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## **On preserving passivity in sampled-data linear systems**

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## Abstract

Passivity is a well known phenomenon in several engineering areas. Due to its interesting properties, it is used in several areas of control engineering. Generally, this property is lost under direct discretization. In this work a new methodology which allows to preserve continuous-time passivity is presented. This methodology is based on choosing a proper output, which preserves the passivity structure, while keeping the continuous-time energy function. Analytic formulation and numerical examples, both for open and closed loop, are provided in the paper.

## I. INTRODUCTION

Since the pioneer work of Willems [12] [13], dissipative systems have been a continued object of study. The concept of dissipativity has traditionally been used to characterize the stability of a given system, and has been applied both for analyzing stability and designing stabilizing controllers. Due to its interesting properties it is still a hot topic in current research [8] [11].

Although passivity has been defined properly both in discrete and continuous-time; it is well known that, in general, the passivity structure is lost under discretization [6] [1], i.e. a discrete-time system describing the behaviour of a sampled-data system is not discrete-time passive even if the initial continuous-time system was passive. Although it is possible to obtain discrete-time models which preserve the passivity structure, such as the ones obtained by applying the backward difference Euler Formula or the Tustin transform[6], they are approximations in the sense that the discrete-time state does not exactly match the sampled-data system states at sampling times.

Strictly speaking, the passivity loss implies that it is not possible to guarantee the closed loop stability when a computer is used to implement a controller, designed in continuous-time by means of the passivity theory. Therefore, in practical applications people tend to use a very small sample time to overcome this problem. Although this approach may be useful in some applications it may not be so when a high performance is needed.

Recently, a different methodology to implement passivity based controllers has been proposed [9][10]. This methodology is based on a passivity observer and a passivity controller, which operate concurrently with the discrete-time controller implementation. The passivity controller injects a variable damping when the passivity observer detects a passivity loss. This approach has been applied with great results in high performance haptic interfaces and teleoperation systems.

It is well known that passivity is not an intrinsic system property, due to the fact that it depends on the selection on a proper input-output couple of variables. Traditionally, when discretizing the system, the continuous-time output is preserved. This output does not, in general, preserve the passivity structure. In this work the selection of a new output is proposed. This new output will assure that the discrete-time equivalent system to the sampled-data system at sampling times will be discrete-time passive if the continuous-time system is passive. This new output has a clear physical meaning and its analytic expression is provided for the case of linear systems.

The paper is organized as follows, section II reviews passivity concepts for continuous time LTI systems, section III reviews passivity concepts for discrete-time LTI systems, section IV describe the traditional procedure applied to obtain discrete systems equivalent to the LTI sampled-data systems, in section V

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the proposed approach is described, in section VI an open loop and closed loop numerical examples are presented in order to illustrate theoretical results, finally in section VII conclusions and future works are presented.

## II. CONTINUOUS TIME PASSIVITY

A LTI continuous-time system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  can be written in a state space form as:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \quad (1)$$

$$y = \mathbf{C}x + \mathbf{D}u \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $y, u \in \mathbb{R}^m$ . Equivalently, this system can be written in an input/output form as:

$$H(s) \triangleq \mathbf{C}(s\mathbf{I}_d - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \quad (3)$$

*Definition 1 (Passivity):* A system (1)-(2) is passive with storage function  $V = \frac{1}{2}x^T\mathbf{P}x$  if :

$$\dot{V} < y^T u \quad (4)$$

usually  $y$  and  $u$  and called power variables.

*Theorem 1 (Theorem 3 in [13]):* Assuming that the system (1)-(2) is a minimal realization. The matrix inequalities

$$\mathbf{H} = \begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} & \mathbf{P}\mathbf{B} - \mathbf{C}^T \\ \mathbf{B}^T\mathbf{P} - \mathbf{C} & -\mathbf{D} - \mathbf{D}^T \end{bmatrix} \leq 0 \quad (5)$$

and

$$\mathbf{P} = \mathbf{P}^T \geq 0 \quad (6)$$

have a solution if and only if (1)-(2) is dissipative with respect to the supply rate  $w = y^T u$ . Moreover, the function  $V = \frac{1}{2}x^T\mathbf{P}x$  defines a quadratic storage function if and only if  $\mathbf{P}$  satisfies these inequalities.

*Remark 1:* By simple algebraic manipulation it can be shown that:

$$\dot{V} = y^T u + \frac{1}{2} \begin{bmatrix} x^T & u^T \end{bmatrix} \mathbf{H} \begin{bmatrix} x \\ u \end{bmatrix} \quad (7)$$

under the condition of theorem 1,  $V$  can be used as a Lyapunov function for the system (1)-(2) and it is passive by inspection of (7)

## III. DISCRETE TIME PASSIVITY

A LTI discrete-time system  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$  can be written in a state space form as:

$$\tilde{x}_{k+1} = \tilde{\mathbf{A}}\tilde{x}_k + \tilde{\mathbf{B}}\tilde{u}_k \quad (8)$$

$$\tilde{y}_k = \tilde{\mathbf{C}}\tilde{x}_k + \tilde{\mathbf{D}}\tilde{u}_k \quad (9)$$

where  $\tilde{x}_k \in \mathbb{R}^n$ ,  $\tilde{y}_k, \tilde{u}_k \in \mathbb{R}^m$ . Equivalently, this system can be written in an input/output way as:

$$\tilde{H}(z) \triangleq \tilde{\mathbf{C}}(z\mathbf{I}_d - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \tilde{\mathbf{D}} \quad (10)$$

*Definition 2 (Discrete Time Passivity):* A system (8)-(9) is passive with storage function  $\tilde{V} = \frac{1}{2}\tilde{x}^T\tilde{\mathbf{P}}\tilde{x}$  if :

$$\Delta\tilde{V}_k \triangleq \tilde{V}_{k+1} - \tilde{V}_k < \tilde{y}_k^T \tilde{u}_k \quad (11)$$

*Proposition 1:* Assuming that the system (8)-(9) is a minimal realization. The matrix inequalities

$$\tilde{\mathbf{H}} \triangleq \begin{bmatrix} \tilde{\mathbf{A}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}} - \tilde{\mathbf{P}} & \tilde{\mathbf{A}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} - \tilde{\mathbf{C}}^T \\ \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}} - \tilde{\mathbf{C}} & (\tilde{\mathbf{D}} + \tilde{\mathbf{D}}^T) - \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} \end{bmatrix} \leq 0 \quad (12)$$

and

$$\tilde{\mathbf{P}} = \tilde{\mathbf{P}}^T \geq 0 \quad (13)$$

have a solution if (8)-(9) is dissipative with respect to the supply rate  $w_k = y_k^T u_k$ . Moreover  $\tilde{V} = \frac{1}{2} \tilde{x}^T \tilde{\mathbf{P}} \tilde{x}$  defines a quadratic storage function if and only if  $\tilde{\mathbf{P}}$  satisfies these inequalities.

*Remark 2:* By simple algebraic manipulation it can be shown that:

$$\Delta \tilde{V}_{k+1} \triangleq \tilde{V}_{k+1} - \tilde{V}_k = \frac{1}{2} x_{k+1}^T \tilde{\mathbf{P}} x_{k+1} - \frac{1}{2} \tilde{x}_k^T \tilde{\mathbf{P}} \tilde{x}_k \quad (14)$$

$$= \tilde{y}_k^T \tilde{u}_k + \frac{1}{2} [\tilde{x}_k^T, \tilde{u}_k^T] \tilde{\mathbf{H}} \begin{bmatrix} \tilde{x}_k \\ \tilde{u}_k \end{bmatrix} \quad (15)$$

under the condition of proposition 1  $\tilde{V}$  can be used as a Lyapunov function for the system (8)-(9) and it is passive by inspection of (15).

#### IV. SAMPLED DATA SYSTEM

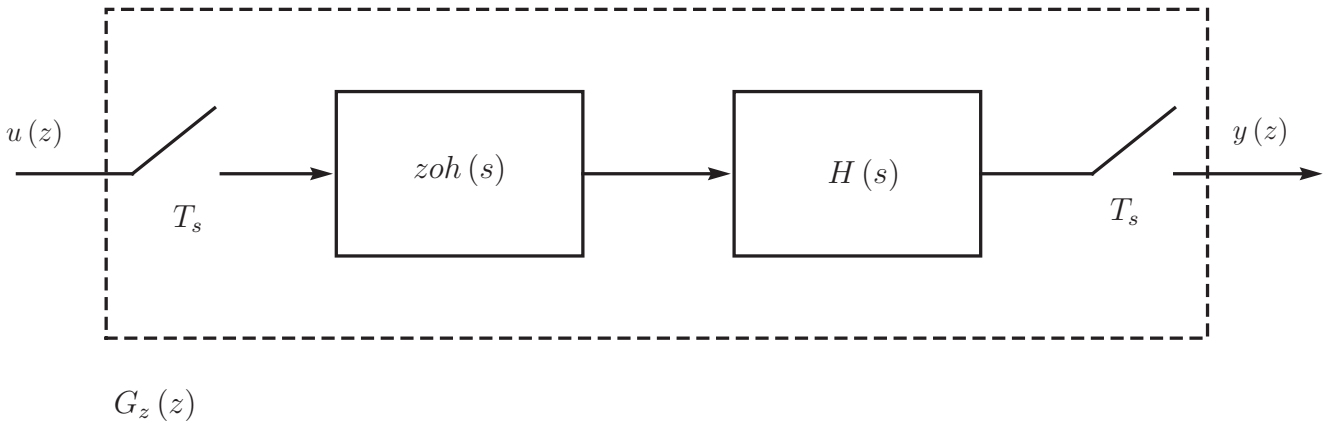


Fig. 1. Analyzed Sampled data system.

A LTI continuous-time system in the form (1)-(2) becomes a sampled-data system when synchronized samplers (period  $T_s$ ) are introduced at the input and the output and a zero order hold is introduced after the input sampler (Fig. 1). For this kind of sampled-data systems it is possible to obtain a discrete-time counterpart in the form (8)-(9) whose states ( $\tilde{x}_k = x(k \cdot T_s)$ ) and outputs ( $\tilde{y}_k = y(k \cdot T_s)$ ) are the same as the sampled continuous-time system ones at sampling times.

It is well known that the sampled-data system state flows between sampling times can be written as:

$$x(t) = \Phi(t) \tilde{x}_k + \Psi(t) \tilde{u}_k \quad (16)$$

where  $\Phi(t) = e^{\mathbf{A}t}$  and  $\Psi(t) = \int_0^t e^{\mathbf{A}(t-\alpha)} \mathbf{B} d\alpha$ .

*Proposition 2 ([3]):* The relationship between the LTI continuous sampled-data system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$

and its discrete-time counterpart  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$  is defined by:<sup>1</sup>

$$\tilde{\mathbf{A}} = \Phi(T_s) = e^{\mathbf{A}T_s} \quad (17)$$

$$\begin{aligned} \tilde{\mathbf{B}} &= \Psi(T_s) = \int_0^{T_s} e^{\mathbf{A}(T_s-\tau)} \mathbf{B} d\tau \\ &= \mathbf{A}^{-1} (\tilde{\mathbf{A}} - \mathbf{I}_d) \mathbf{B} \end{aligned} \quad (18)$$

$$\tilde{\mathbf{C}} = \mathbf{C} \quad (19)$$

$$\tilde{\mathbf{D}} = \mathbf{D} \quad (20)$$

The system  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$  written in an input/output form gives rise to the traditional  $\mathcal{Z}$ -transform:

$$G_c(z) = \tilde{\mathbf{C}} (z\mathbf{I}_d - \tilde{\mathbf{A}})^{-1} \tilde{\mathbf{B}} + \tilde{\mathbf{D}} \quad (21)$$

From now on  $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$  and  $\tilde{\mathbf{D}}$  refer to the ones defined in (17)-(20).

Unfortunately,  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$  is not discrete-time passive [6] (only in particular cases passivity is preserved [1]). An intuitive prove is that passive discrete-time systems are relative degree 0 ( $\tilde{\mathbf{D}} \neq 0$ ) [4] while the discrete-time systems obtained by means of the  $\mathcal{Z}$ -transform will be, in general, relative degree 1 due to the fact that the algebraic input-output coupling is preserved ( $\tilde{\mathbf{C}} = \mathbf{C}, \tilde{\mathbf{D}} = \mathbf{D}$ ), and most continuous-time systems fulfill  $\mathbf{D} = 0$ .

*Remark 3:* Although the system  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$  is not discrete-time passive the discrete-time energy defined by

$$\tilde{V}_k \triangleq \frac{1}{2} \tilde{x}_k^T \mathbf{P} \tilde{x}_k$$

equals the continuous-time energy at sampling times.

## V. PROPOSED METHODOLOGY

In order to preserve the passivity structure it is necessary that the energy evolution in the sampled-data continuous-time and the one in the discrete-time system counterpart system be equal, so the following must be fulfilled:

$$\tilde{V}_k \triangleq \frac{1}{2} \tilde{x}_k^T \mathbf{P} \tilde{x}_k \quad (22)$$

$$\Delta \tilde{V}_k = \tilde{V}_{k+1} - \tilde{V}_k \quad (23)$$

$$= \frac{1}{2} x_{k+1}^T \mathbf{P} x_{k+1} - \frac{1}{2} \tilde{x}_k^T \mathbf{P} \tilde{x}_k \quad (24)$$

$$= \int_{k \cdot T_s}^{(k+1) \cdot T_s} \dot{V}(t) dt \quad (25)$$

As the continuous-time system is assumed to be passive  $\dot{V}$  can be written in the form of (7), so:

$$\begin{aligned} \Delta \tilde{V}_k &= \int_{k \cdot T_s}^{(k+1) \cdot T_s} y(t)^T u(t) \\ &\quad + \frac{1}{2} \begin{bmatrix} x(t)^T, u(t)^T \end{bmatrix} \mathbf{H} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \end{aligned} \quad (26)$$

This expression has two parts, one related to the continuous-time power term and another related to the continuous dissipative term.

<sup>1</sup>Equation (18) can only be used when  $\mathbf{A}$  has no eigenvalues at 0.

### A. Dissipative Term

The dissipative term in (26) is, by the properties of passivity, always negative defined, so it can be written:

$$\int_{k \cdot T_s}^{(k+1) \cdot T_s} \frac{1}{2} \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \mathbf{H} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt < 0 \quad (27)$$

by applying the fact that  $u(t)$  is constant between sampling times and that  $x(t)$  evolution is analytically known, the previous equation can be rewritten as:

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \left[ \int_{k \cdot T_s}^{(k+1) \cdot T_s} \mathbf{J}(\tau)^T \mathbf{H} \mathbf{J}(\tau) d\tau \right] \begin{bmatrix} x_k \\ u_k \end{bmatrix} < 0 \quad (28)$$

where

$$\mathbf{J}(\tau) \triangleq \begin{bmatrix} \Phi(\tau) & \Psi(\tau) \\ 0 & 1 \end{bmatrix} \quad (29)$$

so it is possible to define the discrete-time dissipative matrix as:

$$\mathbf{H}^\star \triangleq \int_{k \cdot T_s}^{(k+1) \cdot T_s} \mathbf{J}(\tau)^T \mathbf{H} \mathbf{J}(\tau) d\tau \quad (30)$$

which is negative defined by construction. So the continuous-time dissipativity implies also discrete-time dissipativity.

### B. Power Term

By assuming that  $\mathbf{H}$  is negative semidefinite in (26) it can be stated that :

$$\Delta \tilde{V}_k < \int_{k \cdot T_s}^{(k+1) \cdot T_s} y(t)^T u(t) dt \quad (31)$$

In order to obtain the discrete-time passive system one would like to write (31) in the same shape as in (11).

In case there was not a holder in the system, the RHS term in (31) would be equal to  $y(k \cdot T_s)^T u(k \cdot T_s) = \tilde{y}_k^T \tilde{u}_k$  in which case the power term would be in the same form as in (11). So the counterpart discrete-time system would be discrete-time passive.

*Remark 4:* The passivity structure is lost due to the effect of the holder, not to the effect of the sampler.

In the case of using a holder (Fig. 1), the RHS term in (31) can be written as

$$\int_{k \cdot T_s}^{(k+1) \cdot T_s} y(t)^T u(t) dt = \left[ \int_{k \cdot T_s}^{(k+1) \cdot T_s} y(t) dt \right]^T \tilde{u}_k \quad (32)$$

comparing this equation with (11), it is clear that in order to force the counterpart discrete-time system to be discrete-time passive it is necessary to choose an output of the form:

$$y_k^\star \triangleq \int_{k \cdot T_s}^{(k+1) \cdot T_s} y(t) dt \quad (33)$$

so equation (31) takes the form

$$\Delta \tilde{V}_k < y_k^{\star T} \tilde{u}_k \quad (34)$$

which is exactly the form of (11). So with the proposed output the counterpart discrete-time system will be discrete-time passive.

*Remark 5:* It is important to note that  $\frac{y_k^\star}{T_s}$  corresponds to the mean value of the continuous-time system output taken within the interval time  $[k \cdot T_s, (k+1) \cdot T_s]$ .

*Remark 6:* Proposed output could also be used to preserve sampled-data nonlinear passive systems.

In [10], a similar definition to the one proposed has been interpreted in terms of state variables variations. As an example, in mechanical systems power variables are force and velocity, so the proposed output can be interpreted in terms of a position variation; in a similar way, in electrical systems power variables are voltage and current, so the proposed output can be interpreted in terms of a voltage variation.

*Proposition 3:* For LTI sampled-data linear systems  $y_k^\star$  can be computed as:

$$y_k^\star = \mathbf{C}^\star \tilde{x}_k + \mathbf{D}^\star \tilde{u}_k \quad (35)$$

where

$$\mathbf{C}^\star \triangleq \mathbf{C} \int_0^{T_s} \Phi(\tau) d\tau \quad (36)$$

$$\mathbf{D}^\star \triangleq \mathbf{C} \int_0^{T_s} \Psi(\tau) d\tau + T_s \mathbf{D} \quad (37)$$

*Proof:*

$$\begin{aligned} y_k^\star &= \int_{kT_s}^{(k+1)T_s} y(\tau) d\tau \\ &= \int_{kT_s}^{(k+1)T_s} [\mathbf{C}x(\tau) + \mathbf{D}\tilde{u}_k] d\tau \\ &= \mathbf{C} \int_{kT_s}^{(k+1)T_s} x(\tau)^T d\tau + T_s \mathbf{D}\tilde{u}_k \\ &= \mathbf{C} \int_0^{T_s} [\Phi(\tau) \tilde{x}_k + \Psi(\tau) \tilde{u}_k] d\tau + T_s \mathbf{D}\tilde{u}_k \\ &= \left[ \mathbf{C} \int_0^{T_s} \Phi(\tau) d\tau \right] \tilde{x}_k \\ &\quad + \left[ \mathbf{C} \int_0^{T_s} \Psi(\tau) d\tau + T_s \mathbf{D} \right] \tilde{u}_k \end{aligned}$$

applying (36) and (37) completes the proof. ■

*Proposition 4:* For those systems without a pole on the origin, the value of  $\mathbf{C}^\star$  and  $\mathbf{D}^\star$  can be computed as:

$$\mathbf{C}^\star = \mathbf{C} \cdot \mathbf{A}^{-1} (\tilde{\mathbf{A}} - \mathbf{I}_d) \quad (38)$$

$$\mathbf{D}^\star = \mathbf{C} \mathbf{A}^{-2} (\tilde{\mathbf{A}} - \mathbf{I}_d - \mathbf{A} T_s) \mathbf{B} + T_s \mathbf{D} \quad (39)$$

*Proof:*

$$\begin{aligned} \mathbf{C}^\star &= \mathbf{C} \int_0^{T_s} \Phi(\tau) d\tau = \mathbf{C} \int_0^{T_s} e^{\mathbf{A}\tau} d\tau \\ &= \mathbf{C} \int_0^{T_s} \sum_{i=0}^{\infty} \frac{\mathbf{A}^i \tau^i}{i!} d\tau = \mathbf{C} \sum_{i=0}^{\infty} \left[ \frac{\mathbf{A}^i \tau^{i+1}}{(i+1)!} \right]_0^{T_s} \\ &= \mathbf{C} \mathbf{A}^{-1} \sum_{i=0}^{\infty} \frac{\mathbf{A}^{i+1} T_s^{i+1}}{(i+1)!} \\ &= \mathbf{C} \mathbf{A}^{-1} \left( \sum_{i=0}^{\infty} \frac{\mathbf{A}^i T_s^i}{(i)!} - \mathbf{I}_d \right) \\ &= \mathbf{C} \mathbf{A}^{-1} (e^{\mathbf{A} T_s} - \mathbf{I}_d) = \mathbf{C} \mathbf{A}^{-1} (\tilde{\mathbf{A}} - \mathbf{I}_d) \end{aligned}$$

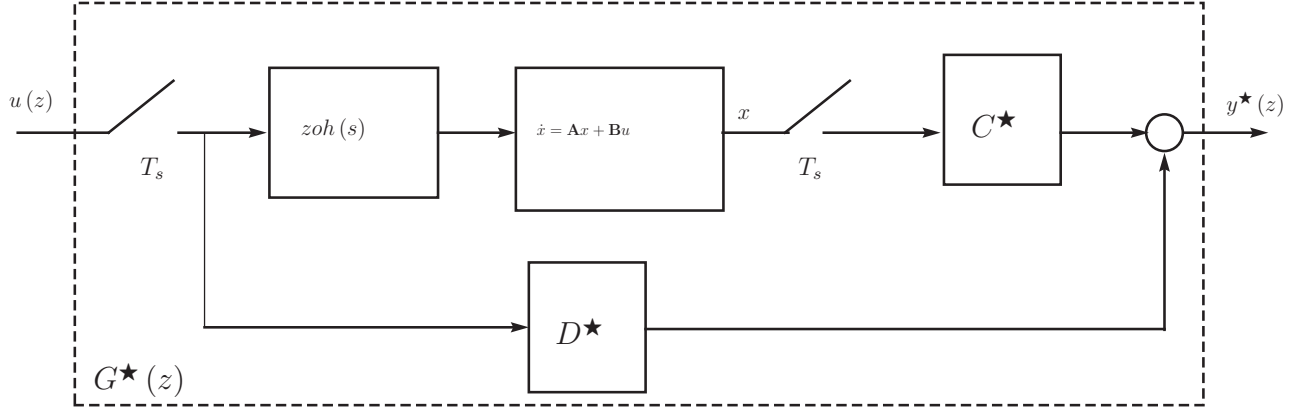


Fig. 2. Proposed Sampled data system.

$$\begin{aligned}
\mathbf{D}^\star &= \mathbf{C} \int_0^{T_s} \Psi(\tau) d\tau + T_s \mathbf{D} \\
&= \mathbf{C} \int_0^{T_s} \int_0^{T_s} e^{\mathbf{A}(T_s-\tau)} \mathbf{B} d\tau d\tau + T_s \mathbf{D} \\
&= \mathbf{C} \int_0^{T_s} \int_0^{T_s} \sum_{i=0}^{\infty} \frac{\mathbf{A}^i (T_s - \tau)^i}{i!} d\tau d\tau \mathbf{B} + T_s \mathbf{D} \\
&= \mathbf{C} \sum_{i=0}^{\infty} \int_0^{T_s} \int_0^{T_s} \frac{\mathbf{A}^i (T_s - \tau)^i}{i!} d\tau d\tau \mathbf{B} + T_s \mathbf{D} \\
&= \mathbf{C} \sum_{i=0}^{\infty} \frac{\mathbf{A}^i T_s^{i+2}}{(i+2)(i+1)!} \mathbf{B} + T_s \mathbf{D} \\
&= \mathbf{C} \mathbf{A}^{-2} \sum_{i=0}^{\infty} \frac{\mathbf{A}^{i+2} T_s^{i+2}}{(i+2)!} \mathbf{B} + T_s \mathbf{D} \\
&= \mathbf{C} \mathbf{A}^{-2} \left( -\mathbf{I}_d - \mathbf{A} T_s + \sum_{i=0}^{\infty} \frac{\mathbf{A}^i T_s^i}{(i)!} \right) \mathbf{B} + T_s \mathbf{D} \\
&= \mathbf{C} \mathbf{A}^{-2} \left( \tilde{\mathbf{A}} - \mathbf{I}_d - \mathbf{A} T_s \right) \mathbf{B} + T_s \mathbf{D}
\end{aligned}$$

This concludes the proof. ■

### C. Proposed System

*Theorem 2:* The discrete-time system  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{C}^\star, \mathbf{D}^\star)$  which can be written in a state space form as:

$$\tilde{x}_{k+1} = \tilde{\mathbf{A}} \tilde{x}_k + \tilde{\mathbf{B}} \tilde{u}_k \quad (40)$$

$$\tilde{y}_k^\star = \mathbf{C}^\star \tilde{x}_k + \mathbf{D}^\star \tilde{u}_k \quad (41)$$

and in an input/output form as

$$\mathbf{G}^\star = \mathbf{C}^\star \left( z\mathbf{I}_d - \tilde{\mathbf{A}} \right)^{-1} \tilde{\mathbf{B}}^\star + \mathbf{D}^\star \quad (42)$$



is discrete-time passive with respect to the supply rate  $(\tilde{y}_k^\star)^T \tilde{u}_k$  and has a storage function which equals the sampled-data continuous-time energy function at sampling times:

$$\tilde{V}_k = \frac{1}{2} \tilde{x}_k^T \mathbf{P} \tilde{x}_k \quad (43)$$

*Proof:* From what has been stated in section V-A and V-B it can be written that:

$$\Delta \tilde{V}_k = (\tilde{y}_k^\star)^T \tilde{u}_k + \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \mathbf{H}^\star \begin{bmatrix} x_k \\ u_k \end{bmatrix} \quad (44)$$

$$< (\tilde{y}_k^\star)^T \tilde{u}_k \quad (45)$$

■

In certain applications it is interesting to use  $\frac{\tilde{y}_k^\star}{T_s}$  as an output instead of using directly  $\tilde{y}^\star$ . This allows to have values at the output with similar range to the ones obtained in the continuous-time case.

*Corollary 1:* The discrete-time system  $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \frac{1}{T_s} \mathbf{C}^\star, \frac{1}{T_s} \mathbf{D}^\star)$ : is discrete-time passive with respect to the supply rate  $\frac{1}{T_s} (\tilde{y}_k^\star)^T \tilde{u}_k$  and the storage function :

$$\tilde{V}_k = \frac{1}{2T_s} \tilde{x}_k^T \mathbf{P} \tilde{x}_k \quad (46)$$

## VI. NUMERICAL EXAMPLE

### A. An open loop case

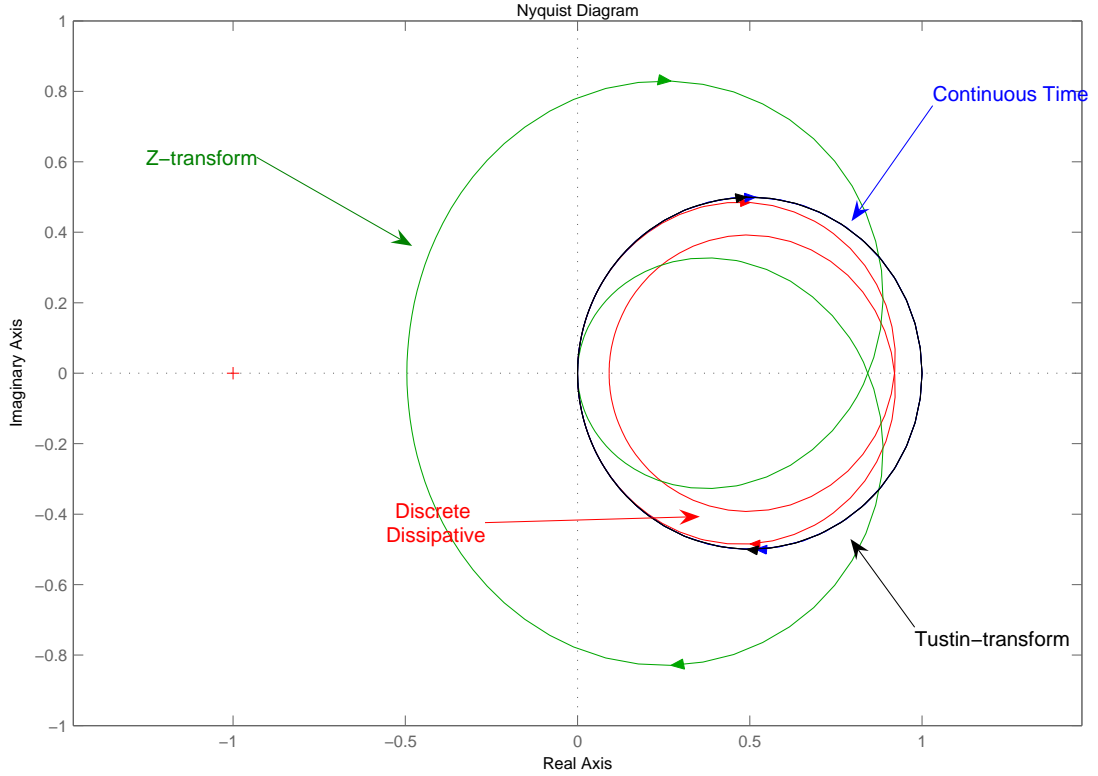


Fig. 3. Open loop example : Nyquist Plot.

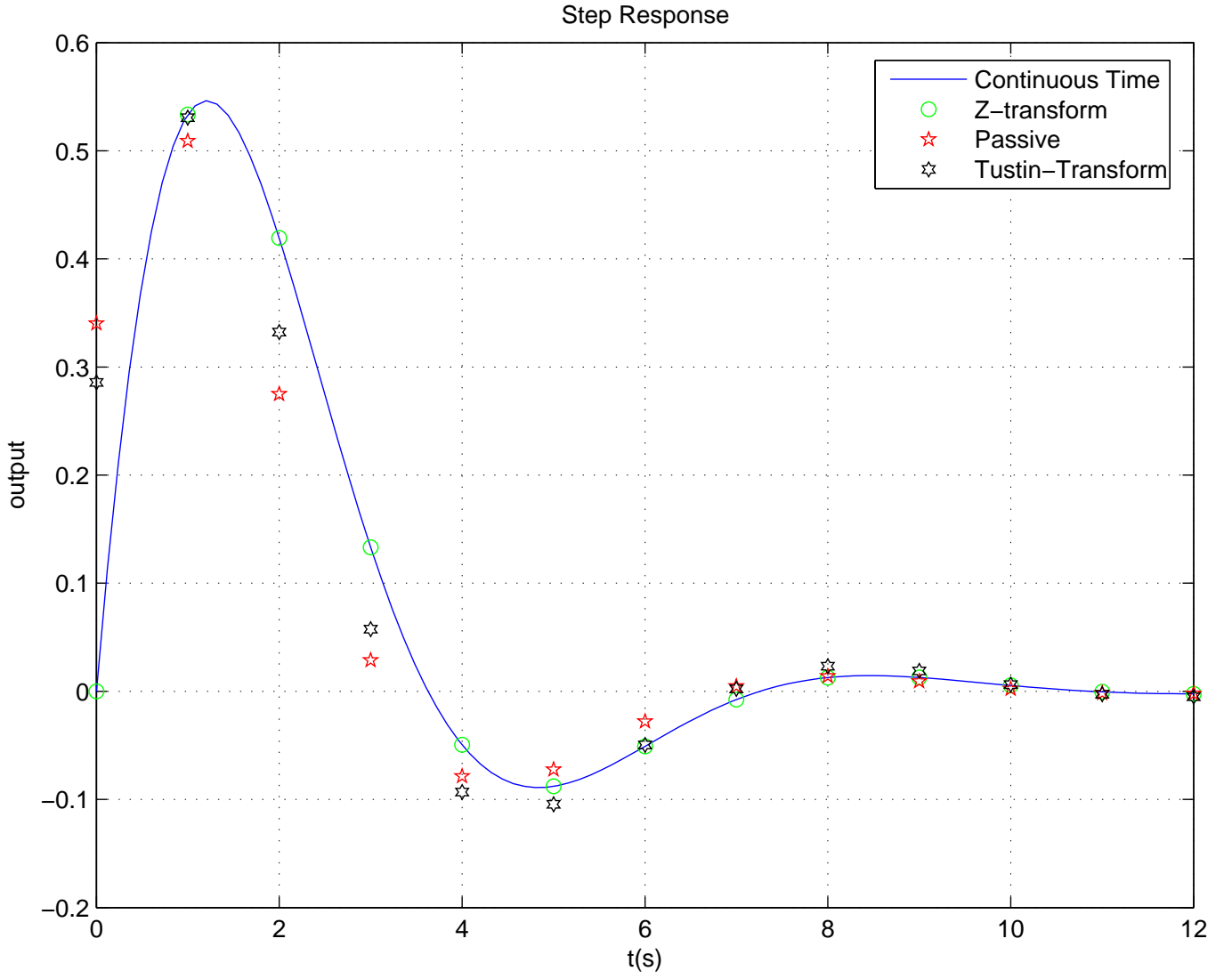


Fig. 4. Open loop example : Step Response.

Given a continuous-time system described by the transfer function:

$$G(s) = \frac{s}{s^2 + s + 1} \quad (47)$$

three discrete-time systems representing the discrete-time counterpart of the sampled-data continuous-time system will be compared. All of them are obtained using the same sampling time ( $T_s = 1$ ). The discretization techniques which will be applied are:

- Tustin transform. Applying the Tustin transform to (47), it is obtained:

$$G_b(z) = \mathcal{T} \{G(s)\} = \frac{0.28571(z-1)(z+1)}{z^2 - 0.8571z + 0.4286} \quad (48)$$

- $\mathcal{Z}$ -transform. Applying the  $\mathcal{Z}$ -transform to (47), it is obtained:

$$G_z(z) = \mathcal{Z} \{G(s)\} = \frac{0.53351(z-1)}{z^2 - 0.7859z + 0.3679} \quad (49)$$

- Proposed approach. Writing the system (42) related to (47), it is obtained:

$$G(z) = \frac{0.3403(z-1)(z+0.7102)}{z^2 - 0.7859z + 0.3679} \quad (50)$$

It is possible to observe that (48) and (50) are relative degree 0 while (49) is relative degree 1, so it can not be discrete-time passive. As expected, traditional discretization does not preserve passivity.

In linear systems passivity is directly related to real positiveness, which has been clearly stated in both continuous [7] and discrete-time [5]. So this measure will be used to analyze the passivity of the discrete-time system compared with the continuous-time passivity. In Fig. 3 it is possible to see the Nyquist plot of all studied transfer functions. As it can be seen, the continuous-time system is positive real defined, so passive. It can be seen that systems obtained by the proposed approach and the Tustin transform are also positive real, while the system obtained by the  $\mathcal{Z}$ -transform has lost this property. As the bilinear transform is a conformal transformation, the frequency response keeps unchanged (so no information about the sampling is introduced, the Continuous Nyquist plot will be preserved independently of  $T_s$ ).

Fig. 4 shows the step response of all presented systems. It is important to note that the only discrete-time system which matches the continuous-time response at sampling times is the one obtained by means of the  $\mathcal{Z}$ -transform.

To conclude this example, it is important to remember that the proposed approach is an exact discretization in the sense that at sampling times the discrete-time state equals the sampled-data system state while the Tustin transform is an approximation and does not fulfill this property. Additionally the time output time response obtained in the Tustin transform does not have a clear physical meaning, while in the proposed approach it has.

### B. A closed loop case

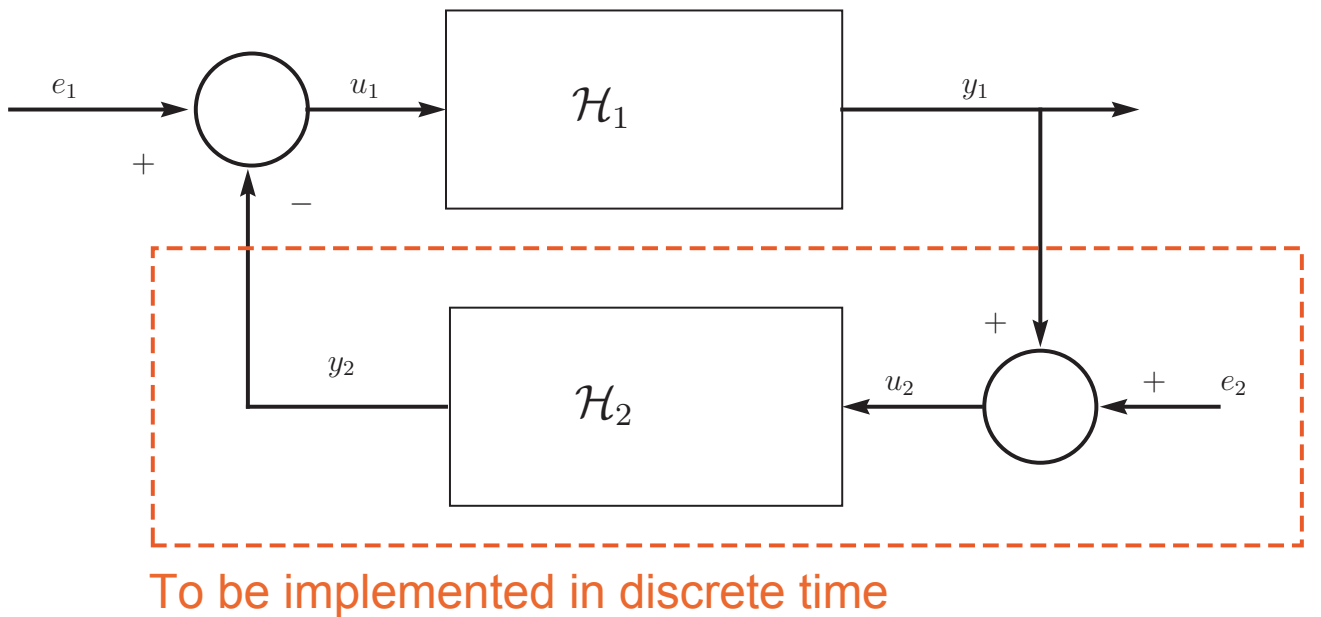


Fig. 5. Closed Loop connection.

Passivity based controllers are based on the fact that two passive systems connected in feedback (Fig. 5) preserve passivity, so stability. In Fig. 5  $\mathcal{H}_1$  plays the role of the plant to be controlled while  $\mathcal{H}_2$  plays the role of the controller. Usually this controller will be implemented in discrete-time, so in order to preserve passivity the proposed methodology will be applied to discretize both the plant and the controller.

As an example, a Buck converter (Fig. 7) used to implement an AC current source will be used.

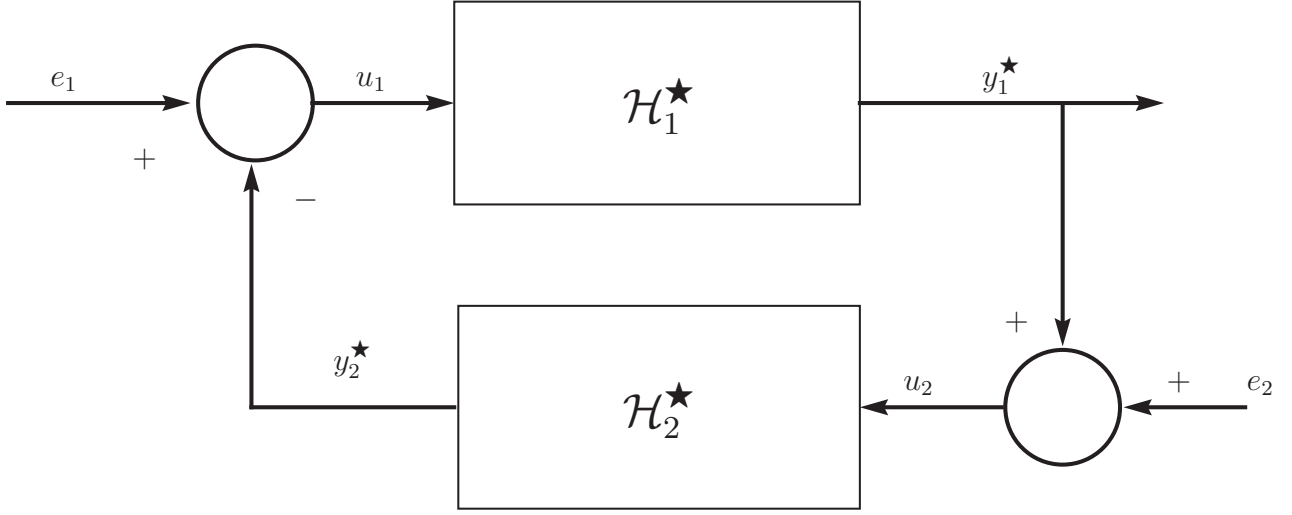


Fig. 6. Closed Loop discrete-time equivalent.

Traditionally this kind of devices is modeled by means of a set of averaged equations:

$$L \frac{di}{dt} = -r_L i - v_o + u \cdot v_i \quad (51)$$

$$C \frac{dv_o}{dt} = i - \frac{v_o}{r_C} \quad (52)$$

where  $L$  stands for the inductance,  $C$  stands for the capacitance,  $r_L$  and  $r_c$  are parasitic resistances,  $i$  is the output current,  $v_i$  is the input voltage which is assumed constant and known,  $v_o$  is the output voltage, and  $u$  is the control action. It is not difficult to check that the relation between  $v \triangleq u \cdot v_i$  and  $i$  is passive with respect to the energy stored in the capacitor and the inductor.

As the goal of the controller is to follow a certain current reference with the shape  $e_2 = I_{ref} \sin(\omega t)$ , according with the internal model principle [2], a resonator system is used as a controller:

$$G_c(s) = \frac{\omega^2 s}{s^2 + \omega^2} \quad (53)$$

this controller has infinite gain at frequency  $\omega \frac{rad}{s}$  so signals in this frequency will be tracked without error. It is not difficult to check that the controller is also passive. A consequence of combining the plant and this controller in feedback is that the closed loop will be stable and perfectly track the reference.

To illustrate the problem the following parameters have been fixed  $L = 6 \cdot 10^{-4} H$ ,  $C = 10^{-4} F$ ,  $r_L = 0.01 \Omega$  and  $r_C = 10^4 \Omega$ . In addition, the reference signal will be defined by  $I_{ref} = 10 A$  and  $\omega = 100\pi \frac{rad}{s}$ . A usual sample time for this kind of system is  $T_s = 5 \cdot 10^{-4} s$ . In order to improve clarity, the definition presented in corollary 1 will be used. The plant with the proposed output can be written as a discrete-time transfer function in the following form :

$$G^\star(z) = \frac{153.7z^2 - 0.4955z - 153.2}{z^2 + 0.9028z + 0.9917} \quad (54)$$

From the practical point of view, it is more relevant how to compute the new output from measurable variables, in this case this can be done in the following way:

$$y_k^\star = c_1 \cdot i(k \cdot T_s) + c_2 \cdot v_o(k \cdot T_s) + d \cdot u(k \cdot T_s) \quad (55)$$

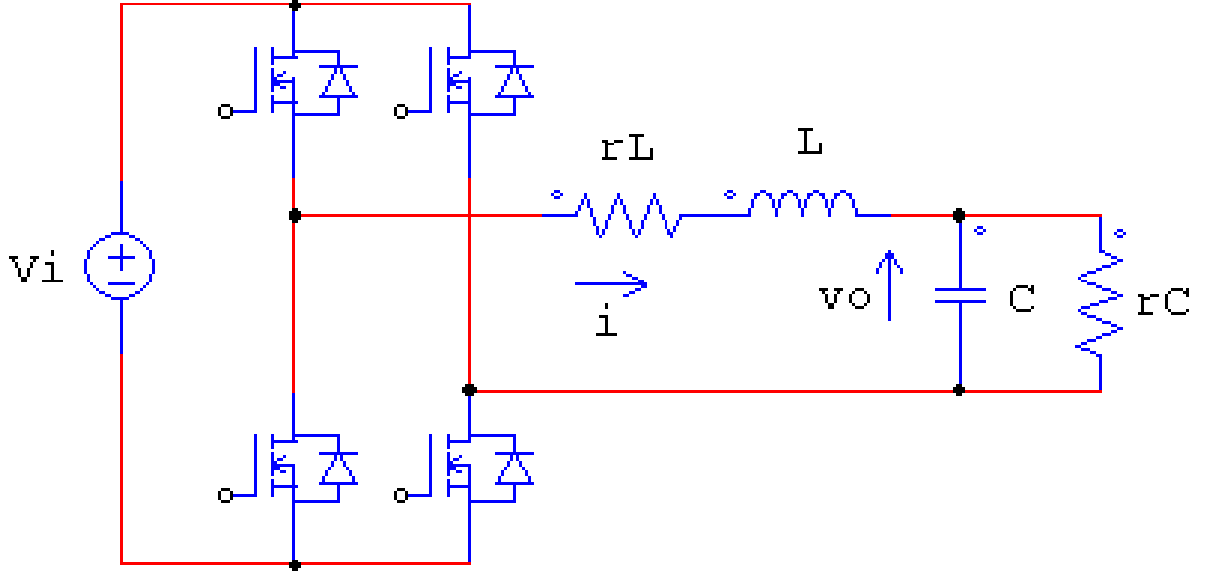


Fig. 7. Buck Converter Schematic.

where,  $c_1 = 0.4349$ ,  $c_2 = -0.2899$  and  $d = 153.6558$ . In this particular application,  $y_k^\star$  corresponds to the mean value of  $i$  in the interval time defined by  $[k \cdot T_s, (k+1) \cdot T_s]$ , additionally it is not difficult to verify that  $\frac{T_s}{C} y_k^\star$  represents the voltage variation in the capacitor in the same time interval.

The same procedure has to be applied to the continuous-time controller, so a discrete-time controller is obtained:

$$G_c^\star(z) = \frac{24.62z^2 - 24.62}{z^2 - 1.975z + 1} \quad (56)$$

it can be checked that both (56) and (54) are discrete-time passive, so their feedback connection will also be discrete-time passive and stable.

It is important to remark that changing the controller output does not imply more complexity, due to the fact that it is not related to physical variables.

In Fig. 8 the system time response is presented for both the continuous-time and discrete-time system, as it can be seen the results are similar in the continuous-time and the discrete-time approach.

## VII. CONCLUSIONS

In this work a new approach to preserve continuous-time passivity in LTI sampled-data systems has been presented. This approach is based on the selection of an appropriate output for the discrete-time equivalent system. This new output has a clear physical meaning, and can be computed as a linear combination of state variables and input variables in a closed form.

The procedure to apply the proposed technique to open and closed loop examples has been presented by means of a couple of examples. Additionally, the proposed transformation may be used in the analysis of systems containing static nonlinearities (ie. extending the hyperstability theory).

From a practical point of view the most important drawback of the proposed approach is the need to measure the complete state, while in traditional discretizations only one measure is needed. To overcome this problem the use of state observers will be analyzed.

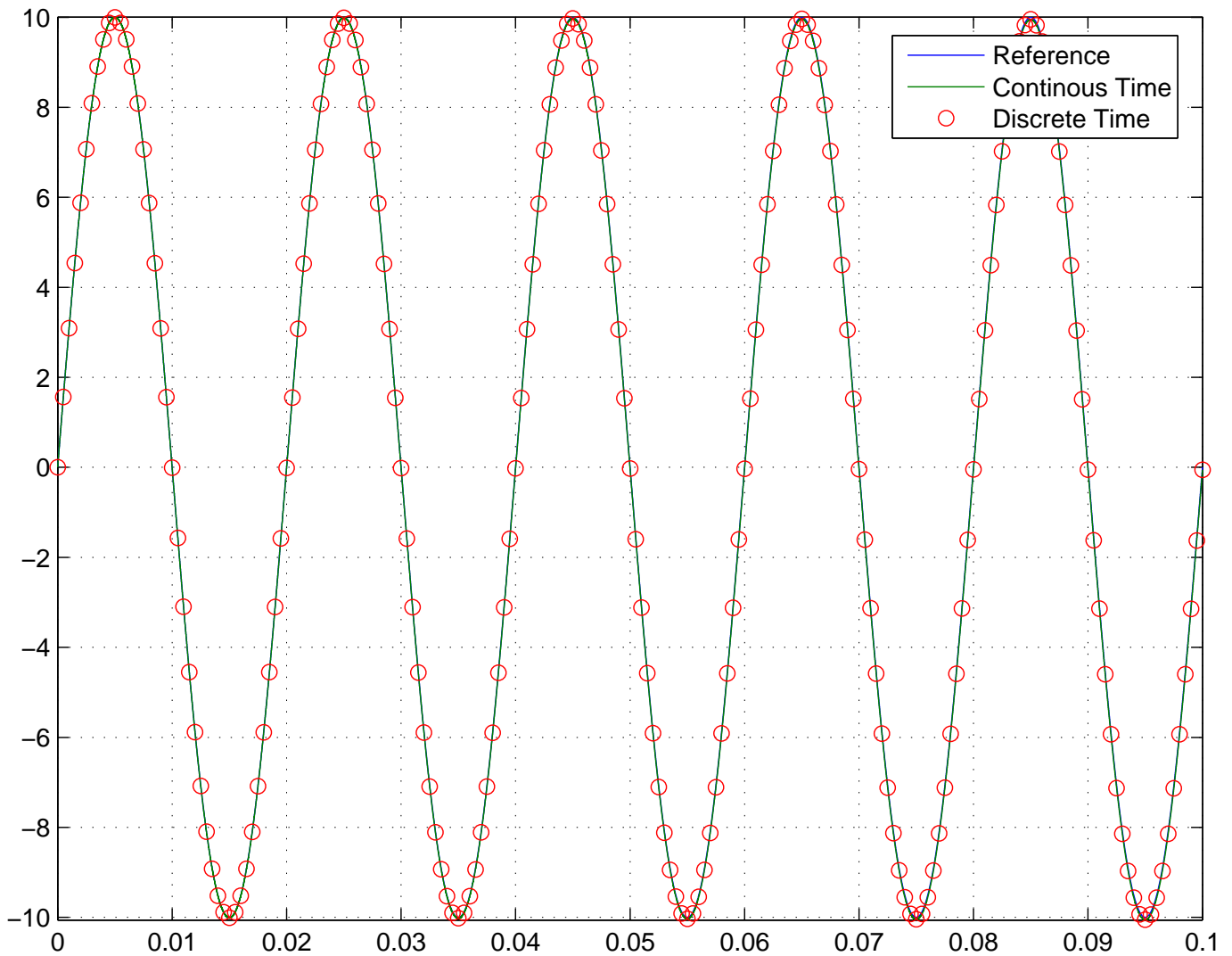


Fig. 8. Closed Loop Time Response.

Although the need to measure the state may seem a hard problem, it is important to remember that in several applications like power converters or mechanical devices it is very common to measure the full state (ie. the current and the voltage in power converters or the position and the velocity in mechanical systems). In this kind of applications the proposed technique could be applied without any difficulty.

The extension of the proposed methodology to nonlinear passive systems is currently under study. Although the proposed output definition applies to generic systems, from the practical point of view most the important problem is finding a method to analytically compute the output from the state variables and the inputs.

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