{i_k})^T$ is the vector of dynamic variables associated to $\mathbf{V}_0$;
- $X_\Omega = \begin{pmatrix} \dot{x}_\Omega \\ xu_\Omega \end{pmatrix}$;
- y_Ω is the vector of the output components associated to $\mathbf{V}_Y$.

By construction of graph $\mathcal{G}(\Sigma_\Lambda)$ representing system (Σ_Λ) , y_Ω is a linear function of the state components associated to $\varphi_e(\mathbf{V}_Y)$ and z_Ω is a linear function of the state components associated to $\bar{\varphi}(\mathbf{V}_0)$. A part of $\bar{\varphi}(\mathbf{V}_0)$ can be directly written in function of y , particularly the part associated to $\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*$ as it is proven in Lemma 1. Then, using our notations, there exist structured matrices C_Ω , E_Ω and $F_{i,\Omega}$ for $i = 1, \dots, m$ such that we can write $\dot{y}_\Omega = C_\Omega X_\Omega$ and $z_\Omega = E_\Omega X_\Omega + \sum_{i=1}^m u_i F_{i,\Omega} y_\Omega$. Yet, we have that $\dim(\text{Im}(E_\Omega^T)) = k$, $X_\Omega \in \mathbb{R}^\nu$ where $\nu = \text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_0)) - \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*)$ and from property **P1** of θ , we have that $\theta(\mathbf{V}_Y) = \dim(\text{Im}(C_\Omega^T))$. Therefore, relation (17) implies that generically

$$\dim(\text{Im}(C_\Omega^T)) + \dim(\text{Im}(E_\Omega^T)) = \nu + 1 \quad (18)$$

Obviously, C_Ω and E_Ω have ν columns and so $\text{Im}(C_\Omega^T)$ and $\text{Im}(E_\Omega^T)$ are subspaces of $\mathbb{R}^\nu$. From equation (18), we can conclude that $\dim(\text{Im}(E_\Omega^T) \cap \text{Im}(C_\Omega^T)) \geq 1$. Consequently, generically there exist matrices P_k^λ and Γ_0^λ such that $P_k^\lambda E_\Omega = \Gamma_0^\lambda C_\Omega \Rightarrow P_k^\lambda z_\Omega = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$. This proves the sufficiency of Lemma 2.

Remark 2 Note that $\theta(\mathbf{V}_Y) + k \geq \text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_0)) + 1 - \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*)$ is also sufficient to prove the existence of vectors $P_k^\lambda \in \mathbb{R}^{1 \times k}$ such that $P_k^\lambda(z_{i_1}, z_{i_2}, \dots, z_{i_k})^T = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$.

Necessity: Assume that there exists vector $P_k^\lambda \in \mathbb{R}^k$, such that $P_k^\lambda z_\Omega = \Gamma_0^\lambda \dot{y} - \sum_{i=1}^m u_i H_{0,i}^\lambda y$ then at least one combination of components associated to $\bar{\varphi}(\mathbf{V}_0)$ can be expressed using $\dot{y}_\Omega$ (y_Ω is the

vector of components of y which intervenes in $\Gamma_0^\lambda \dot{y}$ i.e. $\Gamma_0^\lambda \dot{y} = \Gamma_{0,\Omega}^\lambda \dot{y}_\Omega$). Let $\mathbf{V}_Y$ be the vertex subset associated to the components of y_Ω .

From the construction of the digraph, $P_k^\lambda z_\Omega$ is a linear function of $\dot{x}_j$ where $\mathbf{x}_j \in \bar{\varphi}(\mathbf{V}_0) \cap \mathbf{X}$ and of $x_k u_l$ where $\mathbf{x}_k \mathbf{u}_l \in \bar{\varphi}(\mathbf{V}_0) \cap \mathbf{XU}$. To satisfy $P_k^\lambda z_\Omega = \Gamma_0^\lambda \dot{y} + \sum_{i=1}^m u_i H_{0,i}^\lambda y$, the variables corresponding to $\bar{\varphi}(\mathbf{V}_0)$ are necessarily either eliminated in combination $P_k^\lambda z_\Omega$ or can be expressed only in function y . On the one hand, a combination of k dynamical variables z_i can eliminate at most $k - 1$ elements of $\bar{\varphi}(\mathbf{V}_0)$. On the other hand, the variables associated to the vertices of $\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*$ can be expressed in function of y and u as they are included in $\mathbf{V}_{\mathbf{XU}^0}^*$ (Lemma 1). Let $q = \text{card}[\bar{\varphi}(\mathbf{V}_0) \setminus (\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*)]$. Thus, at least $q - k + 1$ variables corresponding to the vertices included in $\bar{\varphi}(\mathbf{V}_0) \setminus (\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*)$ must be generically expressed in function of y_Ω to satisfy equation $P_k^\lambda z_\Omega = \Gamma_{0,\Omega}^\lambda \dot{y}_\Omega + \sum_{i=1}^m u_i H_{0,i}^\lambda y$. However, y_Ω is a linear function of the state components associated to $\varphi_e(\mathbf{V}_Y)$, which are not necessarily included in $\bar{\varphi}(\mathbf{V}_0)$. Thus, the components of $\varphi_e(\mathbf{V}_Y) \setminus (\bar{\varphi}(\mathbf{V}_0) \cap \varphi_e(\mathbf{V}_Y))$ must be added to the components which must be expressed using y_Ω . The number of state variables we can express using y_Ω is equal to $\theta(\mathbf{V}_Y)$. Thus, $\mathbf{V}_Y$ must satisfy $\theta(\mathbf{V}_Y) \geq \text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_0)) - k + 1 - \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = n_1$.

Furthermore, assume now that there exists a vertex subset $\mathbf{V}'_Y \subseteq \mathbf{Y}$ satisfying $\theta(\mathbf{V}'_Y) = n'_1 > n_1$.

The latter inequality implies that $\theta(\mathbf{V}'_Y) \geq \text{card}(\varphi_e(\mathbf{V}'_Y) \cup \bar{\varphi}(\mathbf{V}_0)) - (k - 1) + 1 - \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*)$. Thus, that for all $\mathbf{V}'_0 \subset \mathbf{V}_0$ such that $\text{card}(\mathbf{V}'_0) = \text{card}(\mathbf{V}_0) - 1 = k - 1$, we have $\theta(\mathbf{V}'_Y) \geq \text{card}(\varphi_e(\mathbf{V}'_Y) \cup \bar{\varphi}(\mathbf{V}_0)) - (k - 1) - \text{card}(\bar{\varphi}(\mathbf{V}'_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*) + 1$. According to the sufficiency of Lemma 2 proved above $\pi(\mathbf{V}'_0)$ is useful (see Remark 2). However, we take as hypothesis $\nexists \mathbf{V}'_0 \subset \mathbf{V}_0$ such that $\pi(\mathbf{V}'_0)$ is useful. Consequently, $\nexists \mathbf{V}'_Y \subseteq \mathbf{Y}$ such that $\theta(\mathbf{V}'_Y) \leq n_1$.

On the one hand, we have proved that the existence of a vector $P_k^\lambda \in \mathbb{R}^{1 \times k}$, such that $P_k^\lambda (z_{i_1}, z_{i_2}, \dots, z_{i_k})^T$ can be expressed in function of $\dot{y}$ and $u_i y$ implies the existence of $\mathbf{V}_Y \subseteq \mathbf{Y}$ such that $\theta(\mathbf{V}_Y) \geq n_1$. On the other hand, the hypothesis of lemma 1 imposes that there cannot exist $\mathbf{V}_Y \subseteq \mathbf{Y}$ such that $\theta(\mathbf{V}_Y) > n_1$. Consequently, the existence of a vector $P_k^\lambda \in \mathbb{R}^{1 \times k}$ such that $P_k^\lambda (z_{i_1}, z_{i_2}, \dots, z_{i_k})^T$ can be expressed in function of $\dot{y}$ and $u_i y$ implies the existence of $\mathbf{V}_Y \subseteq \mathbf{Y}$ such that $\theta(\mathbf{V}_Y) = n_1 = \text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_0)) - k - \text{card}(\bar{\varphi}(\mathbf{V}_0) \cap \mathbf{V}_{\mathbf{XU}^0}^*) + 1$. $\triangle$

The hypothesis of Lemma 2, which is in relation with the notion of structural minimality, is necessary to ensure that the combined vertices verifying conditions of Lemma 2 represent effectively the number of linear independent vectors P^λ such that $P^\lambda z$ can be expressed in function of $\dot{y}$ and $u_i y$. Indeed, assume that $\mathbf{z}_1$ is useful and that $\{\mathbf{z}_1, \mathbf{z}_2\}$ satisfies condition (17). We have then two cases: either $\mathbf{z}_2$ is useful and in this case the vectors representing $\{\mathbf{z}_1\}$, $\{\mathbf{z}_2\}$, $\{\mathbf{z}_1, \mathbf{z}_2\}$ are

not linearly independent. Or $\mathbf{z}_2$ is not useful and so only P_1^λ representing $\mathbf{z}_1$ is such that $P_1^\lambda z$ can be expressed in function of $\dot{y}$ and $u_i y$. In both cases, the fact that $\{\mathbf{z}_1, \mathbf{z}_2\}$ satisfies condition (17) is not significative in the search of useful vertices corresponding to vectors P^λ such that $P^\lambda z$ can be expressed in function of $\dot{y}$ and $u_i y$.

Let us denote by $\mathbf{Z}^1$ the biggest (in the sens of cardinality) structurally minimal set of useful vertices of $\mathbf{Z}_C$ obtained after the first step. Since this set is structurally minimal, it represents all the linearly independent combinations of the first derivative of the output components. $\mathbf{Z}^1$ represents then $\Gamma_0^\lambda \dot{Y}_0 - \sum_{i=1}^m u_i H_{0,i}^\lambda Y_0 = C_1^\lambda x$ and $\mathbf{Y} \cup \mathbf{Z}^1$ represents Y_1 in relations (4).

If $\mathbf{Z}^1 \neq \emptyset$ then the first derivative of the output components can be exploited to give new informations on the state components for any value of u . If $\theta(\mathbf{Y} \cup \mathbf{Z}^1) = \text{card}(\mathbf{X}) = n$ then system (Σ_Λ) is uniformly observable. Otherwise, the output must be differentiated once again. At this aim, a second step is necessary to determinate which vertices of $\mathbf{Z}_C$ are useful for a second differentiation of the output.

In this second step, we consider $\mathbf{Y} \cup \mathbf{Z}^1$ as the new output vertices of the system. This is equivalent in form (4), to the fact that in the second step we differentiate $Y_1 \stackrel{\text{def}}{=} \begin{pmatrix} C^\lambda \\ C_1^\lambda \end{pmatrix} x$ to obtain a relation

on the form $\Gamma_1^\lambda \dot{Y}_1 = C_2^\lambda x + \sum_{i=1}^m u_i H_{1,i}^\lambda Y_1$. As a matter of fact, this second step is similar to the first one but with considering as an output Y_1 instead of $Y_0 = y$. Graphically, this amounts to consider $\mathbf{Y} \cup \mathbf{Z}^1$ as output vertex subset instead of $\mathbf{Y}$.

Therefore, in this second step, we can follow a similar approach than the one presented previously in Lemma 2 by replacing “ $\exists \mathbf{V}_Y \subseteq \mathbf{Y}$ ” by “ $\exists \mathbf{V}_Y \subseteq \mathbf{Y}$ and a structurally minimal subset $\mathbf{V}_Z \subseteq \varphi(\mathbf{Z}^1)$ ”, $\mathbf{V}_{M^0}^*$ by $\mathbf{V}_{M^1}^*$, $\mathbf{V}_{X^0}^*$ by $\mathbf{V}_{X^1}^*$ and $\mathbf{V}_{XU^0}^*$ by $\mathbf{V}_{XU^1}^*$, where $\mathbf{V}_{M^i}^*$, $\mathbf{V}_{X^i}^*$ and $\mathbf{V}_{XU^i}^*$ for $i \geq 0$ are defined by:

Definition 3 For $i \geq 1$, $\mathbf{V}_{M^i}^*$ is the maximal (in the sens of the cardinality) subset of $\mathbf{Y} \cup \varphi(\mathbf{Z}^i)$ such that: $\mathbf{V}_Z \stackrel{\text{def}}{=} \mathbf{V}_{M^i}^* \cap \mathbf{Z}_C$ is structurally minimal and $\theta(\mathbf{V}_{M^i}^*) = \text{card}(\varphi_e(\mathbf{V}_{M^i}^*))$. We take the following notations: $\mathbf{V}_{X^i}^* = \varphi_e(\mathbf{V}_{M^i}^*)$ and $\mathbf{V}_{XU^i}^* = \{\mathbf{s} \in \mathbf{XU} \text{ such that } \alpha(\{\mathbf{s}\}) \subseteq \mathbf{V}_{X^i}^*\}$.

Let us denote by $\mathbf{Z}^2$ the set of useful vertices representing linearly independent combinations of the first and second derivatives of the output components. If $\theta(\mathbf{Y} \cup \mathbf{Z}^2) = \text{card}(\mathbf{X}) = n$ then system (Σ_Λ) is generically uniform observable. Otherwise, we must differentiate the output once again and so on. We present the algorithm which allows us to determine the set $\mathbf{Z}^*$ which is the limit of the structurally minimal subset sequence $(\mathbf{Z}^i)$ and which represents the set of all useful vertices:

Algorithm 1

Initialisation : $\mathbf{Z}^0 = \emptyset$ and $i = 0$

Step i : $\mathbf{Q}_i = \varphi_e(\mathbf{Y} \cup \mathbf{Z}^0)$, $\mathbf{Z}^{i+1} = \emptyset$. Compute $\mathbf{V}_{\mathbf{XU}^i}^*$ according to Definition 3

Determination of the useful vertices:

```

for  $\kappa = 1$  to  $\text{card}(\mathbf{Q}^i)$ 
  for  $j = 1$  to  $\binom{\text{card}(\mathbf{Q}^i)}{\kappa}$ 
    let  $\mathbf{V}_{\kappa,j}$  be the  $j^{\text{th}}$   $\kappa$ -subset of  $\mathbf{Q}^i$ 
    if  $\forall \mathbf{V} \subseteq \mathbf{Z}^{i+1}, \text{card}\left(\bigcup_{\mathbf{v}_r \in \mathbf{V}} \bar{\pi}(\mathbf{v}_r) \cup \mathbf{V}_{\kappa,j}\right) - \text{card}(\mathbf{V}) \geq \kappa, \quad (\text{Cond1})$ 
      then if there exist  $\mathbf{V}_Y \subseteq \mathbf{Y}$  and a structurally minimal vertex subset  $\mathbf{V}_Z \subseteq \wp(\mathbf{Z}^i)$  such that
         $\theta(\mathbf{V}_Y \cup \mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{\kappa,j}) \cap \mathbf{V}_{\mathbf{XU}^i}^*) = \text{card}(\varphi_e(\mathbf{V}_Y \cup \mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{\kappa,j})) + 1 \quad (\text{Cond2})$ 
        then  $\mathbf{Z}^{i+1} = \mathbf{Z}^{i+1} \cup \{\pi(\mathbf{V}_{\kappa,j})\},$ 
      end
    end
  end
end

```

End of step i: If $\theta(\mathbf{Z}^{i+1}) \neq \theta(\mathbf{Z}^i)$, we must differentiate y again: $i \leftarrow i + 1$ and we go to **Step i**

Else $\mathbf{Z}^* = \mathbf{Z}^i$

Remark 3

- Condition (Cond1) above ensures that $\mathbf{Z}^{i+1}$ is structurally minimal. Due to condition (Cond1), if $\mathbf{V}_0 \subseteq \mathbf{Z}$ satisfies condition (Cond2) then either $\pi(\mathbf{V}_0)$ belongs to $\mathbf{Z}^{i+1}$ or $\pi(\mathbf{V}_0) \cup \mathbf{Z}^{i+1}$ is not structurally minimal. This means that if $\mathbf{V}_0 \subseteq \mathbf{Z}$ satisfies condition (Cond2) then $\exists \bar{\mathbf{V}}_0 \subseteq \mathbf{V}_0$ such that $\pi(\bar{\mathbf{V}}_0) \in \wp(\mathbf{Z}^{i+1})$. Furthermore, by construction of $\wp(\mathbf{Z}^{i+1})$, $\forall \mathbf{s} \in \wp(\mathbf{Z}^{i+1})$, $\bar{\pi}(\mathbf{s})$ satisfies condition (Cond2).

- It is easy to see from condition (Cond2) that for $i \geq 0$, if $\mathbf{s} \in \mathbf{Z}^i$ then there exists $\mathbf{s}_0$ such that $\bar{\pi}(\mathbf{s}_0) \subseteq \bar{\pi}(\mathbf{s})$ and $\mathbf{s}_0 \in \mathbf{Z}^{i+1}$.

We illustrate hereafter the previous algorithm on a simple example:

Example 2 Consider the structured system defined by:

$$A^\lambda = \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & \lambda_5 \end{pmatrix}, B_1^\lambda = \begin{pmatrix} 0 & 0 & \lambda_6 & 0 \\ 0 & \lambda_7 & \lambda_8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } C^\lambda = \begin{pmatrix} \lambda_9 & \lambda_{10} & 0 & 0 \\ 0 & \lambda_{11} & 0 & 0 \end{pmatrix}. \text{ To such}$$

a model, we associate the digraph in Figure 2.

For such a system, we apply algorithm 1 :

Step 0: Since $\theta(\mathbf{Y}) = 2$ and $\varphi_e(\mathbf{Y}) = \{\mathbf{x}_1, \mathbf{x}_2\}$ we have that $\mathbf{V}_{\mathbf{M}^0}^* = \mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2\}$ satisfies condition (15). So, we deduce $\mathbf{V}_{\mathbf{X}^0}^* = \varphi(\mathbf{V}_{\mathbf{M}^0}^*) = \{\mathbf{x}_1, \mathbf{x}_2\}$ and $\mathbf{V}_{\mathbf{XU}^0}^* = \{\mathbf{x}_2 \mathbf{u}_1\}$. $\mathbf{Q}_0 = \{\mathbf{z}_1, \mathbf{z}_2\}$.

For $\mathbf{V}_{1,1} = \{\mathbf{z}_1\}$, there does not exist any subset $\mathbf{V}_Y$ of $\mathbf{Y}$ ($\mathbf{Z}^0 = \emptyset$) which allows to satisfy equality (Cond2). Indeed, for $\mathbf{V}_Y = \{\mathbf{y}_1\}$, $\theta(\mathbf{V}_Y) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = 1 + 1 + 0 = 2$ and

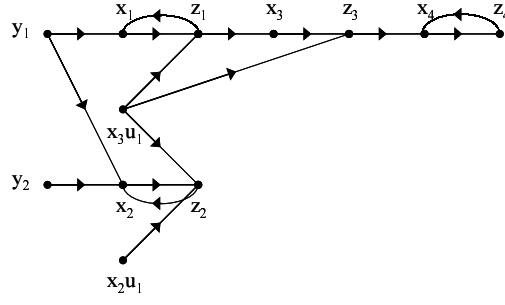


Figure 2: Digraph associated to Example 2

$\text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 3 + 1 = 4$. This is also the case for $\mathbf{V}_Y = \{\mathbf{y}_2\}$ where $\theta(\mathbf{V}_Y) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = 1 + 1 + 0 = 2$ and $\text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 3 + 1 = 4$ and for $\mathbf{V}_Y = \{\mathbf{y}_1, \mathbf{y}_2\}$ where $\theta(\mathbf{V}_Y) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = 2 + 1 + 0 = 3$ and $\text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 3 + 1 = 4$.

We can do exactly the same computations to conclude that for $\mathbf{V}_{1,2} = \{\mathbf{z}_2\}$, there does not exist any subset $\mathbf{V}_Y$ of $\mathbf{Y}$ which allows to satisfy equality **(Cond2)**. On the other hand, for $\mathbf{V}_{2,1} = \{\mathbf{z}_2, \mathbf{z}_2\}$, if we take $\mathbf{V}_Y = \{\mathbf{y}_1, \mathbf{y}_2\}$, condition **(Cond2)** is verified: $\theta(\mathbf{V}_Y) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{2,1}) \cap \mathbf{V}_{\mathbf{XU}^0}^*) = 2 + 2 + 1 = 5$ and $\text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_{2,1})) + 1 = 4 + 1 = 5$.

Thus, $\mathbf{Z}^1 = \{\mathbf{z}_{1,2}\}$. As $\theta(\mathbf{Z}^1) = 1 > \theta(\mathbf{Z}^0) = 0$, we go to step 1.

Step 1: Since, $\theta(\mathbf{Z}^1 \cup \mathbf{Y}) = 3$ and $\text{card}(\varphi_e(\mathbf{Z}^1 \cup \mathbf{Y})) = \text{card}(\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}) = 3$, we have that $\mathbf{V}_{\mathbf{X}^1}^* = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $\mathbf{V}_{\mathbf{XU}^1}^* = \alpha(\mathbf{V}_{\mathbf{X}^1}^*) = \{\mathbf{x}_2\mathbf{u}_1, \mathbf{x}_3\mathbf{u}_1\}$. Furthermore, $\mathbf{Q}_1 = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ and for each one-subset of $\mathbf{Q}_1$ there exist $\mathbf{V}_Y \subseteq \mathbf{Y}$ and a structurally minimal vertex subset $\mathbf{V}_Z \subseteq \varphi(\mathbf{Z}^1)$ such that equality **(Cond2)** is satisfied:

- For $\mathbf{V}_{1,1} = \{\mathbf{z}_1\}$, if we take $\mathbf{V}_Y = \{\mathbf{y}_1, \mathbf{y}_2\}$ and $\mathbf{V}_Z = \emptyset$, we have $\theta(\mathbf{V}_Y \cup \mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^1}^*) = 2 + 1 + 1 = 4$ and $\text{card}(\varphi_e(\mathbf{V}_Y \cup \mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 3 + 1 = 4$.
- For $\mathbf{V}_{1,2} = \{\mathbf{z}_2\}$, if we take $\mathbf{V}_Y = \{\mathbf{y}_2\}$ and $\mathbf{V}_Z = \emptyset$, we have $\theta(\mathbf{V}_Y \cup \mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^1}^*) = 1 + 1 + 2 = 4$ and $\text{card}(\varphi_e(\mathbf{V}_Y \cup \mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 3 + 1 = 3$.
- For $\mathbf{V}_{1,3} = \{\mathbf{z}_3\}$ if we take $\mathbf{V}_Y = \{\mathbf{y}_1, \mathbf{y}_2\}$ and $\mathbf{V}_Z = \{\mathbf{z}_{1,2}\}$ ($\mathbf{V}_Z$ is obviously structurally minimal and is included in $\mathbf{Z}^1$), we have $\theta(\mathbf{V}_Y \cup \mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,1}) \cap \mathbf{V}_{\mathbf{XU}^1}^*) = 3 + 1 + 1 = 5$ and $\text{card}(\varphi_e(\mathbf{V}_Y \cup \mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,1})) + 1 = 4 + 1 = 5$.

Moreover, since all the other subsets of $\mathbf{Q}_1$ do not satisfy condition **(Cond1)** because for any $\mathbf{V}_{\kappa,j} \subseteq \mathbf{Q}_1$, $\kappa \geq 2$, $\{\pi(\mathbf{V}_{\kappa,j})\} \cup \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is not structurally minimal. Thus, $\mathbf{Z}^2 = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$.

Let us now do some comments about algorithm 1. At step 0, $\mathbf{Y}$ represents all output derivatives of order 0 exploitable to reconstruct a part of the state for any value of input value u . Indeed relation $y = C^\lambda x$ does not depend on input value u . We use Lemma 2 to compute $\mathbf{Z}^1$ which

represents all vertices of $\mathbf{Z}_C$ associated to all the linearly independent combinations of dynamic variables which can be expressed in function of the output, its first derivative and the input: $P^\lambda z = \Gamma_0^\lambda \dot{Y}_0 + \sum_{i=1}^m u_i H_{0,i}^\lambda Y_0 = P^\lambda A^\lambda x$. This equation allows us to "observe" $P^\lambda A^\lambda x$ for any value u . The linear independence of such combinations is ensured by condition **(Cond1)** which imposes to $\mathbf{Z}^1$ to be structurally minimal and which ensures that the hypothesis of Lemma 2 is satisfied. $\mathbf{Z}^1$ represents then $\Gamma_0^\lambda \dot{Y}_0 - \sum_{i=1}^m u_i H_{0,i}^\lambda Y_0 = C_1^\lambda x$ defined in relations (4) and $\mathbf{Y} \cup \mathbf{Z}^1$ corresponds to Y_1 . Step i of the algorithm is equivalent to step 0 by enlarging the output components to Y_i and so the vertex set $\mathbf{Y}$ to $\mathbf{Y} \cup \mathbf{Z}^i$ or, roughly speaking, to all the part of state vector observed in the previous steps. Indeed, in form (4), Step i corresponds to the search of matrix Γ_i such that: $\Gamma_i^\lambda \dot{Y}_i = C_{i+1}^\lambda x + \sum_{j=1}^m u_j H_{i,j}^\lambda Y_i$, where $Y_i \stackrel{\text{def}}{=} \begin{pmatrix} C^\lambda \\ C_i^\lambda \end{pmatrix} x$. This step is equivalent to step 0 by considering that the output vector is enlarged to Y_i . So, as in step 0, we mark in step i all the vertices of $\mathbf{Z}_C$ which correspond to linear independent combinations $P^\lambda z$ satisfying relation of the form (10) between the first derivative of the variables associated to $\mathbf{Y} \cup \mathbf{Z}^i$, corresponding to vector Y_i in (4), the input and the state. Consequently, edges starting from $\mathbf{Y} \cup \mathbf{Z}^i$ correspond to all relations, linking the state to the input and the output derivatives of order less or equal to $i + 1$, which allow us to observe a part of the state for any input value. Due to the second item of Remark 3, $\mathbf{Z}^i$ is a non-decreasing sequence of structurally minimal subset w.r.t. the function θ ($\theta(\mathbf{Z}^i) \leq \theta(\mathbf{Z}^{i+1})$). Moreover, since $\theta(\mathbf{Z}^i)$ is bounded by n , there exists an integer k for which $\theta(\mathbf{Z}^{k+j}) = \theta(\mathbf{Z}^k)$, for all $j \geq 0$. So after a finite number of iterations the algorithm stops when $\theta(\mathbf{Z}^{i+1}) = \theta(\mathbf{Z}^i)$ which means that in the $(i + 1)^{th}$ step, the $(i + 1)^{th}$ output differentiation does not give any new relation useful to achieve uniform observability of (Σ_Λ) . In this case, we pose $\mathbf{Z}^* = \mathbf{Z}^k$ and vertex subset $\mathbf{Y} \cup \mathbf{Z}^*$, represents all the vertices of $\mathbf{Y} \cup \mathbf{Z}_C$ which can be associated to output vector $Y_k = \begin{pmatrix} C \\ C_k \end{pmatrix} x$ of relation (4). These vertices can be associated to useful and independent relations between the outputs and their derivatives and the state and which are exploitable to observe x for any input value. The system can be put into the form (4) if and only if $\theta(\mathbf{Y} \cup \mathbf{Z}^*) = n$ which is equivalent, according to the construction of $\mathbf{Z}^*$, to $\text{g_rank} \begin{pmatrix} C^\lambda \\ C_k^\lambda \end{pmatrix} = n$.

4.2 Graphic conditions for generic uniform observability

The first part of this section suggests immediately the following proposition :

Proposition 2 *Consider structured bilinear system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$. Structured system (Σ_Λ) can be put into form (4) if and only if in graph $\mathcal{G}(\Sigma_\Lambda)$, $\theta(\mathbf{Y} \cup \mathbf{Z}^*) = n$, where $\mathbf{Z}^*$ is computed according to Algorithm 1.*

Proof: The proof of (1) $\Leftrightarrow$ (2) is immediate using Lemmas 1 and 2, algorithm 1 and Property **P1** of function θ . $\triangle$

Note that if the system can be put into form (4), then there exists an integer k such that $\mathbf{V}_{\mathbf{x}^k}^* = \mathbf{X}$, and $\mathbf{V}_{\mathbf{x}^k \mathbf{u}}^* = \mathbf{X}\mathbf{U}$. In this case, $\mathbf{Z}^{k+1} = \mathbf{Z}^* = \mathbf{Z}$. Thus, from Proposition 2, we can deduce the following corollary which represents the main result of the paper:

Corollary 1 *Consider structured bilinear system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$.*

(1) (Σ_Λ) is generically uniform observable only if, in graph $\mathcal{G}(\Sigma_\Lambda)$, there exists a path from $\mathbf{Y}$ to each state vertex and there exists a v -disjoint subgraph $\mathcal{S}_\mathcal{G}$ which covers all the state vertices.

(2) (Σ_Λ) is generically uniform observable if $\mathbf{Z}^ = \mathbf{Z}$, where $\mathbf{Z}^*$ is computed using to Algorithm 1.*

Note that the condition (1) of Corollary 1 is necessary because it means also the observability of the generic observability of the system for $u(t) = 0$. Furthermore, from the results of [9], condition (2) is obviously necessary and sufficient in the single output case.

The results of Corollary 1 and Algorithm 1 (used to compute $\mathbf{Z}^*$) show that to achieve the uniform observability, we can either add sensors or put some inputs to zero by removing some actuators. The first solution is obviously more suitable. In the one hand it allows to enlarge $\mathbf{Z}^*$ and to achieve the first condition of Corollary 1, while the second one allows to enlarge $\mathbf{Z}^*$ only. On the second hand, the second solution is not always acceptable from a control point of view.

5 Example

In this section, we illustrate the results presented above with an example. The latter is intentionally not very complicated. Nevertheless, it is clear that the algorithm we propose is well suited to large-scale systems using combinatorial programming techniques.

Example 3 *Consider the SBLs represented in the digraph of Figure 3. This system has 14 state components. The matrix representation of this system is not given because of lack of space. Nevertheless, it can be easily deduced from the digraph. First, note that necessary sufficient condition (1) enounced in Corollary 1 is satisfied as it is shown by the v -disjoint subgraph (Figure 4) which covers all the state vertices.*

We use now algorithm 1 to determine $\mathbf{Z}^$:*

Step 0: *First, we compute $\mathbf{V}_{\mathbf{M}^0}^* = \{\mathbf{y}_3, \mathbf{y}_5\}$, $\mathbf{V}_{\mathbf{X}^0}^* = \{\mathbf{x}_4, \mathbf{x}_8\}$, $\mathbf{V}_{\mathbf{x}^0 \mathbf{u}}^* = \{\mathbf{x}_4 \mathbf{u}_1, \mathbf{x}_4 \mathbf{u}_2\}$ and graphically we have $\mathbf{Q}^0 = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4, \mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_7, \mathbf{z}_8\}$. Because of the presence of $\mathbf{x}_1 \mathbf{u}_1$ in $\bar{\varphi}(\{\mathbf{z}_2\})$ and the fact that $\varphi_e(\{\mathbf{y}_1, \mathbf{y}_2\}) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, we can not find a subset $\mathbf{V}_\mathbf{Y} \subseteq \mathbf{Y}$ ($\mathbf{Z}^0 = \emptyset$) which*

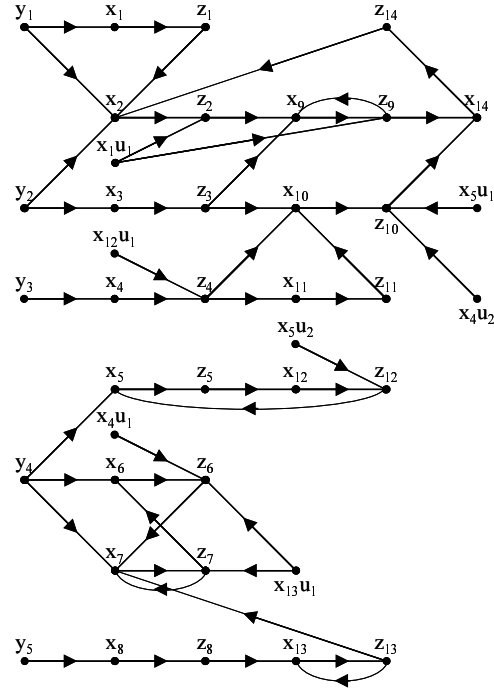


Figure 3: Digraph associated to Example 3

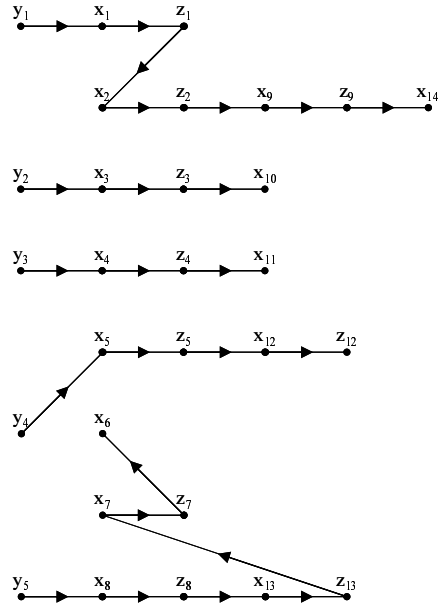


Figure 4: v -disjoint subgraph covering all the state vertices for Example 3

allows to satisfy equality **(Cond2)** for $V_{\kappa,j} = \{z_1\}, \{z_2\}, \{z_3\}, \{z_1, z_2\}$ or $\{z_2, z_3\}$. Nevertheless, for $V_{2,2} = \{z_1, z_3\}$ there exists $V_Y = \{y_1, y_2\}$, which allows to verify condition **(Cond2)**: $\theta(V_Y) + \kappa + \text{card}(\bar{\varphi}(V_{2,2}) \cap V_{XU^0}^*) = 2 + 2 + 0 = 4$ and $\text{card}(\varphi_e(V_Y) \cup \bar{\varphi}(V_{2,1})) + 1 = 3 + 1 = 4$. Similarly, due to the presence of $x_{12}u_1$ in $\bar{\varphi}(\{z_4\})$ and of $x_{13}u_1$ in $\bar{\varphi}(\{z_6\}) \bar{\varphi}(\{z_7\})$, there cannot exist a subset $V_Y \subseteq Y$ which allows to satisfy equality **(Cond2)** for any nonempty subset $V_{\kappa,j} \subseteq \{z_4, z_5, z_6, z_7\}$. Otherwise, for $V_{1,8} = \{z_8\}$, there exists $V_Y = \{y_5\}$, which allows to verify condition **(Cond2)**: $\theta(V_Y) + \kappa + \text{card}(\bar{\varphi}(V_{1,8}) \cap V_{XU^0}^*) = 1 + 1 + 0 = 2$ and $\text{card}(\varphi_e(V_Y) \cup \bar{\varphi}(V_{2,1})) + 1 = 1 + 1 = 2$. Thus, $Z^1 = \{z_8, z_{1,3}\}$. As $\theta(Z^1) = 2 > \theta(Z^0) = 0$, we go to step 1.

Step 1: Simple computations give $V_{M^1}^* = \{y_3, y_5, z_8\}$, $V_{X^1}^* = \{x_4, x_8, x_{13}\}$, $V_{XU^1}^* = \{x_4u_1, x_4u_2, x_{13}u_1\}$, $Q^1 = \varphi_e(Y \cup Z^1) = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{13}\}$.

For the same reasons than in step 0, we cannot find subsets $V_Y \subseteq Y$ and $V_Z \subseteq \varphi(Z^1)$ which allow to satisfy equality **(Cond2)** for $V_{\kappa,j} = \{z_1\}, \{z_2\}, \{z_3\}, \{z_1, z_2\}, \{z_2, z_3\}$. Nevertheless, as previously, for $V_{1,8} = \{z_8\}$ and for $V_{2,2} = \{z_1, z_3\}$, there exist subsets $V_Y \subseteq Y$ and $V_Z \subseteq Z^1$ which allow to satisfy equality **(Cond2)** and so $z_{1,3}$ and z_8 belong to Z^2 . Besides, for $V_{1,13} = \{z_{13}\}$, there exists $V_Y = \emptyset$ and $V_Z = \{z_8\} \subseteq \varphi(Z^1)$ which allow to verify condition **(Cond2)**: $\theta(V_Z) + \kappa + \text{card}(\bar{\varphi}(V_{1,13}) \cap V_{XU^1}^*) = 1 + 1 + 0 = 2$ and $\text{card}(\varphi_e(V_Z) \cup \bar{\varphi}(V_{1,13})) + 1 = 1 + 1 = 2$.

Furthermore, as $x_{12}u_1 \in \bar{\varphi}(\{z_4\})$, $x_1u_1 \in \bar{\varphi}(\{z_9\})$ and $x_5u_1 \in \bar{\varphi}(\{z_{10}\})$, no subset $V_{\kappa,j}$ containing z_4 , z_9 or z_{10} can satisfy **(Cond1)** and **(Cond2)**. Finally, as $x_{13}u_1$ belongs to $V_{XU^1}^*$, for $V_{3,j} = \{z_5, z_6, z_7\}$ there exist $V_Y = \{y_4\}$ and $V_Z = \emptyset$ which allow to verify condition **(Cond2)**: $\theta(V_Y) + \kappa + \text{card}(\bar{\varphi}(V_{3,j}) \cap V_{XU^1}^*) = 1 + 3 + 2 = 6$ and $\text{card}(\varphi_e(V_Y) \cup \bar{\varphi}(V_{3,j})) + 1 = 5 + 1 = 6$. So, after step 1, we have $Z^2 = \{z_8, z_{13}, z_{1,3}, z_{5,6,7}\}$. As $\theta(Z^2) = 4 > \theta(Z^1) = 1$, we go to step 2.

Step 2: We have first compute $V_{M^2}^* = \{y_3, y_5, z_8, z_{13}\}$, $V_{X^2}^* = \{x_4, x_7, x_8, x_{13}\}$, $V_{XU^2}^* = \{x_4u_1, x_4u_2, x_{13}u_1\}$. From the graph, we obtain $Q^2 = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{12}, z_{13}\}$.

Using the same arguments than for step 1, we have that z_8 and z_{13} belongs to Z^3 . As nothing changes for $z_1, z_2, z_3, z_4, z_9, z_{10}$, only $z_{1,3}$ belongs to Z^3 . Otherwise, for $V_{1,7} = \{z_7\}$, there exist $V_Z = \{z_8, z_{13}\} \subseteq \varphi(Z^2)$ and $V_Y = \emptyset$ which allow to verify condition **(Cond2)**: $\theta(V_Z) + \kappa + \text{card}(\bar{\varphi}(V_{2,j}) \cap V_{XU^2}^*) = 2 + 1 + 1 = 4$ and $\text{card}(\varphi_e(V_Z) \cup \bar{\varphi}(V_{1,13})) + 1 = 3 + 1 = 3$.

Furthermore, for $V_{2,j} = \{z_5, z_6\}$, there exist $V_Y = \{y_4\} \subseteq Y$ and a structurally minimal subset $V_Z = \{z_{13}, z_8\} \subseteq \varphi(Z^2)$ such that: $\theta(V_Y \cup V_Z) + \kappa + \text{card}(\bar{\varphi}(V_{1,13}) \cap V_{XU^2}^*) = 3 + 2 + 2 = 7$ and $\text{card}(\varphi_e(V_Y \cup V_Z) \cup \bar{\varphi}(V_{1,13})) + 1 = 6 + 1 = 7$. The previous settings lead to $Z^3 = \{z_7, z_8, z_{13}, z_{1,3}, z_{5,6}\}$ which is structurally minimal because of condition **(Cond1)**. As $\theta(Z^3) = 5 > \theta(Z^2) = 4$, we go to step 3.

Step 3: First, we compute $\mathbf{V}_{M^3}^* = \{y_3, y_4, y_5, z_7, z_8, z_{13}, z_{5,6}\}$, $\mathbf{V}_{X^3}^* = \{x_4, x_5, x_6, x_7, x_8, x_{12}, x_{13}\}$, and $\mathbf{V}_{XU^3}^* = \{x_4u_1, x_4u_2, x_5u_1, x_5u_2, x_{12}u_1, x_{13}u_1\}$. Graphically, we have $\mathbf{Q}^3 = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{12}, z_{13}\}$.

As $x_{12}u_1 \in \mathbf{V}_{XU^3}^*$, we have that z_4 belongs to $\mathbf{Z}^4$: for $\mathbf{V}_{1,4} = \{z_4\}$ and for $\mathbf{V}_Y = \{y_4\}$, we have $\theta(\mathbf{V}_Y) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,4}) \cap \mathbf{V}_{XU^3}^*) = 1 + 1 + 1 = 3$ and $\text{card}(\varphi_e(\mathbf{V}_Y) \cup \bar{\varphi}(\mathbf{V}_{1,4})) + 1 = 2 + 1 = 3$.

Moreover, the presence of z_7 in $\mathbf{Z}^3$ implies that for $\mathbf{V}_{1,5} = \{z_5\}$ there exist $\mathbf{V}_Y = \{y_4\} \subseteq Y$ and $\mathbf{V}_Z = \{z_7, z_{13}, z_8\} \subseteq \wp(\mathbf{Z}^3)$ such that condition **(Cond2)** is satisfied: $\theta(\mathbf{V}_Y \cup \mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,5}) \cap \mathbf{V}_{XU^3}^*) = 4 + 1 + 0 = 5$ and $\text{card}(\varphi_e(\mathbf{V}_Y \cup \mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,5})) + 1 = 4 + 1 = 5$.

Finally, due also to the presence of z_7 in $\mathbf{Z}^3$ we have that for $\mathbf{V}_{1,6} = \{z_6\}$ and for $\mathbf{V}_Z = \{z_7, z_8, z_{13}\} \subseteq \wp(\mathbf{Z}^3)$, $\theta(\mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,6}) \cap \mathbf{V}_{XU^3}^*) = 3 + 1 + 2 = 6$ and $\text{card}(\varphi_e(\mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,6})) + 1 = 5 + 1 = 6$.

Similarly, since $x_5u_1 \in \mathbf{V}_{XU^3}^*$ we have that z_{12} belongs to $\mathbf{Z}^4$: for $\mathbf{V}_{1,12} = \{z_{12}\}$ and for $\mathbf{V}_Z = \{z_8, z_{5,6}, z_{13}\} \subseteq \wp(\mathbf{Z}^3)$, we have $\theta(\mathbf{V}_Z) + \kappa + \text{card}(\bar{\varphi}(\mathbf{V}_{1,12}) \cap \mathbf{V}_{XU^3}^*) = 3 + 1 + 1 = 5$ and $\text{card}(\varphi_e(\mathbf{V}_Z) \cup \bar{\varphi}(\mathbf{V}_{1,12})) + 1 = 4 + 1 = 5$. Therefore, we obtain $\mathbf{Z}^4 = \{z_4, z_5, z_6, z_7, z_8, z_{12}, z_{13}, z_{1,3}\}$. As $\theta(\mathbf{Z}^4) = 7 > \theta(\mathbf{Z}^3) = 5$, we go to step 4.

Step 4: Firstly, we compute $\mathbf{V}_{M^4}^* = \{y_3, y_4, y_5, z_5, z_6, z_7, z_8\}$, $\mathbf{V}_{X^4}^* = \{x_4, x_5, x_6, x_7, x_8, x_{12}, x_{13}\}$, $\mathbf{V}_{XU^4}^* = \{x_4u_1, x_4u_2, x_5u_1, x_5u_2, x_{12}u_1, x_{13}u_1\}$. We obtain graphically $\mathbf{Q}^4 = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}\}$.

$z_4 \in \mathbf{Z}^4$ implies that for both $\mathbf{V}_{2,j} = \{z_{10}, z_{11}\}$ and $\mathbf{V}_{4,j'} = \{z_1, z_2, z_9, z_{10}\}$, there exist $\mathbf{V}_Y \subseteq Y$ and a structurally minimal vertex subset $\mathbf{V}_Z \subseteq \wp(\mathbf{Z}^i)$ such that **(Cond2)** is satisfied. Therefore, $\mathbf{Z}^5 = \{z_4, z_5, z_6, z_7, z_8, z_{12}, z_{13}, z_{1,3}, z_{10,11}, z_{1,2,9,10}\}$. As $\theta(\mathbf{Z}^5) = 9 > \theta(\mathbf{Z}^4) = 7$, we go to step 5.

Step 5: As previously, $\mathbf{V}_{M^5}^* = \{y_3, y_4, y_5, z_5, z_6, z_7, z_8\}$, $\mathbf{V}_{X^5}^* = \{x_4, x_5, x_6, x_7, x_8, x_{12}, x_{13}\}$, $\mathbf{V}_{XU^5}^* = \{x_4u_1, x_4u_2, x_5u_1, x_5u_2, x_{12}u_1, x_{13}u_1\}$ and $\mathbf{Q}^5 = \mathbf{Z}$. Since $z_{1,10,11} \in \mathbf{Z}^5$ and $z_{1,2,9,10} \in \mathbf{Z}^5$, we can find $\mathbf{V}_Y \subseteq Y$ and a structurally minimal vertex subset $\mathbf{V}_Z \subseteq \wp(\mathbf{Z}^5)$ such that **(Cond2)** is satisfied for $\mathbf{V}_{2,j} = \{z_1, z_{14}\}$, $\{z_1, z_3\}$, $\{z_3, z_{11}\}$ and $\{z_{10}, z_{11}\}$. Obviously it is also the case for any 2-subset of $\{z_1, z_3, z_{10}, z_{11}, z_{14}\}$, but we must choose only four of these 2-subsets to satisfy condition **(Cond1)** which ensures the structural minimality of $\mathbf{Z}^6$. Moreover, we can also find $\mathbf{V}_Y \subseteq Y$ and a structurally minimal vertex subset $\mathbf{V}_Z \subseteq \wp(\mathbf{Z}^i)$ such that **(Cond2)** is satisfied for $\mathbf{V}_{3,j} = \{z_1, z_2, z_9\}$. Consequently, $\mathbf{Z}^6 = \{z_4, z_5, z_6, z_7, z_8, z_{12}, z_{13}, z_{1,3}, z_{1,14}, z_{3,11}, z_{10,11}, z_{1,2,9}\}$. As $\theta(\mathbf{Z}^6) = 10 > \theta(\mathbf{Z}^5) = 9$, we go to step 6.

Step 6: We compute $\mathbf{V}_{M^6}^* = \{y_1, y_2, y_3, y_4, y_5, z_4, z_5, z_6, z_7, z_8, z_{1,14}, z_{1,3}, z_{3,11}, z_{10,11}\}$ and as $\mathbf{V}_{X^6}^* = \mathbf{X}$, we can easily deduce that $\mathbf{Z}^7 = \mathbf{Z} = \mathbf{Z}^*$. According to Condition (2) of Corollary 1, we can conclude to the generic uniform observability of the system.

6 Conclusion

In this paper, we propose an analysis tool to study the generic uniform observability of structured bilinear systems. Using a graphic representation dedicated to this class of nonlinear systems two conditions are provided and expressed in graphic terms. The first one is a necessary condition and can be checked directly from the digraph associated to the system. The second one is sufficient and necessitates the use of an iterative algorithm providing the computation of a particular vertex subset. The proposed conditions, which need few information about the system, are very easy to check by means of well-known combinatorial techniques and simply by hand for small systems. That makes our approach particularly suited for large-scale and sparse systems as it is free from numerical difficulties. Indeed, the proposed algorithm and conditions can be easily implemented because they require simple computations on integers and are based on finding edges and sub-graphs in a digraph.

Starting from the presented results, we can study, as in [5] for linear systems, the optimisation of the sensor location to achieve the generic uniform observability. Finally, sensor defects and also the addition of new actuators to the system, for fault tolerant control strategy for example, can lead to the lost of the generic uniform observability property. One of the perspectives of the presented work can be also the study of the actuator placement and sensor redundancy in order to preserve the generic uniform observability property.

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