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Modelling and adaptive control of aerobic continuous stirred tank reactors

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Abstract

A biotechnological aerobic process is modelled as an ordinary differential equation which, under mild assumptions, ensures invariance of the positive orthant and boundedness of the concentrations. An adaptive controller is designed for this general class of processes so that the external substrate can be regulated by the dilution rate into a prespecified arbitrarily small neighbourhood of a constant setpoint reference. The adaptive controller is robust, simple in its design without invoking any identification mechanisms, and is based on output data only. It is shown that the prominent example of a baker's yeast fermentation belongs to this setup, and adaptive tracking is illustrated by simulations.

Keywords: Adaptive control, input saturation, tracking, aerobic processes, yeast fermentation

1 Introduction

The purpose of the paper is threefold. First, it is a contribution to the general modelling of biotechnological aerobic processes including proofs which show that the intuitive assumptions ensure mathematically what is expected from a real process. Secondly, we introduce a simple adaptive controller with saturation which, under mild assumptions, is proved to achieve tracking of an external substrate within a prespecified neighbourhood of a setpoint. Thirdly, a well known example of baker's yeast fermentation is further investigated and shown to be a special case of the proposed general model. Finally, adaptive tracking is illustrated for this example.

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We consider general biotechnological aerobic processes modelled by ordinary differential equations of the form

$$\boxed{\begin{aligned}\dot{x}(t) &= K \varphi(x(t), O(t)) - D(t) x(t) - Q x(t) + D(t) x^{\text{in}}(t), \\ \dot{O}(t) &= K_O \varphi(x(t), O(t)) - D(t) O(t) + \kappa_{\text{La}}[O^* - O(t)],\end{aligned}} \quad (1.1)$$

where, for $n \in \mathbb{N}$ and $m \in \mathbb{N}$, the constants and variables denote

$x(t) = (x_1(t), \dots, x_n(t))^T$	concentrations of the n process variables at time t ,
$O(t)$	concentration of dissolved oxygen at time t ,
$K = [k_1, \dots, k_m] \in \mathbb{R}^{n \times m}$	stoichiometric matrix,
$K_O = [k_{O1}, \dots, k_{Om}] \in \mathbb{R}^{1 \times m}$	stoichiometric oxygen vector,
$Q = \text{diag}\{q_1, \dots, q_n\} \in \mathbb{R}_{\geq 0}^{n \times n}$	proportional gaseous outflow rates,
$\kappa_{\text{La}}[O^* - O(t)]$	oxygen transfer rate $O(t)$ with equilibrium concentration of dissolved oxygen O^* and oxygen mass transfer constant $\kappa_{\text{La}} > 0$,
$\varphi = (\varphi_1, \dots, \varphi_m)^T$	reaction rate vector, where
$\varphi_j(\cdot, \cdot) : \mathbb{R}_{\geq 0}^{n+1} \rightarrow \mathbb{R}_{\geq 0}$	are locally Lipschitz continuous functions,
	$j = 1, \dots, m$,
$x^{\text{in}}(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}^n$	piecewise continuous and bounded function of
$x^{\text{in}}(t) = (x_1^{\text{in}}(t), \dots, x_n^{\text{in}}(t))^T$	n feed concentrations at time t
$D(\cdot) : \mathbb{R}_{\geq 0} \rightarrow [0, D_{\text{max}}]$	piecewise continuous function of dilution rate with
	$D_{\text{max}} > 0$.

Furthermore, the following structural assumptions of (1.1) are assumed.

(A1) There exists $\gamma \in \mathbb{R}_{>0}^n$ such that $\gamma^T k_j \leq 0$ for all columns k_1, \dots, k_m of the stoichiometric matrix K .

(A2) For $j = 1, \dots, m$ we have:

$$\varphi_j(x, O) = \alpha_j(x, O) \cdot \prod_{i \in \text{Aut}_j \cup \text{L}_j} x_i$$

for locally Lipschitz continuous functions $\alpha_j(\cdot, \cdot) : \mathbb{R}_{\geq 0}^{n+1} \rightarrow \mathbb{R}_{\geq 0}$;

if $\varphi_j(x, O) = 0$, then at least one of the components of (x, O) is 0;

$$K_O \varphi(x, 0) = 0.$$

Aut_j and L_j are the autocatalysts and the reactants of the j th reaction, respectively; they are defined in Section 2. Assumptions (A1)-(A2) are discussed in detail in Section 2.

(A1) ensures that $\dot{x}(t) = K \varphi(x(t), O(t))$ is dissipative. This replaces the classical assumption of Conservation of Mass. We do not suppose that the matrix K contains exact stoichiometric coefficients. Our approach should encompass models which contain only the essential reactions and essential substrates, and we also allow for uncertainty of the stoichiometric coefficients.

The term $D(t) x^{\text{in}}(t)$ in (1.1) ensures that the inflow rate is proportional to the dilution

rate. This assumption is essential for proving that all concentrations within the reactor remain bounded.

The decomposition of the reaction rate φ_j into a specific reaction rate α_j and a product of autocatalysts Aut_j and reactants L_j in assumption (A2) is essential for proving that if the process is initialized with positive concentrations, then they stay positive. The remaining conditions are justified by the physical fact that a reaction can only take place if all its activators are present in the reactor.

The **control objective** is to regulate an external substrate $x_l(t)$, $l \in \{1, \dots, n\}$, towards a prespecified neighbourhood of a given constant reference setpoint x_{ref} . This will be achieved by the so called **λ -tracker** (and variations thereof), i.e.

$$\begin{aligned} e(t) &= x_l(t) - x_{\text{ref}}, \\ D(t) &= \text{sat}_{[0, D_{\max}]}(-k(t)e(t) + D^*), \\ \dot{k}(t) &= \delta \begin{cases} (|e(t)| - \lambda)^r, & \text{if } |e(t)| > \lambda \\ 0, & \text{if } |e(t)| \leq \lambda \end{cases} \end{aligned} \tag{1.2}$$

where $r \geq 1$, $\lambda, \delta > 0$, $D_{\max} > D^* \geq 0$, $k(0) \geq 0$ are design parameters, and

$$\text{sat}_{[0, D_{\max}]}(\eta) := \begin{cases} 0, & \text{if } \eta < 0 \\ \eta, & \text{if } \eta \in [0, D_{\max}] \\ D_{\max}, & \text{if } \eta > D_{\max}. \end{cases}$$

These design parameters influence the transient behaviour of the closed-loop system crucially. Their role is discussed in detail in Remark 3.4 and illustrated in the simulations of the baker's yeast process in Section 5.

The λ -tracker (1.2) seems in particular suitable for biotechnological processes since despite their non-linearity, uncertainties, disturbances, and possible unstable multiple equilibria, this controller is only based on structural system data, i.e. (A1)-(A2). It consists of a proportional error feedback with saturation, and the time-varying proportional gain $k(\cdot)$ is determined adaptively by the error measurement only. The idea is that the gain increases as long as the error is outside the λ -strip. Once the gain is sufficiently large, under appropriate assumptions, the error $e(t)$ will converge towards the λ -strip and the gain $k(t)$ is kept constant. That means the control objective is met. The upper bound of the saturation has to meet a feasibility condition which will be made precise below.

The present paper is based on several contributions in different fields. Modelling of the general reactor model has been established by Bastin and Dochain [1], a sufficient condition for dissipativity of mass in terms of the stoichiometric matrix has been developed in Ilchmann and Weirig [11], see also the contribution by Bogaerts et al. [3].

Various control objectives and several industrial implementations are reviewed in Chen et al. [4]. The wide application of adaptive nonlinear techniques for biological reactors lies in the fact that the models include highly nonlinear and slowly time-varying kinetic parameters (see Bastin and Van Impe [2]). Most of the control strategies proposed in

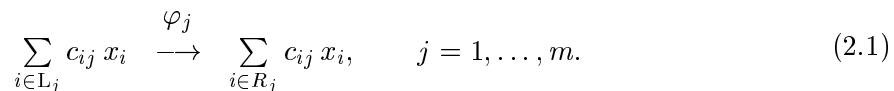
the literature use algorithms to identify the process kinetics and/or reconstruct the non-measured state variables (see for example Pomerleau and Perrier [15] or Ferreira and Feyo de Azevedo [7]).

The adaptive λ -tracker discussed in this paper is in the spirit of Ilchmann and Ryan [10], where it is introduced for linear systems and without any input saturations. In Ilchmann et al. [12] adaptive λ -tracking of an external substrate of a general reactor model was achieved by using the feedrate as the input variable; it also was assumed that the dilution rate is bounded away from zero. However, if aerobic continuous stirred tank reactors are modelled by lumping together the reaction equations in (1.1) to some $\frac{d}{dt}(x, O)^T = \tilde{K}\varphi(x, O)$, then in this general form one cannot derive boundedness of the concentrations of the general model. This is exactly the reason why the oxygen dynamics have to be separated as in (1.1), and a new proof for λ -tracking has to be developed. A first approach in this direction can be found in Weirig [18].

The paper is organised as follows. In Section 2 we introduce and motivate assumptions of the general model (1.1) so that it is sufficiently general to encompass relevant biochemical processes, and sufficiently strict to derive mathematical properties of the process which are intuitively expected. In Section 3 the adaptive feedback strategy to regulate an external substrate to a prespecified neighbourhood of the setpoint reference is introduced and proved to meet the control objective under certain assumptions. In Section 4 a well known model for baker's yeast fermentation is further investigated and shown that it falls into our general setup. This example is also used to illustrate the adaptive controller by some simulations in Section 5.

2 General modelling of bio-chemical aerobic processes

Aerobic biotechnological processes consist of a set of m reactions $\varphi_1, \dots, \varphi_m$ involving $n + 1$ concentrations x_1, \dots, x_{n+1} in the liquid phase of the reactor. Such a process is commonly specified by the following reaction scheme for each j th reaction:



Here

$$L_j \subseteq \{1, \dots, n + 1\}, \quad L_j \neq \emptyset$$

denotes the set of indices of the components x_i which are the *reactants* of the j th reaction,

$$R_j \subseteq \{1, \dots, n + 1\}, \quad R_j \neq \emptyset$$

is the set of indices of the components x_i which are the *reaction products* of the j th reaction.

The quantities of each component involved in the reaction are specified by the nonnegative *stoichiometric coefficients* c_{ij} , sometimes also called *yield coefficients*. The rate of consumption of the reactants, which is equal to the rate of formation of the reaction products, is called the *reaction rate* and denoted by φ_j . For a comprehensive list of reaction rates see for instance the Appendix in Bastin and Dochain [1].

The reaction scheme (2.1) gives rise to describe the process as an ordinary differential equation, see (1.1). The coefficients of the matrix K are given by $\pm c_{ij}$. Models of the form (1.1) have been used throughout the last thirty years in a more or less formal way, and the above formalism was established in the monograph by Bastin and Dochain [1].

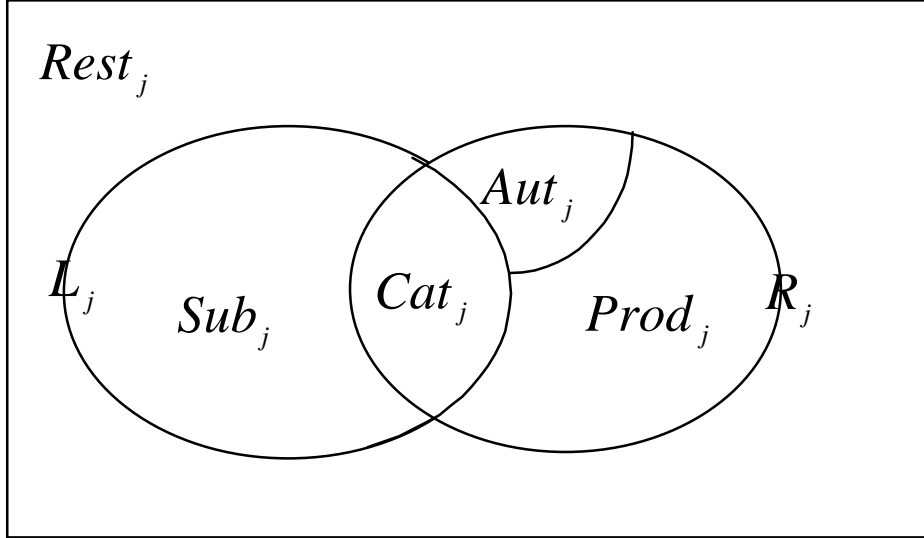


Figure 1: Reaction component sets

Cat_j	$=$	$L_j \cap R_j$	<i>catalysts</i> , i.e. set of the indices of those components which are involved in the j th reaction but maintained by the reaction,
Sub_j	$=$	$L_j \setminus (L_j \cap R_j) \neq \emptyset$	<i>substrates</i> , i.e. the set of those components that are consumed by the j th reaction, they are assumed to be empty,
Prod_j	\subseteq	$R_j \setminus (L_j \cap R_j)$	<i>products</i> , i.e. set of the indices of those components that are produced by the j th reaction,
Aut_j	$=$	$R_j \setminus ((L_j \cap R_j) \cup \text{Prod}_j)$	<i>autocatalysts</i> , i.e. set of the indices of those components that are accumulated by the j th reaction,
Rest_j	$=$	$\{1, \dots, n+1\} \setminus (L_j \cup R_j)$	set of the process components that are not involved in the j th reaction.

Note that, for all $j = 1, \dots, m$,

$$\text{Aut}_j \cup \text{Prod}_j = R_j \setminus (R_j \cap L_j),$$

and $\{1, \dots, n+1\}$ can be represented as the disjoint union

$$\{1, \dots, n+1\} = \text{Cat}_j \dot{\cup} \text{Sub}_j \dot{\cup} \text{Aut}_j \dot{\cup} \text{Prod}_j \dot{\cup} \text{Rest}_j. \quad (2.2)$$

The following characterizations of the catalysts, substrates and products will be useful in the sequel:

$$\left. \begin{aligned} \text{Cat}_j &= \{i \in L_j \cap R_j \mid k_{ij} = 0\} \\ \text{Sub}_j &= \{i \in \{1, \dots, n+1\} \mid k_{ij} < 0\} \\ \text{Prod}_j \cup \text{Aut}_j &= \{i \in \{1, \dots, n+1\} \mid k_{ij} > 0\}. \end{aligned} \right\} \quad (2.3)$$

The reaction rate φ_j is often assumed to be proportional to the microbial specific growth rate μ_j . The most prominent growth rates are the models of Monod or Haldane. Reaction rates φ_j , specific reaction rates α_j , and specific growth rate μ_j are in our setup related as follows.

$$\begin{aligned} \varphi_j(x, O) &= \alpha_j(x, O) \prod_{i \in \text{Aut}_j \cup L_j} x_i = \mu_j(x, O) \prod_{i \in \text{Aut}_j \cup \text{Cat}_j} x_i \\ &= \alpha_j(x, O) \prod_{i \in \text{Sub}_j} x_i \prod_{i \in \text{Aut}_j \cup \text{Cat}_j} x_i \end{aligned} \quad (2.4)$$

A prominent reference on chemical reacting systems is Gavalas [8]. See in particular Section 1.1, where he introduces systems which can be described by an ordinary differential equation as the first equation in (1.1). Although Gavalas does not explicitly say so (see Section 1.1 and also the sentence below equation (1.8.11)), the Principle of Mass Conservation implies the existence of a positive vector $\gamma \in \mathbb{R}_{>0}^n$ so that $\gamma^T k_j = 0$, for all $j = 1, \dots, m$. In this case, and if the dilution rate, feed rate and gaseous outflow rate in the first equation in (1.1) are zero, then

$$\frac{d}{dt} \gamma^T x(t) = \gamma^T K \varphi(x(t), O(t)) = 0,$$

and since all coefficients of γ are positive, this means conservation of mass.

However, if K does not represent the *exact* stoichiometric relationships between the components, then the model does not satisfy the conservation of mass, but might still be relevant since all “essential” reactions are obeyed. For this approach, which was taken in Bastin and Dochain [1], the concept of ‘non-cyclic processes’ has been developed in Ilchmann and Weirig [11]. Cyclicity of (1.1) as defined in [11] means that the process contains a reaction loop, i.e. there exists a subset of reactions S so that every substrate x_i involved in a reaction $j \in S$ is also an autocatalyst or a product of one of the reactions $j \in S$. Most processes in the literature are non-cyclic. In Ilchmann and Weirig [11] we give an algorithm to decide whether a matrix K is non-cyclic or not, and in particular non-cyclic implies (A1). Note also that if the dilution rate, feed rate and gaseous outflow rate in the first

equation in (1.1) are zero, and (A1) is satisfied, then $\frac{d}{dt}\gamma^T x(t) = \gamma^T K \varphi(x(t), O(t)) \leq 0$. Hence the process is dissipative and (A1) generalises conservation of mass.

We are now in a position to state and prove the main result of this section. That is, under the assumptions (A1)-(A2), all concentrations stay within an bounded invariant set for all $t \geq 0$.

Theorem 2.1 Consider the process (1.1) satisfying (A1)-(A2). Then for any initial concentrations $x(0) \in \mathbb{R}_{>0}^n$, $O(0) \in (0, O^*]$, there exists a unique solution of (1.1). This solution does not exhibit a finite escape time, is bounded, and stays within the positive orthant. More precisely,

$$O(t) \in (0, O^*] \quad \text{and} \quad x(t) \in \left\{ x \in \mathbb{R}_{>0}^n \mid \gamma^T x \leq \max\{\gamma^T x(0), \gamma^T \bar{x}^{\text{in}}\} \right\} \quad \forall t \geq 0, \quad (2.5)$$

where

$$\bar{x}^{\text{in}} := (\bar{x}_1^{\text{in}}, \dots, \bar{x}_n^{\text{in}}), \quad \text{and} \quad \bar{x}_i^{\text{in}} := \sup \{x_i^{\text{in}}(t) \mid t \in [0, \infty)\} \quad \text{for } i = 1, \dots, n.$$

Proof: Since the right hand side of the differential equation (1.1) is locally Lipschitz continuous in (x, O) and piecewise continuous in t , it follows from the classical theory of ordinary differential equations that for any $x^{\text{in}}(0) \in \mathbb{R}_{>0}^n$, $O(0) > 0$ there exists a unique solution $(x(\cdot), O(\cdot)) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}^n \times \mathbb{R}_{\geq 0}$ of the initial value problem on a maximally extended interval of existence $[0, \omega)$, where $\omega \in (0, \infty]$.

We show that the zero-axes of the positive orthant $\mathbb{R}_{>0}^{n+1}$ are repelling.

If $O(t_0) = 0$ for some $t_0 > 0$, then by (A2) yields $K_O \varphi(x, 0) = 0$, and thus by (1.1) it follows that $\dot{O}(t_0) = \kappa_{\text{La}} O^* > 0$, whence $O(\cdot) \equiv 0$ is repelling.

To see that the axes $x_i(\cdot) \equiv 0$, $i = 1, \dots, n$, are also repelling, assumption (A2) is essential. For a proof see Proposition 6 in Ilchmann and Weirig [11].

Note that if ω were finite, then this would not be due to the fact that $(x(\cdot), O(\cdot))$ is leaving the positive orthant through the edges, the edges are repelling. Hence a finite ω yields that some components of $(x(\cdot), O(\cdot))$ tend to infinity in finite time.

We prove boundedness of $(x(\cdot), O(\cdot))$ on $[0, \omega)$.

If $O(t) > O^* + \varepsilon$ for some $t \in [0, \omega)$ and $\varepsilon > 0$, then the second equation in (1.1) yields $\dot{O}(t) < -\varepsilon \kappa_{\text{La}}$, and hence $O(\cdot) \equiv O^*$ is repelling from above.

To see boundedness of $x(\cdot)$, suppose there exists $t \in [0, \omega)$ such that $\gamma^T[x(t) - \bar{x}^{\text{in}}] > 0$. Then there exists $\varepsilon > 0$ such that

$$\gamma^T[x(\tau) - \bar{x}^{\text{in}}] > 0 \quad \text{for all } \tau \in [t, t + \varepsilon),$$

and hence, by (A1) and (1.1),

$$\gamma^T[x(t + \varepsilon) - x(t)] = \int_t^{t+\varepsilon} \frac{d}{d\tau} \gamma^T[x(\tau) - \bar{x}^{\text{in}}] d\tau \leq - \int_t^{t+\varepsilon} D(\tau) \gamma^T[x(\tau) - \bar{x}^{\text{in}}] d\tau \leq 0.$$

Therefore, the bounds in (2.5) hold for all $t \in [0, \omega)$.

Finally, since ω was chosen to be maximal and $(x(\cdot), O(\cdot))$ is bounded, it follows from the standard theory of differential equations that $\omega = \infty$. This completes the proof. \square

Note that $x(t)$ in (2.5) belongs to a bounded set which depends only on $x(0)$, \bar{x}^{in} , and γ . If estimates of them are known and of O^* , then Theorem 2.1 yields immediately a bounded set containing any trajectory of the system for any piecewise continuous bounded $D(\cdot)$. This is summarized in the following corollary.

Corollary 2.2 Consider the process (1.1) satisfying (A1)-(A2). If $\widehat{\mathcal{B}} \subset \mathbb{R}_{>0}^n \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}^n \times \mathbb{R}_{>0}^n$ is a bounded set and $(x(0), O^*, \bar{x}^{\text{in}}, \gamma) \in \widehat{\mathcal{B}}$, then this set determines another bounded set $\mathcal{B} \subset \mathbb{R}_{>0}^n \times \mathbb{R}_{>0}$, such that,

$$(x(t), O(t)) \in \mathcal{B} \quad \text{for all} \quad t \geq 0. \quad (2.6)$$

\mathcal{B} is independent of the choice of the piecewise continuous, bounded dilution rate $D(\cdot)$ in (1.1). \square

3 Adaptive λ -setpoint control of external substrates

In this section we study the adaptive λ -setpoint control of an external substrate, the output variable to be controlled. A substrate $x_l(\cdot)$ of the reactor model (1.1) is deemed **external** if, and only if,

$$l \in \bigcup_{j=1}^m \text{Sub}_j \setminus \bigcup_{j=1}^m (\text{Aut}_j \cup \text{Prod}_j). \quad (3.1)$$

We need to assume the following assumptions on the reaction rates with \mathcal{B} as given in (2.6).

$$\textbf{(A3)} \quad \bar{\varphi}_j \geq \sup \{ \varphi_j(x, O) \mid (x, O) \in \mathcal{B} \} \quad \text{are known for all } j = 1, \dots, m.$$

Assumption (A3) is crucial for estimating the saturation bound. The need of this condition is not surprising, the faster the reaction rates, the more flexibility is needed in the input, and since the system parameters are not estimated in our setup, at least a rough upper bound for the reaction rates must be known. The set \mathcal{B} in Corollary 2.2 might be well known in applications, and an upper bound $\bar{\varphi}_j$ can be determined.

We are now in a position to prove the main result of this section.

Theorem 3.1 Consider the process (1.1) satisfying (A1)-(A3) with $\widehat{\mathcal{B}}$ and \mathcal{B} as given in Corollary 2.2. Let $x_l(\cdot)$ be an external substrate and suppose the following feasibility condition holds

$$\inf_{t \geq 0} \{x_l^{\text{in}}(t)\} := \underline{x}_l^{\text{in}} > x_{\text{ref}} - \lambda > 0, \quad D_{\text{max}} > \frac{\sum_{j=1}^m |k_{lj}| \bar{\varphi}_j + q_l [x_{\text{ref}} - \lambda]}{\underline{x}_l^{\text{in}} - [x_{\text{ref}} - \lambda]}. \quad (3.2)$$

Then the application of the λ -tracker (1.2) to (1.1) yields, for any initial data $(x(0), O(0)) \in \widehat{\mathcal{B}}$, $k(0) \geq 0$, a closed-loop system with unique solution

$$(x(\cdot), O(\cdot), k(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \mathcal{B} \times \mathbb{R}_{\geq 0}$$

defined on the whole time axis $\mathbb{R}_{\geq 0}$ and, moreover,

- (i) $\lim_{t \rightarrow \infty} k(t) = k_\infty \in \mathbb{R}_{\geq 0}$, i.e. the gain adaptation converges,
- (ii) $\lim_{t \rightarrow \infty} \text{dist}(x_l(t), [x_{\text{ref}} - \lambda, x_{\text{ref}} + \lambda]) = 0$, i.e. the external substrate $x_l(t)$ tends to the λ -neighbourhood of the reference setpoint x_{ref} as $t \rightarrow \infty$. \square

Proof: Since the right hand side of the closed-loop system (1.1), (1.2) is locally Lipschitz continuous in (x, O) and piecewise continuous in t , it follows from standard theory of ordinary differential equations that there exists a unique solution $(x(\cdot), O(\cdot), k(\cdot))$ on a maximally extended interval of existence $[0, \omega)$, $\omega \in (0, \infty]$.

By Theorem 2.1 $(x(\cdot), O(\cdot))$ is bounded, and so $k(t)$ as the integral of a bounded function cannot exhibit any finite escape time. Therefore, $\omega = \infty$, and applying Theorem 2.1 again yields

$$(x(t), O(t), k(t)) \in \mathbb{R}_{>0}^n \times \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \quad \text{for all } t \geq 0.$$

Next we prove boundedness of $k(\cdot)$.

In passing by note that by (3.1) and (2.3) we have $k_{lj} \leq 0$ for all $j = 1, \dots, m$, and hence (1.1) gives

$$\dot{x}_l(t) = - \sum_{j=1}^m |k_{lj}| \varphi_j(x(t), O(t)) - D(t) x_l(t) - q_l x_l(t) + D(t) x_l^{\text{in}}(t). \quad (3.3)$$

Now suppose that

$$\text{there exists } t' \geq 0 \text{ such that } k(t') > D_{\max}/\lambda. \quad (3.4)$$

We show that there exists a finite time $\hat{t} \geq t'$ such that

$$x_l(t) \in [x_{\text{ref}} - \lambda, x_{\text{ref}} + \lambda] \quad \text{for all } t \geq \hat{t}. \quad (3.5)$$

If $x_l(t) \geq x_{\text{ref}} + \lambda$ and $t \geq t'$, then by (3.4) it follows that $-k(t)[x_l(t) - x_{\text{ref}}] + D^* \leq -k(t)\lambda + D^* < 0$, and thus $D(t) = 0$, so that (3.3) yields,

$$\dot{x}_l(t) = - \sum_{j=1}^m |k_{lj}| \varphi_j(x(t), O(t)) - q_l x_l(t).$$

Since by (3.1) there exists j_0 such that $l \in \text{Sub}_{j_0}$, (2.3) yields $k_{lj_0} < 0$ and hence

$$\dot{x}_l(t) \leq -|k_{lj_0}| \varphi_{j_0}(x(t), O(t)).$$

Now an application of LaSalle's Invariance Principle (see e.g. the version in Knobloch and Kappel [14]) shows that $x_l(t)$ decreases into the λ -strip.

If $x_l(t) \leq x_{\text{ref}} - \lambda$ and $t \geq t'$, then by (3.4) it follows that $-k(t)[x_l(t) - x_{\text{ref}}] + D^* \geq k(t)\lambda + D^* > D_{\text{max}}$, and hence $D(t) = D_{\text{max}}$. Now (3.3) yields

$$\dot{x}_l(t) \geq - \sum_{j=1}^m |k_{lj}| \bar{\varphi}_j - [D_{\text{max}} + q_l] [x_{\text{ref}} - \lambda] + D_{\text{max}} \underline{x}_l^{\text{in}},$$

and by (3.2) it follows that there exists $\varepsilon > 0$ such that $\dot{x}_l(t) \geq \varepsilon$. This proves (3.5).

Now we are in a position to prove boundedness of $k(\cdot)$. If (3.4) is satisfied, then by (3.5), $x_l(t)$ reaches the interval $[x_{\text{ref}} - \lambda, x_{\text{ref}} + \lambda]$ in finite time, and stays within the interval after that. By the gain adaptation (1.2) this implies $k(t) = k(\hat{t})$ for all $t \geq \hat{t}$, whence boundedness of $k(\cdot)$. If (3.2) is not satisfied, then $k(\cdot)$ is obviously bounded.

Claim (i) of the theorem is a simple consequence of monotonicity of $t \mapsto k(t)$ and boundedness of $k(\cdot)$. It remains to prove (ii).

Using the distance function

$$d_\lambda(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, \quad \eta \mapsto d_\lambda(\eta) := \begin{cases} |\eta| - \lambda, & |\eta| \geq \lambda \\ 0, & |\eta| < \lambda, \end{cases}$$

it follows from the gain adaptation in (1.2) that (ii) is equivalent to $d_\lambda(e(\cdot)) \in \mathcal{L}^r(0, \infty; \mathbb{R})$. Since $t \mapsto e(t)$ is absolutely continuous, and $\eta \mapsto d_\lambda(\eta)$ is absolutely continuous and of bounded variation, it follows (see e.g. Hewitt and Stromberg [9]) that $t \mapsto d_\lambda(e(t))$ is absolutely continuous. Hence for almost all $t \geq 0$ we have

$$t \mapsto \frac{d}{dt} d_\lambda(e(t)) \leq |\dot{e}(t)|.$$

Now boundedness of $t \mapsto \frac{d}{dt} d_\lambda(e(t))$ together with $d_\lambda(e(\cdot)) \in \mathcal{L}^r(0, \infty; \mathbb{R})$ allows to apply Barb  lat's lemma (see, e.g., Khalil [13]) to conclude that $\lim_{t \rightarrow \infty} d_\lambda(e(t)) = 0$, whence (ii). This completes the proof of the theorem. \square

Remark 3.2 Note that the assumption of Theorem 3.1 that $x_l(\cdot)$ is an external variable implies that $k_{lj} \leq 0$ for all $j = 1, \dots, m$ and that $k_{lj_0} < 0$ for some j_0 . An immediate consequence of this is that a smoothened version of the “bang-bang” control law

$$D(t) = \begin{cases} D_{\text{max}}, & \text{if } x_l(t) \leq x_{\text{ref}} \\ 0, & \text{if } x_l(t) > x_{\text{ref}} \end{cases}$$

should also meet the control objective. This might help to stress the consequence of the assumption.

Finally, we also consider a non-adaptive version of (1.2) where the time-varying $k(t)$ is replaced by some constant $k' > 0$. Although this non-adaptive strategy is restrictive since k' needs to be sufficiently large, the result is worth knowing due to its simplicity. Furthermore, we give explicit lower bounds in terms of weak systems data, and it is ensured that the external substrate enters and stays within the λ -strip around the reference setpoint after finite time.

Theorem 3.3 Let $D^* \in [0, D_{\max})$ and suppose

$$k' \geq D_{\max}/\lambda \quad (3.6)$$

is known additionally to the assumptions in Theorem 2.1, then the non-adaptive feedback controller

$$D(t) = \text{sat}_{[0, D_{\max}]}(k' e(t) + D^*) \quad (3.7)$$

applied to (1.1) yields, for any initial data $(x(0), O(0)) \in \widehat{\mathcal{B}}$, $k(0) \geq 0$, a closed-loop system with unique solution

$$(x(\cdot), O(\cdot), k(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \mathcal{B} \times \mathbb{R}_{\geq 0}$$

defined on the whole time axis $\mathbb{R}_{\geq 0}$. Moreover, there exists $\hat{t} \geq 0$ such that

$$x_l(t) \in [x_{\text{ref}} - \lambda, x_{\text{ref}} + \lambda] \quad \text{for all } t \geq \hat{t}.$$

Proof: Since (3.6) ensures that the condition in (3.4) is satisfied, the proof is a straightforward simplification of the proof of Theorem 3.1. It is omitted. \square

Remark 3.4 The adaptive λ -tracker (1.2) and the non-adaptive feedback controller (3.7) are simple in its design. However, they contain design parameters which should be carefully chosen when the feedback controller is applied to a real process. D_{\max} depends not only on the feasibility condition (3.2) but also on the physical limitations of the actuator. When both conditions are compatible (i.e. the actuator limit is higher than the bound obtained by (3.2)) one should choose D_{\max} close to the actuator upper bound. This makes the control input smoother. To specify λ appropriately one need to know in advance an estimate about the upper bound for the magnitude of the measurement accuracy and noise. The power r in the gain adaptation in (1.2) determines the speed of the adaptation. If the difference of $(|e(t)| - \lambda)$ is smaller 1, then the bigger $r \geq 1$ is the slower $k(t)$ increases; if the difference is bigger than 1, then the bigger r is the faster $k(t)$ increases. Similar effects, but not in such a nonlinear way, can be achieved by varying δ or the initial gain $k(0)$. D^* is an input reference, an appropriate choice might be known from experiments with constant feedback. The role of all these design parameters is further illustrated in the simulations of the baker's yeast process in Section 5.

4 Baker's yeast fermentation process

The following kinetic model for cellular productivity of a continuous culture of *Saccharomyces cerevisiae*, more commonly known as baker's yeast, was introduced by Sonnleitner and Käppeli [16], and since then it has been used by numerous authors, see [4, 6, 15], and Sweere et al. [17], to name but a few. The dynamical model is obtained from a mass balance of the components, and it is assumed that the reactor is well mixed, the yield coefficients are constant, and the dynamics of the gas phase can be neglected. The yeast

fermentation goes through three pathways: sugar oxidation, ethanol oxidation and sugar fermentation with ethanol as an end product.

We do not model this as a fed-batch process, but in continuous mode operation. Thus this process can be described in the form (1.1) as follows.

$$\frac{d}{dt} \begin{pmatrix} S \\ X \\ C \\ E \end{pmatrix} = \begin{bmatrix} -c_{11} & 0 & -c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ 0 & -c_{42} & c_{43} \end{bmatrix} \begin{pmatrix} \mu_1(S, O) \\ \mu_2(S, O, E) \\ \mu_3(S, O) \end{pmatrix} X - D \begin{pmatrix} S \\ X \\ C \\ E \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \kappa_{CO_2} C \\ 0 \end{pmatrix} + \begin{pmatrix} D S^{\text{in}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.1)$$

$$\frac{d}{dt} O = [-c_{01}, -c_{02}, 0] \begin{pmatrix} \mu_1(S, O) \\ \mu_2(S, O, E) \\ \mu_3(S, O) \end{pmatrix} O - D O + \kappa_{La} [O^* - O],$$

where the state variables are

- $S(t)$ glucose (substrate) concentration in the reactor at time t (the output),
- $X(t)$ yeast concentration in the reactor at time t ,
- $C(t)$ dissolved carbon dioxide concentration in the reactor at time t ,
- $E(t)$ ethanol concentration in the reactor at time t ,
- $O(t)$ dissolved oxygen concentration in the reactor at time t ,

and further variables and constants are

- $D(t)$ dilution rate considered as the input,
- $c_{ij} > 0$ stoichiometric (or yield) coefficients, corresponding to the production of one unit of biomass (i.e. yeast) in each reaction,
- S^{in} glucose concentration in the feed,
- $\kappa_{La} [O^* - O(t)]$ gaseous oxygen transfer rate $O(t)$ with oxygen mass transfer constant κ_{La} and equilibrium concentration of dissolved oxygen O^* ,
- $\kappa_{CO_2} C(t)$ gaseous carbon dioxide outflow rate proportional to $C(t)$.

The main objective is to keep the glucose concentration, which is considered as external substrate, close to the reference value using the dilution rate as manipulating function. For technical reasons, the input must be bounded.

The model is based on a limited oxidation capacity, which is a function of the oxygen concentration in the liquid phase, see Sweere et al. [17]. If the oxidation capacity is sufficiently high to oxidize all glucose consumed, then no ethanol is produced. If in this situation the ethanol is present in the medium as well, then co-consumption of ethanol is possible. If not, then all glucose can be oxidized and the surplus glucose will be consumed according to the reductive metabolism, resulting in ethanol formation.

The process of yeast growth on glucose with ethanol production is described by the following three metabolic reactions. All constants involved are positive.

The reaction rate of the *respiratory growth on glucose* respectively the specific growth rate

is

$$\varphi_1(S, O, X) = \mu_1(S, O) X, \quad \mu_1(S, O) = \begin{cases} c_{11}^{-1} \frac{q_{s,\max} S}{S+K_s} \frac{O}{O+K_c}, & \text{if } \frac{q_{s,\max} S}{S+K_s} \leq \frac{q_{c,\max}}{a} \\ c_{11}^{-1} \frac{q_{c,\max}}{a} \frac{O}{O+K_c}, & \text{if } \frac{q_{s,\max} S}{S+K_s} \geq \frac{q_{c,\max}}{a}, \end{cases}$$

where $q_{s,\max}$ and $q_{c,\max}$ are the maximal specific uptake rates of glucose and oxygen, K_s and K_c are the saturation parameters for glucose uptake and oxygen uptake respectively, $a = c_{01} c_{11}^{-1}$ is the stoichiometric coefficient of the oxygen.

If the oxidation capacity is sufficiently high to oxidise both ethanol and glucose, then their co-consumption is possible, see Sweere et al. [17]. The reaction rate of the *respiratory growth on ethanol* and the specific growth rate is

$$\varphi_2(S, X, E, O) = \mu_2(S, E, O) X, \quad \mu_2(S, E, O) = \frac{\mu_{e,\max} E}{K_e + E} \frac{K_i}{S + K_i} \frac{O}{O + \beta_o},$$

where $\mu_{e,\max}$ is the maximal specific ethanol growth rate, K_i is the inhibition parameter (free glucose inhibits ethanol uptake), K_e is the saturation parameter for growth on ethanol, and β_o is the saturation parameter for the free respiratory capacity available.

Finally, the reaction rate of the *fermentative growth on glucose* respectively the specific growth rate is

$$\varphi_3(S, X, O) = \mu_3(S, O) X, \quad \mu_3(S, O) = \begin{cases} c_{13}^{-1} \frac{q_{s,\max} S}{S+K_s} \frac{K_c}{O+K_c}, & \text{if } \frac{q_{s,\max} S}{S+K_s} \leq \frac{q_{c,\max}}{a} \\ c_{13}^{-1} \left[\frac{q_{s,\max} S}{S+K_s} - \frac{q_{c,\max}}{a} \frac{O}{O+K_c} \right], & \text{if } \frac{q_{s,\max} S}{S+K_s} \geq \frac{q_{c,\max}}{a}. \end{cases}$$

The process consists of 3 reactions involving 5 components $(x, O) = (S, X, C, E, O)$, i.e. the concentrations in the liquid phase of the reactor. Using the notation introduced in Section 2, we see that

$$\begin{aligned} L_1 &= \{1, 5\}, \quad L_2 = \{4, 5\}, \quad L_3 = \{1\}, \quad R_1 = \{2, 3\} = R_2 = \{2, 3\}, \quad R_3 = \{2, 3, 4\}, \\ \text{Cat}_j &= \emptyset, \quad \text{Aut}_j = \{2\} \quad \text{for } j = 1, 2, 3 \\ \text{Sub}_1 &= \{1, 5\}, \quad \text{Sub}_2 = \{4, 5\}, \quad \text{Sub}_3 = \{1\}, \quad \text{Prod}_1 = \text{Prod}_2 = \{3\}, \quad \text{Prod}_3 = \{3, 4\}. \end{aligned}$$

From (3.1) we see that possible external substrates are x_1 and x_5 . Since the oxygen transfer rate is not proportional to the dilution rate which is crucial for Theorem 3.1 (see (3.3)), we choose $S(t)$ as external substrate and $l = 1$.

We are now in a position to factorise the reaction rates as in (2.4). Since $\text{Aut}_1 \cup L_1 = \{1, 2, 5\}$, setting

$$\alpha_1(S, O) = \begin{cases} \frac{c_{11}^{-1}}{O+K_c} \frac{q_{s,\max}}{S+K_s}, & \text{if } \frac{q_{s,\max} S}{S+K_s} \leq \frac{q_{c,\max}}{a} \\ \frac{c_{11}^{-1}}{O+K_c} \frac{q_{c,\max}}{S \cdot a}, & \text{if } \frac{q_{s,\max} S}{S+K_s} \geq \frac{q_{c,\max}}{a} \end{cases}$$

yields

$$\varphi_1(S, O, X) = \alpha_1(S, O) \prod_{i \in \text{Aut}_1 \cup L_1} x_i = \alpha_1(S, O) \cdot S \cdot O \cdot X.$$

Since $\text{Aut}_2 \cup L_2 = \{2, 4, 5\}$, setting

$$\alpha_2(S, O, E) = \frac{\mu_{e,\max}}{K_e + E} \frac{K_I}{S + K_I} \frac{1}{O + \beta_o}$$

yields

$$\varphi_2(S, O, X, E) = \alpha_2(S, O, E) \prod_{i \in \text{Aut}_2 \cup L_2} x_i = \alpha_2(S, O, E) \cdot O \cdot E \cdot X.$$

Finally, since $\text{Aut}_3 \cup L_3 = \{1, 2\}$, setting

$$\alpha_3(S, O) = \begin{cases} \frac{c_{13}^{-1} K_c q_{s,\max}}{O + K_c S + K_s}, & \text{if } \frac{q_{s,\max} S}{S + K_s} \leq \frac{q_{c,\max}}{a} \\ c_{13}^{-1} \left(\frac{q_{s,\max}}{S + K_s} - \frac{O}{O + K_c} \frac{q_{c,\max}}{S \cdot a} \right), & \text{if } \frac{q_{s,\max} S}{S + K_s} \geq \frac{q_{c,\max}}{a} \end{cases}$$

yields

$$\varphi_3(S, O, X) = \alpha_3(S, O) \prod_{i \in \text{Aut}_3 \cup L_3} x_i = \alpha_3(S, O) \cdot S \cdot X.$$

We check assumptions (A1)-(A3):

(A1) is immediate from the special form of K in (4.1).

(A2) follows from the above factorisations and since $K_O = [-c_{01}, -c_{02}, 0]$.

(A3) requires that for each $\varphi_j(x, O)$ an upper bound is known. The three reactions of the process (4.1) are autocatalytic and therefore the reaction rates are of the form $\varphi_j(x, O) = \mu_j(x, O)X$, $j = 1, 2, 3$. Since the growth capacity of a population of microorganisms is strongly limited, the specific growth rates are bounded. The upper bounds are

$$\mu_1(S, O) \leq \bar{\mu}_1 := c_{11}^{-1} \frac{q_{c,\max}}{a}, \quad \mu_2(S, E, O) \leq \bar{\mu}_2 := \mu_{e,\max}, \quad \mu_3(S, O) \leq \bar{\mu}_3 := c_{13}^{-1} \frac{q_{c,\max}}{a}.$$

Usually, the exact values of these parameters are not available but the range of their variations is well known, see Sonnleitner and Käppeli [16]. Therefore the maximal growth capacity of the yeast population in each reaction is known. Furthermore the upper bound of the biomass concentration \bar{X} is usually known in applications, see [4, 6]. Hence, $\bar{\varphi}_j \geq \bar{\mu}_j \bar{X}$ are known for $j = 1, 2, 3$.

By the above findings, the model of the baker's yeast fermentation process is a special case of the general model of bio-chemical aerobic processes analysed in Section 1 and 2, and meets the assumptions required for the adaptive setpoint control introduced in Section 3. Therefore, in the following Section 5 we will illustrate how the λ -tracker works when applied to (4.1).

5 Simulations

In this section we simulate the application of the λ -tracker (1.2) to the baker's yeast fermentation process (4.1). The output variable to be regulated within a neighbourhood of a constant concentration is the glucose concentration. The following kinetic data are taken from Sonnleitner and Käppeli [16].

$$\begin{aligned} q_{s,\max} &= 3.5 [g_{\text{gluc}} g_{\text{biomass}}^{-1} \text{h}^{-1}], & q_{c,\max} &= 0.256 [g_{\text{O}_2} g_{\text{biomass}}^{-1} \text{h}^{-1}], & \mu_{e,\max} &= 0.17 [\text{h}^{-1}], \\ K_s &= 0.2 [g_{\text{gluc}}/\text{l}], & K_c &= 0.0001 [g_{\text{O}_2}/\text{l}], & K_e &= 0.1 [g/\text{l}], \\ K_i &= 0.1 [g/\text{l}], & a &= 0.4142 [g_{\text{O}_2}/g_{\text{gluc}}], & \beta_o &= 0.003 [\text{mg}/\text{l}], \end{aligned}$$

where g_{gluc} , g_{O_2} and g_{biomass} denote gram glucose, gram oxygen, and gram biomass respectively.

The constant yield coefficients are chosen as in Pomerleau and Perrier [15], so that the stoichiometric matrix K and the vector K_O are

$$K = \begin{bmatrix} -2.04 & 0 & -20 \\ 1 & 1 & 1 \\ 1.23 & 0.9 & 9.09 \\ 0 & -1.39 & 10 \end{bmatrix}, \quad K_O = [-0.83, -1.56, 0].$$

Following Feyo de Azevedo et al. (1992), the other constant process parameters are set

$$\kappa_{\text{La}} = 100 [\text{h}^{-1}], \quad O^* = 0.007 [g/\text{l}], \quad S_{\text{in}} = 10 [g/\text{l}].$$

The initial values of the state variables are

$$S(0) = 0.95, \quad O(0) = 0.0066, \quad X(0) = 0.1, \quad C(0) = 0.000325, \quad E(0) = 0.0001 [g/\text{l}].$$

The control objective is to regulate the glucose concentration $S(t)$ into a λ -neighbourhood of the reference concentration $S_{\text{ref}} = 0.05$. The tolerated error around the reference should be below 5%, and hence we set

$$\lambda = 0.0025.$$

According to Theorem 3.1 we need to determine an upper bound of the input saturation D_{\max} . Recall that $l = 1$. Hence by the zero entries of K we need to determine upper bounds of the first and third reaction rate. Sonnleitner and Käppeli [16] allow the parameters to vary within the following ranges

$$0.24 \leq q_{c,\max} \leq 0.264, \quad 0.47 \leq c_{11}^{-1} \leq 0.5, \quad 0.05 \leq c_{13}^{-1} \leq 0.1.$$

An upper bound for the biomass concentration, taken from Feyo de Azevedo et al. (1992), is $\overline{X} = 3 [g/\text{l}]$. Hence the reaction rates are bounded by

$$\varphi_1(S, O, X) \leq 0.3187, \quad \varphi_3(S, X, O) \leq 0.0637.$$

Now it is easy to see that the fraction on the right hand side in (3.2) is 0.3842. Therefore, we may choose $D_{\max} = 0.385$ satisfying (3.2).

If the design parameters of the λ -tracker (1.2) are $\lambda \ll 1$ and $r = \delta = 1$, then for small error the growth of $t \mapsto k(t)$ is “slow”; it is even slower if $r > 1$. To fasten this up, one has to increase δ . For this reason we choose $\delta = 45$, $r = 1$.

Adaptive λ -tracker (1.2) with different offsets applied to (4.1): The simulations show that the λ -tracker is successful and what the effect of different design parameters is. In the first run of simulations, depicted in Figure 2, we choose the offset to be $D^* = 0$ (solid line). The gain increases rapidly until it is sufficiently large after 2 1/2 hours so that the substrate is forced into the λ -strip (dotted line) around the reference setpoint. The simulations are performed over a period of one day, and the figures are divided into an initial phase of 4 hours and the remaining 4-24 hours. Since $S(t)$ remains inside the λ -strip after $t = 3$ hours, the gain stays constant and also longer simulations have shown $k(3) = k(200) = 49.73$. Note also that $S(t)$ as well as the control action $D(t)$ behave smoothly without any overshoots. Moreover, $D(t)$ does not reach the upper saturation bound. In a second run we change the offset to $D^* = 0.2$ (dashed line). One may think that this should give a better behaviour since $D^* = 0.2$, also depicted in Figure 2, is close to the steady state value observed in the previous simulations. Although the results are not significantly different, the large substrate concentration at the beginning leads to a fast decrease of the dilution rate within 1/2 hour, and from then on the behaviour is similar to the first simulation. This different initial period has the effect that the error is larger, leading to a slightly larger gain $k(24) = k(200) = 50.42$, and this larger gain forces the substrate closer to the reference setpoint than in the first simulation. The same effect could also be achieved by a higher initial adaptive gain $k(0)$, but this leads to a considerably larger terminal gain. The dashed line in Figure 3 shows the trajectories of $k(t)$, $S(t)$ and $D(t)$ when $k(0) = 25$ and $D^* = 0$. For comparison the solid line depicts the case when $k(0) = 0$ and $D^* = 0.2$.

The effect of varying r depends on $(|e(t)| - \lambda)$. If this difference is bigger than 1, the gain $k(t)$ increases rapidly when r increases and the terminal gain might become much larger than actually needed for the control objective. If the difference is smaller than one, the increase of $k(t)$ is more moderate. This is illustrated in Figure 3 where satisfies $(|S(t)| - \lambda) < 1$ for all t and the output $S(t)$ enters the λ -strip at about $t = 7$ hours if $r = 1$ and at $t = 11.5$ hours if $r = 3$.

The other variables - biomass, ethanol, oxygen, carbon dioxide - reach a 5% neighbourhood of their steady states within 17, 30, 15, 15 hours, respectively (see Figure 4).

Non-adaptive tracker (3.7) applied to (4.1): In Figure 5 we show the simulations for the non-adaptive controller (3.7) when applied to (4.1). Again, we choose $D^* = 0$ and all the other data as in Figure 2. According to (3.6), the constant gain parameter is set $k' = D_{\max}/\lambda = 154$. This conservative bound is more than three times higher than the terminal high-gain parameter $k(\infty) = 49.73$ found adaptively. The undesirable effect of a “large” gain is the higher frequency (chattering) in the dilution rate, see Figure 5. This observation motivates the use of the adaptive gain controller in preference to the fixed gain controller.

Adaptive λ -tracker (1.2) with noise corrupting the output and applied to (4.1): The λ -tracker (1.2) can cope with noise corrupting the output measurement, provided the

amplitude of this noise is sufficiently small in terms of the λ -strip and the feasibility condition. In this case the measured error becomes

$$e(t) = x_l(t) - x_{\text{ref}} + n(t),$$

where the noise $n(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a bounded continuous function. If we set $\bar{n} := \sup_{t \geq 0} |n(t)|$, then it can be shown that $x_l(t)$ tends to $[x_{\text{ref}} - (\lambda + \bar{n}), x_{\text{ref}} + (\lambda + \bar{n})]$ as t goes to infinity. We omit the exact statement and proof of an analogous Theorem 3.1, it is very technical. Instead, we illustrate robustness of the λ -tracker with respect to stochastic noise (normal distribution, zero mean, variance one) corrupting the output measurement.

If the model and design parameters are as in Figure 2, then the λ -tracker is not able to force the substrate into the λ -strip, and hence $k(t)$ grows unboundedly. The reason is that the stochastic measurement noise is too large in amplitude and too vivid. We do not depict these simulations. However, if we allow 10% tracking error, i.e. $\lambda = 0.005$, then the simulation results are quite satisfactory, see Figure 6. Certainly, since the noise is corrupting the measurement, the control action is directly affected and hence correspondingly vivid. The λ -tracker regulates, as desired, the external substrate around the setpoint. In this set of simulations we have also shown the effect of slowing down the gain adaptation by decreasing $\delta = 45$ (solid line) to $\delta = 33$ (dashed line). $\delta = 45$ leads to a higher $k(t)$, and this amplifies the noise so that the control action is more vivid as well as temporary spikes in the glucose concentration occur. These observations hold also true over the longer period of 48 hours, what can be readily seen in Figure 7 and 8. Compared to the simulations without noise, the gain terminates at the same order of magnitude and the transient behaviour of the substrate, although not quite smooth, is kept in the λ -strip. Note that $D(t)$ is depicted in Figures 7 and 8 over a shorter time (24-30 hours) to give a better view on the actual input trajectory. Although the control effort is considerably more vivid than in the noise free case, which is not surprising since the measurement noise has a direct influence on the control, it is still realisable as a physical actuator. The other variables, i.e. biomass, ethanol, oxygen, carbon dioxide are depicted in Figure 9.

6 Conclusions

In this paper, control of a wide class of aerobic continuous stirred tank reactors has been achieved by a proportional error feedback controller with input saturations, where the gain is found adaptively. It is proved that regulation of external substrates to a neighborhood of a constant reference concentration is possible under mild conditions. We have also worked out structural conditions of the general process model which are essential when exploiting them mathematically. As a side result we show that proportional non-adaptive error feedback subjected to saturation is possible for the class of systems provided the system data satisfy a crude estimate. However, adaptive λ -tracking results in a much lower gain.

λ -tracking requires only very limited information of the system data and it readily tolerates noise corrupting the output measurement. The only price to be paid is that setpoint tracking is not achieved asymptotically but in a neighbourhood of the setpoint. However,

the neighbourhood is prespecified and arbitrarily small, which suffices for practical purposes.

Another advantage of the λ -tracker over other approaches on control of biotechnological processes, such as PI or PID controllers as for example in Dairaku et al. [5], or adaptive linearising control relying on system parameters or invoking estimators for the system parameters (see for example Chen et al. [4]), is its simplicity. However, the "closer" one comes to reality the more tuning of the design parameters in (1.2) is required.

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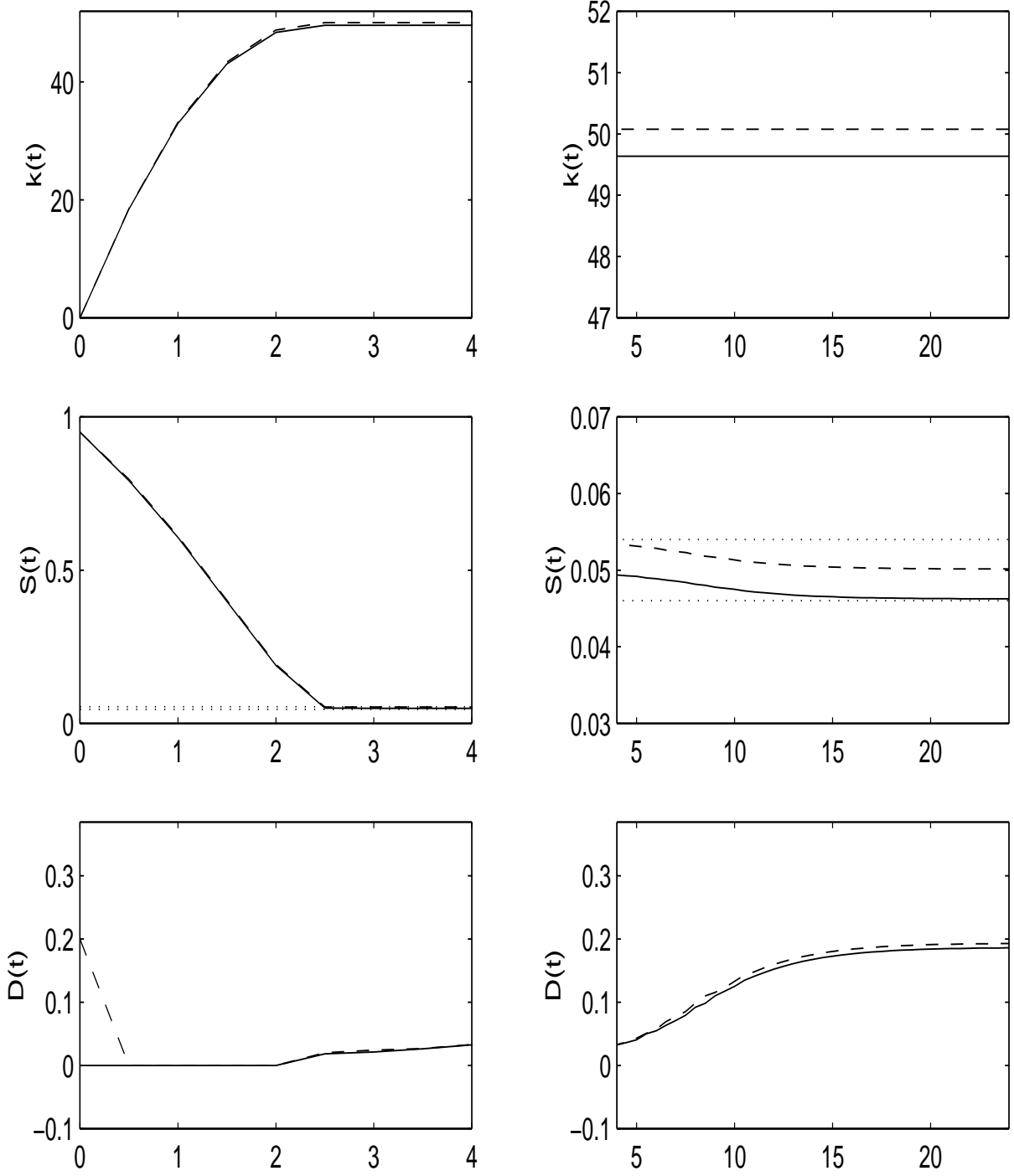


Figure 2: $k(t)$, $S(t)$ and $D(t)$ for (1.2) applied to (4.1), $\delta = 45$, $r = 1$, $\lambda = 0.0025$, $D^* = 0.2$ (dashed line), $D^* = 0$ (solid line)

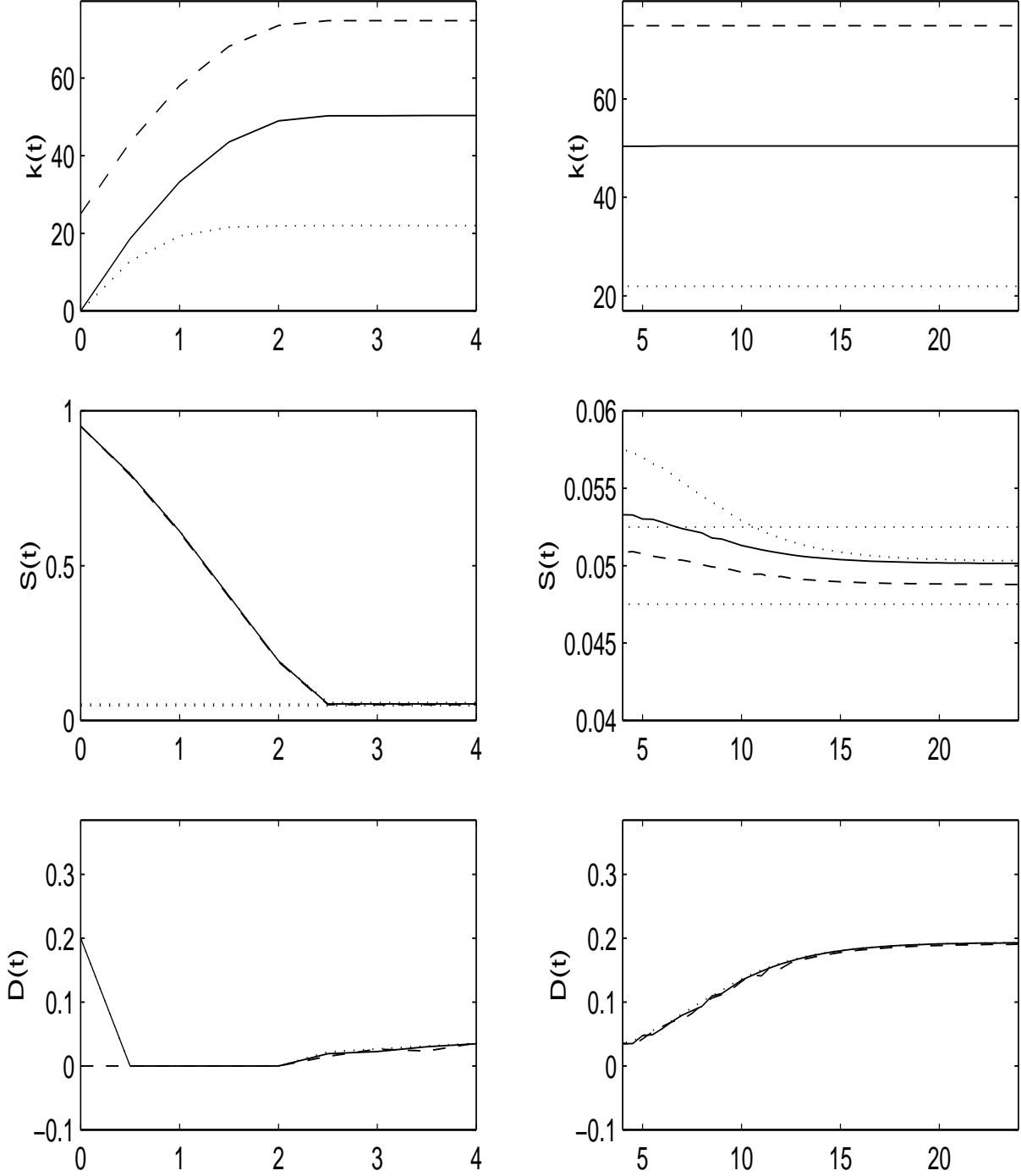


Figure 3: $k(t)$, $S(t)$ and $D(t)$ for (1.2) applied to (4.1), $\delta = 45$, $\lambda = 0.0025$, $r = 1$, $D^* = 0.2$, $k(0) = 0$ (solid line), $r = 1$, $D^* = 0$, $k(0) = 25$ (dashed line), $r = 3$, $D^* = 0.2$, $k(0) = 0$ (dotted line)

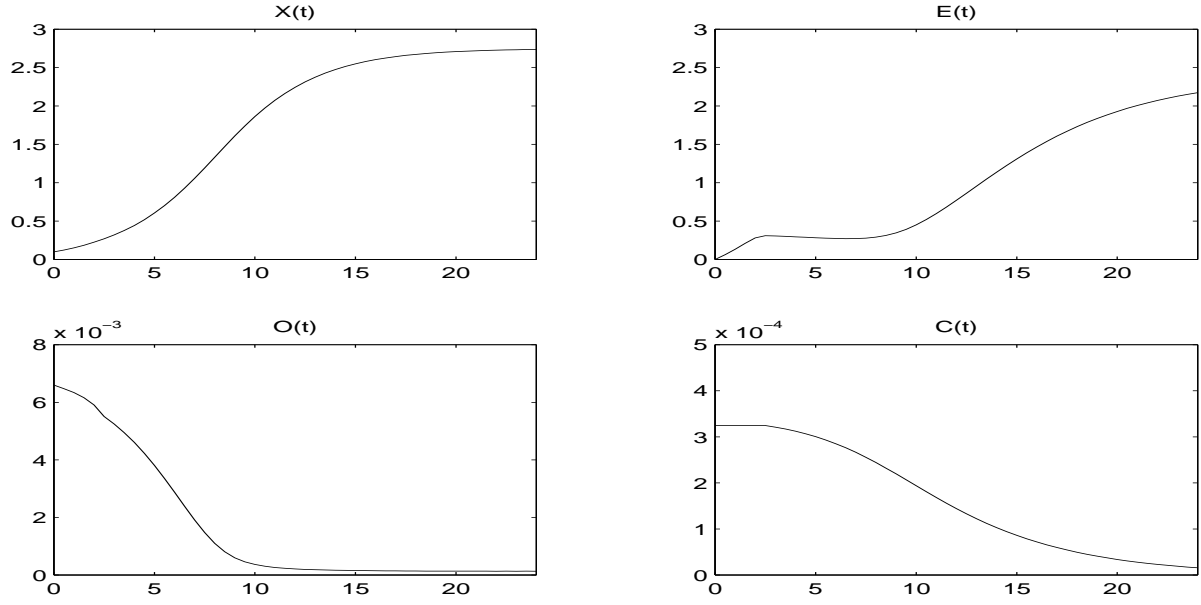


Figure 4: $X(t)$, $E(t)$, $O(t)$ and $C(t)$ for (1.2) applied to (4.1), $\delta = 45$, $r = 1$, $\lambda = 0.0025$, $D^* = 0.2$

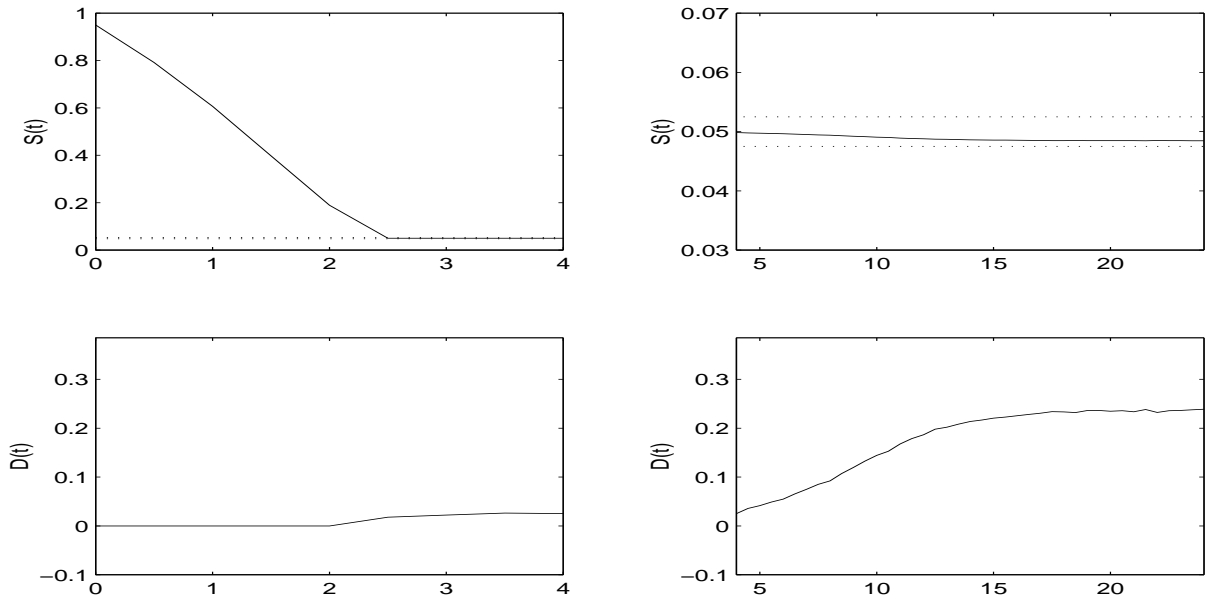


Figure 5: $S(t)$ and $D(t)$ for (3.7) applied to (4.1), $\lambda = 0.0025$, $D^* = 0$, and $k' = 154$

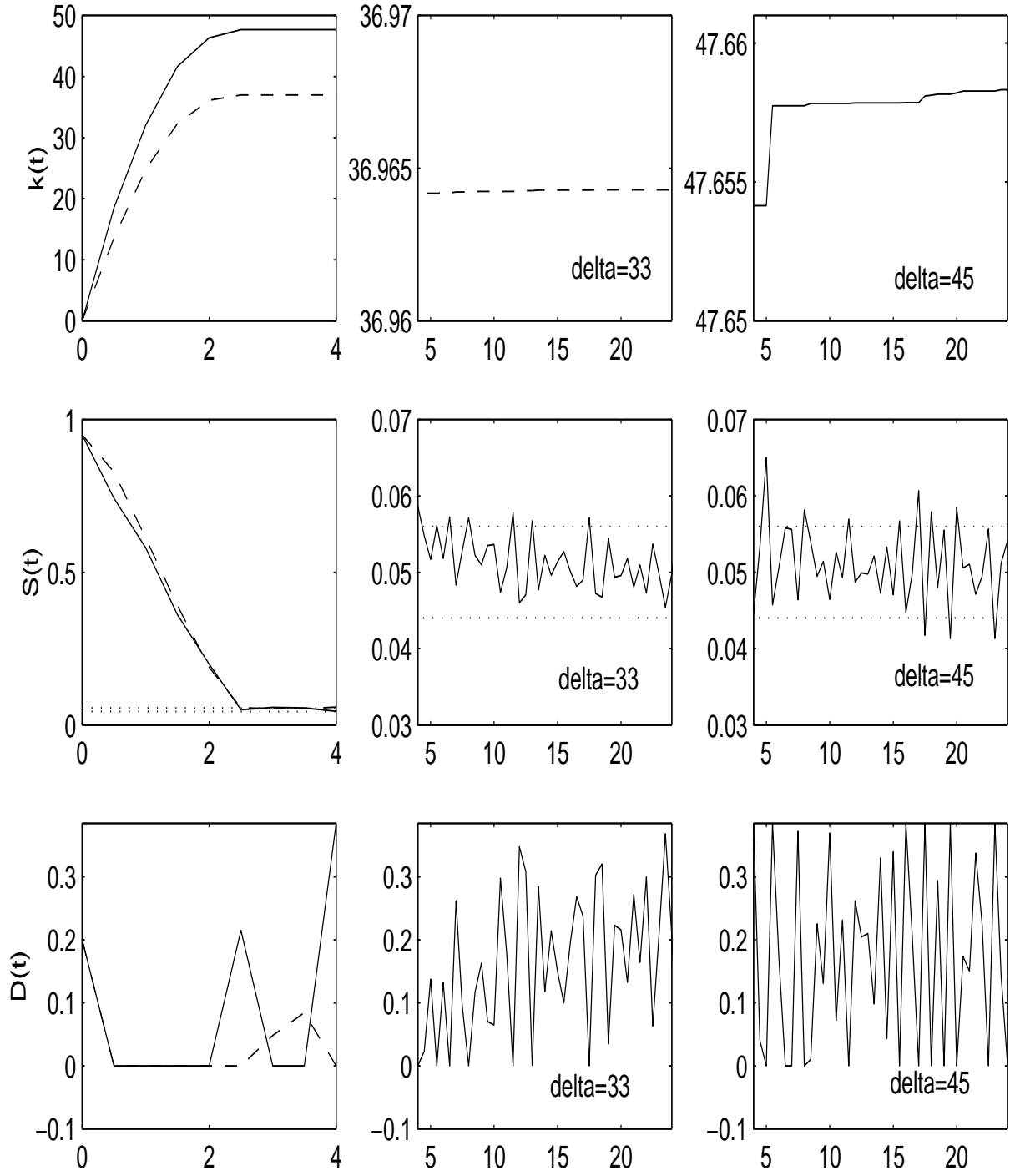


Figure 6: $k(t)$, $S(t)$ and $D(t)$ for (1.2) applied to (4.1) in the presence of noise, $r = 1$, $\lambda = 0.005$, $\delta = 45$ (solid line), $\delta = 33$ (dashed line)

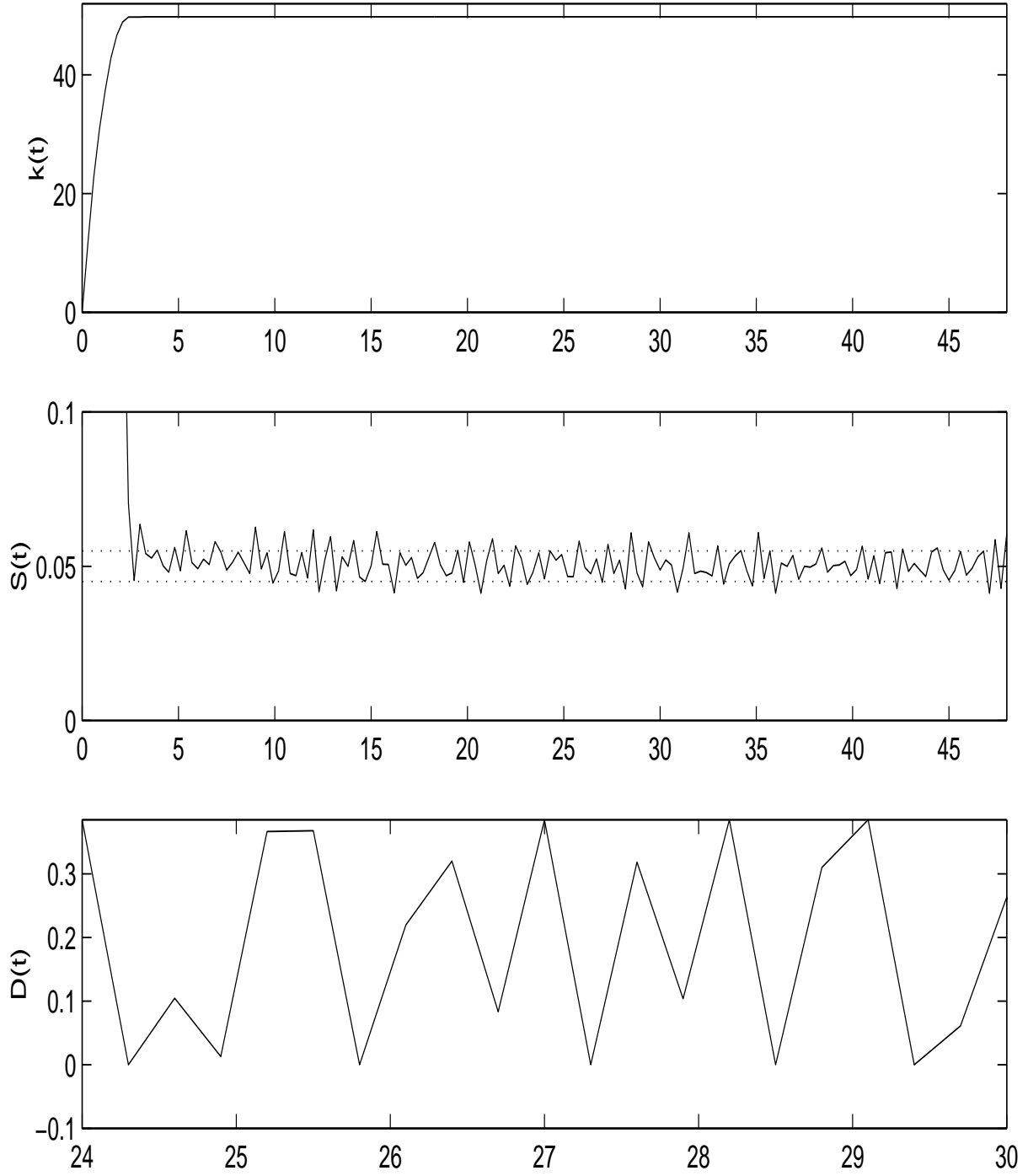


Figure 7: $k(t)$, $S(t)$ and $D(24 - 30)$ for (1.2) applied to (4.1) in the presence of noise, $r = 1$, $\lambda = 0.005$, $\delta = 45$, $t = 48$ [h]

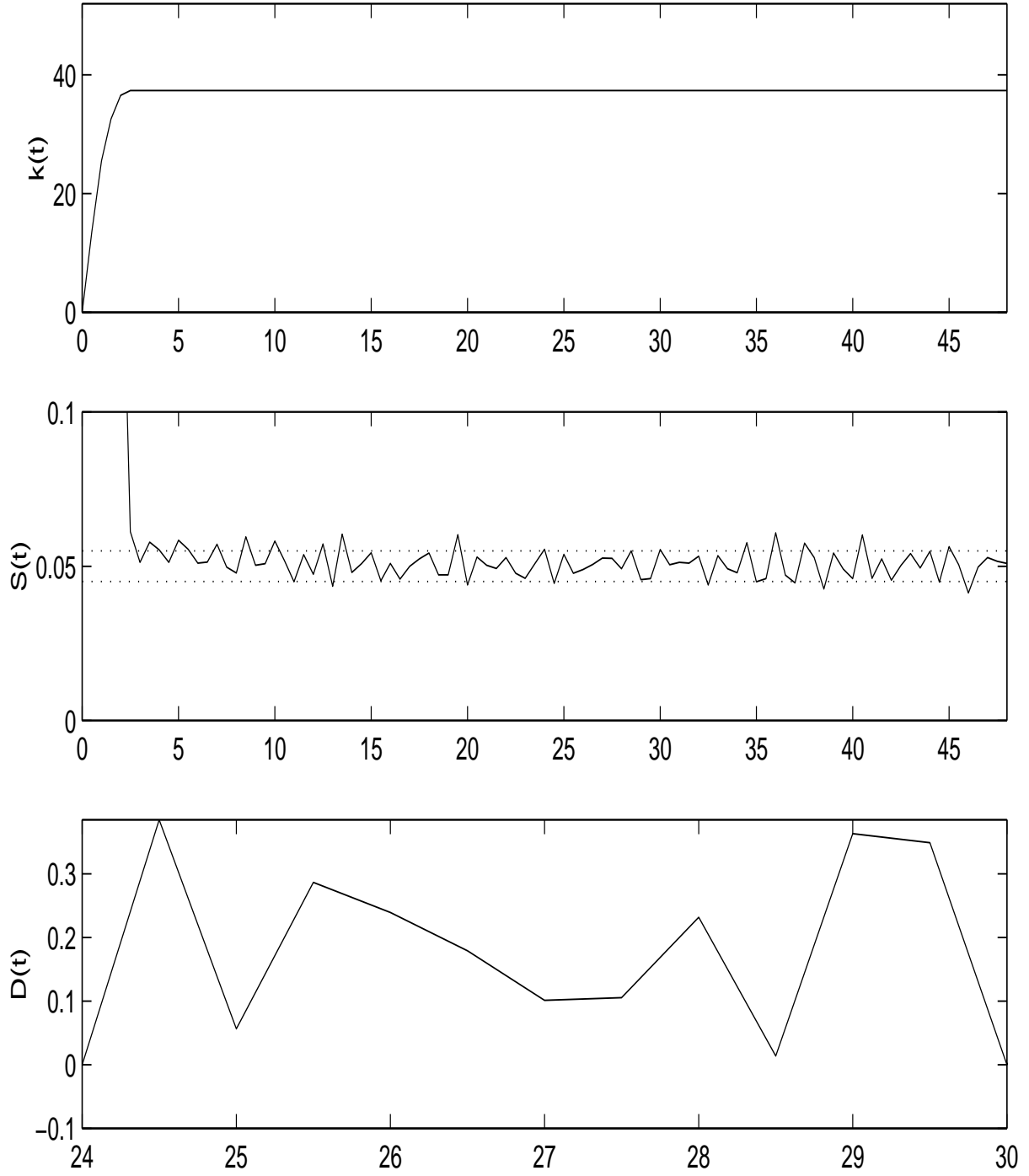


Figure 8: $k(t)$, $S(t)$ and $D(24 - 30)$ for (1.2) applied to (4.1) in the presence of noise, $r = 1$, $\lambda = 0.005$, $\delta = 33$, $t = 48$ [h]

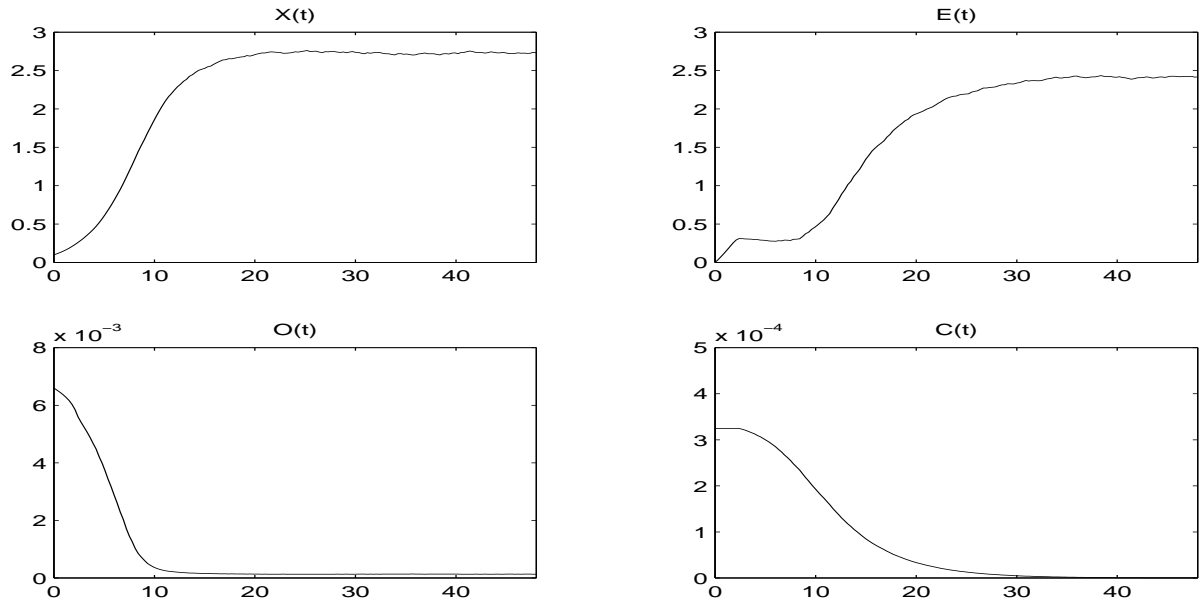


Figure 9: $X(t)$, $E(t)$, $O(t)$ and $C(t)$ for (1.2) applied to (4.1) in the presence of noise, $r = 1$, $\lambda = 0.005$, $\delta = 33$, $t = 48[\text{h}]$