An extension of Kracht's theorem to generalized Sahlqvist formulas

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ABSTRACT. Sahlqvist formulas are a syntactically specified class of modal formulas proposed by Hendrik Sahlqvist in 1975. They are important because of their first-order definability and canonicity, and hence axiomatize complete modal logics. The first-order properties definable by Sahlqvist formulas were syntactically characterized by Marcus Kracht in 1993. The present paper extends Kracht's theorem to the class of 'generalized Sahlqvist formulas' introduced by Goranko and Vakarelov and describes an appropriate generalization of Kracht formulas.

KEYWORDS: modal logic, Sahlqvist formulas, Kracht formulas, first-order definability, safe expressions.

DOI:10.3166/JANCL.18.1-25 © 0 Lavoisier, Paris

1. Introduction

The Sahlqvist theorem is a hard working horse in modal logic. It describes a large class of first-order definable canonical modal formulas. A standard proof of completeness results boils down to finding relevant first-order properties and corresponding Sahlqvist formulas and next — to applying Sahlqvist completeness theorem. Also Sahlqvist formulas are often applied for proofs of negative results such as non-finite axiomatizability.

Kracht's theorem is an important addition to the Sahlqvist theorem. It explicitly describes the class of first-order correspondents to Sahlqvist formulas (Kracht, 1993), (Kracht, 1999). Moreover, it gives an algorithm constructing a Sahlqvist formula from its first-order analogue.

Journal of Applied Non-Classical Logics. Volume 0 - No. 0/0, page Pages undefined

So when we encode a first-order condition into a Sahlqvist formula, we implicitly use Kracht's algorithm. That is why for axiomatizing modal logics Kracht's theorem is not less important than the Sahlqvist theorem.

In (Goranko *et al.*, 2000), (Goranko *et al.*, 2006) the Sahlqvist theorem was further generalized. These results turned out to be at the intersection of at least two research lines.

The first line came from attempts at axiomatizing many-dimensional modal logics. Probably, the first known generalized Sahlqvist formula was cub_1 (see page 19 of this paper) for the first time published in (Shehtman, 1978), expressing the 'cubifying' property of 3-dimensional product frames (see Figure 1)

$$(\forall x_0)\forall x_1\forall x_2\forall x_3 (x_0R_1x_1 \land x_0R_2x_2 \land x_0R_3x_3 \rightarrow \\ \exists y(y \in R_3(R_2(x_1) \cap R_1(x_2)) \land y \in R_2(R_3(x_1) \cap R_1(x_3)) \land \\ \land y \in R_1(R_2(x_3) \cap R_3(x_2)))).$$

$$(1)$$

Modifications of this formula were used by A. Kurucz in the proof of some negative results on \geq 3-dimensional products (Kurucz, 2000), (Kurucz, 2008). Let us also mention that generalized Sahlqvist formulas appear in axiomatizing 2-dimensional squares with extinguished diagonal (Kikot, n.d.).

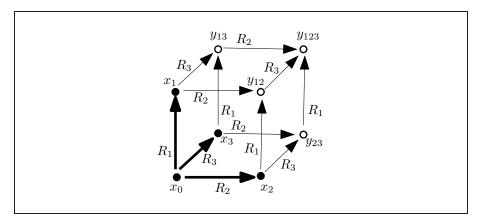


Figure 1.

First-order conditions like (1) can be illustrated by pictures with black and white points, and bold and simple arrows, as in Figure 1. A formal analogue of such a picture is the notion of a *diagram*. It turns out that under some natural conditions, the corresponding first-order $\forall \exists$ -formula is modally definable if and only if the diagram does not have non-oriented cycles consisting of white points and simple arrows, and, in the case of modal definability, the $\forall \exists$ -formula always corresponds to a generalized Sahlqvist formula (Kikot, 2005).

The second line of research arises from the natural problem — to find sufficient conditions for first-order definability and canonicity of modal formulas. The relevant

part starts with (Goranko *et al.*, 2000) extending the Sahlqvist theorem to polyadic modal languages. The same paper gives an example of first-order definable and canonical modal formulas that are not Sahlqvist (namely, formula D_2 from Example 31 below). However, the question if these new formulas have Sahlqvist equivalents, was remaining unsolved for some time. This question was solved by V. Goranko and D. Vakarelov who introduced the notion of a-persistence and showed that all Sahlqvist formulas are a-persistent while D_2 is not (Goranko *et al.*, 2006). It was in (Goranko *et al.*, 2006) that the notion of a 'generalized Sahlqvist formula', that lies in the center of the present paper, was introduced as a partial case of so called 'inductive formula'.

Then algorithms were proposed (see (Conradie *et al.*, 2004), (Conradie *et al.*, 2006) and references therein), for computing first-order equivalents of some modal formulas. The Sahlqvist theorem was further generalized in (Vakarelov, 2003) and (Vakarelov, 2002), yielding the class of complex Sahlqvist formulas, but they are actually semantically equivalent to standard Sahlqvist formulas.

Another challenging problem is: 'given a first-order formula, find the modal logic of the corresponding elementary class'. Let us mention that the problems 'given a first-order formula, determine if it is modally definable' and 'given a modal formula, determine if it is first-order definable' are undecidable due to Chagrova's theorem (Chagrov *et al.*, 2006). That is why any sufficient condition for modal (or f. o.) definability is very interesting by itself. In this context besides the above cited Kracht's result (Kracht, 1993), (Kracht, 1999) and the study of diagram formulas (Kikot, 2005) we can make especially mention the brilliant work (Hodkinson, 2006) giving an explicit infinite axiomatization for any elementary class. In some particular cases more concise (although also infinite) axiomatizations are constructed in (Balbiani *et al.*, 2006). However, we still do not have a criterion of finite axiomatizability for the logics from (Hodkinson, 2006) and (Balbiani *et al.*, 2006).

The present paper continues the study on modal logics of elementary classes. We extend the class of Kracht formulas to the class of 'generalized Kracht formulas'. Then we propose an algorithm constructing a modal correspondent for a given generalized Kracht formula. This modal correspondent is a generalized Sahlqvist formula, and therefore it is canonical (and, a fortiori, Kripke complete).

Our terminology slightly differs from (Goranko *et al.*, 2006); in particular, the notion 'regular formula' has a different meaning. Also, the term 'safe expression' is not the same as in (Blackburn *et al.*, 2002).

2. Regular box-formulas.

We consider the modal language \mathcal{ML}_{Λ} with countably many propositional variables, unary modalities \Diamond_{λ} and their duals \Box_{λ} , where $\lambda \in \Lambda$, boolean connectives $\land, \lor, \neg, \rightarrow$ and boolean constants \top, \bot . A formula in this language is called *positive* if it does not contain \neg and \rightarrow (but may contain \bot .)

Recall that in the Sahlqvist theorem 'boxed atoms' (i.e. the expressions of the form $\Box^n p$) are crucial, because they allow us to obtain the minimal valuation for an antecedent. In generalized Sahlqvist formulas 'boxed atoms' are replaced by 'regular box-formulas'.

DEFINITION 1 (GORANKO et al., 2006). — A box-formula is defined by recursion:

- a variable p_i is a box-formula;

- if POS is a positive modal formula and BF is a box-formula then $POS \rightarrow BF$ is a box-formula;

- if BF is a box-formula then $\Box_{\lambda}BF$ is a box-formula.

Thus a box-formula is equivalent to one of the form

 $POS_1 \rightarrow \Box^{\alpha_1}(POS_2 \rightarrow \Box^{\alpha_2}(POS_3 \rightarrow \ldots \rightarrow p_i)\ldots),$

where \Box^{α_j} are sequences of boxes, POS_j are positive.

The last variable p_i of this formula is called its head. $BF \succ p_i$ denotes that p_i is the head of a box-formula BF.

Let A be a set of box-formulas. The dependency graph of A is an oriented graph $G = (V_A, E_A)$, where the set of vertices $V_A = \{p_1, \ldots, p_n\}$ consists of all variables occurring in A, and the adjacency relation is

 $p_i E_A p_j \iff p_i$ occurs (not as a head) in some formula $\phi \in A$ with the head p_j .

A set of box-formulas A is called regular if its dependency graph is acyclic, i.e., it does not contain oriented cycles.

We will use a more convenient technical version of Definition 1.

The set of propositional variables is split into countably many groups $p_1^0, p_2^0, p_3^0, \ldots$, $p_1^1, p_2^1, p_3^1, \ldots, p_1^2, p_2^2, p_3^2, \ldots$ and so on. The upper index (called the *rank*) is the number of the group and the lower index is the number of a variable within a group. Put $\bar{p}^i = \{p_1^i, p_2^i, p_3^i, \ldots\}.$

DEFINITION 2. — A regular box-formula of rank k is defined by recursion:

– a variable p_i^k is a box-formula of rank k, – if $POS(\bar{p}^0, \bar{p}^1, \dots, \bar{p}^{k-1})$ is a positive modal formula, depending only on the variables of rank < k and REG is a regular box-formula of rank k then $POS(\bar{p}^0, \bar{p}^1, \dots, \bar{p}^{k-1}) \to REG$ is a regular box-formula of rank k,

- if REG is a regular box-formula of rank k then $\Box_{\lambda} REG$ is a regular boxformula of rank k.

LEMMA 3. — Let A be a set of modal formulas. Then

(1) if A is a set of regular box-formulas (in the sense of Definition 2), then A is a regular set of box-formulas (in the sense of Definition 1).

(2) if A is a regular set of box-formulas, then we can range the propositional variables (i.e. choose the upper indices) so that A becomes a set of regular box-formulas.

PROOF. — (1) is trivial. In fact, if A is a set of regular formulas and $p_i^s E_A p_j^t$, then s < t. So the dependency graph does not contain oriented cycles.

We prove (2) by induction on the number of vertices in V_A . If it has a single vertex, the statement is trivial. Suppose it has n vertices. Since our graph does not have oriented cycles, there is a vertex v in V_A without successors. Suppose v corresponds to a variable p_l for some $l \le n$. We eliminate this vertex (and of course, all entering edges) and obtain the graph $G'_A = (V'_A, E'_A)$. Since v does not have successors, p_l can occur only in the heads of box-formulas from A. If A' is obtained from Aby eliminating box-formulas with the head p_l , then $G_{A'} = G'_A$. By the induction hypothesis we can range the vertices of G'_A so that all formulas in A' become regular. For $i \ne l$ let r(i) be the rank of p_i . Put the rank of p_l to be max r(i) + 1. Then A is a set of regular formulas, since all formulas in $A \setminus A'$ have p_l as their head, and the rank of p_l is maximal.

Besides the modal language \mathcal{ML}_{Λ} , we need additional languages $L_k^{\#}, L_k^P$ and L. Their vocabularies are

 $\begin{array}{l} - \mbox{ for } L_k^P : P_i^l \, (l < k), \cap, \cup, R_\lambda^{-1}, R_\lambda^\Box, \top, \bot; \\ - \mbox{ for } L_k^\# : \#, P_i^l \, (l < k), \cap, \cup, R_\lambda^{-1}, R_\lambda^\Box, R_\lambda, \top, \bot; \\ - \mbox{ for } L : x_i, \cap, \cup, R_\lambda^{-1}, R_\lambda^\Box, R_\lambda, \top, \bot. \end{array}$

Here \bot, \top are constants, $P_i^l, \#, x_i$ are variables, $R_{\lambda}^{-1}, R_{\lambda}^{\Box}, R_{\lambda}$ are unary function symbols, \cap, \cup are binary function symbols. We call the terms of these languages expressions.

To every regular box-formula ϕ of rank k we assign an $L_k^{\#}$ -expression KV^{ϕ} . (Later we shall see that KV^{ϕ} is the operator for the relative minimal valuation for the head of ϕ .)

First we assign an expression $KP^{POS} \in L_k^P$ to every positive formula POS:

$$\begin{split} KP^\top &:= \top, \quad KP^\perp := \bot, \\ KP^{p_i^l} &:= P_i^l, \text{ where } l < k, \\ KP^{POS_1 \land POS_2} &:= KP^{POS_1} \cap KP^{POS_2}, \\ KP^{POS_1 \lor POS_2} &:= KP^{POS_1} \cup KP^{POS_2}, \\ KP^{\diamondsuit_\lambda POS} &:= R_\lambda^{-1}(KP^{POS}), \\ KP^{\Box_\lambda POS} &:= R_\lambda^{\Box}(KP^{POS}). \end{split}$$

This definition obviously corresponds to the truth definition in the standard Kripke semantics. If we have a frame $F = (W, (R_{\lambda} : \lambda \in \Lambda))$ and θ is a valuation for the

variables p_i^l , where l < k, then $\theta(POS)$ is the value of KP^{POS} under the interpretation I sending \top to W, \perp to \emptyset , P_i^l to $\theta(p_i^l)$, $R_{\lambda}^{-1}(A)$ to $\{x \mid \exists y \, xRy \text{ and } y \in A\}$, $R_{\lambda}^{\Box}(A)$ to $\{x \mid \forall y \text{ if } xRy \text{ then } y \in A\}$

Now we assign an $L_k^{\#}$ -expression KV^{ϕ} to any regular box-formula ϕ of rank k. DEFINITION 4. — We set

$$KV^{p_i} := \#,$$

$$KV^{POS \to \psi} := KV^{\psi} (\# \cap KP^{POS})$$

$$KV^{\Box_{\lambda}\psi} := KV^{\psi} (R_{\lambda}(\#)).$$

Here $KV^{\phi}(t)$ denotes the substitution instance $[t/\#]KV^{\phi}$. That is to obtain $KV^{POS \to \psi}$, we substitute the term $\# \cap KP^{POS}$ for # in KV^{ψ} , and to obtain $KV^{\Box_{\lambda}\psi}$, we substitute the term $R_{\lambda}(\#)$ for # in KV^{ψ} .

EXAMPLE 5. —

1) Let
$$\phi = \Box_{\lambda}^{l} p_{0}^{1}$$
. Then $KV^{\phi} = R_{\lambda}^{l}(\#)$.
2) If $\phi = \Box_{1}(\Diamond_{2} p_{0}^{0} \to \Box_{3} p_{0}^{1})$, then $KV^{\phi} = R_{3}(R_{2}^{-1}(P_{0}^{0}) \cap R_{1}(\#))$.

In a model $M = (W, R_{\lambda}, \theta)$, where $x \in W$, we can evaluate $KV^{\phi}(x)$ under the interpretation I described above and identify it with a certain subset of W.

LEMMA 6 (ON MONOTONICITY OF KV^{ϕ}). — $KV^{\phi}(x)$ is monotonic with respect to P_i^l .

PROOF. — This is trivial, since all operations $\cap, \cup, R_{\lambda}^{-1}, R_{\lambda}^{\Box}, R_{\lambda}$ are monotonic.

The next lemma shows that the operator KV^{ϕ} really defines the 'relative minimal valuation' for the truth of ϕ in the standard Kripke semantics.

LEMMA 7 (ON MINIMALITY OF KV^{ϕ}). — Let ϕ be a regular box-formula with a head p_i^k . Consider a Kripke model $M = (W, (R_{\lambda} : \lambda \in \Lambda), \theta)$ where $\theta(p_i^l) = P_i^l$ $(l \leq k)$. Then

$$M, x \models \phi \Longleftrightarrow P_i^k \supseteq KV^{\phi}(x).$$

PROOF. — The proof is by induction on the length of ϕ . If ϕ is a variable, there is nothing to prove.

Let $\phi = POS \rightarrow \psi$. Then

$$x \models \phi \iff x \models POS \rightarrow \psi \iff$$

$$(\text{ if } x \models POS, \text{ then } x \models \psi) \iff$$

$$(\text{ if } x \models POS, \text{ then } P_i^k \supseteq KV^{\psi}(x)) \iff$$

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$$(x \neq POS \text{ or } P_i^k \supseteq KV^{\psi}(x)) \iff$$

$$\{x\} \cap KP^{POS} = \emptyset \text{ or } P_i^k \supseteq KV^{\psi}(x)$$
(2)

There are only two possible values of $\{x\} \cap KP^{POS}$, viz. $\{x\}$ and \emptyset . A simple induction argument shows that $KV^{\psi}(\emptyset) = \emptyset$. So by an easy study of cases (2) is equivalent to

$$P_i^k \supseteq KV^{\psi}(\{x\} \cap KP^{POS}) \iff P_i^k \supseteq KV^{\phi}(x).$$

Let $\phi = \Box_{\lambda} \psi$.

$$\begin{aligned} x \vDash \phi &\iff x \vDash \Box_{\lambda} \psi \iff \\ \forall y (x R_{\lambda} y \Rightarrow y \vDash \psi) \iff \\ \forall y (x R_{\lambda} y \Rightarrow P_i^k \supseteq K V^{\psi}(y)) \iff ^1 \\ P_i^k \supseteq K V^{\psi}(R_{\lambda}(x)) \iff \\ P_i^k \supseteq K V^{\phi}(x). \end{aligned}$$

Let A be a finite set of regular box-formulas, $\mathcal{P}(A)$ be the set of all subsets of A.

Consider a set $V = \{x_1, \ldots, x_n\}$, and a function $f : V \to \mathcal{P}(A)$.

DEFINITION 8. — To every variable p_i^k we assign an expression $KF_f^{p_i^k}$ of our language L (see above) by induction on k. We put

$$KF_f^{p_i^{\kappa}} = \bigcup_{\phi \succ p_i^{\kappa}, \phi \in f(x_i)} KVF_f^{\phi}(x_i),$$

where $KVF_f^{\phi}(x_i)$ is obtained by substituting expressions $KF_f^{p_i^l}$ for P_i^l for all l < k in the expression $KV^{\phi}(x_i)$, that we can denote by

$$KVF_f^{\phi}(x_i) = [KV_f^{p_i^l}/P_i^l]_{l < k}KV^{\phi}(x_i).$$

In particular,

$$KF_f^{p_i^0} = \bigcup_{\phi \succ p_i^0, \phi \in f(x_i)} KV_f^{\phi}(x_i),$$

where $KV_f^{\phi}(x_i)$ does not contain P's.

EXAMPLE 9. — If $V = \{x_1, x_2\}$, $f(x_1) = \{\Box_4 p_0^0\}$ and $f(x_2) = \Box_1(\Diamond_2 p_0^0 \rightarrow \Box_3 p_0^1)$, then

$$KF_f^{p_0^o} = R_4(x_1),$$

^{1.} Here we use the fact that KV^{ψ} is destributive over arbitrary unions.

$$KF_f^{p_0^1} = R_3 \left(R_2^{-1}(R_4(x_1)) \cap R_1(x_2) \right).$$

The next lemma shows that the operator $KF_f^{p_i^k}$ corresponds to the absolute minimal valuation for a variable p_i^k .

LEMMA 10. — Among all valuations θ such that for all $j x_j \models f(x_j)^2$ there is the smallest one θ_{\min} , and $\theta_{\min}(p_i^l) = KF_f^{p_i^l}$.

PROOF. — Put

$$\operatorname{rank} (f) = \max_{\substack{x_j \in V \\ \phi \in f(x_j)}} \operatorname{rank} (\phi),$$

where rank (ϕ) denotes the rank of its head. Let us introduce a new function $f^-:V\to A,$ as follows:

$$f^{-}(x_j) = f(x_j) \cap \{\phi \mid \operatorname{rank}(\phi) < \operatorname{rank}(f)\}$$

It is clear that

$$\operatorname{rank}(f^{-}) \leq \operatorname{rank}(f) - 1$$

We argue by induction on rank f.

The base: rank f = 0. Then

$$\theta_{\min}(p_i^0) = \bigcup_{\phi \succ p_i^0, \phi \in f(x_j)} KV^{\phi}(x_j) = KF_f^{p_i^0}.$$

The induction step. Suppose rank f = k. Consider the map f^- . Then by the induction hypothesis there exists $\theta^-_{\min}(p_i^l) = KF_{f^-}^{p_i^l}$ for l < k, such that for any valuation θ^- , given on the variables of rank < k

$$\forall j \ \theta^-, x_j \models f^-(x_j) \to \theta^- \supseteq \theta^-_{\min}.$$

Put

$$\theta_{\min}(p_i^k) = KF_f^{p_i^k} = \bigcup_{\phi \succ p_i^k, \phi \in f(x_j)} KVF_f^{\phi}(x_j).$$

Suppose that for some θ

$$\forall j \ F, x_j, \theta \models f(x_j).$$

^{2.} Strictly speaking, in this lemma we mean that we have a frame $F = (W, (R_{\lambda} : \lambda \in \Lambda))$ and a valuation of object variables $g : V \to W$, so this formula must be read as $F, g(x_j), \theta \models f(x_j)$, but following Kracht (Kracht, 1999) we will identify x_i with $g(x_i)$, and will not take care of the frame F.

Let us prove that

$$\theta \supseteq \theta_{\min}.$$

Let θ^- be a restriction of θ to variables of rank < k. By the induction hypothesis

$$\theta^- \supseteq \theta^-_{\min}.$$

Consider an arbitrary $\phi \in f(x_j)$ with the head p_i^k . By Lemma 7 (on the minimality of KV^{ϕ})

$$\theta(p_i^k) \supseteq KV^\phi(x_j)$$

and by Lemma 6 (on the monotonicity of KV^{ϕ})

$$KV^{\phi}(x_j) \supseteq KVF_f^{\phi}(x_j),$$

hence

$$\theta(p_i^k) \supseteq KVF_f^{\phi}(x_j).$$

So

$$\theta(p_i^k) \supseteq \bigcup_{\phi \succ p_i^k, \phi \in f(x_j)} KVF_f^{\phi}(x_j) = \theta_{\min}(p_i^k)$$

3. Safe expressions

In this section we study the values of KVF and KF.

DEFINITION 11. — Let B be a set of L-expressions. A positive combination of B (denoted POS(B)) is any L-expression built from the members of B using only $\cap, \cup, R_{\lambda}^{-1}, R_{\lambda}^{\Box}, \top, \bot$ (i. e. all operations of L excepting R_{λ}).

DEFINITION 12. — Let \mathcal{K} be the minimal class of L-expressions satisfying the conditions:

$$- \{x_1, \dots, x_n\} \subseteq \mathcal{K}, \\ - \text{ if } S \in \mathcal{K}, \text{ then } R_{\lambda}(S) \in \mathcal{K}, \end{cases}$$

- if $B \subseteq \mathcal{K}$ and $S \in \mathcal{K}$ then $S \cap POS(B) \in \mathcal{K}$,

where POS(B) denotes any positive combination of B.

Now we give another description of \mathcal{K} .

DEFINITION 13. — Let ψ be a subexpression of $\phi \in L$. We say that a subexpression ψ is safe for ϕ if one of the following holds:

1) $\psi = x_i$; 2) $\psi = R_{\lambda}(\psi')$, where ψ' is safe for ϕ ; 3) $\psi = \psi' \cap \psi''$, where either ψ' or ψ'' is safe for ϕ .

Let $Sub(\phi)$ denote the set of all subexpressions of ϕ . We say that an expression ϕ is safe if

1) ϕ is safe for ϕ ;

2) for every subexpression $R_{\lambda}(\psi)$ of ϕ , ψ is safe for ϕ . Some examples of safe expressions are x_i , R(x), $R(R(x) \cap R^{-1}R(x))$, $R(R(x) \cap R^{-1}(\top))$, $R\left(\left(R(x) \cap R^{-1}R(x)\right) \cap \left(R^{-1}(x) \cap R^{-1}(R(x))\right)\right)$.

The Figure 2 shows the dependency tree of the latter expression.

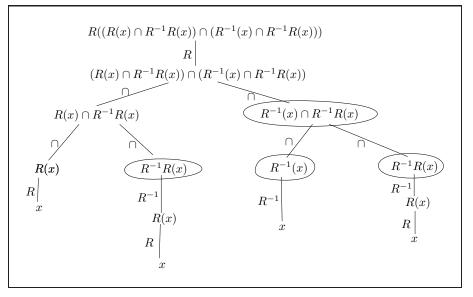


Figure 2.

One can easily check that this expression is safe. Denote it by ϕ . In fact, all subexpressions on the left branch are safe for ϕ , hence ϕ is safe for itself. However, some of its subexpressions are not safe for ϕ ; they are circled in the picture. But the operator R is applied only to the nodes, that are safe for ϕ .

Examples of non-safe expressions are $R^{-1}(x)$, $R(R^{-1}(x))$, $R(\top)$.

LEMMA 14. — For any L-expression ϕ

$$\phi \in \mathfrak{K} \iff \phi \text{ is safe.}$$

PROOF. — By induction on the length of ϕ . The base is trivial.

Suppose ϕ is of the form $\phi_1 \cup \phi_2$. Then ϕ is not safe, and $\phi \notin \mathcal{K}$. The same holds if $\phi = R_{\lambda}^{-1}(\psi)$ or $\phi = R_{\lambda}^{\Box}(\psi)$.

Suppose $\phi = \phi_1 \cap \phi_2$.

If ϕ is safe, then either ϕ_1 or ϕ_2 is safe. Without any loss of generality assume that ϕ_2 is safe. Then by the induction hypothesis $\phi_2 \in \mathcal{K}$. Consider ϕ_1 . Since all subexpressions of the form $R_{\lambda}(\psi)$ are safe, ϕ_1 is of the form $POS(\psi_1, \ldots, \psi_k)$, where all ψ_i are safe. By the inductive hypothesis $\psi_i \in \mathcal{K}$, hence $\phi = POS(\psi_1, \ldots, \psi_k) \cap \phi_2 \in \mathcal{K}$.

The other way round, if $\phi_1 \cap \phi_2 \in \mathcal{K}$, then ϕ_1 or ϕ_2 is in \mathcal{K} , so either ϕ_1 or ϕ_2 is safe, and the other expression is a positive combination of safe expressions. So $\phi_1 \cap \phi_2$ is safe.

Suppose $\phi = R_{\lambda}(\psi)$. If ϕ is safe, then ψ is safe, so $\psi \in \mathcal{K}$, hence $\phi \in \mathcal{K}$. The other way round, if $\phi \in \mathcal{K}$, then $\psi \in \mathcal{K}$, and ψ is safe; hence ϕ is safe.

LEMMA 15. — There is a linear algorithm, which for any given L-expression ϕ decides, whether ϕ is safe (or, according to Lemma 14, whether ϕ is in \mathcal{K}).

PROOF. — We run through the syntactic tree of ϕ starting from its leaves and assign the value 'safe for ϕ ', or 'not safe for ϕ ' to every node (that is, to a subexpression of ϕ) according to Definition 13. If we see that R_{λ} is applied to a node, which is not safe, we stop and conclude that ϕ is not safe. Otherwise, we look whether ϕ is safe for ϕ , and return the result.

This algorithm takes time proportional to the number of nodes in the syntactical tree of the expression, hence, it is linear with respect to the length of the given expression ϕ .

COROLLARY 16. — Let ϕ and ψ be safe expressions. After replacing any occurrence of x_i in ϕ with ψ we obtain a safe expression ϕ' .

LEMMA 17 (SOUNDNESS OF \mathcal{K} WITH RESPECT TO KVF). — Let ϕ be a regular box-formula of rank k with a head p_i^k , and let A be the set of all regular box-formulas of ranks $\leq k$, let f be a map $\{x_1, \ldots, x_n\} \rightarrow \mathcal{P}(A)$. Then $KVF_f^{\phi}(x_j)$ is in \mathcal{K} , and hence, $KF_f^{p_i^k}$ is a union of elements of \mathcal{K} .

PROOF. — By induction on the length of ϕ within the induction on k. The case k = 0 is trivial. So suppose $0 \le l < k$.

Case 1. Let $\phi = p_i^k$. Then $KV^{\phi} = \#$ and $KVF_f^{\phi}(x_j) = x_j \in \mathcal{K}$.

Case 2. Let $\phi = POS \rightarrow \psi$. Then $KV^{\phi} = KV^{\psi}(\# \cap KP^{POS})$ and

$$KVF_f^{\phi}(x_j) = [KF_f^{p_i^l}/P_i^l]KV^{\psi}(x_j \cap KP^{POS}) =$$
$$= [KF_f^{p_i^l}/P_i^l]KV^{\psi}(x_j \cap [KF_f^{p_i^l}/P_i^l]KP^{POS}) \in \mathcal{K}.$$

In fact, $[KF_f^{p_i^l}/P_i^l]KV^{\psi}(x_j) = KVF_f^{\psi}(x_j) \in \mathcal{K}$ by the induction hypothesis. Consider $x_j \cap [KF_f^{p_i^l}/P_i^l]KP^{POS}$. Since l < k, by the induction hypothesis $KF_f^{p_i^l}$ is

a union of safe expressions. Hence $[KF_f^{p_i^l}/P_i^l]KP^{POS}$ is a positive combination of safe expressions. That is $x_j \cap [KF_f^{p_i^l}/P_i^l]KP^{POS} \in \mathcal{K}$ and it is sufficient to apply Corollary 16.

Case 3. Let $\phi = \Box_{\lambda}\psi$. Then $KV^{\phi} = KV^{\psi}(R_{\lambda}(\#))$. Similarly, $KVF_{f}^{\phi}(x_{j})$ is the result of replacing a single occurrence of x_{j} with $R_{\lambda}(x_{j})$ in $KVF_{f}^{\psi}(x_{j})$, which is safe by the induction hypothesis. So Corollary 16 implies that $KVF_{f}^{\phi}(x_{j}) \in \mathcal{K}$.

LEMMA 18 (COMPLETENESS OF \mathcal{K} WITH RESPECT TO KVF). — Let $E(x_1, \ldots, x_k)$ be a safe L-expression and A be the set of all regular formulas. Then there exists a function

$$f^E: \{x_1,\ldots,x_k\} \to \mathcal{P}(A),$$

and a formula $\phi \in \bigcup_i f^E(x_i)$ with the head p_i^l such that $E(x_1, \ldots, x_k) = KF_{f^E}^{p_i^l} = KVF_{f^E}^{\phi}$.

PROOF. — Induction on the length of E.

The case $E = x_i$ is trivial:

$$f(x_j) = \begin{cases} \emptyset, & \text{if } j \neq i \\ P_i^0, & \text{if } j = i \end{cases}$$

Consider an arbitrary safe E. Then in the syntactical tree of E there is a path connecting E with some x_i , and passing only through safe subexpressions of E. We denote the subexpressions on this path by $E_0 = \{x_i\}, E_1, \ldots, E_b = E$.

Consider the case $E_1 = R_\lambda(x_i)$. Then consider E' obtained from E by replacing the subexpression E_1 with an expression E_0 (that is we replace $R_\lambda(x_i)$ with x_i). Now we apply the induction hypothesis to E' and obtain a function $f^{E'}$, and a formula ϕ with the head p_i^l . Then we replace ϕ by $\Box_\lambda \phi$ in $f^{E'}$, leaving p_i^l and others components of $f^{E'}$ as they are. This yields us a function f^E , since $KV^{\Box_\lambda\psi} = KV^{\psi}(R_\lambda(\#))$ and the substitution, transforming KV into KVF is the same for E and E'.

Now consider the case $E_1 = \{x_t\} \cap POS(\psi_1, \ldots, \psi_k)$, where all ψ_j are safe. By the induction hypothesis, for any ψ_j there exist functions f^{ψ_j} and variables $p_j^{l_j}$. Let E' be an expression obtained from E by replacing E_1 with E_0 . By the induction hypothesis there exists a function $f^{E'}$ and a formula ϕ with the head p_m^l . Without any loss of generality we may assume that the functions f^{ψ_j} and $f^{E'}$ do not have common propositional variables and that $l > l_j$ for all j from 1 to k. Take $f^{E'}$, and replace ϕ by $POS' \to \phi$, where POS' is obtained from an L-expression POS by replacing each of subexpressions ψ_j with $p_j^{l_j}, \forall$ with \cup, \land with \cap, R_λ^{-1} with $\Diamond_\lambda, R_\lambda^{\Box}$ by \Box_λ . We denote the result by $f_r^{E'}$. Put $f^E(x_i) = f^{\psi_1}(x_i) \cup \ldots \cup f^{\psi_k}(x_i) \cup f_r^{E'}(x_i)$ and the variable p_m^l . COROLLARY 19. — Let \mathcal{E} be a set of safe expressions. Then there exists a function $f^{\mathcal{E}} : \{x_0, \ldots, x_k\} \to \mathcal{P}(A)$ and the collection of variables $\{p^E | E \in \mathcal{E}\}$ ³ such that for all $E \in \mathcal{E}$

$$E = KF_{f^{\mathcal{E}}}^{p_E}.$$

PROOF. — By Lemma 18 for each $E \in \mathcal{E}$ there exist f^E and p_E such that $E = KF_{f^E}^{p_E}$. Without any loss of generality we may assume that for different $E f^E$ do not have common propositional variables. Then we can put

$$f^{\mathcal{E}}(x_i) = \bigcup_{E \in \mathcal{E}} f^E(x_i).$$

Now we see that the class \mathcal{K} describes the values of KVF. So the values of KF are in the closure of \mathcal{K} under \cup .

REMARK 20. — This definition of safety does not coincide with the notion of 'safety under bisimulations' from (Blackburn *et al.*, 2002).

DEFINITION 21 (BLACKBURN et al., 2002). — A first-order formula $\alpha(x, y)$ is called safe under bisimulation if for all Kripke models M and M', bisimulation Zbetween them and points $x_0 \in M$, $x'_0 \in M$ such that xZx' for all y_0 if $M \models \alpha[x \setminus x_0, y \setminus y_0]$ then there is $y'_0 \in M'$ such that $M' \models \alpha[x \setminus x'_0, y \setminus y'_0]$ and yZy'.

One can generalize this definition to the following.

DEFINITION 22. — A first-order formula $\alpha(x_1, \ldots, x_n, y)$ is called safe under bisimulations if for for all Kripke models M and M', bisimulation Z between them and points $x_i \in M$, $x'_i \in M$ ($1 \leq i \leq n$) such that $x_i Z x'_i$ for all y_0 if $M \models \alpha[x_i \setminus x_i^0, y \setminus y^0]$ then there is $y'^0 \in M'$ such that $M' \models \alpha[x_i \setminus (x_i^0)', y \setminus y'_0]$ and yZy'.

We may conjecture that these two definitions of safety (the syntactic safety from this paper and safety under bisimulation) coincide. However, this is not the case. Indeed, the formula $y \in R(R(x_1) \cap R(x_2))$ is safe according to our definition, but not safe under bisimulations.

4. Generalized Sahlqvist formulas

DEFINITION 23 (GORANKO et al., 2006). — A generalized Sahlqvist implication is a formula $GSA \rightarrow \bot$, where GSA^4 is built from regular box-formulas and negative formulas (that is, negations of positive formulas) using only $\land, \lor, \diamondsuit_{\lambda}$. If we

^{3.} According to our notation, p_E actually denotes $p_{i_E}^{l_E}$

^{4.} Generalized Sahlqvist Antecedent

prohibit the use \lor in GSA, we obtain the definition of a generalized simple Sahlqvist implication.

A generalized Sahlqvist formula⁵ is a formula built up from generalized Sahlqvist implications by applying boxes and conjunctions, and by applying disjunctions only to formulas without common proposition letters.

The reduction of a generalized Sahlqvist formula to a generalized simple Sahlqvist implication is standard (Blackburn *et al.*, 2002). So without any loss of generality we may consider a generalized simple Sahlqvist implication $GSA \rightarrow \bot$, where GSA is built from regular box-formulas and negative formulas using only \land and \Diamond_{λ} . It is convenient to represent such formulas with labelled trees of a special kind, similar to syntactical trees.

DEFINITION 24. — Consider a structure $\hat{T} = (W, (R_{\lambda} : \lambda \in \Lambda))$. A path from x_1 to x_n in \hat{T} is a sequence $x_1\lambda_1x_2\lambda_2x_3\ldots x_n$, where $x_i \in W$, $\lambda_i \in \Lambda$ and $x_iR_{\lambda_i}x_{i+1}$ in \hat{T} . Two paths $x_1\lambda_1x_2\lambda_2x_3\ldots x_n$ and $x'_1\lambda'_1x'_2\lambda'_2x'_3\ldots x'_n$ are called equal if for all $1 \le i \le n x_i = x'_i$ and for all $1 \le i \le n - 1$ $\lambda_i = \lambda'_i$.

A pair (\hat{T}, r) is called a tree with a root r if the following holds

1) $r \in W$,

2) $R_{\lambda}^{-1}(r) = \emptyset$ for all $\lambda \in \Lambda$,

3) for all $x \neq r$ there is a unique path from r to x.

Let A be a set of modal formulas. A labelled tree with a root r is a tuple $T = (W, (R_{\lambda} : \lambda \in \Lambda), r, f)$, where $(W, (R_{\lambda} : \lambda \in \Lambda), r)$ is a tree with a root r and f (a label function) is a map from W to $\mathcal{P}(A)$.

DEFINITION 25. — Let ϕ be built up from formulas of A by applying only diamonds and conjunction. A reduced syntactical tree of a formula ϕ is a labelled tree defined by induction on the length of ϕ .

Case 1: $\phi = a$, where $a \in A$. Then T^{ϕ} contains a single point x. The map f^{ϕ} takes x to $\{a\}$ and the relations R^{ϕ}_{λ} are empty.

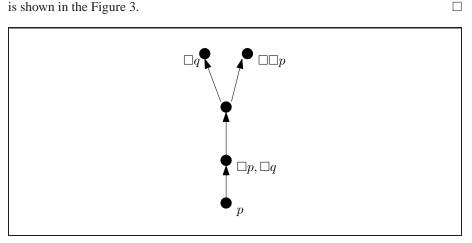
Case 2: $\phi = \chi \land \psi$. Then put $W^{\phi} = (W^{\chi} \setminus \{r^{\chi}\}) \cup (W^{\psi} \setminus \{r^{\psi}\}) \cup \{r^{\phi}\}$, where r^{ϕ} is some new point. The relations R_{λ} on W^{χ} and W^{ψ} remain the same, and $r^{\phi}R_{\lambda}w$ iff $w \in W_{\chi}$ and $r^{\chi}R_{\lambda}^{\chi}w$ or $w \in W_{\psi}$ and $r^{\psi}R_{\lambda}^{\psi}w$. The map f^{ϕ} sends r^{ϕ} to $f^{\chi}(r^{\chi}) \cup f^{\psi}(r^{\psi})$ and is equal to f^{χ} or f^{ψ} in all other points.

Case 3: $\phi = \Diamond_{\lambda}\psi$. Then $W^{\phi} = W^{\psi} \cup \{r^{\phi}\}$, where r^{ϕ} is a new point. The R_{μ} for $\mu \neq \lambda$ we leave untouched, and to R_{λ} we add an arrow, joining r^{ϕ} with r^{ψ} . We put $f(r^{\phi}) = \emptyset$, and do not change f in all other points.

EXAMPLE 26. — The reduced syntactical tree of the formula

$$\phi = \Diamond (\Box p \land \Box q \land \Diamond (\Diamond \Box q \land \Diamond \Box \Box p)) \land p$$

^{5.} In subsequent publications Goranko and Vakarelov refer to these formulas as the monadic inductive formulas





LEMMA 27. — Let A be an arbitrary set of modal formulas and let ϕ be built from formulas of A using only \wedge and \Diamond_{λ} . Let $T^{\phi} = (W^{\phi}, (R^{\phi}_{\lambda} : \lambda \in \Lambda), r^{\phi}, f^{\phi})$ be a reduced syntactical tree of ϕ . Then for all frames $F = (W, R_{\lambda} : \lambda \in \Lambda)$ for any valuation $\theta F, x, \theta \models \phi$ iff there exists a monotonic map $h : T^{\phi} \to F$ (that is for all $x, y \in W^{\phi}$ if $xR_{\lambda}^{\phi}y$ then $h(x)R_{\lambda}h(y)$ such that $h(r^{\phi}) = x$ and for any $w \in W^{\phi}$, $a \in A$ if $a \in f^{\phi}(w)$ then $F, h(w), \theta \models a$.

The proof of the Lemma 27 trivially follows from the semantics of \wedge and \Diamond_{λ} .

For Sahlqvist formulas A is the set of all boxed atoms and negative formulas. For generalized Sahlqvist formulas A is the set of all regular box-formulas and negative formulas.

The next lemma shows the standard second-order quantifier elimination in a simple generalized Sahlqvist implication.

LEMMA 28. — (cf. (Goranko et al., 2006) and (Blackburn et al., 2002), Section 3.6) Let ϕ be a simple generalized Sahlqvist implication ϕ with a reduced syntactical tree $T = (\{y_0, y_1, \dots, y_n\}, (R_{\lambda}^T : \lambda \in \Lambda), y_0, f).$ Let $f_{REG}(y_i) = f(y_i) \cap REG$, and $f_{NEG}(y_i) = f(y_i) \cap NEG$ where REG and NEG are respectively the sets of all regular box-formulas and all negative formulas. Then the first-order correspondent of ϕ is of the form

$$\begin{aligned}
& (x_j \in KF_{f_{REG}}^{p_i^k})^{\#} / P_i^k(x_j)] \forall x_1 \dots \forall x_n \left(\bigwedge_{y_i R_{\lambda}^T y_j} x_i R_{\lambda} x_j \to \right) \\ & \bigvee_{\psi \in f_{NEG}(y_j)} (x_j \models \neg \psi)^* \end{aligned} \right).
\end{aligned}$$
(3)

Here $KF_{f_{REG}}^{p_i^k}$ is the minimal valuation (see Definition 8), # denotes the first-order transcription of $x_j \in KF_{f_{REG}}^{p_i^k}$, defined on the page 17, and * means the standard first-order translation of a modal formula.

PROOF. — The proof is standard. As in the Sahlqvist theorem, we can eliminate the second-order quantifiers by substituting appropriate (minimal) valuations.

Let ϕ be a simple generalized Sahlqvist implication with a reduced syntactical tree $T = (\{y_0, y_1, \ldots, y_n\}, (R_{\lambda}^T : \lambda \in \Lambda), y_0, f)$. Then for any frame $F, F, x_0 \models \phi$ is equivalent to the universal second order formula

$$\forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \left(\exists x_1 \dots \exists x_n \left(\bigwedge_{y_i R_{\lambda}^T y_j} x_i R_{\lambda} x_j \wedge \bigwedge_i (x_i \models f(y_i))^* \right) \to \bot \right),$$

where for a set of modal formulas $f(y_i)$ the notation $x_i \models f(y_i)$ means that in the point x_i all members of $f(y_i)$ are true.

Now we can put the existential quantifiers in the prefix. Since they are in the antecedent of the implication, they become universal:

$$\forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \forall x_1 \dots \forall x_n \left(\left(\bigwedge_{y_i R_{\lambda}^T y_j} x_i R_{\lambda} x_j \wedge \bigwedge_i (x_i \models f(y_i))^* \right) \to \bot \right)$$

Then let us swap them with the second-order quantifiers:

$$\forall x_1 \dots \forall x_n \forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \left(\left(\bigwedge_{y_i R_{\lambda}^T y_j} x_i R_{\lambda} x_j \wedge \bigwedge_i (x_i \models f(y_i))^* \right) \to \bot \right).$$

Now we apply the equivalence $A \wedge B \rightarrow C \equiv A \rightarrow (B \rightarrow C)$, yielding

$$\forall x_1 \dots \forall x_n \forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \left(\bigwedge_{y_i R_{\lambda}^T y_j} x_i R_{\lambda} x_j \to \left(\bigwedge_i (x_i \models f(y_i))^* \to \bot \right) \right).$$

Let us move the second-order universal quantifiers to the consequent:

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{y_i R^T y_j} x_i R x_j \to \forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \left(\left(\bigwedge_i (x_i \models f(y_i))^* \right) \to \bot \right) \right)$$

Now let us recall that $f(y_i) = f_{NEG}(y_i) \cup f_{REG}(y_i)$. Let us move the formulas of $f_{NEG}(y_i)$ from the antecedent to the consequent of the inner implication:

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{y_i R^T y_j} x_i R x_j \to \forall P_{i_1}^{k_1} \dots \forall P_{i_m}^{k_m} \left(\left(\bigwedge_{i \ \psi \in f_{REG}(y_i)} (x_i \models \psi)^* \right) \to \right) \right)$$

$$\bigvee_{i} \bigvee_{\psi \in f_{NEG}(y_i)} \neg (x_i \models \psi)^*) \right) \right).$$

According to Lemma 10, there is the smallest valuation verifying the antecedent

$$\bigwedge_{i} \bigwedge_{\psi \in f_{REG}(y_i)} (x_i \models \psi)^*.$$

The negation of a negative formula is positive. So we can eliminate the second-order quantifiers by substituting the minimal valuation and obtain the formula (3).

5. Generalized Kracht Formulas

Now we will extend Kracht's theorem to generalized Sahlqvist formulas. To this end we need an extension of our first-order language. The only contribution of this work is the usage of \mathcal{K} . All other definitions from this section (restricted quantification, inherently universality) are taken from (Blackburn *et al.*, 2002) and originate from Kracht.

We abbreviate the first-order formula $\forall y(xR_{\lambda}y \to \alpha(y))$ to $(\forall y \triangleright_{\lambda} x)\alpha(y)$. Likewise $\exists y(xR_{j}y \land \alpha(y))$ is abbreviated to $(\exists y \triangleright_{\lambda} x)\alpha(x)$. We shall use only formulas, in which variables do not occur both as free and bound, and in which two distinct occurrences of quantifiers do not bind the same variable; we call such formulas *clean*.

Let \mathcal{K} be the class of all safe expressions from Section 3. We add new (k + 1)-ary predicates $x_l \in E(x_1, \ldots, x_k)$ for any expression $E \in \mathcal{K}$. Depending on the context, they can also be considered as abbreviations for the corresponding first-order formulas with free variables x, x_1, \ldots, x_k .

More precisely, for any *L*-expression *E* (not necessary safe) we define a first-order formula $(x_l \in E)^{\#}$ by the recursion on the length of *E*:

$$(x_{l} \in x_{i})^{\#} := (x_{l} = x_{i});$$

$$(x_{l} \in \top)^{\#} := (x_{l} = x_{l});$$

$$(x_{l} \in \bot)^{\#} := \neg (x_{l} = x_{l});$$

$$(x_{l} \in E_{1} \cap E_{2})^{\#} := (x_{l} \in E_{1})^{\#} \land (x_{l} \in E_{2})^{\#};$$

$$(x_{l} \in E_{1} \cup E_{2})^{\#} := (x_{l} \in E_{1})^{\#} \lor (x_{l} \in E_{2})^{\#};$$

$$(x_{l} \in R_{\lambda}^{-1}(E))^{\#} := \exists y(x_{l}R_{\lambda}y \land (y \in E)^{\#});$$

$$(x_{l} \in R_{\lambda}^{\Box}(E))^{\#} := \exists y(y_{R}\lambda x_{l} \land (y \in E)^{\#});$$

$$(x_{l} \in R_{\lambda}(E))^{\#} := \exists y(y_{R}\lambda x_{l} \land (y \in E)^{\#}).$$

This translation obviously corresponds to the standard set-theoretic semantics.

DEFINITION 29. — (cf (Blackburn et al., 2002), p. 172) We call a formula restrictedly positive if it is built up from formulas $y \in E(x_1, \ldots, x_k)$, using \land , \lor and restricted quantifiers.

We say that a variable x in a clean formula α is inherently universal if either x is free, or x is bound by a restricted universal quantifier which is not in the scope of an existential quantifier.

A formula α in the extended first-order language is called a generalized Kracht formula with free variables if α is clean, restrictedly positive and in every subformula of the form $y \in E(v_1, \ldots, v_k)$ ($E \in \mathcal{K}$), the variables v_1, \ldots, v_k are inherently universal. A formula α is called a generalized Kracht formula if it is a generalized Kracht formula with free variables and it contains exactly one free variable.

The definition of ordinary Kracht formulas is obtained from this definition by replacing \mathcal{K} with $\{R_{\lambda_1} \dots R_{\lambda_n}(x_j)\} \cup \{R_{\lambda_1}^{-1} \dots R_{\lambda_n}^{-1}(x_j)\}$.

Now we are ready to state the main theorem.

THEOREM 30. — A first-order formula ϕ is a first-order correspondent of a generalized Sahlqvist formula iff ϕ is a generalized Kracht formula.

Note, that every ordinary Kracht formula can be rewritten as a generalized Kracht formula. Namely, instead of $xR_{\lambda_1} \dots R_{\lambda_k} y$, where x is inherently universal, we write $y \in R_{\lambda_k} \dots R_{\lambda_1}(x)$ (obviously, $R_{\lambda_k} \dots R_{\lambda_1}(x)$ is a safe expression). Instead of $yR_{\lambda_k} \dots R_{\lambda_1} x$, where x is inherently universal, we write

$$(\exists z_1 \triangleright_{\lambda_1} y)(\exists z_2 \triangleright_{\lambda_2} z_1) \dots (\exists z_k \triangleright_{\lambda_k} z_{k-1})(z_k \in x).$$

EXAMPLE 31. — Consider the formula

$$D_2 = p \land \Box(\Diamond p \to \Box q) \to \Diamond \Box \Box q$$

from (Goranko et al., 2006). Its first-order correspondent is a generalized Kracht formula

$$FO(D_2) = \exists y \triangleright x \left(\forall z' \triangleright y \forall z \triangleright z' z \in R(R(x) \cap R^{-1}(x)) \right),$$

or, in a more standard form,

$$FO(D_2) = \exists y \left(xRy \land \forall z \left(yR^2 z \to z \in R(R(x) \cap R^{-1}(x)) \right) \right).$$

In (Goranko *et al.*, 2006) the authors show that it is not equivalent to any standard Sahlqvist formula.

Consider the formula

$$ns = p \land \Box_1(\Diamond_1 p \to \Box_3 r) \to \Diamond_2(\Diamond_2 p \land \Diamond_3 r).$$

Then

$$FO(ns) = \exists y \triangleright_1 x \left(y \in R_1^{-1}(x) \land \exists v \triangleright_3 y \left(v \in R_3(R_2(x) \cap R_2^{-1}(x)) \right) \right).$$

This generalized Kracht formula is equivalent to

$$\exists y \exists z \exists v (xR_1y \wedge yR_1x \wedge xR_2z \wedge zR_2x \wedge yR_3v \wedge zR_3v).$$

The formula cub_1 is a theorem of \mathbf{K}^3 (Shehtman, 1978), see also (Gabbay *et al.*, 2003), p. 397

 $\begin{aligned} cub_1 &= \left[\Diamond_1(\Box_2 p_{12} \land \Box_3 p_{13}) \land \Diamond_2(\Box_1 p_{21} \land \Box_3 p_{23}) \land \Diamond_3(\Box_1 p_{31} \land \Box_2 p_{32}) \land \\ \Box_1 \Box_2(p_{12} \land p_{21} \rightarrow \Box_3 q_3) \land \Box_1 \Box_3(p_{13} \land p_{31} \rightarrow \Box_2 q_2) \land \Box_2 \Box_3(p_{23} \land p_{32} \rightarrow \Box_1 q_1) \right] \\ &\to \Diamond_1 \Diamond_2 \Diamond_3(q_1 \land q_2 \land q_3). \end{aligned}$

Its first-order correspondent is a generalized Kracht formula

$$\forall x_1 \triangleright_1 x \forall x_2 \triangleright_2 x \forall x_3 \triangleright_3 x \exists y' \triangleright_1 x \exists y'' \triangleright_2 y' \exists y \triangleright_3 y''$$

$$(y \in R_3(R_2(x_1) \cap R_1(x_2)) \land y \in R_2(R_3(x_1) \cap R_1(x_3)) \land$$

$$\land y \in R_1(R_2(x_3) \cap R_3(x_2))).$$

This formula is equivalent to (1).

Examples of generalized Kracht formulas applied to many-dimensional modal logics can be found in (Kurucz, 2000), (Kurucz, 2008) and (Kikot, n.d.).

The rest of the paper will be devoted to the proof of this theorem.

6. Quasi-safe expressions

DEFINITION 32. — An L-expression is called quasi-safe if it is a positive combination of safe expressions.

The expression \top , \bot , $R^{-1}(\top)$ are here considered as quasi-safe but not safe.

If we extend our first-order language with atomic formulas $x \in E$ where E is a quasi-safe expression, we obtain a quantifier elimination in the scope of the existential quantifier.

LEMMA 33. — Let ψ be a generalized Kracht formula with free variables, such that all atomic formulas of ϕ are of the form $y \in E$ where all variables occuring in E are free. Then ψ is equivalent to a quantifier free formula ψ' in the extended language (cf. (Blackburn et al., 2002), p. 175).

PROOF. — We apply the induction on the number of quantifiers in ψ .

Consider the case $\psi = \exists y \triangleright_{\lambda} x\phi$. By the induction hypothesis, ϕ is a quantifier free formula. Hence we can assume that it is of the form $\phi = K_1 \vee \ldots \vee K_n$, where

 K_i are conjunctions of atomic formulas. But then $\psi \equiv \exists y \triangleright_{\lambda} x K_1 \lor \ldots \lor \exists y \triangleright_{\lambda} x K_n$. Then, since all E_i do not contain y, we can transform each of the disjuncts as follows

$$\exists y \triangleright_{\lambda} x(\alpha_1 \in E_1 \land \ldots \land \alpha_m \in E_m) \equiv \bigwedge_{\alpha_i \neq y} \alpha_i \in E_i \land x \in R_{\lambda}^{-1} \left(\bigcap_{\alpha_i = y} E_i\right),$$

and obtain a quantifier free equivalent of ψ .

Similarly, let $\psi = \forall y \triangleright_{\lambda} x \phi$. By the induction hypothesis, ϕ is quantifier free, so it can be presented in the form $\phi = D_1 \land \ldots \land D_n$, where D_i are disjunctions of atomic formulas. But then ψ is equivalent to $\forall y \triangleright_{\lambda} x D_1 \land \ldots \land \forall y \triangleright_{\lambda} x D_n$. Then each of conjucts can be transformed as follows

$$\forall y \triangleright_{\lambda} x(\alpha_1 \in E_1 \lor \ldots \lor \alpha_m \in E_m) \equiv \bigvee_{\alpha_i \neq y} \alpha_i \in E_i \lor x \in R_{\lambda}^{\square} \left(\bigcup_{\alpha_i = y} E_i \right).$$

COROLLARY 34. — Let ψ be a generalized Kracht formula, beginning with an existential quantifier. Then ψ is equivalent to a quantifier free formula ψ' in the language with quasi-safe atoms.

7. Proof of the theorem.

'Only if'. If ϕ is a simple generalized Sahlqvist implication, then the statement follows from (3). It is sufficient to note that

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{y_i R_\lambda^T y_j} x_i R_\lambda x_j \to C \right)$$

is equivalent to

$$\forall x_1 \triangleright_\lambda x_{p(1)} \dots \forall x_n \triangleright_\lambda x_{p(n)} C,$$

where $y_{p(i)}$ is the unique predecessor of y_i in T. The variables x_1, \ldots, x_n are inherently universal, the disjunction C is built from atomic formulas using \lor, \land and restricted quantifiers, and every atomic formula is of the form $v \in E(x_{i_1}, \ldots, x_{i_k})$, since we substitute the disjunctions of such formulas for all P_i^k in the standard translation of positive formulas.

The general case follows from Lemma 3.53 of (Blackburn et al., 2002) stating that

- if ϕ and $\alpha(x)$ are locally correspondents, so are $\Box_{\lambda}\phi$ and $\forall y \triangleright_{\lambda} x\alpha(y)$,

- if ϕ locally corresponds to $\alpha(x)$ and ψ locally corresponds to $\beta(x)$ then $\phi \wedge \psi$ locally corresponds to $\alpha(x) \wedge \beta(x)$,

- if ϕ locally corresponds to α , ψ locally corresponds to $\beta(x)$ and ϕ and ψ do not have propositional letters in common, then $\phi \lor \psi$ locally corresponds to $\alpha(x) \lor \beta(x)$,

and it remains to note that the class of generalized Kracht formulas is closed under disjunction, conjuntion and necessitation.

To prove 'if', we need to generalize the notion of modal definability to first-order formulas with many free variables (cf. (Kracht, 1999), p. 193).

We say that a first-order formula $\phi(x_1, \ldots, x_n)$ is definable if there is a sequence of modal formulas ϕ_1, \ldots, ϕ_n such that for any frame $F = (W, (R_\lambda : \lambda \in \Lambda))$ for any points $x_1^0, \ldots, x_n^0 \in W$

$$F \models \Phi(x_1, \dots, x_n)[x_1^0, \dots, x_n^0] \iff$$
for any valuation θ there exists *i* such that $F, x_i^0, \theta \models \phi_i$.

Here the left hand \models means the truth in F considered as a classical first-order structure.

For example, a formula x_1Rx_2 is definable by the sequence $\Diamond \neg p, p$. Clearly, that if ϕ has a single variable, then this definition coinsides with the standard modal definability.

Now we show that the formula $(x_l \in E)^{\#}$ is definable for all quasi-safe E.

To this end, consider the following translation T from quasi-safe expressions to modal language. Let E be a quasi-safe expression. Let \mathcal{E} be the set of all safe subexpressions occuring in E.

Now we define E^T by the induction on the length of E:

if E is safe then $E^T = p_E$; if $E = E_1 \cap E_2$ then $E^T = E_1^T \wedge E_2^T$; if $E = E_1 \cup E_2$ then $E^T = E_1^T \vee E_2^T$; if $E = R_\lambda^{-1} E_1$ then $E^T = \Diamond_\lambda E_1^T$; if $E = R_\lambda^{\Box} E_1$ then $E^T = \Box_\lambda E_1^T$.

LEMMA 35. — Let E be quasi-safe and let \mathcal{E} be the set of all safe subexpressions occuring in E. Let $f^{\mathcal{E}}$ be the function from Corollary 19 for the set \mathcal{E} . Then $(x_l \in E)^{\#}$ is definable by the sequence $\bar{\phi} = \phi_1, \phi_2, \dots, \phi_m$, such that

$$\phi_i = \left\{ \begin{array}{ll} \bigvee_{\phi \in f^{\varepsilon}(x_i)} \neg \phi, & i \neq l; \\ \bigvee_{\phi \in f^{\varepsilon}(x_i)} \neg \phi \lor E^T, & i = l. \end{array} \right.$$

PROOF. — Suppose that we have a frame $F = (W, (R_{\lambda} : \lambda))$, and the variables x_i are identified with points of W. Then we can evaluate E and regard it as a subset of W.

Let us call a valuation θ admissible if for all $i x_i, \theta \models f^{\mathcal{E}}(x_i)$.

Consider the following statements:

- (1) $x_l \in E$
- (2) $x_l, \theta_{\min} \models E^T$, where θ_{\min} is the valuation from Lemma 10.
- (3) for all admissible valuations $\theta, x_l \models E^T$.

Then due to the form of ϕ_i , the statement of the lemma can be rephrased as (1) \iff (3). But Lemma 10 ensures that (2) \iff (3).

Let us prove $(1) \iff (2)$ by induction on the length of a quasi-safe E.

The base. Suppose E is safe. In this case $E^T = p_E$ and

$$\theta_{\min}(p_E) = KF_{f^{\mathcal{E}}}^{p_E} = E.$$

The first equality holds by Lemma 10 and the second one by Corollary 19, and the statement is clear.

The induction step trivially follows from the interpretation of \lor , \land , \diamondsuit_i , and \Box_i in Kripke semantics.

In fact, let $E = E_1 \cap E_2$, that is $E^T = E_1^T \wedge E_2^T$.

$$x_l \in E_1 \cap E_2 \iff x_l \in E_1 \text{ and } x_l \in E_2 \iff$$

$$\iff heta_{\min}, x_l \models E_1^T \text{ and } heta_{\min}, x_l \models E_2^T \iff heta_{\min}, x_l \models E_1^T \land E_2^T$$

The case of the disjunction is similar.

Let $E = R_{\lambda}^{-1}(E_1)$. Then $x_l \in R_{\lambda}^{-1}(E_1) \iff \exists y(x_l R_{\lambda} y \land (y \in E_1)) \iff$ $\iff \exists y(x_l R_{\lambda} y \text{ and } \theta_{\min}, y \models E_1^T) \iff \theta_{\min}, x_l \models \Diamond_{\lambda} E_1^T.$

Let
$$E = R_{\lambda}^{\Box}(E_1)$$
. Then
 $x_l \in R_{\lambda}^{\Box}(E_1) \iff \forall y(x_l R_{\lambda} y \to (y \in E_1)) \iff$
 $\iff \forall y(x_l R_{\lambda} y \to \theta_{\min}, y \models E_1^T) \iff \theta_{\min}, x_l \models \Box_{\lambda} E_1^T$

We also need a dual version of Theorem 5.6.4 from (Kracht, 1999):

THEOREM 36 (KRACHT, 1999). — If $\alpha(x_0)$ is obtained from definable formulas using conjunction, disjunction and restricted universal quantification, then $\alpha(x_0)$ is definable.

Now we are ready to prove the main theorem.

LEMMA 37. — Let $\alpha(x_0)$ be a first-order formula with the only free variable x_0 . Then the following statements are equivalent:

(1) $\alpha(x_0)$ is a first-order correspondent of a generalized Sahlqvist formula;

(2) $\alpha(x_0)$ is a generalized Kracht formula;

(3) $\alpha(x_0)$ is obtained from formulas of the form $x_l \in E$, where E is quasi-safe, using conjuction, disjunction and restricted universal quantification.

Proof. —

 $(1) \rightarrow (2)$ was proved at the beginning of Section 7, in the 'only if' part.

 $(2) \rightarrow (3)$. Given a generalized Kracht formula ϕ , we apply the quantifier elimination from Corollary 34 to its maximal subformulas beginning with existential quantifiers. Then we obtain a formula satisfying (3).

 $(3) \rightarrow (1)$. Apply Lemma 35 and Theorem 36 to $\alpha(x_0)$.

It is clear that Lemma 37 implies Theorem 30

8. Discussion

1. The papers (Goranko *et al.*, 2000), (Goranko *et al.*, 2006) deal mainly with 'inductive' formulas, that are, in brief, generalized Sahlqvist formulas in polyadic modal languages. The theory of inductive formulas is in some sense more elegant, than the theory of generalized Sahlqvist formulas. So it would be interesting to extend Kracht's theorem to inductive formulas in polyadic modal languages. D. Vakarelov made a conjecture that their characterization may be nicer.

2. Note that there is a certain asymmetry between R and R^{-1} in the definition of safe expressions. In temporal language this asymmetry disappears, and, as Gorando and Vakarelov show in (Goranko *et al.*, 2006), every generalized Sahlqvist formula is semantically equivalent to the standard Sahlqvist one.

3. Traditionally the correspondence between Sahlqvist and Kracht formulas and their generalization is considered from the viewpoint of definability. We have several answers to the natural question "what first-order formulas are modally definable?" For example there is a sufficient syntactic condition given by the class of Kracht formulas and their generalization, and there is also a semantical characterization given by Goldblatt-Thomason theorem (Goldblatt *et al.*, 1974). But we can also ask when the modal logic of an elementary class is finitely axiomatizable. Kracht formulas and their generalization give a sufficient syntactic condition in this case too, but we do not have a semantical characterization. It would be interesting to look for other elementary classes with finitely axiomatizable modal logics. For example, it is known (Balbiani *et al.*, 2006) that the modal logic of the elementary class of the formula $\exists y(xRy \land R(y) \subset \{y\})$ is finitely axiomatizable.

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