On the relationships between theories of time granularity and the monadic second-order theory of one successor

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On the relationships between theories of time granularity and the monadic second-order theory of one successor

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ABSTRACT. *In this paper we explore the connections between the monadic second-order theory of one successor (*$\mathrm{MSO}[<]$ *for short) and the theories of* ω*-layered structures for time granularity. We first prove that the decision problem for* $\mathrm{MSO}[<]$ *and that for a suitable first-order theory of the upward unbounded layered structure are inter-reducible. Then, we show that a similar result holds for suitable chain variants of the MSO theory of the totally unbounded layered structure (this allows us to solve a decision problem about theories of time granularity left open by Franceschet et al. [FRA 06]).*

KEYWORDS: *time granularity, expressiveness, decidability.*

1. Introduction

In this paper we explore the connections between the monadic second-order (MSO) theory of one successor and the theories of ω-layered structures for time granularity. We first prove that the decision problem for the MSO theory of one successor and that for a suitable first-order theory of the upward unbounded layered structure are inter-reducible. Then, we show that a similar result holds for suitable chain variants of the MSO theory of the totally unbounded layered structure.

The ability of representing the same situation and/or different situations at various time granularities and of relating these different representations in a principled way is recognized as a meaningful research theme for temporal logic and a major requirement for a number of applications in different areas of computer science, including formal specification and verification, artificial intelligence, and temporal databases, e.g., [BET 00, DYR 95, FIA 94, LAD 86, LAM 85]. We focus our attention on the area of formal methods for the specification and verification of complex sys-

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tems [FRA 03a, MON 99, MON 02, MON 96b]. In this area, the addition of a notion of time granularity makes it possible to specify in a concise way reactive systems whose behaviour can be naturally modeled with respect to a (possibly infinite) set of differently-grained temporal domains. A logical framework for *time granularity* has been systematically developed in [MON 96a] and later extended in [FRA 01] and [PUP 06]. It is based on a many-level view of temporal structures that replaces the flat temporal domain of standard linear and branching temporal logics by a temporal universe consisting of a (possibly infinite) set of differently-grained temporal domains.

The MSO theory of the n-layered (there are exactly n temporal domains) k-refinable (each time point can be refined into k time points of the immediately finer temporal domain, if any) temporal structure for time granularity, with matching decidability results, has been investigated in [MON 96b]. The MSO theory of the k-refinable *upward unbounded layered structure* (UULS, for short), that is, the ω-layered structure consisting of a finest temporal domain together with an infinite number of coarser and coarser domains (a portion of the 2-refinable UULS is depicted in Figure 1), has been studied in [MON 99]. In the same paper, the authors deal with the MSO theory of the k-refinable *downward unbounded layered structure* (DULS), that is, the ω-layered structure consisting of a coarsest domain together with an infinite number of finer and finer domains (a portion of the 2-refinable DULS is depicted in Figure 2). Finally, the MSO theory of the k-refinable *totally unbounded layered structure* (TULS), which merges the UULS and the DULS, has been studied in [PUP 06]. The decidability of the MSO theory of the UULS can be proved by reducing the satisfiability problem for MSO logic over the UULS to the emptiness problem for systolic tree automata, while the decidability of the MSO theories of the DULS and the TULS can be proved by reducing the satisfiability problem for MSO logic over them to the emptiness problem for Rabin tree automata. The structure of the decidability proofs for the UULS and the DULS is briefly summarized in [EUZ 05]. The proof for the TULS is an easy adaptation of the one for the DULS. The proof for the DULS exploits an embedding technique that appends the infinite sequence of k-refinable infinite trees to the rightmost full path of the k-ary tree. The same technique can be applied to the case of the TULS, provided that we reverse the edges on the leftmost full path from a given node upward. In [MON 04], Montanari and Puppis shows that one can embed both the UULS and the DULS into the TULS by adding a unary predicate that identifies a distinguished layer of the structure, namely, the bottom (resp., top) layer of the UULS (resp, DULS). The decision problem for such an expanded structure has been solved by reducing it to the acceptance problem for Rabin tree automata.

In this paper, we establish some interesting connections between the MSO theory of one successor, denoted by $\mathrm{MSO}[<]$, and suitable fragments of (variants of) MSO theories of ω-layered structures. In particular, we take into consideration ω-layered structures expanded with the equi-level and equi-column predicates. The equi-level predicate constrains two time points to belong to the same layer, while the equi-column predicate constrains them to be at the same distance from the origin of the layers they belong to. Definability and decidability issues for ω-layered structures expanded with the equi-level and equi-column predicates have been systematically investigated by Franceschet et al. in [FRA 06][1]. In this paper we broaden the scope of such an investigation. First, we introduce a notion of reducibility via interpretations and we exploit it to compare the expressiveness of first-order (FO) and MSO logics interpreted over the discrete linear order $\langle \mathbb{N}, < \rangle$ and the 2-refinable UULS. One can easily show that the logics of the latter structure are strictly more expressive than the logics of the former one. However, we prove that the expansion of $\langle \mathbb{N}, < \rangle$ with the binary predicate (actually a function) $flip$ [MON 00b], which expresses properties of the binary representations of numbers, makes the resulting FO (resp., MSO) logic at least as expressive as the FO (resp. MSO) logic over the UULS. Next, we introduce a relaxed notion of reducibility, which allows us to define a mapping of formulas from one logic to another one where each variable can be mapped into several variables, instead of a single one, of possibly different types. By exploiting such a reduction, and by encoding finite sets with natural numbers, we show how to translate formulas of MSO logic over $\langle \mathbb{N}, < \rangle$ into equi-satisfiable formulas of the FO logic over the 2-refinable UULS expanded with a suitable predicate $Path$. We also provide the converse reduction, thus showing that the satisfiability problems for the two logics are actually inter-reducible.

As a matter of fact, the effective translation from FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ **to** MSO$[<]$ **has a nontrivial practical impact, since it allows one to map verification problems for UULSs to verification problems for** MSO$[<]$**, thus making it possible to exploit the wide spectrum of techniques available for that logic.**

We then consider the TULS equipped with the *layer* 0 predicate and either the equi-level or the equi-column predicate. We exploit a different encoding of (possibly infinite) chains (that is, subsets of paths) in order to reduce the satisfiability problem for the chain fragment of MSO logic over the 2-refinable TULS to the satisfiability problem for (full) MSO logic over $\langle \mathbb{N}, < \rangle$. The converse reduction is accomplished by embedding $\langle \mathbb{N}, < \rangle$ inside the leftmost branch of the TULS. All together these results provide alternative characterizations of the MSO theory of one successor in terms of suitable theories of ω-layered structures. In addition, the characterization of the chain fragment of MSO logic over the TULS equipped with the equi-column predicate positively answers to a decision problem left open in [FRA 06], namely, the problem of establishing whether the satisfiability problem for the chain/path fragment of MSO logic over the 2-refinable DULS is decidable.

The rest of the paper is organized as follows. In Section 2 we introduce basic concepts and notation, and we provide background knowledge on MSO logics of ω-layered structures. In Section **??** we compare the expressiveness of FO and MSO logics interpreted over the discrete linear order and the UULS. In Section **??** we show that the satisfiability problems for MSO logic over $\langle \mathbb{N}, < \rangle$ and for FO logic over the 2-refinable UULS expanded with the predicate $Path$ are inter-reducible. Finally, in Section **??** we prove that the satisfiability problems for MSO logic over $\langle \mathbb{N}, < \rangle$ and for the chain fragment of MSO logic interpreted over the UULS are inter-reducible.

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1. In [FRA 06], the authors also provide a succinct account of existing results about definability and decidability problems for k-ary trees expanded with equi-level and equi-column predicates.

2. MSO logics of ω-layered structures

In this section we introduce classical monadic logics and interpret them over layered structures for time granularity.

DEFINITION 1. — *(*The language of monadic second-order logic*) Let $\tau = c_1, \ldots, c_r, u_1, \ldots, u_s, b_1, \ldots, b_t$ be a finite alphabet of relational symbols, where $c_1, \ldots, c_r$ (resp., $u_1, \ldots, u_s$, $b_1, \ldots, b_t$) are constant symbols (resp., unary relational symbols, binary relational symbols), and let $\mathcal{P}$ be an alphabet of (uninterpreted) unary relational symbols. The language* $\mathrm{MSO}[\tau \cup \mathcal{P}]$ *of monadic-second order logic over τ and $\mathcal{P}$ is defined as follows:*

– atomic formulas *are of the forms $x = y$, $x = c_i$, with $1 \leq i \leq r$, $u_i(x)$, with $1 \leq i \leq s$, $b_i(x, y)$, with $1 \leq i \leq t$, $x \in X$, and $P(x)$, where x, y are individual variables, X is a set variable, and $P \in \mathcal{P}$;*

– formulas *are built up from atomic formulas by means of the Boolean connectives $\neg$ and $\wedge$, and the quantifier $\exists$ ranging over both individual and set variables.*

In the following, we shall write $\mathrm{MSO}_{\mathcal{P}}[\tau]$ for $\mathrm{MSO}[\tau \cup \mathcal{P}]$ and we shall write $\mathrm{MSO}[\tau]$ when $\mathcal{P}$ is meant to be the empty set. The symbols belonging to the signature τ are interpreted over a suitable relational structure, such as, for instance, the set $\mathbb{N}$ of natural numbers or an infinite tree, in the obvious way. Details can be found in [THO 97].

The satisfiability problem for $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) with respect to a given relational structure is the problem of establishing, for any given $\mathrm{MSO}[\tau]$-formula (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$-formula) $\phi(x_1, \ldots, x_m, X_1, \ldots, X_n)$, whether there exists a valuation of free variables in $\{x_1, \ldots, x_m, X_1, \ldots, X_n\}$ (resp., of free variables in $\{x_1, \ldots, x_m, X_1, \ldots, X_n\}$ and symbols in $\mathcal{P}$) that satisfies ϕ. The $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) theory of a given relational structure (resp., $\mathcal{P}$-labeled relational structure) is the set of all and only the $\mathrm{MSO}[\tau]$-sentences (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$-sentences) that hold in the structure (resp., $\mathcal{P}$-labeled structure). The decision problem for the $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) theory of a given structure (resp., $\mathcal{P}$-labeled structure) can be easily reduced to the satisfiability problem for $\mathrm{MSO}[\tau]$ (resp., $\mathrm{MSO}_{\mathcal{P}}[\tau]$) over such a structure (resp., $\mathcal{P}$-labeled structure). For this reason, hereafter we shall concentrate our attention on the latter problem.

In the following, we shall also take into consideration the first-order fragment $\mathrm{FO}[\tau]$ of $\mathrm{MSO}[\tau]$, over ω-layered structures, as well as its path (resp., chain) fragment $\mathrm{MPL}[\tau]$ (resp., $\mathrm{MCL}[\tau]$), which is obtained by constraining set variables to be evaluated over paths (resp., chains), together with their $\mathcal{P}$-variants $\mathrm{FO}_{\mathcal{P}}[\tau]$ and $\mathrm{MPL}_{\mathcal{P}}[\tau]$ (resp., $\mathrm{MCL}_{\mathcal{P}}[\tau]$)[2]. It is worth pointing out that, while free set variables in the path (resp., chain) fragments are evaluated over the set of paths (resp., chains), there are no constraints on the valuation of symbols $\mathcal{P}$ in the first-order, path, and chain fragments. As a consequence, we have that the satisfiability problem for $\mathrm{FO}_{\mathcal{P}}[\tau]$, $\mathrm{MPL}_{\mathcal{P}}[\tau]$, and $\mathrm{MCL}_{\mathcal{P}}[\tau]$ is more difficult than that for $\mathrm{FO}[\tau]$, $\mathrm{MPL}[\tau]$, and $\mathrm{MCL}[\tau]$.

2. The definitions of path and chain differ from one ω-layered structure to the other and they will be formally stated below.

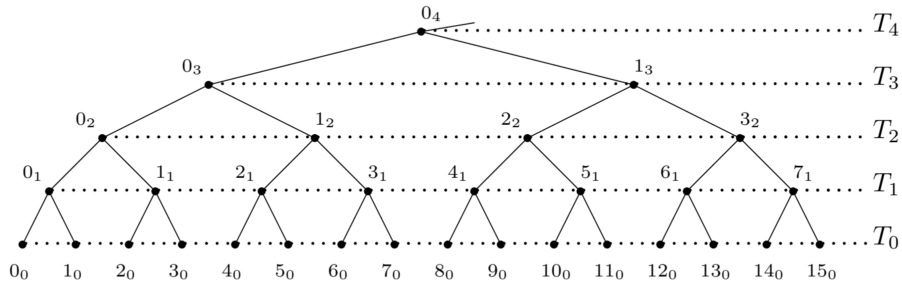

Figure 1. *The* 2*-refinable upward unbounded layered structure.*

To compare the various logics, we take advantage of a suitable notion of *reducibility*. We say that (the satisfiability problem for) a logic $\mathcal{L}$ is *reducible* to (the satisfiability problem for) a logic $\mathcal{L}'$, denoted $\mathcal{L} \to \mathcal{L}'$, if there exists an effective translation of $\mathcal{L}$-formulas into equi-satisfiable $\mathcal{L}'$-formulas (notice that the number and the types of free variables in the former formulas may not coincide with the number and the types of free variables in the latter formulas). Moreover, we say that $\mathcal{L}$ and $\mathcal{L}'$ are *inter-reducible*, denoted $\mathcal{L} \rightleftarrows \mathcal{L}'$, if both $\mathcal{L} \to \mathcal{L}'$ and $\mathcal{L}' \to \mathcal{L}$. It is immediate to see that if $\mathcal{L} \to \mathcal{L}'$ and $\mathcal{L}'$ is decidable (resp., $\mathcal{L}$ is undecidable), then $\mathcal{L}$ is decidable (resp., $\mathcal{L}'$ is undecidable) as well. As a matter of fact, a well-known method to reduce the satisfiability problem for a logic $\mathcal{L}$ to the satisfiability problem for a logic $\mathcal{L}'$ is to define an *interpretation* of $\mathcal{L}'$ into $\mathcal{L}$, namely, to find (i) a mapping ι from elements in the relational structure of $\mathcal{L}'$ to elements in the relational structure of $\mathcal{L}$ and (ii) a mapping τ from atomic $\mathcal{L}'$-formulas to $\mathcal{L}$-formulas with the same free variables in such a way that an $\mathcal{L}'$-formula $\phi(x_1, ..., x_m, X_1, ..., X_n)$ holds with a valuation $(c_1, ..., c_m, b_1, ..., b_n)$ if and only if the corresponding formula $\tau(\phi(x_1, ..., x_m, X_1, ..., X_n))$ holds with a valuation $(\iota(c_1), ..., \iota(c_m), \iota(b_1), ..., \iota(b_n))$ (here the mapping ι is extended in the natural way to sets b_i of elements and the mapping τ is extended to boolean combinations and existential closures of atomic formulas). Notice that if there is an interpretation of a logic $\mathcal{L}'$ into a logic $\mathcal{L}$, then $\mathcal{L}'$ is trivially reducible to $\mathcal{L}$. In such a particular case, we say that $\mathcal{L}$ is *at least as expressive as* $\mathcal{L}'$. If there is also a converse reduction from $\mathcal{L}$ to $\mathcal{L}'$ obtained via interpretation, then we say that $\mathcal{L}$ is *as expressive as* $\mathcal{L}'$. Let α be a unary (resp. binary) relational symbol. We say that α is *definable* in $\mathrm{MSO}[\tau]$ if there is an interpretation of $\mathrm{MSO}[\tau \cup \{\alpha\}]$, in particular, of atomic formulas of the form $\alpha(x)$ (resp. $\alpha(x, y)$), into $\mathrm{MSO}[\tau]$. Clearly, if α is definable in $\mathrm{MSO}[\tau]$, then $\mathrm{MSO}[\tau \cup \{\alpha\}]$ is as expressive as $\mathrm{MSO}[\tau]$. The notion of definability naturally transfers to any fragment of $\mathrm{MSO}[\tau]$.

Upward unbounded layered structures.. Let $\mathcal{U} = \bigcup_{i \geq 0} T_i$. For any $k \geq 2$, the k-refinable *upward unbounded layered structure* (UULS) is a triplet $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, which intuitively represents a complete k-ary infinite tree generated from the leaves (cf. Figure 1). The set $\mathcal{U}$ is the domain of the structure, defined as the union of

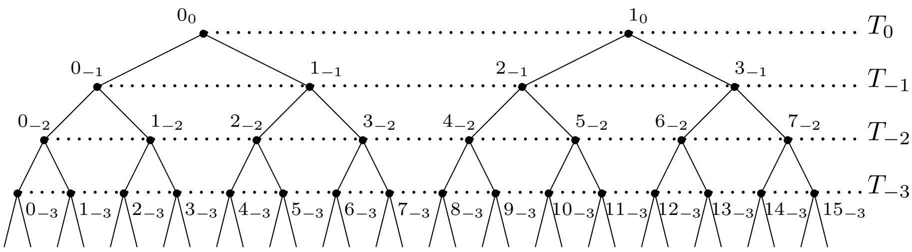

Figure 2. *The* 2*-refinable downward unbounded layered structure.*

all non-negative layers, $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_0) = \perp$ for all $a \in \mathbb{N}$ and $\downarrow_i (a_b) = c_d$ if and only if $b > 0$, $d = b-1$, and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{U}$ given by the *inorder* (left-root-right) visit of the tree shaped structure. A *path* over an UULS is a subset of the domain whose elements can be written as a possibly infinite sequence $x_0, x_1, \ldots$ such that, for every $i > 0$, there exists $0 \leq j < k$ such that $x_{i-1} =\downarrow_j (x_i)$. Notice that every pair of infinite paths over an UULS may differ on a finite prefix only. A *chain* is any subset of a path.

Downward unbounded layered structures.. Let $\mathcal{D} = \bigcup_{i \leq 0} T_i$. For any $k \geq 2$, the k-refinable *downward unbounded layered structure* (DULS) is a triplet $\langle \mathcal{D}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, which can be viewed as an infinite sequence of complete k-ary infinite trees (cf. Figure 2). The set $\mathcal{D}$ is the domain of the structure, defined as the union of all non-positive layers, $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_b) = c_d$ if and only if $d = b-1$ and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{D}$ induced by the natural ordering on the top layer T_0 (i.e. $0_0 < 1_0 < 2_0 < \ldots$) and by the *preorder* (root-left-right) visit of the elements belonging to the same tree. A *path* over a DULS is a subset of the domain $\mathcal{D}$ whose elements can be written as a possibly infinite sequence $x_0, x_{-1}, \ldots$ such that, for every $i \leq 0$, there exists $0 \leq j < k$ such that $x_{i-1} =\downarrow_j (x_i)$. A *chain* is any subset of a path.

Totally unbounded layered structures.. Let $\mathcal{T} = \bigcup_{i \in \mathbb{Z}} T_i$. For any $k \geq 2$, the k-refinable *totally unbounded layered structure* (TULS) is simply the union of the k-refinable DULS and the k-refinable UULS (cf. Figure 3). It can be formally defined as the triplet $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, where $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_b) = c_d$ if and only if $d = b-1$ and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{T}$ given by the *inorder* (left-root-right) visit of the tree shaped structure. A *path* over a TULS is a subset of the domain whose elements can be written as a possibly bi-infinite sequence $\ldots, x_{-1}, x_0, x_1, \ldots$ such that, for every $i \in \mathbb{Z}$, there exists $0 \leq j < k$ such that $x_{i-1} =\downarrow_j (x_i)$. A *chain* is any subset of a path.

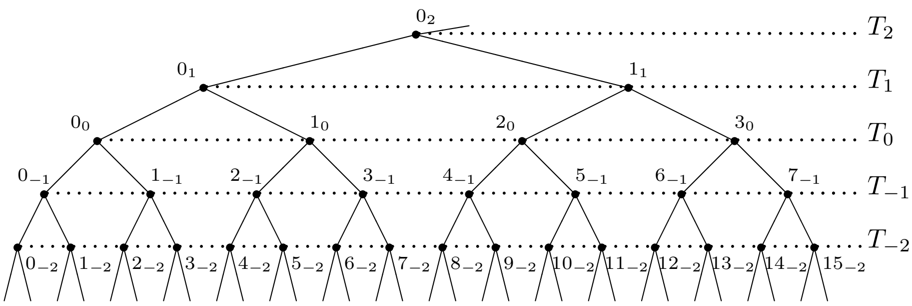

Figure 3. *The* 2*-refinable totally unbounded layered structure.*

A $\mathcal{P}$-labeled UULS (resp. DULS, TULS) is obtained by augmenting, for each predicate in $\mathcal{P}$, a UULS (resp. DULS, TULS) with a set $P \subseteq \mathcal{U}$ (resp. $P \subseteq \mathcal{D}$, $P \subseteq \mathcal{T}$), which represents all elements where the predicate holds.

The theories $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ of UULSs (resp. DULSs, TULSs) are monadic second-order theories with the equality and the binary relational symbols $<, \downarrow_0, \ldots, \downarrow_{k-1}$. Notice that the theories of UULSs and TULSs are equivalent to their corresponding theories devoid of the ordering relation $<$ (this because $<$ can be defined both in the UULSs and in the TULSs by suitable $\mathrm{MSO}[(\downarrow_i)_{i=0}^{k-1}]$-formulas which use the projection functions only). On the other hand, the theories of DULSs devoid of the ordering relation $<$ are strictly less expressive than the corresponding plain theories, because the ordering on the top layer T_0 can not be defined on the grounds of the projection functions only. Moreover, the theories of DULSs and UULSs are embeddable into the theories of TULSs expanded with the unary predicate T_0.

The decidability of the theories of DULSs, UULSs, and TULSs (possibly extended with the predicate T_0) has been proved by reducing each the underlying relational structures to suitable 'collapsed' structures. In particular, the theory of the k-refinable DULS is embeddable into the monadic second-order theory of the infinite complete k-ary tree, and the theory of the k-refinable UULS is embeddable into the monadic second-order theory of the k-ary systolic tree [MON 99, MON 00a, MON 02, MON 04].

THEOREM 2. — *The satisfiability problem for* $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over the* k*-refinable DULS, UULS, and TULS is (nonelementarily) decidable.*

Figure 4 summarizes the relationships between the considered logics induced by reducibility (an arrow from $\mathcal{L}$ to $\mathcal{L}'$ stands for $\mathcal{L} \rightarrow \mathcal{L}'$). From Theorem 2 it follows that all the satisfiability problem for all logics in Figure 4, when interpreted over UULSs, DULSs, and TULSs, are decidable.

Figure 4. *The hierarchy of logics over layered structures.*

3. On the expressiveness of logics of linear and layered structures

In this section, we discuss reducibility relationships between logics over (discrete) linear temporal structures and logics over upward unbounded layered ones. Here, we focus on reductions obtained via interpretations.

In [**?**], Montanari et al. describe in detail how the basic temporal operators for time granularity, namely, the displacement, contextualization, and projection operators, can be defined in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$. As an example, we report the definition of the unary predicate Δ_0 holding at the origin of each layer. The predicate Δ_0 is interpreted as the set of all and only the elements belonging to the leftmost branch of an upward unbounded layered structure, which is defined as the least set containing the element 0_0 and all its ancestors 0_1, 0_2, ... (cf. Figure **??**). This predicate can be defined in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ as follows. Given a second-order formula $\phi(X)$, with a free set variable X, let $\mu(\phi(X))(x)$ be the following second-order formula, with a free individual variable x:

$$\exists X \ (x \in X \ \wedge \ \phi(X) \ \wedge \ \forall Y \ (\phi(Y) \rightarrow \forall y (y \in X \rightarrow y \in Y))).$$

$\mu(\phi(X))(x)$ evaluates to true if and only if the valuation for x belongs to the smallest valuation for X for which $\phi(X)$ holds true. Using the operator μ, $\Delta_0(x)$ can be expressed as follows:

$$\mu \, (0_0 \in X \ \wedge \ \forall y, z \ ((z \in X \wedge \ \downarrow_0 (y) = z) \rightarrow y \in X)) \, (x),$$

where $0_0 \in X$ is a shorthand for $\exists y \ (y \in X \ \wedge \ \forall z \ (y \leq z))$. It is easy to verify that such a formula captures the smallest valuation for X which contains 0_0 and it is closed parent-wise. This shows that Δ_0 is definable in $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$.

Now, we show that $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ is reducible (via interpretation) to $\mathrm{MSO}[<, flip]$, which is a proper extension of $\mathrm{MSO}[<]$ with the binary relation symbol $flip$ [MON 99, MON 00b, THO 03]. The language of $\mathrm{MSO}[<, flip]$ is defined in the standard way. The domain of the underlying relational structure is the set of natural numbers $\mathbb{N}$. The relational symbol $<$ is interpreted as the usual ordering over $\mathbb{N}$, while the relational symbol $flip$ is interpreted as a unary function, which, for any natural

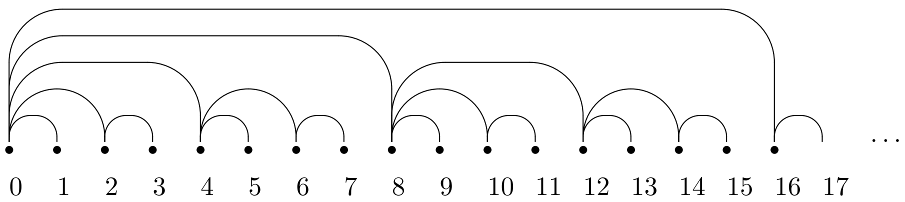

Figure 5. *The structure of the function $flip$.*

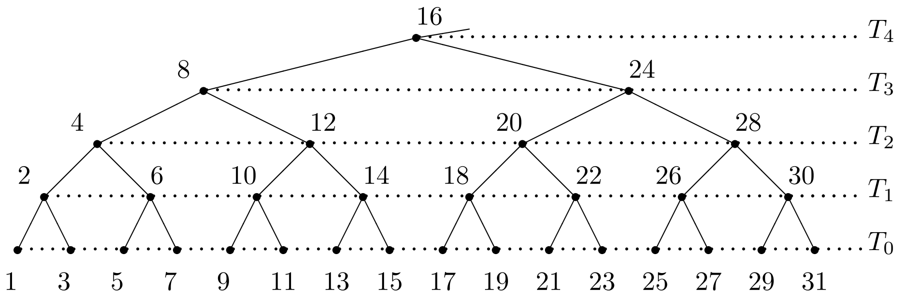

Figure 6. *The concrete 2-refinable upward unbounded layered structure.*

number $x > 0$, returns the natural number $x - x'$, where x' is the least power of 2, with a non-null coefficient, that occurs in the binary representation of x.

DEFINITION 3 (THE FUNCTION $flip$). — *The function $flip : \mathbb{N}^+ \to \mathbb{N}$ is such that, for all $x \in \mathbb{N}^+$,*

$$flip(x) = y \quad \textit{iff} \quad x = \sum_{j=0}^{n} 2^{i_j}, \textit{ with } i_n > i_{n-1} > \ldots > i_0 \geq 0, \textit{ and } y = x - 2^{i_0}.$$

The function $flip$ is not defined for $x = 0$; however, totality can be recovered by extending it with $flip(0) = 0$. (Notice that $flip(x) < x$, for all $x \in \mathbb{N}^+$, and $flip(x) \leq x$, for all $x \in \mathbb{N}$. Later, we will often use these properties of $flip$ to simplify definitions.) Furthermore, it is useful to add a maximum element ∞ to $\mathbb{N}$, with $flip(\infty) = 0$. A graphical representation of the function $flip$ is given in Figure 5.

An interpretation of $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$ into $\mathrm{MSO}[<, flip]$ can be defined as follows [**?**]. First, it is possible to rename each node a_b of the 2-refinable upward unbounded layered structure by a positive natural number $\iota(a_b) = 2^b + a2^{b+1}$. The resulting structure is called *concrete* 2-refinable upward unbounded layered structure and it can be viewed as the (discrete) linear order $\langle \mathbb{N}^+, < \rangle$ expanded with two functions $\downarrow_0$ and $\downarrow_1$ such that, for every $x = 2^b + a2^{b+1} \in \mathbb{N}^+$, $\downarrow_0 (x) = x - 2^{b-1}$ and $\downarrow_1 (x) = x + 2^{b-1}$. A fragment of this concrete structure is depicted in Figure 6. Notice that all odd numbers are associated with layer T_0, while even numbers are distributed over the remaining layers. Notice also that the labeling of the concrete structure does not include the number 0. In the following, we will see that it is convenient to consider 0 as the image of the first node of an imaginary additional finest layer, whose remaining nodes have no corresponding number in $\mathbb{N}$ (notice that the node corresponding to 0 turns out to be the left son of the node corresponding to 1). It is worth to remark that the addition/removal of a (definable) node in a structure preserves the expressiveness of the corresponding logic. For such a reason, in the following, we do not focus on the encoding of the element 0.

The binary relations $\downarrow_0$ and $\downarrow_1$ of the concrete 2-refinable UULS cannot be defined neither in FO$[<]$ nor in MSO$[<]$ (this can be easily seen by considering, for instance, the relation $\downarrow_0$ restricted to the elements of the leftmost branch, which coincides with the relation $\{(2x, x) : x \in \mathbb{N}^+\}$). This implies that MSO$[<, \downarrow_0, \downarrow_1]$ (resp. FO$[<, \downarrow_0, \downarrow_1]$) is strictly more expressive than MSO$[<]$ (resp. FO$[<]$). However, both relations $\downarrow_0$ and $\downarrow_1$ can be defined in terms of the function $flip$, as shown below. For every even natural number x (the relations $\downarrow_0$ and $\downarrow_1$ are not defined on odd natural numbers), we have:

$$\downarrow_0 (x) = max\{y : y < x, flip(y) = flip(x)\}, \text{ and} \tag{1}$$

$$\downarrow_1 (x) = max\{y : flip(y) = x\}. \tag{2}$$

Such a correspondence can be translated into suitable first-order logical formulas, thus implying that both relations $\downarrow_0$ and $\downarrow_1$ are definable in FO$[<, flip]$.

THEOREM 4. — *MSO$[<, flip]$ (resp. FO$[<, flip]$) is at least as expressive as MSO$[<, \downarrow_0, \downarrow_1]$ (resp. FO$[<, \downarrow_0, \downarrow_1]$) over the (concrete) UULS.*

PROOF 5. — In order to prove the claim it suffices to provide suitable FO$[<, flip]$-formulas $\tau(\downarrow_0 (x, y))$ and $\tau(\downarrow_1 (x, y))$ for the interpretation of the two atomic formulas $\downarrow_0 (x, y)$ and $\downarrow_1 (x, y)$, belonging to both the logical languages MSO$[<, flip]$ and FO$[<, flip]$. Such formulas are defined as follows:

$$\begin{array}{rcl} \tau(\downarrow_0 (x, y)) & := & y < x \ \wedge \ flip(y) = flip(x) \ \wedge \\ & & \forall z \ ((z < x \ \wedge \ flip(z) = flip(x)) \ \rightarrow \ (z = y \ \vee \ z < y)); \\ \tau(\downarrow_1 (x, y)) & := & flip(y) = flip(x) \ \wedge \\ & & \forall z \ ((flip(z) = flip(x)) \ \rightarrow \ (z = y \ \vee \ z < y)). \end{array}$$

■

In [MON 00b], show that the satisfiability problem for MSO$[<, flip]$ is (non-elementarily) decidable. From such a result and from Theorem 4 it immediately follows that the satisfiability problem for MSO$[<, \downarrow_0, \downarrow_1]$ is decidable. The same result holds among the first-order fragments of the considered logics.

We now want to give an interpretation of $\mathrm{MSO}[<, flip]$ (resp. $\mathrm{FO}[<, flip]$) into a suitable monadic second-order (resp. first-order) logical language over the 2-refinable (concrete) UULS. In order to do that we denote by $\downarrow_0^*$ the reflexive and transitive closure of $\downarrow_0$ and we define the function $flip$ in terms of $\downarrow_0^*$ and $\downarrow_1$:

$$flip(x) = y \quad \text{iff} \quad (\downarrow_1(y), x) \in \downarrow_0^* \;\text{ or }\; (y = 0 \,\wedge\, (x, 0) \in \downarrow_0^*). \tag{3}$$

From such a correspondence it follows that the relation $flip$ is definable in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$.

THEOREM 6. — $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$ *(resp.* $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$*) over the (concrete) UULS is at least as expressive as* $\mathrm{MSO}[<, flip]$ *(resp.* $\mathrm{FO}[<, flip]$*).*

PROOF 7. — We simply need to translate the atomic $\mathrm{FO}[<, flip]$-formula $flip(x, y)$ (holding true if and only if $y = flip(x)$) into an equivalent $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$-formula $\sigma(flip(x, y))$. This can be done by defining the mapping σ as follows:

$$\begin{aligned} \sigma(flip(x, y)) \;&:=\; (\exists z\, (\downarrow_1(y, z) \wedge \downarrow_0^*(z, x))) \;\vee \\ &\;\; (\forall z\, (y = z \,\vee\, y < z) \wedge \downarrow_0^*(x, y)). \end{aligned}$$

■

As a concluding remark, it is worth to notice that the projection function $\downarrow_0$ can be defined in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$:

$$\downarrow_0(x) = y \quad \text{iff} \quad \downarrow_0^*(x, y) \,\wedge\, \neg\exists z\, (y < z \wedge \downarrow_0^*(x, z)).$$

This allows us to conclude that $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ (resp. $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$) is at least as expressive as $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$ (resp. $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$). In fact, one can easily show that $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$ is less expressive than $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ (for instance, the unary predicate $\{2^n : n \in \mathbb{N}\}$ is definable in $\mathrm{FO}[<, \downarrow_0^*, \downarrow_1]$ but not in $\mathrm{FO}[<, \downarrow_0, \downarrow_1]$). On the other hand, since the reflexive and transitive closure of a binary predicate is always in monadic second-order logic, we have that $\mathrm{MSO}[<, \downarrow_0^*, \downarrow_1]$ is as expressive as $\mathrm{MSO}[<, \downarrow_0, \downarrow_1]$.

The relationships among the various first-order and monadic second-order logics over linear and layered structures are summarized in Figure 7, where a bold arrow from a logic $\mathcal{L}$ to another logic $\mathcal{L}'$ means that $\mathcal{L}'$ is at least as expressive as $\mathcal{L}$ (in other words, bold arrows represent reducibility relations obtained via logical interpretations).

4. An alternative characterization of $\mathrm{MSO}[<]$

In the following, we provide a characterization of $\mathrm{MSO}[<]$ in terms of the first-order logic over the expanded 2-refinable UULS $(\mathcal{U}, <, \downarrow_0, \downarrow_1, Path^<, D_0)$. The predicate $Path^<$ subsumes both the equi-level predicate T and the ancestor predicate $\downarrow^\star$, while D_0 holds at all and only the elements belonging to the leftmost branch

Figure 7. *The expressiveness of logics over linear and layered structures.*

of the tree. From the point of view of expressiveness, such a result defines the precise relationship that holds between the logic MSO$[<]$ over the flat structure of natural numbers and the logic MSO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ over the UULS, showing the rather surprising fact that the satisfiability problem for MSO$[<]$ is reducible to the satisfiability problem for a suitable first-order logic over the UULS. As a matter of fact, the effective translation from FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ to MSO$[<]$ has a nontrivial practical impact, since it allows one to map verification problems for UULSs to verification problems for MSO$[<]$, thus making it possible to exploit the wide spectrum of techniques available for that logic.

We start by defining the predicate $Path^<$. The predicate $Path^<(x, y, z, w)$ over the UULS holds true if and only if $T(x, z)$ (x and z belong to the same layer), $T(y, w)$ (y and w belong to the same layer), and there exist two finite downward paths, one from x to y and the other from z to w, such that, for each right projection in the path from x to y, there exists a corresponding right projection in the path from z to w. More formally, we require that

– $T(x, z)$ and $T(y, w)$;

– there are two paths $c_0, \ldots, c_n$ and $b_0, \ldots, b_n$ such that $x = c_0$, $y = c_n$, $z = b_0$, $w = b_n$, $c_{i+1} = \downarrow_{i_c} (c_i)$, and $b_{i+1} = \downarrow_{i_b} (b_i)$, with $i_b, i_c \in \{0, 1\}$ for $0 \leq i \leq n-1$;

– $\downarrow_{i_c} = \downarrow_1$ implies $\downarrow_{i_b} = \downarrow_1$, for all $0 \leq i \leq n-1$.

It is immediate to see that the predicate $Path^<(x, y, z, w)$ subsumes the equi-level predicate $T(x, y)$, since $T(x, y)$ is equivalent to $Path^<(x, x, y, y)$. Moreover it also subsumes the ancestor predicate $\downarrow^\star (x, y)$. By definition, $\downarrow^\star (x, y)$ holds true if and only if either x is equal to y or x is an ancestor of y, that is, there exists a finite path $c_0, \ldots, c_n$ such that $c_0 = x$, $c_n = y$, and $c_{i+1} = \downarrow_i (c_i)$, for $0 \leq i \leq n-1$, and thus $\downarrow^\star (x, y)$ is equivalent to $Path^<(x, y, x, y)$ (in the following we will often use $\downarrow^\star (x, y)$ as a shorthand for $Path^<(x, y, x, y)$).

THEOREM 8. — FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ *over the* 2*-refinable UULS is inter-reducible to* MSO$[<]$.

PROOF 9. — We first prove that the logic MSO$[<]$ can be reduced to the logic FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$. As a first step, we replace MSO$[<]$ with *Weak* MSO$[<]$

(WMSO$[<]$), where second-order quantification refers to finite sets only. As shown by McNaughton [THO 90a], MSO$[<]$ and WMSO$[<]$ have the same expressive power, and thus such a replacement is legitimate. Moreover, we replace WMSO$[<]$ with the simpler, but equivalent, formalism WMSO$_0[<]$ where only second-order variables occur and the atomic formulas are of the forms $X \subseteq Y$ (X is a subset of Y) and $Succ(X, Y)$ (X and Y are the singletons $\{x\}$ and $\{y\}$, respectively, and $y = x + 1$).

The reduction is based on a suitable coding of (finite) sets of natural numbers into elements of the concrete 2-refinable UULS. More precisely, any second-order variable X of WMSO$_0[<]$ is replaced by a first-order variable $\overline{x}$ of FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ and any interpretation $\nu(X)$ of X is mapped into an interpretation $\mu_\nu(\overline{x})$ of $\overline{x}$ as follows:

– if $\nu(X) = \emptyset$, then $\mu_\nu(\overline{x})$ is the origin (i.e. the lower left element) of the UULS;

– if $\nu(X) = \{n_0, n_1, \ldots, n_s\}$, then $\mu_\nu(\overline{x})$ is the element a_b of the UULS such that $2^b + a2^{b+1} = 2^{n_s} + \ldots + 2^{n_1} + 2^{n_0}$.

It is worth pointing out that in the logic WMSO$_0[<]$ the interpretation $\nu(X)$ of a second-order variable X is finite, and thus rule 2 is effective. An intuitive account of the mapping ν can be given in terms of the concrete 2-refinable UULS depicted in Figure 6: the set $\nu(X)$ is the set of positions of the non-zero coefficients of the binary representation of $\mu_\nu(\overline{x})$.

Later in the proof, we will take advantage from the following interpretation of the set $\nu(X)$ as a path over the concrete UULS. First notice that, since in WMSO$_0[<]$ the interpretation of set variables is finite, $\nu(X)$ has not only a least element $min(\nu(X))$, but also a greatest element $max(\nu(X))$. We associate $\nu(X)$ with the path from the origin of the layer $T_{max(\nu(X))}$ to the element $\mu_\nu(\overline{x})$, belonging to the layer $T_{min(\nu(X))}$. Such a path provides an encoding of the elements of $\nu(X)$ as follows: $max(\nu(X))$ (i.e. the index of the layer of the first element in the path) and $min(\nu(X))$ (i.e. the index of the layer of the last element in the path) belong to $\nu(X)$; moreover, if the element a_b of the UULS, with $min(\nu(X)) < b \leq max(\nu(X))$, belongs to the path, then $\downarrow_1 (a_{b-1})$ belongs to the path if (and only if) $b-1 \in \nu(X)$ and $\downarrow_0 (a_{b-1})$ belongs to the path if (and only if) $b - 1 \notin \nu(X)$.

On the ground of the above defined correspondence, we can translate every of WMSO$_0[<]$-formula ϕ into an FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$-formula $\tau(\phi)$, where the mapping τ is inductively defined as follows:

$$
\begin{array}{rcl}
\tau(Succ(X,Y)) & := & D_0(\overline{x}) \ \wedge \ D_0(\overline{y}) \wedge \ \downarrow_0 (\overline{y}) = \overline{x}; \\
\tau(X \subseteq Y) & := & (\overline{x} = \overline{y}) \vee \\
 & & (\overline{x} < \overline{y} \ \wedge \ \exists z, w \ (D_0(z) \wedge (Path^<(z, \overline{x}, w, \overline{y}) \vee \\
 & & \ \exists h, k \ (Path^<(z, \overline{x}, w, k) \wedge h = \downarrow_1 (k) \wedge \downarrow^\star (h, \overline{y}))))); \\
\tau(\phi \ \wedge \ \psi) & := & \tau(\phi) \ \wedge \ \tau(\psi); \\
\tau(\neg\phi) & := & \neg\tau(\phi); \\
\tau(\exists X \ \phi) & := & \exists \overline{x} \ \tau(\phi).
\end{array}
$$

The rules for atomic formulas can be explained by taking into account the relationship that holds between interpretations of set variables in WMSO$_0[<]$ and interpretations of the corresponding individual variables in FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ as well as the structure of the concrete 2-refinable UULS. As for the formula $Succ(X, Y)$, it suffices to notice that singletons are mapped into elements which are powers of two, and thus belong to the leftmost branch of the concrete UULS, and that the successor relation can be directly captured by the left projection. The translation of the formula $X \subseteq Y$ is more involved. The case in which $X = Y$ is trivial, and thus we concentrate our attention on the case $X \subset Y$. As anticipated, we take advantage from the interpretation of $\nu(X)$ and $\nu(Y)$ as paths over the concrete UULS. In order to guarantee that $X \subset Y$ we have to check that at each layer T_i, with $min(\nu(X)) < i \leq max(\nu(X))$, if the path associated with $\nu(X)$ follows a right projection, then the path associated with $\nu(Y)$ must follow a right projection as well (notice that in general the path associated with $\nu(Y)$ may be longer than the one associated with $\nu(X)$). This can be ensured by exploiting predicate $Path^<$.

From the given translation of WMSO$_0[<]$ into FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ and from the correspondence between the interpretation of set variables in WMSO$_0[<]$ and the interpretation of the corresponding individual variables in FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$, it is not difficult to show that a WMSO$_0[<]$-formula ϕ is satisfiable, with an interpretation ν, if and only if $\tau(\phi)$ is satisfiable, with the interpretation μ_ν.

Let us consider now the opposite reduction, namely, from the logic FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ to the logic MSO$[<]$. Also in this case we assume to have an injective mapping each first order variable x of FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ to a second order variable $\overline{X}$ of MSO$[<]$. The encoding of the elements of the UULS into natural numbers is exactly the reverse of the encoding considered in the opposite case, which induces, for an interpretation μ over the UULS, an interpretation ν_μ over the natural numbers. A formula ϕ of FO$[<, \downarrow_0, \downarrow_1, Path^<, D_0]$ is translated into a formula $\sigma(\phi)$ of FO$[<]$ by a function σ inductively defined as follows:

$$
\begin{array}{rcl}
\sigma(x = y) & := & \overline{X} = \overline{Y}; \\
\sigma(D_0(x)) & := & \exists z\ (z \in \overline{X}\ \wedge\ \forall h\ (h \in \overline{X} \to h = z)); \\
\sigma(x < y) & := & \exists z\ (z \in \overline{Y}\ \wedge\ z \notin \overline{X} \wedge \\
& & \qquad \forall w\ ((w \in \overline{X} \wedge w \notin \overline{Y}) \to w < z)); \\
\sigma(Path^<(x, y, z, w)) & := & \neg \exists z\ (z \in \overline{X} \vee z \in \overline{Y}\ \vee z \in \overline{Z} \vee z \in \overline{W}) \vee \\
& & \exists h, k\ (h \in \overline{X}\ \wedge\ h \in \overline{Z}\ \wedge k \in \overline{Y}\ \wedge\ k \in \overline{W} \wedge \\
& & \qquad \forall v\ (v \in \overline{X} \to h \leq v \wedge v \in \overline{Z} \to h \leq v) \wedge \\
& & \qquad \forall v\ (v \in \overline{Y} \to k \leq v \wedge v \in \overline{W} \to k \leq v) \wedge \\
& & \qquad \forall v\ ((v \in \overline{X} \wedge v > h) \to v \in \overline{Y} \wedge \\
& & \qquad\qquad (v \in \overline{Y} \wedge v > h) \to v \in \overline{X}) \wedge \\
& & \qquad \forall v\ ((v \in \overline{Z} \wedge v > k) \to v \in \overline{W} \wedge \\
& & \qquad\qquad (v \in \overline{W} \wedge v > k) \to v \in \overline{Z}) \wedge \\
& & \qquad \forall v\ ((v < h \wedge k \leq v \wedge v \in \overline{Y}) \to v \in \overline{W}));
\end{array}
$$

$$
\begin{array}{rcl}
\sigma(\downarrow_0 (x) = y) & := & \exists z\ (z \in \overline{X}\ \wedge\ z \notin \overline{Y} \wedge \\
 & & \quad \forall w\ (w \in \overline{X} \rightarrow z \leq w) \wedge \\
 & & \quad \forall w\ ((w \in \overline{X} \wedge z < w) \rightarrow w \in \overline{Y} \wedge \\
 & & \qquad (w \in \overline{Y} \wedge z \leq w) \rightarrow w \in \overline{X} \wedge \\
 & & \qquad (w \in \overline{Y} \wedge w < z) \rightarrow z = w + 1) \wedge \\
 & & \quad \exists w\ (w \in \overline{Y} \wedge w < z)); \\
\sigma(\downarrow_1 (x) = y) & := & \exists z\ (z \in \overline{X} \wedge \\
 & & \quad \forall w\ (w \in \overline{X} \rightarrow z \leq w) \wedge \\
 & & \quad \forall w\ ((w \in \overline{X} \wedge z \leq w) \rightarrow w \in \overline{Y} \wedge \\
 & & \qquad (w \in \overline{Y} \wedge z \leq w) \rightarrow w \in \overline{X} \wedge \\
 & & \qquad (w \in \overline{Y} \wedge w < z) \rightarrow z = w + 1) \wedge \\
 & & \quad \exists w\ (w \in \overline{Y} \wedge w < z)); \\
\sigma(\phi \wedge \psi) & := & \sigma(\phi)\ \wedge\ \sigma(\psi); \\
\sigma(\neg \phi) & := & \neg \sigma(\phi); \\
\sigma(\exists x\ \phi) & := & \exists \overline{X}\ \sigma(\phi).
\end{array}
$$

The translation σ can be better understood by considering the concrete UULS structure over natural numbers. The predicate $D_0(x)$ holds if x is interpreted over a power of two, namely if $\mu_\nu(\overline{X})$ is a singleton.

As for the predicate $Path^<(x, y, z, w)$, we have that the elements x and z (resp., y and w) belong to the same level if their corresponding sets $\overline{X}$ and $\overline{Z}$ (resp. $\overline{Y}$ and $\overline{W}$) have the same least element. Moreover, the predicate $\downarrow^\star (x, y)$ holds if the path from the leftmost branch to x, described by the set $\overline{X}$, is a prefix of the path from the leftmost branch to y, described by $\overline{Y}$.
The translation of the projections $\downarrow_0$ and $\downarrow_1$ exploits the fact that, iff $x = 2^{k_n} + 2^{k_{n-1}} + \ldots + 2^{k_0}$, with $k_n > k_{n-1} > \ldots > k_0 > 0$, then $\downarrow_0 (x) = y$ if $y = x - 2^{k_0} + 2^{k_0 - 1}$ and $\downarrow_1 (x) = y$ if $y = x + 2^{k_0} + 2^{k_0 - 1}$. ■

5. Another characterization of MSO$[<]$

In this section, we provide another characterization of MSO$[<]$ in terms of the chain fragments of monadic second-order logics interpreted over TULSs expanded with the unary predicate T_0 and with either the *equi-level* predicate T or the *equi-column* predicate D. The binary equi-level predicate T allows one to check whether two given elements of a layered structure belong to the same layer, while the equi-column predicate D allows one to check whether two given elements are at the same distance from the origin of the layer they belong to. In the case of TULSs, we can formally define the predicate T and D as follows:

$$
\begin{array}{rcl}
T & := & \{(a_b, c_b) : a \in \mathbb{N}, c \in \mathbb{N}, b \in \mathbb{Z}\}; \\
D & := & \{(a_b, a_d) : a \in \mathbb{N}, b \in \mathbb{Z}, d \in \mathbb{Z}\}.
\end{array}
$$

The results presented in this section imply that the satisfiability problem for the chain fragments of monadic second-order logics interpreted over the structures $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, T\rangle$ and $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, D\rangle$ is decidable. Such results are taken from [tes] and partly based on a proof method introduced by Thomas in [THO 90b], which allows one to reduce the chain fragment of a monadic second-order logic over a tree-shaped strcuture to a monadic second-order logic over a discrete linear structure. As usual, we consider, for simplicity, 2-refinable layered structures.

THEOREM 10. — MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ *over the TULS is inter-reducible to* MSO$[<]$.

PROOF 11. — We show that the logic MSO$[<]$ over the natural numbers is reducible to the logic MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ over the TULS. We first replace MSO$[<]$ with the equivalent framework MSO$_0[<]$ where only second-order variables occur and the atomic subformulas are of the forms $X \subseteq Y$ and $Succ(X, Y)$. We denote by D_0 the unary predicate that consists of all and only the elements belonging to the left-most upward branch of the 2-refinable TULS. Such a predicate can be easily defined by a formula in the chain fragment of monadic second-order logic over the TULS. Thus, we can translate a given MSO$_0[<]$-formula ϕ into an equi-satisfiable MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula $\tau(\phi)$ by constraining each second-order variable to be interpreted with elements from D_0. Formally, the mapping τ is inductively defined as follows:

$$\begin{array}{rcl}
\tau(X \subseteq Y) & := & Y \subseteq D_0 \wedge X \subseteq Y; \\
\tau(Succ(X, Y)) & := & X \subseteq D_0 \wedge \forall x, y(x \in X \wedge y \in Y \rightarrow x =\downarrow_0 (y)); \\
\tau(\phi \wedge \psi) & := & \tau(\phi) \wedge \tau(\psi); \\
\tau(\neg\phi) & := & \neg\tau(\phi); \\
\tau(\exists X \phi) & := & \exists X \ \tau(\phi).
\end{array}$$

Since D_0 is definable in MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$, we have that MSO$_0[<]$ (and hence MSO$[<]$) is reducible to MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$.

As for the converse result, we have to translate a given MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula into an equi-satisfiable MSO$[<]$-formula. We define such a translation in two steps. We first translate an MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula into an equi-satisfiable monadic second-order formula over the bi-infinite linear structure $\langle \mathbb{Z}, < \rangle$. Then, we exploit standard constructions in logic to map the latter formula to an equi-satisfiable MSO$[<]$-formula over $\langle \mathbb{N}, < \rangle$. As for the first step, we encode chain variables with suitable pairs of second-order variables and then we give rules to rewrite atomic formulas. Notice that the ordering $<$ of the TULS can be easily defined by a formula in the chain fragment of its monadic second-order logic. Moreover, we can assume, without loss of generality, that second-order variables of MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formulas are interpreted by *non-empty* chains. Therefore, we can restrict ourselves to the equivalent setup of MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ where variables are instanciated by non-empty chains and atomic formulas are of the forms $X \subseteq Y$ (chain X is included in chain Y), $\downarrow_i (X, Y)$ (X and Y are singletons $\{x\}$ and $\{y\}$, respectively, and $y =\downarrow_i (x)$), $T_0(X)$ (X is a singleton $\{x\}$, with $x \in T_0$), and $T(X, Y)$ (X and Y are singletons $\{x\}$ and $\{y\}$, respectively, and x and y belong to the same layer).

As a preliminary remark, notice that, for every non-empty chain C over the TULS and for every $b \in \mathbb{Z}$, there is at most one $a \in \mathbb{N}$ such that a_b is an element of C. Now, we explain how one can encode a generic non-empty chain C with two subsets Z_C and W_C of $\mathbb{Z}$. We say that $P \subseteq \mathcal{T}$ is a *cover* of a non-empty chain C if P is a maximal path including C, namely, if $C \subseteq P$ and for every $b \in \mathbb{Z}$, there is exactly one $a \in \mathbb{N}$ such that $a_b \in P$. We denote by P_C the *leftmost* cover of C, that is, the (unique) cover P_C such that, whenever b is the least integer for which there is $a \in \mathbb{N}$ satisfying $a_b \in C$, then every descendant of a_b along P_C is of the form c_d, with $c = 2^{b-d}a$ and $d \leq b$. Then, we define Z_C and W_C in such a way that, for every $b \in \mathbb{Z}$,

– $b \in Z_C$ iff there is a (unique) *odd* index $a \in \mathbb{N}$ such that $a_b \in P_C$ (namely, a_b is a $\downarrow_1$-successor in the path P_C);

– $b \in W_C$ iff there is a (unique) index $a \in \mathbb{N}$ such that $a_b \in C$ (namely, C intersects the layer T_b).

Intuitively, Z_C represents those layers which are reached by right-hand side projections along the path P_C, while W_C selects only those layers which intersect the chain C. Notice that the encoding (Z_C, W_C) determines in an unambiguous way the non-empty chain C. Moreover, we can map the above construction in the logic. Precisely, for each chain variable X, we introduce two set variables Z_X and W_X (to be instantiated by sets of integers) and we translate a given MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-formula ϕ to an equi-satisfiable MSO$[<]$-formula $\sigma(\phi)$. For the sake of simplicity, we existentially close the formula ϕ, thus obtaining an equivalent MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-sentence. The mapping σ is inductively defined as follows (for the sake of readability, we introduce various obvious shorthands):

$$
\begin{array}{rcl}
\sigma(X \subseteq Y) & := & W_X \subseteq W_Y \ \wedge (Z_X = Z_Y \vee \\
 & & \quad \exists w \ (w \in W_X \ \wedge \ \forall w' \ (w' \in W_X \rightarrow w' \geq w) \ \wedge \\
 & & \quad\quad \forall z \ (z \geq w \rightarrow (z \in Z_X \leftrightarrow z \in Z_y)))); \\
\sigma(\downarrow_0 (X, Y)) & := & \exists w \ (Z_X = Z_Y \wedge W_X = \{w\} \wedge W_Y = \{w-1\}); \\
\sigma(\downarrow_1 (X, Y)) & := & \exists w \ (Z_X \cup \{w-1\} = Z_Y \wedge W_X = \{w\} \wedge W_Y = \{w-1\}); \\
\sigma(T_0(X)) & := & W_X = \{0\}; \\
\sigma(T(X, Y)) & := & \exists w \ (W_X = W_Y = \{w\}); \\
\sigma(\phi \ \wedge \ \psi) & := & \sigma(\phi) \ \wedge \ \sigma(\psi); \\
\sigma(\neg\phi) & := & \neg\sigma(\phi); \\
\sigma(\exists X \ \phi) & := & \exists Z_X, W_X \ (\sigma(\phi) \ \wedge \ W_X \neq \emptyset \ \wedge \forall w \ (w \in W_X \rightarrow \\
 & & (\forall w'(w' \notin W_X \vee w' \geq w) \rightarrow \forall z(z \notin Z_X \vee z \geq w)))).
\end{array}
$$

It is routine to check that the MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$-sentence ϕ holds in the structure $\langle \mathcal{T}, \downarrow_0, \downarrow_1, <, T_0, T \rangle$ if and only if $\sigma\phi$ holds in $\langle \mathbb{Z}, < \rangle$.

It remains to prove that MSO$[<]$ over $\langle \mathbb{Z}, < \rangle$ is reducible to MSO$[<]$ over $\langle \mathbb{N}, < \rangle$. In order to do that, we denote by $even$ and odd the (definable) unary predicates $\{2n : n \in \mathbb{N}\}$ and $\{2n+1 : n \in \mathbb{N}\}$, respectively, and then we translate any given formula over $\langle \mathbb{Z}, < \rangle$ into an equi-satisfiable formula over $\langle \mathbb{N}, < \rangle$ by replacing every atomic formula of the form $x < y$ with the formula $(odd(x) \wedge even(y)) \vee (even(x) \wedge even(y) \wedge x < y) \vee (odd(x) \wedge odd(y) \wedge y < x)$. This shows that MCL$[<, \downarrow_0, \downarrow_1, T_0, T]$ is reducible to MSO$[<]$. ■

THEOREM 12. — MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ *over the TULS is inter-reducible to* MSO$[<]$.

PROOF 13. — We first show that the logic MSO$[<]$ over the natural numbers is reducible to the logic MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ over the TULS. In a way similar to the proof of Theorem 10, we replace MSO$[<]$ with the equivalent framework MSO$_0[<]$ where only second-order variables occur and the atomic subformulas are of the forms $X \subseteq Y$ and $Succ(X, Y)$. Then, we denote by D_0 the leftmost upward branch of the 2-refinable TULS. Such a predicate can be defined by a suitable formula in the chain fragment of monadic second-order logic over the TULS. Finally, we translate a given MSO$_0[<]$-formula ϕ into an equi-satisfiable MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-formula $\tau(\phi)$, by following the same construction provided in the proof of Theorem 10. Since D_0 is definable in MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$, we have that MSO$_0[<]$ (and hence MSO$[<]$) is reducible to MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$.

As for the converse result, in order to make it possible to check whether two elements of the TULS lie on the same column, we need to encode non-empty chains by suitable values and sets over a discrete linear structure. Notice that the ordering $<$ of the TULS can be easily defined by a formula in the chain fragment of monadic second-order logic. Thus, we can restrict ourselves to the equivalent setup of MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ where variables are instanciated with non-empty chains over the 2-refinable TULS and atomic formulas are of the forms $X \subseteq Y$ (chain X is included in chain Y), $\downarrow_i (X, Y)$ (X and Y are singletons $\{x\}$ and $\{y\}$, respectively, and $y = \downarrow_i (x)$), $T_0(X)$ (X is a singleton $\{x\}$, with $x \in T_0$), and $D(X, Y)$ (X and Y are singletons $\{x\}$ and $\{y\}$, respectively, and x and y belong to the same column). Then, we existentially close an MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-formula to be interpreted over $\langle \mathcal{T}, (\downarrow_i)_{i=0}^{k-1}, <, T_0, D \rangle$ and we translate the corresponding sentence ϕ into an equivalent MSO$[<, neg]$-sentence $\sigma(\phi)$ over the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$, where ∞ denotes a special element not belonging to $\mathbb{Z}$ and neg denotes the binary relation $\{(z, -z) : z \in \mathbb{Z}\}$. Later, we will show that the logic MSO$[<, neg]$ over the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ is reducible, in its turn, to the logic MSO$[<]$ over $\langle \mathbb{N}, < \rangle$.

The idea for the translation from MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ to MSO$[<, neg]$ is to encode a non-empty chain C by an integer s_C and three subsets Z_C, W_C, Q_C of $\mathbb{N}$. We denote by P_C the rightmost cover of C, formally, the superset of C that contains exactly one element a_b of the TULS for each $b \in \mathbb{Z}$ and such that, whenever b is the least integer for which there is a satisfying $a_b \in C$, then every descendant of a_b along P_C is of the form c_d, with $c = 2^{b-d}(a + 1) - 1$ and $d \leq b$. We now distinguish between two cases: either P_C coincides with the leftmost branch of the TULS (this happens when C is a downward infinite chain lying entirely on the leftmost branch), or there is a minimum index $i \in \mathbb{Z}$ such that $0_i \in P_C$. In the former case, we set $s_C = \infty$, $Z_C = \emptyset$, $W_C = \{i \geq 0 : 0_{-i} \in C\}$, and $Q_C = \{i > 0 : 0_i \in C\}$. In the latter case, we define s_C as the minimum $i \in \mathbb{Z}$ such that $0_i \in P_C$ and we define $Z_C, W_C, Q_C \subseteq \mathbb{N}$ as follows:

– $b \in Z_C$ iff there is a (unique) odd index $a \in \mathbb{N}$ such that $a_{s_C - b} \in P_C$ (namely, $a_{s_C - b}$ is a $\downarrow_1$-successor in the path P_C);

– $b \in W_C$ iff there is a (unique) index $a \in \mathbb{N}$ such that $a_{s_C - b} \in C$ (namely, C intersects the layer $T_{s_C - b}$);

– $b \in Q_C$ iff $b > 0$ and $0_{s_C + b} \in C$ (namely, C intersects the layer $T_{s_C + b}$).

Notice that, in both cases, the encoding (s_C, Z_C, W_C, Q_C) uniquely determines the non-empty chain C. Switching to logic, we introduce, for each chain variable X, a first-order variable s_X and three second-order variables Z_X, W_X, and Q_X. Then, we translate an MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$-sentence ϕ to an MSO$[<, neg]$-sentence $\sigma(\phi)$ inductively as follows:

$$
\begin{array}{rcl}
\sigma(X \subseteq Y) & := & (W_X \subseteq W_Y) \ \wedge \ (Q_X \subseteq Q_Y) \wedge \\
& & ((s_X = s_Y = \infty) \ \vee \ (s_X = s_Y \neq \infty \wedge Z_X = Z_Y) \vee \\
& & \ (s_X = s_Y \neq \infty \wedge \\
& & \ \exists w \ (w \in W_X \wedge \forall w'(w' \in W_X \rightarrow w \leq w') \wedge \\
& & \qquad \forall z (z \geq w \rightarrow (z \in Z_X \leftrightarrow z \in Z_Y))))); \\
\sigma(\downarrow_0 (X,Y)) & := & (s_Y = s_X - 1 \ \wedge \ Z_X = Z_Y \ \wedge \\
& & \ W_X = W_Y = \{0\} \ \wedge \ Q_X = Q_Y = \emptyset) \vee \\
& & \exists w \ (s_X = s_Y \neq \infty \ \wedge \ Z_X = Z_Y \cup \{w-1\} \ \wedge \\
& & \qquad W_X = \{w\} \ \wedge \ W_Y = \{w-1\} \ \wedge \ Q_X = Q_Y = \emptyset); \\
\sigma(\downarrow_1 (X,Y)) & := & (s_X = s_Y \neq \infty) \ \wedge \ (Z_X = Z_Y) \wedge \\
& & \exists w \ (W_X = \{w\} \ \wedge \ W_Y = \{w-1\}) \wedge \\
& & (Q_X = Q_Y = \emptyset); \\
\sigma(T_0(X)) & := & (W_X = \{neg(s_X)\}) \ \wedge \ (Q_X = \emptyset); \\
\sigma(D(X,Y)) & := & (s_X \neq \infty) \ \wedge \ (s_Y \neq \infty) \ \wedge \ (Z_X = Z_Y) \wedge \\
& & \exists w \ (W_X = W_Y = \{w\}) \ \wedge \ (Q_X = Q_Y = \emptyset); \\
\sigma(\phi \wedge \psi) & := & \sigma(\phi) \ \wedge \ \sigma(\psi); \\
\sigma(\neg\phi) & := & \neg\sigma(\phi); \\
\sigma(\exists X \ \phi) & := & \exists s_X, Z_X, W_X, Q_X \\
& & \sigma(\phi) \ \wedge \ (Z_X \cup W_X \cup Q_X \subseteq \mathbb{N}) \ \wedge \ (W_X \cup Q_X \neq \emptyset) \wedge \\
& & ((s_X = \infty \wedge Z_C = \emptyset) \vee \\
& & \ (s_X \neq \infty \wedge \forall w ((w \in W_x \wedge \forall w'(w' \in W_X \rightarrow w \geq w')) \rightarrow \\
& & \qquad\qquad \forall z (z > w \rightarrow z \in Z_X)))).
\end{array}
$$

It is routine to check that for every sentence ϕ, ϕ holds in the TULS expanded with the predicates T_0 and D if and only if $\sigma(\phi)$ holds in the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$.

It remains to prove that MSO$[<, neg]$ over $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ is reducible to MSO$[<]$ over $\langle \mathbb{N}, < \rangle$. In fact, the structure $\langle \mathbb{Z} \cup \{\infty\}, <, neg \rangle$ can be embedded in $\langle \mathbb{N}, < \rangle$. Precisely, we can denote by $even$ and by odd the (definable) unary predicates $\{2n : n \in \mathbb{N}\}$ and $\{2n+1 : n \in \mathbb{N}\}$, respectively, and we can translate any MSO$[<, neg]$-formula ψ into an MSO$[<]$-formula $\rho(\psi)$ by replacing every atomic formula of the form $x < y$ with the formula $(x = 1 \wedge y = 2) \vee (odd(x) \wedge odd(y) \wedge x < y) \vee (x \neq 0 \wedge even(x) \wedge even(y) \wedge y < x)$ and every atomic formula of the form $neg(x,y)$ with the formula $(x = y = 1) \vee (odd(x) \wedge x = y + 1) \vee (x \neq 0 \wedge even(x) \wedge y = x + 1)$. This shows that MCL$[<, \downarrow_0, \downarrow_1, T_0, D]$ over the TULS is reducible to MSO$[<]$. ■

In [FRA 03b] Franceschet et al. show that the satisfiability problems for the (weak) MSO logics over the DULS and the UULS expandend with either the equi-level or the equi-column predicates are not decidable. These undecidability results are proved by reducing several undecidable problems (e.g., the tiling problem over the two-dimensional infinite grid) to satisfiability problems for the corresponding structures. On the positive side, they prove the decidability of the satisfiability problem for the chain fragment of MSO logic interpreted over the DULS and the UULS expanded with the equi-level predicate and over UULSs expanded with the equi-column predicate, but they leave open the problem for the DULS expanded with the equi-column predicate. Since the MSO-definability of the DULS and the UULS in terms of the TULS, equipped with the predicate T_0, holds even if we restrict ourselves to interpretations with chain quantifiers only, Theorems 10 and 12 allow us to positively solve such a decision problem.

COROLLARY 14. — *The satisfiability problem for* $\mathrm{MCL}[<, \downarrow_0, \downarrow_1, T_0, D]$ *over the DULS is decidable.*

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