# Global solution to the drift-diffusion-Poisson system for semiconductors with nonlinear recombination-generation rate 

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#### Abstract

In this paper, we study the Cauchy problem of a time-dependent drift-diffusionPoisson system for semiconductors. Existence and uniqueness of global weak solutions are proven for the system with a higher-order nonlinear recombination-generation rate $R$. We also show that the global weak solution will converge to a unique equilibrium as time tends to infinity.


Keywords: drift-diffusion-Poisson system; global weak solution; uniqueness; long-time behavior.
AMS Subject Classification: 35G25, 35J20, 35B40, 35B45

## 1 Introduction

We consider the following drift-diffusion-Poisson model for semiconductors that is a coupled system of parabolic-elliptic equations:

$$
\left\{\begin{array}{l}
n_{t}=\operatorname{div}\left(\nabla n+n \nabla\left(\psi+V_{n}\right)\right)-R(n, p, x)  \tag{1.1}\\
p_{t}=\operatorname{div}\left(\nabla p+p \nabla\left(-\psi+V_{p}\right)\right)-R(n, p, x) \\
-\varepsilon^{2} \Delta \psi=n-p-D(x)
\end{array}\right.
$$

System (1.1) models the transport of the electrons and holes in semiconductor and plasma devices (cf. [28, 29]). $n=n(x, t)$ is the spatial distribution of electrons (negatively charged) and $p=p(x, t)$ is the spatial distribution of holes (positively charged). $\psi=\psi(x, t)$ is the self-consistent electrostatic potential created by the two charge carrier species (electrons and holes) and by the doping profile $D=D(x)$ of the semiconductor device. The charge carriers are assumed to be confined by the external potentials $V_{n}$ and $V_{p}$. This replaces the usual assumption of a bounded domain (cf. [9, 15, 29] and the references therein). The function $R=R(n, p, x)$ represents the so-called recombination-generation rate for electrons and holes. The parameter $\varepsilon$ appearing in the Poisson equation is the scaled Debye length of

[^0]the semiconductor device that stands for the screening of the hole and electron particles. In this paper, we are interested in the Cauchy problem to system (1.1) and assume that (1.1) is subject to the following initial data
\[

$$
\begin{equation*}
\left.n(x, t)\right|_{t=0}=n_{I}(x) \geq 0,\left.\quad p(x, t)\right|_{t=0}=p_{I}(x) \geq 0 \tag{1.2}
\end{equation*}
$$

\]

The generation and recombination of electrons and holes in a semiconductor play an important role in their electrical and optical behavior [14]. Recombination is a process by which both carriers annihilate each other: the electrons fall in one or multiple steps into the empty state that is associated with the hole. Generation can be viewed as its inverse process whereby electrons and holes are created. There are several typical recombination mechanisms that the energy of carriers will be dissipated during these processes by different ways (cf. e.g., [14, 28, 29]). For instance,

1. Band-to-band recombination (also referred to as direct thermal recombination). The energy is emitted in the form of a photon. The recombination rate depends on the density of available electrons and holes and it can be expressed as

$$
\begin{equation*}
R(n, p)=C\left(n p-n_{i}^{2}\right) \tag{1.3}
\end{equation*}
$$

where $n_{i}$ denotes the intrinsic carrier density of the semiconductor.
2. Shockley-Read-Hall (SRH) recombination (also called the trap-assisted recombination). A two-step transition of an electron from the conduction band to the valence band occurs and $R$ is in the form

$$
\begin{equation*}
R(n, p)=\frac{\left(n p-n_{i}^{2}\right)}{r_{1} n+r_{2} p+r_{3}} \tag{1.4}
\end{equation*}
$$

where $r_{1}, r_{2}, r_{3}$ are proper positive functions.
3. Auger recombination. An electron and a hole recombine in a band-to-band transition, but the resulting energy is given off to another electron or hole in the form of kinetic energy. The corresponding recombination rate is similar to that of band-to-band recombination, but involves a third particle:

$$
\begin{equation*}
R(n, p)=\left(C_{n} n+C_{p} p\right)\left(n p-n_{i}^{2}\right) \tag{1.5}
\end{equation*}
$$

Extensive mathematical study of the drift-diffusion-Poisson system has been developed in the literature. For the initial boundary value problem of (1.1) in a bounded domain $\Omega \subset \mathbb{R}^{N}$ with various boundary conditions (e.g., the Neumann type, Robin type, or mixed boundary conditions), existence and uniqueness as well as long-time behavior have been investigated by many authors, see for instance, [3, 4, 6, $8,-10,17,23,25,26,32,37]$ and reference therein. On the other hand, for the sake of modeling simplicity and for the particularly interesting mathematical features, it would also be interesting to consider the Cauchy problem of (1.1) (cf. e.g., $[1,[5,11,30,31)$. Existence and uniqueness results and stability of strong solutions in $L^{p}\left(\mathbb{R}^{N}\right)$ spaces $(N \geq 2)$ were proven in [31] for a system analogous to (1.1). However, in their system there were no external potentials and the recombination-generation rate $R$ was
replaced by a given function $f=f(x, t) \in L^{\theta}\left(0, T ; W^{1, \eta}\right)$ with $1 \leq \theta<2, \frac{N}{2}<\eta<N$, which expressed the variation of the charge by the external current. As far as the long-time behavior of global solutions to the Cauchy problem is concerned, when the recombination-generation term $R$ is absent, exponential convergence to equilibrium with a confining potential and an algebraic rate towards a self-similar state without confinement have been obtained in [1]. The analysis therein is based on the well-known entropy approach for diffusion and diffusionconvection equations that has been extensively studied in recent years (cf. [2, 7, 34] and the references therein). We also refer to [30] in which an optimal $L^{p}$ decay estimate of solutions to the Cauchy problem was obtained via a time weighted energy method (without confinement and recombination-generation rate $R$ ). When the recombination-generation process is taken into account, the situation is more complicated. In [36], the authors proved the global existence and uniqueness of weak solutions of problem (1.1) in $\mathbb{R}^{3}$ with an (unbounded) external confining potential $V_{n}=V_{p}=V$ and under the restrictive assumption that $R$ has a linear growth (which, however, recovers the Shockley-Read-Hall recombination, cf. (1.4)). Besides, existence and uniqueness of the steady state and partial result on the convergence to equilibrium were obtained. Recently, exponential $L^{1}$ convergence to equilibrium was proved in [11] via entropy method for global solutions to a simplified convection-diffusion-reaction model with confinement and Shockley-Read-Hall recombination-generation rate but neglecting the influence of the self-consistent potential $\psi$. It would be interesting to study the well-posedness as well as long time behavior of the full convection-diffusion-reaction-Poisson system (1.1)(1.2) with more general recombination-generation rate $R$ including the higher nonlinear cases (1.3) and (1.5).

For the sake of simplicity, we consider the whole-space case posed on $\mathbb{R}^{3}$. Similar results can be obtained for the two-dimensional whole space case with some minor modifications, due to the different properties of the Newtonian potential.

In order to formulate our assumptions and results, we first introduce some notations on the functional settings. $H^{m}\left(\mathbb{R}^{3}\right)(m \in \mathbb{N})$ is used to denote the Sobolev space $W^{m, 2}\left(\mathbb{R}^{3}\right)$, and $\|\cdot\|_{H^{m}\left(\mathbb{R}^{3}\right)}$ is its corresponding norm. We denote $L^{r}\left(\mathbb{R}^{3}\right)(r \geq 1)$ with norm $\|\cdot\|_{L^{r}\left(\mathbb{R}^{3}\right)}$ and the vector space $\mathbf{L}^{r}\left(\mathbb{R}^{3}\right)=\left(L^{r}\left(\mathbb{R}^{3}\right)\right)^{3}(r \geq 1)$ with norm $\|\cdot\|_{\mathbf{L}^{r}\left(\mathbb{R}^{3}\right)}$. Moreover, for a potential function $V$, we define the weighted $L^{r}(r>1)$ space as follows

$$
\begin{equation*}
L^{r}\left(\mathbb{R}^{3}, e^{V(x)} d x\right):=\left\{\left.u \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{3}\right)\left|\int_{\mathbb{R}^{3}}\right| u(x)\right|^{r} e^{V(x)} d x<\infty\right\} \tag{1.6}
\end{equation*}
$$

with norm $\|u\|_{L^{r}\left(\mathbb{R}^{3}, e^{V(x)} d x\right)}:=\left(\int_{\mathbb{R}^{3}}|u(x)|^{r} e^{V(x)} d x\right)^{1 / r}$. We define the weighted vector space and its norm in a similar way, which are denoted by $\mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V(x)} d x\right)$ and $\|\cdot\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V(x)} d x\right)}$, respectively. For any Hilbert space $H$, we denote its subspace

$$
H_{+}=\left\{f(x) \in H \mid f(x) \geq 0, \text { a.e. } x \in \mathbb{R}^{3}\right\} .
$$

Throughout this paper, we use $C, C_{i}(i \in \mathbb{N})$ to denote genetic constants that may vary in different places (even in the same estimate). Particular dependence of those constants will be explained in the text if necessary.

Next, we make the following assumptions on confining potentials $V_{n}, V_{p}$, the recombinationgeneration rate $R$ and the doping profile $D$ :
(H1a) There exist constants $\rho_{n}, \rho_{p}>0$ such that

$$
\frac{\partial^{2} V_{n}}{\partial x^{2}} \geq \rho_{n} \mathbb{I}, \quad \frac{\partial^{2} V_{p}}{\partial x^{2}} \geq \rho_{p} \mathbb{I}, \quad \forall x \in \mathbb{R}^{3},
$$

in the sense of positive-defined matrix. Moreover, there exists $K>0$ such that

$$
\begin{equation*}
\left\|\Delta V_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq K, \quad\left\|\Delta V_{p}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq K \tag{1.7}
\end{equation*}
$$

(H1b) There exists $K^{\prime}>0$ such that

$$
\begin{equation*}
\left\|V_{n}(x)-V_{p}(x)\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq K^{\prime}<+\infty \tag{1.8}
\end{equation*}
$$

(H2a) The recombination-generation rate $R=R(n, p, x)$ is of the form

$$
R(n, p, x)=F(n, p)\left(n p-\delta^{2} \mu_{n} \mu_{p}\right)
$$

where $\mu_{n}(x)=e^{-V_{n}(x)}, \mu_{p}(x)=e^{-V_{p}(x)} . \delta$ is a positive constant standing for the scaled average intrinsic carrier density of the semiconductor. Without loss of generality, we assume that $\delta=1$ in the remaining part of this paper.
(H2b) The scalar function $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with linear growth, namely, there exist constants $c_{1}, c_{2}>0$ independent of $n, p$ such that

$$
\begin{gathered}
\left|F\left(n_{1}, p_{1}\right)-F\left(n_{2}, p_{2}\right)\right| \leq c_{1}\left(\left|n_{1}-n_{2}\right|+\left|p_{1}-p_{2}\right|\right), \quad \forall n_{1}, p_{1}, n_{2}, p_{2} \in \mathbb{R}, \\
|F(n, p)| \leq c_{2}(1+|n|+|p|) \quad \forall n, p \in \mathbb{R} .
\end{gathered}
$$

Moreover, $F(n, p) \geq 0$ if $n, p \geq 0$.
(H3) $D(x) \in L^{1}\left(\mathbb{R}^{3}\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right)$.
Remark 1.1. It easily follows from (H1a) that the confining potentials $V_{n}(x)$ and $V_{p}(x)$ are uniformly convex and can be bounded from below by a finite number $V_{b} \in \mathbb{R}$. Thus, $\left\|\mu_{n}\right\|_{L^{\infty}} \leq e^{-V_{b}}$ and $\left\|\mu_{p}\right\|_{L^{\infty}} \leq e^{-V_{b}}$. Without loss of generality, we assume in (1.1) that the diffusion coefficients and carrier mobilities are equal to one. Moreover, for the sake of simplicity, we set

$$
\int_{\mathbb{R}^{3}} \mu_{n} d x=\int_{\mathbb{R}^{3}} \mu_{p} d x=1 .
$$

The above simplifications do not affect the subsequent analysis. We also infer from (1.8) that the norms on $L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)$ and $L^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)$ are equivalent.

A typical example for the confining potential is $\frac{|x|^{2}}{2}$ (cf. e.g., [1, [2, [11]). We remark that the confining potential is introduced in the Cauchy problem (although somewhat unphysical) in order to prevent the particles from escaping to infinity as time progresses. For the Cauchy problem of drift-diffusion-Poisson system without generation-recombination term, different types of large time behavior for the cases with or without confining potential have been illustrated in [1] (see also [5] for a related system modeling the bipolar plasma).

Now we state the the main results of this paper.

Theorem 1.1 (Well-posedness). Suppose that (H1a)-(H3) are satisfied. For any initial data $n_{I} \in L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right), p_{I} \in L^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right), n_{I}, p_{I} \geq 0$, problem (1.1)(1.2) admits a unique global weak solution $(n, p, \psi)$ such that for any $T>0$,

$$
\begin{aligned}
& n \in L^{\infty}\left(0, T ; L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right)\right), \quad \nabla n \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)\right), \\
& p \in L^{\infty}\left(0, T ; L^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right)\right), \quad \nabla p \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)\right), \\
& n_{t}, p_{t} \in L^{2}\left(0, T ;\left(H^{1}\left(\mathbb{R}^{3}\right)\right)^{\prime}\right), \\
& n(t) \geq 0, \quad p(t) \geq 0, \quad t \in[0, T], \quad \text { a.e. } x \in \mathbb{R}^{3}, \\
& \nabla \psi \in L^{\infty}\left(0, T ; \mathbf{L}^{\infty}\left(\mathbb{R}^{3}\right)\right), \Delta \psi \in L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right),
\end{aligned}
$$

where $\psi=\psi(x, t)$ is given by the Newtonian potential

$$
\psi(x, t)=\frac{1}{S_{3}} \int_{\mathbb{R}^{3}} \frac{n(y, t)-p(y, t)-D(y)}{|x-y|} d y
$$

with $S_{3}=\frac{2 \pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}$ being the surface area of the 2D unit ball.
Theorem 1.2 (Long-time behavior). Under the assumptions in Theorem 1.1, we have the following uniform-in-time estimate for the global solutions:

$$
\begin{equation*}
\|n(t)\|_{L^{\infty}\left(0,+\infty ; L^{r}\left(\mathbb{R}^{3}\right)\right)}+\|p(t)\|_{L^{\infty}\left(0,+\infty ; L^{r}\left(\mathbb{R}^{3}\right)\right)}<\infty, \quad \forall r \in[1,+\infty] \tag{1.9}
\end{equation*}
$$

Moreover, for every fixed $t^{*}>0$, the global shifted solution $(n(t+s), p(t+s), \psi(t+s))$ $\left(s \in\left(0, t^{*}\right)\right)$ of problem (1.1) -(1.2) converges to the unique steady state $\left(n_{\infty}, p_{\infty}, \psi_{\infty}\right)$ that satisfies (4.2) as $t \rightarrow+\infty$ in the following sense:

$$
\begin{aligned}
& n(t+\cdot) \rightarrow n_{\infty}, \quad p(t+\cdot) \rightarrow p_{\infty} \quad \text { in } L^{1}\left(\left(0, t^{*}\right) \times \mathbb{R}^{3}\right), \\
& \nabla \psi(t+\cdot) \rightharpoonup \nabla \psi_{\infty} \quad \text { in } L^{2}\left(0, t^{*} ; \mathbf{H}^{1}\left(\mathbb{R}^{3}\right)\right), \\
& \psi(t+\cdot) \rightharpoonup \psi_{\infty} \quad \text { in } L^{2}\left(0, t^{*} ; L^{6}\left(\mathbb{R}^{3}\right)\right) .
\end{aligned}
$$

Remark 1.2. Our results holds for arbitrary but fixed $\varepsilon>0$. In the following analysis, we just set $\varepsilon=1$ without of loss of generality. We note that the quasi-neutral limit (namely, zero-Debye-length limit $\varepsilon \rightarrow 0$ ) of the drift-diffusion-Poisson system is a challenging and physically complex modeling problem for bipolar kinetic models of semiconductors, which has been analyzed by several authors, see, e.g. [27, 35] and the references cited therein.

As we have mentioned before, for a class of recombination-generation rate with at most linear growth, existence of global weak solutions to problem (1.1)-(1.2) in $\mathbb{R}^{3}$ has been obtained in (36. However, argument therein fails to apply in our present case due to the higher-order nonlinear reaction term $R$ that includes both the band-to-band and the Auger recombination (cf. (H2a)-(H2b)). On the other hand, well-posedness results for drift-diffusion-Poisson system with higher-order recombination-generation rate have been proved in the bounded domain in $\mathbb{R}^{N}, N \leq 3$ (see [8]10] for the case with $F$ being bounded, and 37] for the case that $F$ has a linear growth). Since now we are considering the Cauchy problem in the whole space that is unbounded and the carriers are confined by unbounded external potentials, the methods for the initial boundary value problem cannot be applied directly. We need
to exploit and employ several techniques in the literature to prove the global existence and uniqueness of solutions to problem (1.1)-(1.2). In order to overcome the difficulties from the higher-order reaction term $R$, we first introduce a $L^{\infty}$ cut-off to the unknowns $n, p$ in $R$ and study an approximation problem associated with our original system (1.1)-(1.2). To deal with the unbounded confining potentials, we then transform the approximate problem into a new form by introducing some new variables with proper weight functions. After obtaining the well-posedness of the approximate problem, we try to derive proper uniform estimates based on a Stampacchia-type $L^{\infty}$ estimation technique (cf. [9]) that enable us to pass to limit and show the existence of global weak solutions to the original system (1.1)-(1.2). Finally, we get uniform-in-time $L^{r}(r \in[1,+\infty])$ estimates for the global solutions under more general assumptions by extending the methods in [11,36] and investigate the long-time behavior of global solutions.

The remaining part of the paper is organized as follows. In Section 2, we prove the well-posedness of an approximate problem and obtain some uniform estimates that are independent of the approximate parameter. In Section 3, we prove the existence of global solutions to the original problem (1.1)-(1.2) by passing to the limit and show the uniqueness of the solution. In Section 4, we obtain some uniform-in-time estimates of the solutions and show that as time tends to infinity the global solutions will converge to a unique steady state.

## 2 Well-posedness of the Approximate System

In order to overcome the difficulty brought by the higher-order nonlinearity $R$, we introduce and study the following approximate problem in this section. For any $\sigma>0$, consider

$$
(A P)\left\{\begin{array}{l}
\partial_{t} n_{\sigma}=\operatorname{div}\left(\nabla n_{\sigma}+n_{\sigma} \nabla\left(\psi_{\sigma}+V_{n}\right)\right)-\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)  \tag{2.1}\\
\partial_{t} p_{\sigma}=\operatorname{div}\left(\nabla p_{\sigma}+p_{\sigma} \nabla\left(-\psi_{\sigma}+V_{p}\right)\right)-\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right), \\
-\Delta \psi_{\sigma}=n_{\sigma}-p_{\sigma}-D(x)
\end{array}\right.
$$

subject to the initial data

$$
\begin{equation*}
\left.n_{\sigma}(x, t)\right|_{t=0}=n_{I},\left.\quad p_{\sigma}(x, t)\right|_{t=0}=p_{I} \tag{2.2}
\end{equation*}
$$

The approximated recombination-generation rate $\tilde{R}$ in (2.1) is given by

$$
\begin{align*}
\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right) & =R\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}, x\right) \\
& =F\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right)\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}} \frac{p_{\sigma}}{1+\sigma p_{\sigma}}-\mu_{n} \mu_{p}\right) . \tag{2.3}
\end{align*}
$$

Now we state the main result of this section.
Theorem 2.1. Suppose that assumptions (H1a)-(H3) are satisfied. For any $\sigma>0, n_{I} \in$ $L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right) \cap L^{4}\left(\mathbb{R}^{3}\right), p_{I} \in L^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right) \cap L^{4}\left(\mathbb{R}^{3}\right), n_{I}, p_{I} \geq 0$, problem (2.1)-(2.2) admits a unique global weak solution $\left(n_{\sigma}, p_{\sigma}, \psi_{\sigma}\right)$ such that for any $T>0$,

$$
\begin{array}{lc}
n_{\sigma} \in C\left([0, T] ; L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)\right), & \nabla n_{\sigma} \in L^{2}\left((0, T) ; \mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)\right), \\
p_{\sigma} \in C\left([0, T] ; L^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)\right), & \nabla p_{\sigma} \in L^{2}\left((0, T) ; \mathbf{L}^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)\right),
\end{array}
$$

$$
n_{\sigma}(t) \geq 0, \quad p_{\sigma}(t) \geq 0, \quad t \in[0, T], \quad \text { a.e. } \quad x \in \mathbb{R}^{3} .
$$

$\psi_{\sigma}=\psi_{\sigma}(x, t)$ is the Newtonian potential with respect to $n_{\sigma}(x, t), p_{\sigma}(x, t)$ given by

$$
\psi_{\sigma}(x, t)=\frac{1}{S_{3}} \int_{\mathbb{R}^{3}} \frac{n_{\sigma}(y, t)-p_{\sigma}(y, t)-D(y)}{|x-y|} d y, \quad \text { with } S_{3}=\frac{2 \pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}
$$

The proof of Theorem 2.1 consists of several steps. First, we derive some properties for the new reaction term $\tilde{R}$ under assumptions (H2a)-(H2b).

Lemma 2.1. Under assumptions (H2a)-(H2b), the function $\tilde{R}=\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)$ satisfies the following properties
(i) $\tilde{R}$ has at most a linear growth for any $n_{\sigma}, p_{\sigma} \geq 0$, i.e.,

$$
\begin{equation*}
\left|\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)\right| \leq C_{\sigma}\left(a(x)+n_{\sigma}+p_{\sigma}\right), \tag{2.4}
\end{equation*}
$$

where $0 \leq a(x) \in L^{1}\left(\mathbb{R}^{3}\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right)$ and $a(x) \in L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right) \cap L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)$.
(ii) $\tilde{R}$ is Lip-continuous in $L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right), i=\{n, p\}$, such that for any $n_{\sigma}^{(1)}$, $n_{\sigma}^{(2)} \in$ $L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right), p_{\sigma}^{(1)}, p_{\sigma}^{(2)} \in L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)$, it holds

$$
\begin{align*}
& \left\|\tilde{R}\left(n_{\sigma}^{(1)}, p_{\sigma}^{(1)}, \cdot\right)-\tilde{R}\left(n_{\sigma}^{(2)}, p_{\sigma}^{(2)}, \cdot\right)\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)} \\
\leq & \tilde{K}\left(\left\|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)}+\left\|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)}\right), \tag{2.5}
\end{align*}
$$

where the constant $\tilde{K}$ may depend on $c_{1}, c_{2}, \sigma$ and $V_{b}$ in Remark 1.1.
Proof. We observe that for any $\varphi \geq 0$, it holds

$$
\begin{equation*}
0 \leq \frac{\varphi}{1+\sigma \varphi} \leq \frac{1}{\sigma} \quad \text { and } \quad \frac{\varphi}{1+\sigma \varphi} \leq \varphi, \quad \forall \sigma>0 . \tag{2.6}
\end{equation*}
$$

Due to the above simple facts and assumptions (H2a)-(H2b), we can verify that

$$
\begin{align*}
\left|\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)\right| & \leq c_{2}\left(1+\frac{n_{\sigma}}{1+\sigma n_{\sigma}}+\frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right)\left|\frac{n_{\sigma}}{1+\sigma n_{\sigma}} \frac{p_{\sigma}}{1+\sigma p_{\sigma}}-\mu_{n} \mu_{p}\right| \\
& \leq c_{2}\left(1+\frac{2}{\sigma}\right) \mu_{n} \mu_{p}+c_{2}\left(\frac{1}{\sigma}+\frac{1}{\sigma^{2}}\right)\left(n_{\sigma}+p_{\sigma}\right) . \tag{2.7}
\end{align*}
$$

Then we can simply set $C_{\sigma}=c_{2}\left(1+\frac{2}{\sigma}+\frac{1}{\sigma^{2}}\right)$ and $a(x)=\mu_{n} \mu_{p}$, which obviously satisfies the required conditions by assumption (H1a).

For any $n_{\sigma}^{(1)}, n_{\sigma}^{(2)} \in L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right), p_{\sigma}^{(1)}, p_{\sigma}^{(2)} \in L_{+}^{2}\left(\mathbb{R}^{3}, e^{V_{p}(x)} d x\right)$, we infer from (2.6) that

$$
\begin{equation*}
\left|\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}}-\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}}\right| \leq\left|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right|, \quad\left|\frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}-\frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right| \leq\left|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right|, \tag{2.8}
\end{equation*}
$$

and as a result,

$$
\left|\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}} \frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}-\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}} \frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right|
$$

$$
\begin{align*}
& \leq \frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}\left|\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}}-\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}}\right|+\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}}\left|\frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}-\frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right| \\
& \leq \frac{1}{\sigma}\left(\left|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right|+\left|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right|\right) \tag{2.9}
\end{align*}
$$

Denote

$$
F_{j}=F\left(\frac{n_{\sigma}^{(j)}}{1+\sigma n_{\sigma}^{(j)}}, \frac{p_{\sigma}^{(j)}}{1+\sigma p_{\sigma}^{(j)}}\right), \quad j=1,2
$$

Then we get

$$
\begin{align*}
& \left\|\tilde{R}\left(n_{\sigma}^{(1)}, p_{\sigma}^{(1)}, \cdot\right)-\tilde{R}\left(n_{\sigma}^{(2)}, p_{\sigma}^{(2)}, \cdot\right)\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)} \\
\leq & \left\|\left(F_{1}-F_{2}\right) \mu_{n} \mu_{p}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)} \\
& +\left\|F_{1}\left(\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}} \frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}-\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}} \frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right)\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)} \\
& +\left\|\left(F_{1}-F_{2}\right) \frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}} \frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{i}(x)} d x\right)} \\
:= & I_{1}+I_{2}+I_{3}, \quad i=\{n, p\} . \tag{2.10}
\end{align*}
$$

For the case $i=n$, it follows from (2.6), (2.8), (2.9), (H2a)-(H2b) and Remark 1.1 that

$$
\begin{aligned}
& I_{1} \leq\left\|c_{1}\left(\left|\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}}-\frac{n_{\sigma}^{(2)}}{1+\sigma n_{\sigma}^{(2)}}\right|+\left|\frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}-\frac{p_{\sigma}^{(2)}}{1+\sigma p_{\sigma}^{(2)}}\right|\right) \mu_{n} \mu_{p}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)} \\
& \leq\left\|c_{1}\left(\left|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right|+\left|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right|\right) \mu_{n} \mu_{p}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)} \\
& \leq 2 c_{1} e^{-2 V_{b}}\left(\left\|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}+\left\|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}\right), \\
& I_{2} \leq\left\|\frac{c_{2}}{\sigma}\left(1+\frac{n_{\sigma}^{(1)}}{1+\sigma n_{\sigma}^{(1)}}+\frac{p_{\sigma}^{(1)}}{1+\sigma p_{\sigma}^{(1)}}\right)\left(\left|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right|+\left|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right|\right)\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)} \\
& \leq \frac{2 c_{2}}{\sigma}\left(1+\frac{2}{\sigma}\right)\left(\left\|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}+\left\|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}\right), \\
& \leq \frac{I_{3}}{} \leq \frac{1}{\sigma^{2}}\left\|c_{1}\left(\left|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right|+\left|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right|\right)\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)} \\
& \leq \frac{2 c_{1}}{\sigma^{2}}\left(\left\|n_{\sigma}^{(1)}-n_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}+\left\|p_{\sigma}^{(1)}-p_{\sigma}^{(2)}\right\|_{L^{2}\left(\mathbb{R}^{3}, e^{V_{n}(x)} d x\right)}\right),
\end{aligned}
$$

where $V_{b}$ is the lower bound of $V_{n}, V_{p}$ (see Remark 1.1). Collecting the above estimates together, we see that $(\overline{2.5})(i=n)$ holds with

$$
\tilde{K}=2 c_{1} e^{-2 V_{b}}+\frac{2 c_{2}}{\sigma}\left(1+\frac{2}{\sigma}\right)+\frac{2 c_{1}}{\sigma^{2}}
$$

The case $i=p$ can be treated in the same way. The proof is complete.

Next, for any fixed $\sigma>0$, we introduce the following transformation of unknown variables (cf. [2, 36])

$$
\begin{equation*}
u(x, t):=n_{\sigma}(x, t) e^{\frac{V_{n}(x)}{2}}, \quad v(x, t):=p_{\sigma}(x, t) e^{\frac{V_{p}(x)}{2}} \tag{2.11}
\end{equation*}
$$

then it follows from system (2.1) and a direct computation that $u$ and $v$ satisfy the following transformed approximate system

$$
(T A P)\left\{\begin{array}{l}
u_{t}-\Delta u+A_{n}(x) u=f_{1}\left(u, v, \psi_{\sigma}\right)  \tag{2.12}\\
v_{t}-\Delta v+A_{p}(x) v=f_{2}\left(u, v, \psi_{\sigma}\right) \\
-\Delta \psi_{\sigma}=u e^{-\frac{V_{n}(x)}{2}}-v e^{-\frac{V_{p}(x)}{2}}-D(x)
\end{array}\right.
$$

where

$$
A_{n}(x)=\frac{1}{4}\left|\nabla V_{n}(x)\right|^{2}-\frac{1}{2} \Delta V_{n}(x)+K, \quad A_{p}(x)=\frac{1}{4}\left|\nabla V_{p}(x)\right|^{2}-\frac{1}{2} \Delta V_{p}(x)+K
$$

with $K$ being the constant in (H1a) (see (1.7)). System (2.12) is subject to the initial data

$$
\begin{equation*}
\left.u(x, t)\right|_{t=0}=n_{I}(x) e^{\frac{V_{n}(x)}{2}}:=u_{I},\left.\quad v(x, t)\right|_{t=0}=p_{I}(x) e^{\frac{V_{p}(x)}{2}}:=v_{I} \tag{2.13}
\end{equation*}
$$

It follows from (1.7) that $A_{n}(x)$ and $A_{p}(x)$ are bounded from below, i.e.

$$
A_{n}(x) \geq \frac{K}{2}, \quad A_{p}(x) \geq \frac{K}{2}, \quad \text { a.e. } x \in \mathbb{R}^{3}
$$

Under the transformation (2.11), the right-hand side of (2.12) are given by:

$$
\begin{aligned}
f_{1}\left(u, v, \psi_{\sigma}\right)= & K u+e^{\frac{V_{n}(x)}{2}}\left(\operatorname{div}\left(n_{\sigma} \nabla \psi_{\sigma}\right)-\tilde{R}\left(n_{\sigma}, p_{\sigma}\right)\right) \\
= & K u+\nabla u \cdot \nabla \psi_{\sigma}-\frac{1}{2} u \nabla \psi_{\sigma} \cdot \nabla V_{n}-u^{2} e^{-\frac{V_{n}(x)}{2}}+u v e^{-\frac{V_{p}(x)}{2}}+D(x) u \\
& \quad-e^{\frac{V_{n}(x)}{2}} \tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right) \\
f_{2}\left(u, v, \psi_{\sigma}\right)= & K v+e^{\frac{V_{p}(x)}{2}}\left(\operatorname{div}\left(-p_{\sigma} \nabla \psi_{\sigma}\right)-\tilde{R}\left(n_{\sigma}, p_{\sigma}\right)\right) \\
= & K v-\nabla v \cdot \nabla \psi_{\sigma}+\frac{1}{2} v \nabla \psi_{\sigma} \cdot \nabla V_{p}-v^{2} e^{-\frac{V_{p}(x)}{2}}+u v e^{-\frac{V_{n}(x)}{2}}-D(x) v \\
& -e^{\frac{V_{p}(x)}{2}} \tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)
\end{aligned}
$$

In what follows, we first prove the local well-posedness of the transformed approximate problem (2.12)-(2.13).

Proposition 2.1. Suppose that (H1a)-(H3) are satisfied and $u_{I}, v_{I} \in L_{+}^{2}\left(\mathbb{R}^{3}\right)$. Then for any $\sigma>0$, there exists $T_{\sigma}>0$ such that problem (2.12) -(2.13) admits a unique solution $\left(u, v, \psi_{\sigma}\right)$ on $\left[0, T_{\sigma}\right]$, which satisfies

$$
\begin{gathered}
u, v \in C\left(\left[0, T_{\sigma}\right] ; L_{+}^{2}\left(\mathbb{R}^{3}\right)\right), \quad \nabla u, \nabla v \in L^{2}\left(0, T_{\sigma} ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right) \\
u \nabla V_{n}, v \nabla V_{p} \in L^{2}\left(0, T_{\sigma} ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right)
\end{gathered}
$$

The potential $\psi_{\sigma}$ is given by

$$
\psi_{\sigma}(x, t)=\frac{1}{S_{3}} \int_{\mathbb{R}^{3}} \frac{u(y, t) e^{-\frac{V_{n}(y)}{2}}-v(y, t) e^{-\frac{V_{p}(y)}{2}}-D(y)}{|x-y|} d y
$$

Proof. We consider the following auxiliary linear problem of the transformed approximated problem (2.12)-(2.13), such that for any $\tilde{u}, \tilde{v} \in C\left([0, T] ; L^{2}\left(\mathbb{R}^{3}\right)\right), \nabla \tilde{u}, \nabla \tilde{v} \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right)$, $\tilde{u} \nabla V_{n}, \tilde{v} \nabla V_{p} \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right)$,

$$
(A T A P)\left\{\begin{array}{l}
u_{t}-\Delta u+A_{n}(x) u=f_{1}^{+}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right)  \tag{2.14}\\
v_{t}-\Delta v+A_{p}(x) v=f_{2}^{+}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right) \\
\left.u(x, t)\right|_{t=0}=u_{I},\left.\quad v(x, t)\right|_{t=0}=v_{I}
\end{array}\right.
$$

where $\tilde{\psi}_{\sigma}$ satisfies

$$
\begin{equation*}
-\Delta \tilde{\psi}_{\sigma}=\tilde{u} e^{-\frac{V_{n}(x)}{2}}-\tilde{v} e^{-\frac{V_{p}(x)}{2}}-D(x), \tag{2.15}
\end{equation*}
$$

and the nonlinearities $f_{1}^{+}, f_{2}^{+}$are given by

$$
\begin{gathered}
f_{1}^{+}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right)=K \tilde{u}^{+}+e^{\frac{V_{n}(x)}{2}}\left[\operatorname{div}\left(\tilde{u}^{+} e^{-\frac{V_{n}(x)}{2}} \nabla \tilde{\psi}_{\sigma}\right)-\tilde{R}\left(\tilde{u}^{+} e^{-\frac{V_{n}(x)}{2}}, \tilde{v}^{+} e^{-\frac{V_{p}(x)}{2}}, x\right)\right], \\
f_{2}^{+}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right)=K \tilde{v}^{+}+e^{\frac{V_{p}(x)}{2}}\left[-\operatorname{div}\left(\tilde{v}^{+} e^{-\frac{V_{p}(x)}{2}} \nabla \tilde{\psi}_{\sigma}\right)-\tilde{R}\left(\tilde{u}^{+} e^{-\frac{V_{n}(x)}{2}}, \tilde{v}^{+} e^{-\frac{V_{p}(x)}{2}}, x\right)\right], \\
\text { with } \quad \tilde{u}^{+}:=\max \{0, \tilde{u}\}, \quad \tilde{v}^{+}:=\max \{0, \tilde{v}\} .
\end{gathered}
$$

Since now the nonlinearity $\tilde{R}$ in the approximate problem (2.1) satisfies the properties in Lemma 2.1, using assumptions (H1a)-(H3) we are able to prove the local well-posedness of problem (2.12)-(2.13) by the contraction mapping principle as in [36, Theorem 2.2]. Since the proof is the same, we only sketch it here. Denote

$$
\begin{aligned}
\Sigma_{T}=\{ & (u, v) \in C\left([0, T] ; L^{2}\left(\mathbb{R}^{3}\right) \times L^{2}\left(\mathbb{R}^{3}\right)\right), \\
& (\nabla u, \nabla v) \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right) \times \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right): \\
& u(0)=u_{I} \geq 0, v(0)=v_{I} \geq 0, \max _{0 \leq t \leq T}\left(\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) \leq 2 M, \\
& \int_{0}^{T}\left(\|\nabla u(t)\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla v(t)\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d t \leq 2 M, \\
& \left.\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\left|u(t) \nabla V_{n}\right|^{2}+\left|v(t) \nabla V_{p}\right|^{2}\right) d t \leq 2 M .\right\}
\end{aligned}
$$

where

$$
M:=\left\|u_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|v_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}
$$

Then we can prove that there exists a sufficiently small $T_{\sigma}>0$ such that the mapping $\mathcal{G}:(\tilde{u}, \tilde{v}) \mapsto(u, v)$ defined by (2.14) maps $\Sigma_{T_{\sigma}}$ to itself and is a strict contraction. Hence, the contraction principle entails that $\mathcal{G}$ has a unique fixed point in $\Sigma_{T_{\sigma}}$ such that $\mathcal{G}(u, v)=(u, v)$. Next, due to the special structure of the approximated recombination-generation rate $\tilde{R}$, using the idea in [15], one can show the nonnegativity of the fixed point $(u, v)$ of $\mathcal{G}$, provided that the initial data $u_{I}, v_{I}$ are nonnegative (cf. [36, Theorem 2.2]). Since $\tilde{u}^{+}=\tilde{u}, \tilde{v}^{+}=\tilde{v}$ if $\tilde{u}, \tilde{v} \geq 0$, we see that $f_{i}^{+}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right)=f_{i}\left(\tilde{u}, \tilde{v}, \tilde{\psi}_{\sigma}\right)(i=1,2)$ for $\tilde{u}, \tilde{v} \geq 0$. Thus, $(u, v)$ is the local solution of problem (2.12)-(2.13). The details are omitted here.

Lemma 2.2. Assume that (H1a)-(H3) are satisfied. For any $T>0$, if $n_{I}, p_{I} \in L^{r}\left(\mathbb{R}^{3}\right) \cap$ $L^{1}\left(\mathbb{R}^{3}\right), r \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\|n_{\sigma}(t)\right\|_{L^{s}\left(\mathbb{R}^{3}\right)}+\left\|p_{\sigma}(t)\right\|_{L^{s}\left(\mathbb{R}^{3}\right)} \leq C_{T}, \quad 0 \leq t \leq T, \quad s=1, \ldots, r . \tag{2.16}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\left\|\nabla \psi_{\sigma}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq C_{T}, \quad 0 \leq t \leq T \tag{2.17}
\end{equation*}
$$

provided that $n_{I}, p_{I} \in L^{4}\left(\mathbb{R}^{3}\right) \cap L^{1}\left(\mathbb{R}^{3}\right)$. In particular, the constant $C_{T}$ is independent of $\sigma>0$.

Proof. Integrating the equations for $n_{\sigma}$ and $p_{\sigma}$ in (2.1) on $\mathbb{R}^{3}$, we infer from (2.6), (H2a)(H2b) that

$$
\begin{aligned}
& \frac{d}{d t}\left(\left\|n_{\sigma}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}+\left\|p_{\sigma}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}\right) \\
= & -2 \int_{\mathbb{R}^{3}} R\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}, x\right) d x \\
= & -2 \int_{\mathbb{R}^{3}}\left(\frac{n_{\sigma} p_{\sigma}}{\left(1+\sigma n_{\sigma}\right)\left(1+\sigma p_{\sigma}\right)}-\mu_{n} \mu_{p}\right) F\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right) d x \\
\leq & 2 \int_{\mathbb{R}^{3}} \mu_{n} \mu_{p} F\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right) d x \\
\leq & 2 c_{2} \int_{\mathbb{R}^{3}} \mu_{n} \mu_{p}\left(1+n_{\sigma}+p_{\sigma}\right) d x \\
\leq & C\left(1+\left\|n_{\sigma}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}+\left\|p_{\sigma}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}\right),
\end{aligned}
$$

which yields

$$
\begin{equation*}
\left\|n_{\sigma}(t)\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}+\left\|p_{\sigma}(t)\right\|_{L^{1}\left(\mathbb{R}^{3}\right)} \leq\left(\left\|n_{I}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}+\left\|p_{I}\right\|_{L^{1}\left(\mathbb{R}^{3}\right)}+1\right) e^{C T}, \quad \forall t \in[0, T] \tag{2.18}
\end{equation*}
$$

Next, multiplying the equations for $n_{\sigma}$ and $p_{\sigma}$ by $n_{\sigma}^{r}, p_{\sigma}^{r}(r \in \mathbb{N})$ respectively, integrating on $\mathbb{R}^{3}$ and adding the resultants, we obtain that

$$
\begin{aligned}
& \frac{1}{r+1} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x+\frac{4 r}{(r+1)^{2}} \int_{\mathbb{R}^{3}}\left(\left|\nabla\left(n_{\sigma^{\frac{r+1}{2}}}\right)\right|^{2}+\left|\nabla\left(p_{\sigma^{\frac{r+1}{2}}}\right)\right|^{2}\right) d x \\
= & \frac{r}{r+1} \int_{\mathbb{R}^{3}} \Delta \psi_{\sigma}\left(n_{\sigma}^{r+1}-p_{\sigma}^{r+1}\right)+\frac{r}{r+1} \int_{\mathbb{R}^{3}}\left(\Delta V_{n} n_{\sigma}^{r+1}+\Delta V_{p} p_{\sigma}^{r+1}\right) d x \\
& -\int_{\mathbb{R}^{3}} R\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}, x\right)\left(n_{\sigma}^{r}+p_{\sigma}^{r}\right) d x .
\end{aligned}
$$

Since $n_{\sigma}, p_{\sigma} \geq 0$, we observe that

$$
\begin{aligned}
& -\int_{\mathbb{R}^{3}} R\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}, x\right)\left(n_{\sigma}^{r}+p_{\sigma}^{r}\right) d x \\
= & -\int_{\mathbb{R}^{3}} \frac{n_{\sigma} p_{\sigma}}{\left(1+\sigma n_{\sigma}\right)\left(1+\sigma p_{\sigma}\right)} F\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right)\left(n_{\sigma}^{r}+p_{\sigma}^{r}\right) d x \\
& +\int_{\mathbb{R}^{3}} \mu_{n} \mu_{p} F\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}\right)\left(n_{\sigma}^{r}+p_{\sigma}^{r}\right) d x \\
\leq & C \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r}+p_{\sigma}^{r}+n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x .
\end{aligned}
$$

On the other hand, from the Poisson equation and the elementary calculation

$$
\left(a^{r+1}-b^{r+1}\right)(b-a)=-\sum_{k=0}^{r} a^{r-1} b^{k}(a-b)^{2} \leq 0, \quad \forall a, b \geq 0,
$$

we infer from (H3) that

$$
\begin{aligned}
\frac{r}{r+1} \int_{\mathbb{R}^{3}} \Delta \psi_{\sigma}\left(n_{\sigma}^{r+1}-p_{\sigma}^{r+1}\right) d x & =\frac{r}{r+1} \int_{\mathbb{R}^{3}}\left(p_{\sigma}-n_{\sigma}+D(x)\right)\left(n_{\sigma}^{r+1}-p_{\sigma}^{r+1}\right) d x \\
& \leq\|D(x)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x .
\end{aligned}
$$

Besides, it follows from (H1a) that

$$
\frac{r}{r+1} \int_{\mathbb{R}^{3}}\left(\Delta V_{n} n_{\sigma}^{r+1}+\Delta V_{p} p_{\sigma}^{r+1}\right) d x \leq K \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x .
$$

Summing up, we have

$$
\begin{aligned}
& \frac{1}{r+1} \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x+\frac{4 r}{(r+1)^{2}} \int_{\mathbb{R}^{3}}\left(\left|\nabla\left(n_{\sigma}^{\frac{r+1}{2}}\right)\right|^{2}+\left|\nabla\left(p_{\sigma}^{\frac{r+1}{2}}\right)\right|^{2}\right) d x \\
\leq & C \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{r}+p_{\sigma}^{r}+n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x \\
\leq & C \int_{\mathbb{R}^{3}}\left(1+n_{\sigma}^{r+1}+p_{\sigma}^{r+1}\right) d x .
\end{aligned}
$$

Then it follows from the Gronwall inequality that

$$
\left\|n_{\sigma}\right\|_{L^{r+1}\left(\mathbb{R}^{3}\right)}^{r+1}+\left\|p_{\sigma}\right\|_{L^{r+1}\left(\mathbb{R}^{3}\right)}^{r+1}+\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\left|\nabla\left(n_{\sigma}^{\frac{r+1}{2}}\right)\right|^{2}+\left|\nabla\left(p_{\sigma^{\frac{r+1}{2}}}\right)\right|^{2}\right) d x d t \leq C_{T}, \quad r \in \mathbb{N},
$$

provided that $\int_{\mathbb{R}^{3}}\left(n_{I}^{r+1}+p_{I}^{r+1}\right) d x<\infty$.
Since

$$
\psi_{\sigma}=\frac{1}{S_{3}} \int_{\mathbb{R}^{3}} \frac{n_{\sigma}(y, t)-p_{\sigma}(y, t)-D(y)}{|x-y|} d y
$$

we have

$$
\nabla \psi_{\sigma}(x, t)=\frac{1}{S_{3}} \int_{\mathbb{R}^{3}} \frac{\left(n_{\sigma}(y, t)-p_{\sigma}(y, t)-D(y)\right)(x-y)}{|x-y|^{3}} d y .
$$

For any $x \in \mathbb{R}^{3}$,

$$
\left|\nabla \psi_{\sigma}(x, t)\right| \leq C \int_{\mathbb{R}^{3}} \frac{\left|n_{\sigma}(y, t)-p_{\sigma}(y, t)-D(y)\right|}{|x-y|^{2}} d y .
$$

It follows from [30, Corollary 2.2] (a direct consequence of the Hardy-Littlewood-Sobolev inequality [38]) that for any $1<q^{\prime}<q<\infty$ with $\frac{1}{q}=\frac{1}{q^{\prime}}-\frac{1}{3}$,

$$
\left\|\nabla \psi_{\sigma}\right\|_{\mathbf{L}^{q}\left(\mathbb{R}^{3}\right)} \leq C\left\|n_{\sigma}-p_{\sigma}-D(x)\right\|_{L^{q^{\prime}}\left(\mathbb{R}^{3}\right)}
$$

Besides, if $n_{\sigma}, p_{\sigma} \in L^{\infty}\left(0, T ; L^{q^{\prime}}\left(\mathbb{R}^{3}\right) \cap L^{1}\left(\mathbb{R}^{3}\right)\right)$ with $q^{\prime}>3$, then we have

$$
\nabla \psi_{\sigma} \in L^{\infty}\left(0, T ; \mathbf{L}^{\infty}\left(\mathbb{R}^{3}\right)\right) .
$$

The proof is complete.

Based on the a priori estimates obtained in Lemma 2.2, we can prove existence of global solutions to problem (2.1)-(2.2).

Proposition 2.2. Suppose that all assumptions in Proposition 2.1 are satisfied. Assume in addition that $n_{I}, p_{I} \in L^{4}\left(\mathbb{R}^{3}\right)$. Then the local solution $\left(n_{\sigma}, p_{\sigma}, \psi_{\sigma}\right)$ obtained in Proposition 2.1 is global.

Proof. Multiplying the first two equations in (2.12) by $u$ and $v$ respectively, integrating on $\mathbb{R}^{3}$, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla u\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\int_{\mathbb{R}^{3}} A_{n}(x) u^{2} d x \\
= & \int_{\mathbb{R}^{3}} f_{1}\left(u, v, \psi_{\sigma}\right) u d x \\
= & K \int_{\mathbb{R}^{3}} u^{2} d x+\frac{1}{2} \int_{\mathbb{R}^{3}} u^{2} \Delta \psi_{\sigma} d x-\frac{1}{2} \int_{\mathbb{R}^{3}} u^{2} \nabla \psi_{\sigma} \cdot \nabla V_{n} d x \\
& -\int_{\mathbb{R}^{3}} u e^{\frac{V_{n}(x)}{2}} R\left(\frac{u e^{-\frac{V_{n}(x)}{2}}}{1+\sigma u e^{-\frac{V_{n}(x)}{2}}}, \frac{v e^{-\frac{V_{p}(x)}{2}}}{1+\sigma v e^{-\frac{V_{p}(x)}{2}}}, x\right) d x, \\
& \frac{1}{2} \frac{d}{d t}\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla v\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\int_{\mathbb{R}^{3}} A_{p}(x) v^{2} d x \\
= & \int_{\mathbb{R}^{3}} f_{2}\left(u, v, \psi_{\sigma}\right) u d x \\
= & K \int_{\mathbb{R}^{3}} v^{2} d x-\frac{1}{2} \int_{\mathbb{R}^{3}} v^{2} \Delta \psi_{\sigma} d x+\frac{1}{2} \int_{\mathbb{R}^{3}} v^{2} \nabla \psi_{\sigma} \cdot \nabla V_{p} d x \\
& -\int_{\mathbb{R}^{3}} v e^{\frac{V_{p}(x)}{2}} R\left(\frac{u e^{-\frac{V_{n}(x)}{2}}}{1+\sigma u e^{-\frac{V_{n}(x)}{2}}}, \frac{v e^{-\frac{V_{p}(x)}{2}}}{1+\sigma v e^{-\frac{V_{p}(x)}{2}}}, x\right) d x .
\end{aligned}
$$

From the Poisson equation for $\psi_{\sigma}$, Lemma [2.2, the Gagliardo-Nirenberg inequality

$$
\|u\|_{L^{\frac{8}{3}\left(\mathbb{R}^{3}\right)}} \leq C\|\nabla u\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{\frac{3}{8}}\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{\frac{5}{8}}
$$

and the Young inequality, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}}\left(u^{2}-v^{2}\right) \Delta \psi_{\sigma} d x \\
= & \int_{\mathbb{R}^{3}}\left(u^{2}-v^{2}\right)\left(-n_{\sigma}+p_{\sigma}\right) d x+\int_{\mathbb{R}^{3}} D(x)\left(u^{2}-v^{2}\right) d x \\
\leq & \left(\|u\|_{L^{\frac{8}{3}}\left(\mathbb{R}^{3}\right)}^{2}+\|p\|_{L^{\frac{8}{3}\left(\mathbb{R}^{3}\right)}}^{2}\right)\left(\left\|n_{\sigma}\right\|_{L^{4}\left(\mathbb{R}^{3}\right)}+\left\|p_{\sigma}\right\|_{L^{4}\left(\mathbb{R}^{3}\right)}\right) \\
& +\|D(x)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\left(\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) \\
\leq & \frac{1}{2}\left(\|\nabla u\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla v\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right)+C\left(\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) .
\end{aligned}
$$

We infer from the assumption $u_{I}, v_{I} \in L_{+}^{2}\left(\mathbb{R}^{3}\right)$ and the Cauchy-Schwarz inequality that

$$
\int_{\mathbb{R}^{3}} n_{I} d x \leq \frac{1}{2} \int_{\mathbb{R}^{3}} \mu_{n} d x+\frac{1}{2} \int_{\mathbb{R}^{3}} u_{I}^{2} d x<+\infty, \quad \int_{\mathbb{R}^{3}} p_{I} d x \leq \frac{1}{2} \int_{\mathbb{R}^{3}} \mu_{p} d x+\frac{1}{2} \int_{\mathbb{R}^{3}} v_{I}^{2} d x<+\infty,
$$

namely, $n_{I}, p_{I} \in L^{1}\left(\mathbb{R}^{3}\right)$. Lemma 2.2 implies that if the initial data satisfy $n_{I}, p_{I} \in L^{4}\left(\mathbb{R}^{3}\right) \cap$ $L^{1}\left(\mathbb{R}^{3}\right)$, then the estimate (2.17) holds. As a consequence, we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{3}} u^{2} \nabla \psi_{\sigma} \cdot \nabla V_{n} d x\right| & \leq\left\|\nabla \psi_{\sigma}\right\|_{\mathbf{L}^{\infty}\left(\mathbb{R}^{3}\right)}\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}\left(\int_{\mathbb{R}^{3}} u^{2}\left|\nabla V_{n}\right|^{2} d x\right)^{\frac{1}{2}} \\
& \leq \frac{1}{8} \int_{\mathbb{R}^{3}} u^{2}\left|\nabla V_{n}\right|^{2} d x+C_{T}\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{3}} v^{2} \nabla \psi_{\sigma} \cdot \nabla V_{p} d x\right| & \leq\left\|\nabla \psi_{\sigma}\right\|_{\mathbf{L}^{\infty}\left(\mathbb{R}^{3}\right)}\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}\left(\int_{\mathbb{R}^{3}} v^{2}\left|\nabla V_{p}\right|^{2} d x\right)^{\frac{1}{2}} \\
& \leq \frac{1}{8} \int_{\mathbb{R}^{3}} v^{2}\left|\nabla V_{p}\right|^{2} d x+C_{T}\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} .
\end{aligned}
$$

It follows from the nonnegativity of $(u, v)$ and assumptions (H1b), (H2b) that

$$
\begin{aligned}
& -\int_{\mathbb{R}^{3}}\left(u e^{\frac{V_{n}(x)}{2}}+v e^{\frac{V_{p}(x)}{2}}\right) R\left(\frac{u e^{-\frac{V_{n}(x)}{2}}}{1+\sigma u e^{-\frac{V_{n}(x)}{2}}}, \frac{v e^{-\frac{V_{p}(x)}{2}}}{1+\sigma v e^{-\frac{V_{p}(x)}{2}}}, x\right) d x \\
\leq & \int_{\mathbb{R}^{3}}\left(u e^{\frac{V_{n}(x)}{2}}+v e^{\frac{V_{p}(x)}{2}}\right) F\left(\frac{u e^{-\frac{V_{n}(x)}{2}}}{\left.1+\sigma u e^{-\frac{V_{n}(x)}{2}}, \frac{v e^{-\frac{V_{p}(x)}{2}}}{1+\sigma v e^{-\frac{V_{p}(x)}{2}}}\right) \mu_{n} \mu_{p} d x}\right. \\
\leq & C \int_{\mathbb{R}^{3}}\left(u e^{\frac{V_{n}(x)}{2}}+v e^{\frac{V_{p}(x)}{2}}\right)\left(\mu_{n} \mu_{p}+u e^{-\frac{V_{n}(x)}{2}}+v e^{-\frac{V_{p}(x)}{2}}\right) d x \\
\leq & C\left(1+\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) .
\end{aligned}
$$

Recalling the definitions of $A_{n}, A_{p}$ and (1.7), we have

$$
\begin{align*}
\int_{\mathbb{R}^{3}} A_{n}(x) u^{2} d x & =\int_{\mathbb{R}^{3}}\left(\frac{1}{4}\left|\nabla V_{n}\right|^{2}-\frac{1}{2} \Delta V_{n}+K\right) u^{2} d x \\
& \geq \frac{1}{4} \int_{\mathbb{R}^{3}} u^{2}\left|\nabla V_{n}\right|^{2} d x \tag{2.19}
\end{align*}
$$

and

$$
\begin{align*}
\int_{\mathbb{R}^{3}} A_{p}(x) v^{2} d x & =\int_{\mathbb{R}^{3}}\left(\frac{1}{4}\left|\nabla V_{p}\right|^{2}-\frac{1}{2} \Delta V_{p}+K\right) v^{2} d x \\
& \geq \frac{1}{4} \int_{\mathbb{R}^{3}} v^{2}\left|\nabla V_{p}\right|^{2} d x \tag{2.20}
\end{align*}
$$

As a result, we have

$$
\begin{aligned}
& \frac{d}{d t}\left(\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right)+\|\nabla u\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla v\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& +\frac{1}{4} \int_{\mathbb{R}^{3}} u^{2}\left|\nabla V_{n}\right|^{2} d x+\frac{1}{4} \int_{\mathbb{R}^{3}} v^{2}\left|\nabla V_{p}\right|^{2} d x \\
\leq & C_{T}\left(1+\|u\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|v\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) .
\end{aligned}
$$

It follows from the Gronwall inequality that for any $T>0$ and $t \in[0, T]$,

$$
\begin{equation*}
\|u(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}+\|v(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)} \leq C_{T}, \tag{2.21}
\end{equation*}
$$

$$
\begin{align*}
& \int_{0}^{T}\left(\|\nabla u\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla v\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d t \leq C_{T}  \tag{2.22}\\
& \int_{0}^{T} \int_{\mathbb{R}^{3}}\left(u^{2}\left|\nabla V_{n}\right|^{2} d x+v^{2}\left|\nabla V_{p}\right|^{2}\right) d x d t \leq C_{T} \tag{2.23}
\end{align*}
$$

where $C_{T}$ is a constant depending on $T, c_{2}, V_{b},\left\|u_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)},\left\|v_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)},\left\|n_{I}\right\|_{L^{4}\left(\mathbb{R}^{3}\right)},\left\|p_{I}\right\|_{L^{4}\left(\mathbb{R}^{3}\right)}$ but it is independent of $\sigma$.

Then we are able to extend the local solution $\left(n_{\sigma}, p_{\sigma}, \psi_{\sigma}\right)$ obtained in Proposition 2.1 to the interval $[0, T]$ for arbitrary $T>0$. The proof is complete.

Proof Theorem [2.1, Recalling the transformation (2.11), we conclude Theorem 2.1 from Propositions 2.1, 2.2. The proof is complete.

## 3 Well-posedness of the Original Problem

In this section, we prove the existence and uniqueness of global solutions to the Cauchy problem of original system (1.1). For this purpose, we shall derive some a priori estimates on the solutions $\left(n_{\sigma}, p_{\sigma}, \psi_{\sigma}\right)$ of the approximate problem that are uniform in the parameter $\sigma>0$. Then, we pass to the limit as $\sigma \rightarrow 0$ to achieve our goal. In Lemma [2.2, we have already shown the uniform estimates on $\left\|n_{\sigma}(t)\right\|_{L^{r}\left(\mathbb{R}^{3}\right)},\left\|p_{\sigma}(t)\right\|_{L^{r}\left(\mathbb{R}^{3}\right)}$ on arbitrary interval $[0, T]$. Next, we proceed to obtain uniform estimates on the $L^{\infty}$ norms of $n_{\sigma}$ and $p_{\sigma}$ via a Stampacchia-type $L^{\infty}$ estimation technique (cf. [9, 33]). The following technical lemma plays an important role in the proof. It shows that a nonnegative, non-increasing function will vanish at some finite value under suitable growth condition that indicates certain rapid decay of the function.

Lemma 3.1. Suppose that $\omega(k)$ is a nonnegative non-increasing function on $\left[k_{0},+\infty\right)$, and there are positive constants $\gamma, \beta$ such that

$$
\omega(\hat{k}) \leq M(k)(\hat{k}-k)^{-\gamma} \omega(k)^{1+\beta}, \quad \forall \hat{k}>k \geq k_{0}
$$

where the function $M(k)$ is non-decreasing and satisfies

$$
0 \leq k^{-\gamma} M(k) \leq M_{0}, \quad \forall k \in\left[k_{0},+\infty\right)
$$

Then

$$
\omega\left(k^{*}\right)=0 \quad \text { with } \quad k^{*}=2 k_{0}\left(1+2^{\frac{1+2 \beta}{\beta^{2}}} M_{0}^{\frac{1+\beta}{\beta \gamma}} \omega\left(k_{0}\right)^{\frac{1+\beta}{\gamma}}\right) .
$$

Remark 3.1. Readers may refer to [9, Lemma 2.3] for the proof of Lemma 3.1. Besides, we note that the conclusion of Lemma 3.1 implies that $\omega(k)=0$ for all $k \geq k^{*}$.

Lemma 3.2. Suppose that all assumptions in Theorem 2.1 are satisfied. Assume in addition that $n_{I}, p_{I} \in L^{\infty}\left(\mathbb{R}^{3}\right)$. Then for any $T>0$, we have

$$
\begin{equation*}
\left\|n_{\sigma}(t)\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq C_{T}, \quad\left\|p_{\sigma}(t)\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq C_{T}, \quad 0 \leq t \leq T \tag{3.1}
\end{equation*}
$$

where the constant $C_{T}$ is independent of $\sigma>0$.

Proof. Denote

$$
\begin{equation*}
k_{0}:=\max \left\{\left\|n_{I}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)},\left\|p_{I}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right\} \geq 0 . \tag{3.2}
\end{equation*}
$$

For any $k \geq k_{0}$, we introduce the sets

$$
B_{n k}(t)=\left\{x \in \mathbb{R}^{3}, n_{\sigma}(x, t)>k\right\}, \quad B_{p k}(t)=\left\{x \in \mathbb{R}^{3}, p_{\sigma}(x, t)>k\right\},
$$

and for arbitrary $T>0$, we set

$$
\omega_{T}(k)=\sup _{0 \leq t \leq T}\left(\left|B_{n k}(t)\right|+\left|B_{p k}(t)\right|\right) .
$$

It is obvious that $\omega_{T}(k)$ is a nonnegative, non-increasing function on $\left[k_{0},+\infty\right)$. Moreover, we infer from the $L^{1}$-estimate (2.18) that for arbitrary but fixed $T>0, \omega_{T}(k)$ is bounded.

For any $k \geq k_{0}$ and $f(\cdot) \in H^{1}\left(\mathbb{R}^{3}\right)$, it follows from $k \geq 0$ and $(f-k)^{+} \leq|f|$ that $(f-k)^{+} \in L^{2}\left(\mathbb{R}^{3}\right)$. Moreover, for the weak derivative of $(f-k)^{+}$, we have (see [24, Lemma 7.6] or [33, Theorem 1.56])

$$
\nabla(f-k)^{+}= \begin{cases}\nabla f & \text { if } f>k \\ 0 & \text { if } f \leq k\end{cases}
$$

Hence, $(f-k)^{+} \in H^{1}\left(\mathbb{R}^{3}\right)$. Recalling Theorem 2.1 and the lower boundedness of $V_{n}$ and $V_{p}$, we find that

$$
n_{\sigma}, \quad p_{\sigma} \in L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{3}\right)\right),
$$

which indicates

$$
\left(n_{\sigma}-k\right)^{+}, \quad\left(p_{\sigma}-k\right)^{+} \in L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{3}\right)\right) .
$$

Now, multiplying the first and second equation in (2.1) by $\left(n_{\sigma}-k\right)^{+}$and $\left(p_{\sigma}-k\right)^{+}\left(k \geq k_{0}\right)$, respectively, integrating on $\mathbb{R}^{3} \times(0, t)$ and adding the resultants together, we obtain

$$
\begin{align*}
& \frac{1}{2}\left(\left\|\left(n_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) \\
& +\int_{0}^{t}\left(\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
= & \int_{0}^{t} \int_{\mathbb{R}^{3}}\left[\operatorname{div}\left(n_{\sigma} \nabla\left(\psi_{\sigma}+V_{n}\right)\right)\left(n_{\sigma}-k\right)^{+}+\operatorname{div}\left(p_{\sigma} \nabla\left(-\psi_{\sigma}+V_{p}\right)\right)\left(p_{\sigma}-k\right)^{+}\right] d x d \tau \\
& -\int_{0}^{t} \int_{\mathbb{R}^{3}} R\left(\frac{n_{\sigma}}{1+\sigma n_{\sigma}}, \frac{p_{\sigma}}{1+\sigma p_{\sigma}}, x\right)\left[\left(n_{\sigma}-k\right)^{+}+\left(p_{\sigma}-k\right)^{+}\right] d x d \tau \\
=: & \int_{0}^{t}\left(I_{1}+I_{2}\right) d \tau \tag{3.3}
\end{align*}
$$

where we have used the facts that $\left\|\left(n_{I}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}=\left\|\left(p_{I}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}=0$ due to the assumptions $n_{I}, p_{I} \in L^{\infty}\left(\mathbb{R}^{3}\right)$ and $k \geq k_{0}$ (cf. (3.2)).

Integrating by parts and using the Poisson equation for $\psi_{\sigma}$, we expand the term $I_{1}$ as follows

$$
\begin{aligned}
I_{1}= & \int_{\mathbb{R}^{3}} \operatorname{div}\left[\left(n_{\sigma}-k\right) \nabla\left(\psi_{\sigma}+V_{n}\right)\right]\left(n_{\sigma}-k\right)^{+} d x+k \int_{\mathbb{R}^{3}} \Delta\left(\psi_{\sigma}+V_{n}\right)\left(n_{\sigma}-k\right)^{+} d x \\
& +\int_{\mathbb{R}^{3}} \operatorname{div}\left[\left(p_{\sigma}-k\right) \nabla\left(-\psi_{\sigma}+V_{p}\right)\right]\left(p_{\sigma}-k\right)^{+} d x+k \int_{\mathbb{R}^{3}} \Delta\left(-\psi_{\sigma}+V_{p}\right)\left(p_{\sigma}-k\right)^{+} d x
\end{aligned}
$$

$$
\begin{align*}
= & k \int_{\mathbb{R}^{3}}\left[\Delta\left(\psi_{\sigma}+V_{n}\right)\left(n_{\sigma}-k\right)^{+}+\Delta\left(-\psi_{\sigma}+V_{p}\right)\left(p_{\sigma}-k\right)^{+}\right] d x \\
& -\int_{\mathbb{R}^{3}}\left(n_{\sigma}-k\right) \nabla\left(\psi_{\sigma}+V_{n}\right) \cdot \nabla\left(n_{\sigma}-k\right)^{+} d x \\
& -\int_{\mathbb{R}^{3}}\left(p_{\sigma}-k\right) \nabla\left(-\psi_{\sigma}+V_{p}\right) \cdot \nabla\left(p_{\sigma}-k\right)^{+} d x \\
= & k \int_{\mathbb{R}^{3}} \Delta \psi_{\sigma}\left[\left(n_{\sigma}-k\right)^{+}-\left(p_{\sigma}-k\right)^{+}\right] d x+k \int_{\mathbb{R}^{3}}\left[\Delta V_{n}\left(n_{\sigma}-k\right)^{+}+\Delta V_{p}\left(p_{\sigma}-k\right)^{+}\right] d x \\
& -\int_{\mathbb{R}^{3}}\left(n_{\sigma}-k\right)^{+} \nabla\left(\psi_{\sigma}+V_{n}\right) \cdot \nabla\left(n_{\sigma}-k\right)^{+} d x \\
& -\int_{\mathbb{R}^{3}}\left(p_{\sigma}-k\right)^{+} \nabla\left(-\psi_{\sigma}+V_{p}\right) \cdot \nabla\left(p_{\sigma}-k\right)^{+} d x \\
= & k \int_{\mathbb{R}^{3}}\left(-n_{\sigma}+p_{\sigma}+D(x)\right)\left[\left(n_{\sigma}-k\right)^{+}-\left(p_{\sigma}-k\right)^{+}\right] d x \\
& +k \int_{\mathbb{R}^{3}}\left[\Delta V_{n}\left(n_{\sigma}-k\right)^{+}+\Delta V_{p}\left(p_{\sigma}-k\right)^{+}\right] d x \\
& -\frac{1}{2} \int_{\mathbb{R}^{3}} \nabla\left(\psi_{\sigma}+V_{n}\right) \cdot \nabla\left[\left(n_{\sigma}-k\right)^{+}\right]^{2} d x-\frac{1}{2} \int_{\mathbb{R}^{3}} \nabla\left(-\psi_{\sigma}+V_{p}\right) \cdot \nabla\left[\left(p_{\sigma}-k\right)^{+}\right]^{2} d x \\
= & k \int_{\mathbb{R}^{3}}\left(-n_{\sigma}+p_{\sigma}\right)\left[\left(n_{\sigma}-k\right)^{+}-\left(p_{\sigma}-k\right)^{+}\right] d x \\
& +k \int_{\mathbb{R}^{3}} D(x)\left[\left(n_{\sigma}-k\right)^{+}-\left(p_{\sigma}-k\right)^{+}\right] d x \\
& +k \int_{\mathbb{R}^{3}}\left(\Delta V_{n}\left(n_{\sigma}-k\right)^{+}+\Delta V_{p}\left(p_{\sigma}-k\right)^{+}\right) d x \\
& +\frac{1}{2} \int_{\mathbb{R}^{3}}\left(-n_{\sigma}+p_{\sigma}\right)\left(\left[\left(n_{\sigma}-k\right)^{+}\right]^{2}-\left[\left(p_{\sigma}-k\right)^{+}\right]^{2}\right) d x \\
& +\frac{1}{2} \int_{\mathbb{R}^{3}} D(x)\left(\left[\left(n_{\sigma}-k\right)^{+}\right]^{2}-\left[\left(p_{\sigma}-k\right)^{+}\right]^{2}\right) d x \\
& +\frac{1}{2} \int_{\mathbb{R}^{3}}\left(\Delta V_{n}\left[\left(n_{\sigma}-k\right)^{+}\right]^{2}+\Delta V_{p}\left[\left(p_{\sigma}-k\right)^{+}\right]^{2}\right) d x \\
:= & J_{1}+\ldots+J_{6} . \tag{3.4}
\end{align*}
$$

It is easy to verify that

$$
\begin{equation*}
J_{1} \leq 0, \quad J_{4} \leq 0, \quad \forall k \geq k_{0} \tag{3.5}
\end{equation*}
$$

Besides, it follows from (H2b) that

$$
\begin{equation*}
I_{2} \leq c_{2} \int_{\mathbb{R}^{3}} \mu_{n} \mu_{p}\left(1+n_{\sigma}+p_{\sigma}\right)\left[\left(n_{\sigma}-k\right)^{+}+\left(p_{\sigma}-k\right)^{+}\right] d x:=J_{7} \tag{3.6}
\end{equation*}
$$

Then we infer from (3.3)-(3.6) that

$$
\begin{aligned}
& \frac{1}{2}\left(\left\|\left(n_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) \\
& +\int_{0}^{t}\left(\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
\leq & \int_{0}^{t}\left(J_{2}+J_{3}+J_{5}+J_{6}+J_{7}\right) d \tau .
\end{aligned}
$$

By assumptions (H1b) and (H3), we have

$$
\begin{aligned}
& \left|\int_{0}^{t} J_{5} d \tau\right|+\left|\int_{0}^{t} J_{6} d \tau\right| \\
\leq & \frac{1}{2}\|D(x)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\left\|\Delta V_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau+\left\|\Delta V_{p}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \int_{0}^{t}\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
\leq & C \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau .
\end{aligned}
$$

Let $\eta>0$ be a small constant to be chosen later. Using the Hölder inequality and GagliardoNirenberg inequality, we get

$$
\begin{aligned}
& \left|\int_{0}^{t} J_{2} d \tau\right|+\left|\int_{0}^{t} J_{3} d \tau\right| \\
\leq & k \int_{0}^{t} \int_{\mathbb{R}^{3}}\left(|D(x)|+\left|\Delta V_{n}\right|\right)\left(n_{\sigma}-k\right)^{+} d x d \tau \\
& +k \int_{0}^{t} \int_{\mathbb{R}^{3}}\left(|D(x)|+\left|\Delta V_{p}\right|\right)\left(p_{\sigma}-k\right)^{+} d x d \tau \\
\leq & k\left(\|D(x)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta V_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{3}\left(\mathbb{R}^{3}\right)}\left|B_{n k}\right|^{\frac{2}{3}} d \tau \\
& +k\left(\|D(x)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta V_{p}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{0}^{t}\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{3}\left(\mathbb{R}^{3}\right)}\left|B_{p k}\right|^{\frac{2}{3}} d \tau \\
\leq & C k \omega_{T}(k)^{\frac{2}{3}} \int_{0}^{t}\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{\frac{1}{2}}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{\frac{1}{2}} d \tau \\
& +C k \omega_{T}(k)^{\frac{2}{3}} \int_{0}^{t}\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{\frac{1}{2}}\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{\frac{1}{2}} d \tau \\
\leq & \eta T \int_{0}^{t}\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\eta T \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\frac{C}{\eta} k^{2} \omega_{T}(k)^{\frac{4}{3}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\int_{0}^{t} J_{7} d \tau\right| \\
\leq & \int_{0}^{t} \int_{\mathbb{R}^{3}} \mu_{n} \mu_{p}\left[\left(n_{\sigma}-k\right)^{+}+\left(p_{\sigma}-k\right)^{+}+2 k+1\right]\left[\left(n_{\sigma}-k\right)^{+}+\left(p_{\sigma}-k\right)^{+}\right] d x d \tau \\
\leq & C \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +C(2 k+1) \int_{0}^{t} \int_{\mathbb{R}^{3}}\left(n_{\sigma}-k\right)^{+} d x d \tau+C(2 k+1) \int_{0}^{t} \int_{\mathbb{R}^{3}}\left(p_{\sigma}-k\right)^{+} d x d \tau \\
\leq & C \int_{0}^{t}\left(\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau
\end{aligned}
$$

$$
\begin{aligned}
& +C(2 k+1) \omega_{T}(k)^{\frac{2}{3}} \int_{0}^{t}\left(\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{3}\left(\mathbb{R}^{3}\right)}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{3}\left(\mathbb{R}^{3}\right)}\right) d \tau \\
\leq & \eta T \int_{0}^{t}\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +(C+\eta T) \int_{0}^{t}\left\|\left(n_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\frac{C}{\eta}(2 k+1)^{2} \omega_{T}(k)^{\frac{4}{3}}
\end{aligned}
$$

Taking

$$
\eta=\frac{1}{2 T}
$$

in the above estimates, we obtain that

$$
\begin{aligned}
& \left\|\left(n_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& \quad+\int_{0}^{t}\left(\left\|\nabla\left(n_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\nabla\left(p_{\sigma}-k\right)^{+}\right\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
& \leq \\
& C_{1} \int_{0}^{t}\left(\left\|\left(n_{\sigma}(\tau)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}(\tau)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
& \quad+T C_{2}\left(k^{2}+1\right) \omega_{T}(k)^{\frac{4}{3}} .
\end{aligned}
$$

It follows from the Gronwall inequality that for $t \in[0, T]$

$$
\begin{equation*}
\left\|\left(n_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|\left(p_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} \leq e^{C_{1} T} T C_{2}\left(k^{2}+1\right) \omega_{T}(k)^{\frac{4}{3}} . \tag{3.7}
\end{equation*}
$$

On the other hand, for any $t \in[0, T]$ and $\hat{k}>k \geq k_{0}$,

$$
\begin{align*}
\left\|\left(n_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} & =\int_{B_{n k}(t)}\left|\left(n_{\sigma}(t, \cdot)-k\right)^{+}\right|^{2} d x \\
& \geq \int_{B_{n \hat{k}}(t)}\left(n_{\sigma}(t, \cdot)-k\right)^{2} d x \\
& \geq(\hat{k}-k)^{2}\left|B_{n \hat{k}}(t)\right|,  \tag{3.8}\\
\left\|\left(p_{\sigma}(t)-k\right)^{+}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} & =\int_{B_{p k}(t)}\left|\left(p_{\sigma}(t, \cdot)-k\right)^{+}\right|^{2} d x \\
& \geq \int_{B_{p \hat{k}}(t)}\left(p_{\sigma}(t, \cdot)-k\right)^{2} d x \\
& \geq(\hat{k}-k)^{2}\left|B_{p \hat{k}}(t)\right| . \tag{3.9}
\end{align*}
$$

We deduce from (3.7)-(3.9) that

$$
\omega_{T}(\hat{k}) \leq e^{C_{1} T} T C_{2}\left(k^{2}+1\right)(\hat{k}-k)^{-2} \omega_{T}(k)^{\frac{4}{3}}, \quad \forall \hat{k}>k \geq k_{0} .
$$

Now in Lemma 3.1, we set

$$
M(k)=e^{C_{1} T} T C_{2}\left(k^{2}+1\right) \geq 0, \quad \gamma=2, \quad \beta=\frac{1}{3} .
$$

The function $M(k)$ has the following property

$$
\frac{M(k)}{k^{2}}=\frac{k^{2}+1}{k^{2}} e^{C_{1} T} T C_{2} \leq\left(1+\frac{1}{k_{0}^{2}}\right) e^{C_{1} T} T C_{2}:=M_{0}, \quad \forall k \in\left[k_{0},+\infty\right)
$$

Therefore, there exists a constant

$$
k^{*}=2 k_{0}\left(1+2^{15} M_{0}^{2} \omega_{T}\left(k_{0}\right)^{\frac{2}{3}}\right)>k_{0}
$$

which is independent of $\sigma$ such that

$$
\omega_{T}\left(k^{*}\right)=0 .
$$

Namely,

$$
n_{\sigma}(x, t) \leq k^{*} \quad \text { and } \quad p_{\sigma}(x, t) \leq k^{*}, \quad \text { a.e. in } \mathbb{R}^{3} \times[0, T] .
$$

The proof is complete.
Proof of Theorem 1.1. It follows from Lemma [2.2, (2.22) and Lemma 3.2 that the following uniform estimates (independent of the parameter $\sigma>0$ ) hold:

$$
\begin{gather*}
\left\|n_{\sigma}\right\|_{L^{\infty}\left(0, T ; L^{q}\left(\mathbb{R}^{3}\right)\right)}+\left\|p_{\sigma}\right\|_{L^{\infty}\left(0, T ; L^{q}\left(\mathbb{R}^{3}\right)\right)} \leq C_{T},  \tag{3.10}\\
\left\|n_{\sigma}\right\|_{L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{3}\right)\right)}+\left\|p_{\sigma}\right\|_{L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{3}\right)\right)} \leq C_{T},  \tag{3.11}\\
\left\|\nabla \psi_{\sigma}\right\|_{L^{\infty}\left(0, T ; \mathbf{L}^{q^{\prime}}\left(\mathbb{R}^{3}\right)\right)}+\left\|\Delta \psi_{\sigma}\right\|_{L^{\infty}\left(0, T ; L^{q}\left(\mathbb{R}^{3}\right)\right)} \leq C_{T}, \tag{3.12}
\end{gather*}
$$

where $q \in[1,+\infty], q^{\prime} \in\left[\frac{3}{2},+\infty\right]$.
Besides, we infer from (2.23), (2.11) and (H1a) that

$$
\begin{align*}
& \int_{0}^{T} \int_{\mathbb{R}^{3}}\left(n_{\sigma}^{2}\left|\nabla V_{n}\right|^{2} d x+p_{\sigma}^{2}\left|\nabla V_{p}\right|^{2}\right) d x d t \\
\leq & e^{-V_{b}} \int_{0}^{T} \int_{\mathbb{R}^{3}}\left(u^{2}\left|\nabla V_{n}\right|^{2} d x+v^{2}\left|\nabla V_{p}\right|^{2}\right) d x d t \\
\leq & C_{T} . \tag{3.13}
\end{align*}
$$

Then by the equations for $n_{\sigma}$ and $p_{\sigma}$ in (2.1) and (3.10)-(3.13), we obtain that

$$
\left\|\partial_{t} n_{\sigma}\right\|_{L^{2}\left(0, T ;\left(H^{1}\left(\mathbb{R}^{3}\right)\right)^{\prime}\right)}+\left\|\partial_{t} p_{\sigma}\right\|_{L^{2}\left(0, T ;\left(H^{1}\left(\mathbb{R}^{3}\right)\right)^{\prime}\right)} \leq C_{T}
$$

From the uniform estimates (3.10)-(3.12), we deduce that there exist

$$
n, p \in L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)
$$

with

$$
\nabla u, \nabla p \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right), \quad \partial_{t} u, \partial_{t} p \in L^{2}\left(0, T ;\left(H^{1}\left(\mathbb{R}^{3}\right)\right)^{\prime}\right)
$$

and $\psi$ with $\Delta \psi \in L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right), \nabla \psi \in L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right)$ such that for a sequence $\left\{\sigma_{j}\right\} \searrow$ 0 as $j \rightarrow+\infty$ (not relabeled when taking a subsequence),

$$
\begin{aligned}
& n_{\sigma_{j}} \rightharpoonup n, \quad p_{\sigma_{j}} \rightharpoonup p, \quad \text { weakly-star in } L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right), \\
& \nabla n_{\sigma_{j}} \rightharpoonup \nabla n, \quad \nabla p_{\sigma_{j}} \rightharpoonup \nabla p, \quad \text { weakly in } L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right),
\end{aligned}
$$

$$
\begin{array}{ll}
\partial_{t} n_{\sigma_{j}} \rightharpoonup \partial_{t} n, & \partial_{t} p_{\sigma_{j}} \rightharpoonup \partial_{t} p, \quad \text { weakly in } L^{2}\left(0, T ;\left(H^{1}\left(\mathbb{R}^{3}\right)\right)^{\prime}\right), \\
\Delta \psi_{\sigma_{j}} \rightharpoonup \Delta \psi, & \text { weakly-star in } L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right), \\
\nabla \psi_{\sigma_{j}} \rightharpoonup \nabla \psi, & \text { weakly in } L^{2}\left(0, T ; \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right) .
\end{array}
$$

Moreover, on account of the compact embedding theorem we have (up to a subsequence if necessary and without relabelling for the sake of simplicity)

$$
n_{\sigma_{j}} \rightarrow n, \quad p_{\sigma_{j}} \rightarrow p, \quad \text { strongly in } L^{2}\left(0, T ; L_{l o c}^{2}\left(\mathbb{R}^{3}\right)\right), \quad \text { thus also a.e. in } \mathbb{R}^{3} \times(0, T) .
$$

Then, for any $\varphi \in L^{2}\left(0, T ; C_{c}^{\infty}\left(\mathbb{R}^{3}\right)\right)$, we have

$$
\begin{aligned}
& \left|\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(n_{\sigma_{j}} \nabla \psi_{\sigma_{j}}-n \nabla \psi\right) \nabla \varphi d x d t\right| \\
\leq & \left|\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(n_{\sigma_{j}}-n\right) \nabla \psi_{\sigma_{j}} \nabla \varphi d x d t\right|+\left|\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\nabla \psi_{\sigma_{j}}-\nabla \psi\right) n \nabla \varphi d x d t\right| \\
\leq & \left\|\nabla \psi_{\sigma_{j}}\right\|_{L^{\infty}\left(0, T ; L^{2}\left(\mathbb{R}^{3}\right)\right)}\left\|n_{\sigma_{j}}-n\right\|_{L^{2}\left(0, T ; L^{2}(\operatorname{supp} \varphi)\right)}\|\nabla \varphi\|_{L^{2}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)} \\
& +\left|\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\nabla \psi_{\sigma_{j}}-\nabla \psi\right)(n \nabla \varphi) d x d t\right| \\
\rightarrow & 0, \quad \text { as } \sigma_{j} \rightarrow 0,
\end{aligned}
$$

similarly,

$$
\left|\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(p_{\sigma_{j}} \nabla \psi_{\sigma_{j}}-p \nabla \psi\right) \nabla \varphi d x d t\right| \rightarrow 0, \quad \text { as } \sigma_{j} \rightarrow 0
$$

Next, we study the convergence of the recombination-generation rate. The uniform bound (3.10) and (H2b) yield that

$$
\left\|\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right)\right\|_{L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{3}\right)\right)} \leq C_{T} .
$$

Thus there exists $G \in L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{3}\right)\right)$ such that (up to a subsequence)

$$
\begin{equation*}
\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right) \rightarrow G, \text { weakly in } L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{3}\right)\right) \quad \text { as } \sigma_{j} \rightarrow 0 . \tag{3.14}
\end{equation*}
$$

For any bounded domain $\Omega \subset \mathbb{R}^{3}$, there holds

$$
\begin{aligned}
\int_{0}^{T} \int_{\Omega}\left(\frac{n_{\sigma_{j}}}{1+\sigma_{j} n_{\sigma_{j}}}-n\right)^{2} d x d t & \leq \int_{0}^{T} \int_{\Omega}\left(n_{\sigma_{j}}-n\right)^{2} d x d t+\int_{0}^{T} \int_{\Omega}\left(\sigma_{j} n_{\sigma_{j}} n\right)^{2} d x d t \\
& \rightarrow 0, \text { as } \sigma_{j} \rightarrow 0,
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\frac{n_{\sigma_{j}}}{1+\sigma_{j} n_{\sigma_{j}}} \rightarrow n \quad \text { strongly in } L^{2}\left(0, T ; L_{l o c}^{2}\left(\mathbb{R}^{3}\right)\right) . \tag{3.15}
\end{equation*}
$$

In the same manner, we have

$$
\begin{equation*}
\frac{p_{\sigma_{j}}}{1+\sigma_{j} p_{\sigma_{j}}} \rightarrow p \quad \text { strongly in } L^{2}\left(0, T ; L_{l o c}^{2}\left(\mathbb{R}^{3}\right)\right) . \tag{3.16}
\end{equation*}
$$

Since $F$ is Lip-continuous (see (H2b)), we infer from (3.15) and (3.16) that on any bounded domain $\Omega \subset \mathbb{R}^{3}$, the following convergence (up to a subsequence)

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega}\left[F\left(\frac{n_{\sigma_{j}}}{1+\sigma_{j} n_{\sigma_{j}}}, \frac{p_{\sigma_{j}}}{1+\sigma_{j} p_{\sigma_{j}}}\right)-F(n, p)\right]^{2} d x d t \\
\leq & C \int_{0}^{T} \int_{\Omega}\left(\frac{n_{\sigma_{j}}}{1+\sigma_{j} n_{\sigma_{j}}}-n\right)^{2}+\left(\frac{p_{\sigma_{j}}}{1+\sigma_{j} p_{\sigma_{j}}}-p\right)^{2} d x d t \\
\rightarrow & 0, \quad \text { as } \sigma_{j} \rightarrow 0,
\end{aligned}
$$

namely,

$$
\begin{align*}
& F\left(\frac{n_{\sigma_{j}}}{1+\sigma_{j} n_{\sigma_{j}}}, \frac{p_{\sigma_{j}}}{1+\sigma_{j} p_{\sigma_{j}}}\right) \rightarrow F(n, p), \\
& \quad \text { strongly in } L^{2}\left(0, T ; L_{l o c}^{2}\left(\mathbb{R}^{3}\right)\right) \text { and a.e. in } \mathbb{R}^{3} \times(0, T) . \tag{3.17}
\end{align*}
$$

As a result, we have the point-wise convergence of $\tilde{R}$ (up to a subsequence)

$$
\begin{equation*}
\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right) \rightarrow R(n, p, x), \quad \text { a.e. in } \mathbb{R}^{3} \times(0, T), \tag{3.18}
\end{equation*}
$$

which together with (3.14) implies that $G=R(n, p, x)$ and

$$
\tilde{R}\left(n_{\sigma}, p_{\sigma}, x\right) \rightharpoonup R(n, p, x), \quad \text { weakly in } L^{2}\left(0, T ; L^{2}\left(\mathbb{R}^{3}\right)\right) \quad \text { as } \sigma_{j} \rightarrow 0
$$

Based on the above convergent results, now we are able to pass to the limit by letting $\sigma_{j} \rightarrow 0$ in the approximate problem (2.1) and obtain a global weak solution ( $n, p, \psi$ ) of problem (1.1)-(1.2). The system (1.1) is satisfied in the following sense that for any $\varphi \in$ $C_{0}^{\infty}\left([0, T) \times \mathbb{R}^{3}\right)$,

$$
\begin{align*}
& -\int_{0}^{T} \int_{\mathbb{R}^{3}} n \varphi_{t} d x d t+\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\nabla n+n \nabla\left(\psi+V_{n}\right)\right) \cdot \nabla \varphi d x d t \\
& \quad+\int_{0}^{T} \int_{\mathbb{R}^{3}} R(n, p, x) \varphi d x d t=\int_{\mathbb{R}^{3}} n_{I} \varphi(x, 0) d x,  \tag{3.19}\\
& -\int_{0}^{T} \int_{\mathbb{R}^{3}} p \varphi_{t} d x d t+\int_{0}^{T} \int_{\mathbb{R}^{3}}\left(\nabla p+p \nabla\left(-\psi+V_{p}\right)\right) \cdot \nabla \varphi d x d t \\
& \quad+\int_{0}^{T} \int_{\mathbb{R}^{3}} R(n, p, x) \varphi d x d t=\int_{\mathbb{R}^{3}} p_{I} \varphi(x, 0) d x,  \tag{3.20}\\
& \int_{0}^{T} \int_{\mathbb{R}^{3}} \nabla \psi \cdot \nabla \varphi d x d t=\int_{0}^{T} \int_{\mathbb{R}^{3}}(n-p-D) \varphi d x d t . \tag{3.21}
\end{align*}
$$

Finally, we prove the uniqueness of global solutions to problem (1.1)-(1.2). Let $\left(n_{i}, p_{i}, \psi_{i}\right)$ $(i=1,2)$ be two solutions to problem (1.1)-(1.2) with initial data $n_{I i}, p_{I i}$. Set now

$$
n=n_{1}-n_{2}, \quad p=p_{1}-p_{2}, \quad \psi=\psi_{1}-\psi_{2}, \quad n_{I}=n_{I 1}-n_{I 2}, \quad p_{I}=p_{I 1}-p_{I 2}
$$

Taking the difference of the equations for $n_{1}$ and $n_{2}$, testing the resultant by $n$, we find that

$$
\frac{1}{2}\|n(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\int_{0}^{t}\|\nabla n\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau
$$

$$
\begin{align*}
= & \frac{1}{2}\left\|n_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}-\int_{0}^{t} \int_{\mathbb{R}^{3}} n \nabla n \cdot \nabla V_{n} d x d \tau-\int_{0}^{t} \int_{\mathbb{R}^{3}}\left(n \nabla \psi_{1}+n_{2} \nabla \psi\right) \nabla n d x d \tau \\
& -\int_{0}^{t} \int_{\mathbb{R}^{3}}\left(R\left(n_{1}, p_{1}, x\right)-R\left(n_{2}, p_{2}, x\right)\right) n d x d \tau \\
:= & \frac{1}{2}\left\|n_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+E_{1}+E_{2}+E_{3} . \tag{3.22}
\end{align*}
$$

Using the uniform estimates for $n_{i}, p_{i}, \psi_{i}$, we have

$$
\begin{align*}
\left|E_{1}+E_{2}\right| \leq & \frac{1}{2}\left(\left\|\Delta V_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta \psi_{1}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{0}^{t}\|n\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\left\|n_{2}\right\|_{L^{\infty}\left(0, T ; L^{3}\left(\mathbb{R}^{3}\right)\right)} \int_{0}^{t}\left(\epsilon\|\nabla n\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+C_{\epsilon}\|\nabla \psi\|_{\mathbf{L}^{6}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau . \tag{3.23}
\end{align*}
$$

On the other hand, by (H2a)-(H2b), we deduce that

$$
\begin{align*}
\left|E_{3}\right| \leq & \left|\int_{0}^{t} \int_{\mathbb{R}^{3}}\left(F\left(n_{1}, p_{1}\right)-F\left(n_{2}, p_{2}\right)\right) n \mu_{n} \mu_{p} d x d \tau\right| \\
& +\left|\int_{0}^{t} \int_{\mathbb{R}^{3}}\left(F\left(n_{1}, p_{1}\right)-F\left(n_{2}, p_{2}\right)\right) n_{1} p_{1} n d x d \tau\right| \\
& +\left|\int_{0}^{t} \int_{\mathbb{R}^{3}} F\left(n_{2}, p_{2}\right)\left(n_{1} p+n p_{2}\right) n d x d \tau\right| \\
\leq & C\left(V_{b},\left\|n_{i}\right\|_{L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)},\left\|p_{i}\right\|_{L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)}\right) \\
& \times \int_{0}^{t}\left(\|n\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau . \tag{3.24}
\end{align*}
$$

In a similar manner, we have the following estimate for $p$

$$
\begin{align*}
& \frac{1}{2}\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\int_{0}^{t}\|\nabla p\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
\leq & \frac{1}{2}\left\|p_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\frac{1}{2}\left(\left\|\Delta V_{p}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta \psi_{1}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{0}^{t}\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} d \tau \\
& +\left\|p_{2}\right\|_{L^{\infty}\left(0, T ; L^{3}\left(\mathbb{R}^{3}\right)\right)} \int_{0}^{t}\left(\epsilon\|\nabla p\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+C_{\epsilon}\|\nabla \psi\|_{\mathbf{L}^{6}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
& +C\left(V_{b},\left\|n_{i}\right\|_{L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)},\left\|p_{i}\right\|_{L^{\infty}\left(0, T ; L^{\infty}\left(\mathbb{R}^{3}\right)\right)}\right) \\
& \times \int_{0}^{t}\left(\|n\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau . \tag{3.25}
\end{align*}
$$

Since $\psi$ satisfies the Poisson equation $-\Delta \psi=n-p$, then it follows from [30, Corollary 2.2] that

$$
\begin{equation*}
\|\nabla \psi\|_{\mathbf{L}^{6}\left(\mathbb{R}^{3}\right)}^{2} \leq C\left(\|n\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) . \tag{3.26}
\end{equation*}
$$

Therefore, taking $\epsilon$ sufficiently small satisfying

$$
0<\epsilon \leq \frac{1}{2} \min \left\{1,\left\|n_{2}\right\|_{L^{\infty}\left(0, T ; L^{3}\left(\mathbb{R}^{3}\right)\right)}^{-1},\left\|p_{2}\right\|_{L^{\infty}\left(0, T ; L^{3}\left(\mathbb{R}^{3}\right)\right)}^{-1}\right\},
$$

we deduce from (3.22)-(3.26) that

$$
\begin{aligned}
& \|n(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\int_{0}^{t}\left(\|\nabla n\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|\nabla p\|_{\mathbf{L}^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau \\
\leq & \left\|n_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|p_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+C_{T} \int_{0}^{t}\left(\|n\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) d \tau .
\end{aligned}
$$

From the Gronwall inequality, we can conclude the continuous dependence on the initial data that

$$
\|n(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\|p(t)\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2} \leq\left(\left\|n_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}+\left\|p_{I}\right\|_{L^{2}\left(\mathbb{R}^{3}\right)}^{2}\right) e^{C_{T^{t}}}, \quad \forall t \in[0, T]
$$

which yields the uniqueness. The proof is complete.

## 4 Long-time Behavior

In section 3 we have proved the existence and uniqueness of global solutions to problem (1.1)-(1.2). However, the global-in-time estimates for the solution $(n, p, \psi)$ depends on $T$ that can be chosen arbitrary. In this section, we extend the results in [11,36] to our current problem (1.1)-(1.2). For this purpose, we first need to obtain some uniform-in-time estimates on the global solution.

Let $\alpha=\int_{\mathbb{R}^{3}}\left(n_{I}-p_{I}\right) d x$. We easily see from (1.1) that the difference of charges is conserved for all $t>0$ :

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}(n(t, \cdot)-p(t, \cdot)) d x=\alpha . \tag{4.1}
\end{equation*}
$$

The relative entropy associated with (1.1) is as follows:

$$
\begin{aligned}
e(t):= & \int_{\mathbb{R}^{3}}\left[n\left(\ln \frac{n}{n_{\infty}}-1\right)+n_{\infty}\right] d x+\int_{\mathbb{R}^{3}}\left[p\left(\ln \frac{p}{p_{\infty}}-1\right)+p_{\infty}\right] d x \\
& +\frac{1}{2} \int_{\mathbb{R}^{3}}\left|\nabla \psi-\nabla \psi_{\infty}\right|^{2} d x
\end{aligned}
$$

where $\left(n_{\infty}, p_{\infty}, \psi_{\infty}\right)$ is the steady state of system (1.1) that satisfies

$$
\left\{\begin{array}{l}
n_{\infty}(x)=D_{n} e^{-\psi_{\infty}} \mu_{n}, \quad D_{n} \in \mathbb{R}^{+},  \tag{4.2}\\
p_{\infty}(x)=D_{p} e^{\psi_{\infty}} \mu_{p}, \quad D_{n} \in \mathbb{R}^{+}, \\
n_{\infty} p_{\infty}=\mu_{n} \mu_{p}, \quad \int_{\mathbb{R}^{3}} n_{\infty} d x-\int_{\mathbb{R}^{3}} p_{\infty} d x=\alpha, \\
-\Delta \psi_{\infty}=n_{\infty}-p_{\infty}-D(x) .
\end{array}\right.
$$

Remark 4.1. Denote $I=\int_{\mathbb{R}^{3}} e^{-\psi_{\infty}-V_{n}(x)} d x, J=\int_{\mathbb{R}^{3}} e^{\psi_{\infty}-V_{p}(x)} d x$. Then the coefficients $D_{n}$ and $D_{p}$ in (4.2) are given by (cf. [36, Lemma 3.1])

$$
D_{n}=\frac{\alpha+\sqrt{\alpha^{2}+4 I J}}{2 I}, \quad D_{p}=\frac{\sqrt{\alpha^{2}+4 I J}-\alpha}{2 J}
$$

satisfying $D_{n} D_{p}=1$.
Following the argument in [36, Theorem 3.1], where the special case $V_{n}=V_{p}=V$ was considered, we can still prove the existence and uniqueness of $\left(n_{\infty}, p_{\infty}, \psi_{\infty}\right)$.

Proposition 4.1. Suppose that assumptions (H1a), (H1b) and (H3) are satisfied. Then the stationary problem (4.2) admits a unique solution $\left(\psi_{\infty}, n_{\infty}, p_{\infty}\right)$ such that

$$
\psi_{\infty} \in D^{1,2}\left(\mathbb{R}^{3}\right)=\left\{\phi(x) \in L^{6}\left(\mathbb{R}^{3}\right) \mid \nabla \phi \in \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)\right\}
$$

Moreover, $\psi_{\infty} \in L^{\infty}\left(\mathbb{R}^{3}\right)$ and $\nabla \psi_{\infty} \in \mathbf{L}^{\infty}\left(\mathbb{R}^{3}\right) \cap \mathbf{L}^{2}\left(\mathbb{R}^{3}\right)$.

Then we have
Lemma 4.1. Suppose the assumptions of Theorem 1.1 are satisfied. The global solution $(n, p)$ of problem (1.1)-(1.2) satisfies

$$
\sup _{t \geq 0}\left[\|n(t)\|_{L^{r}\left(\mathbb{R}^{3}\right)}+\|p(t)\|_{L^{r}\left(\mathbb{R}^{3}\right)}\right]<\infty, \quad \forall r \in[1,+\infty] .
$$

Proof. By a straightforward calculation, we have the dissipation of the relative entropy

$$
\begin{equation*}
\frac{d}{d t} e(t)=-\mathcal{D}(t) \leq 0 \tag{4.3}
\end{equation*}
$$

with the entropy dissipation

$$
\begin{aligned}
\mathcal{D}(t)= & -\int_{\mathbb{R}^{3}} n\left|\nabla \ln \left(\frac{n}{N}\right)\right|^{2} d x-\int_{\mathbb{R}^{3}} p\left|\nabla \ln \left(\frac{p}{P}\right)\right|^{2} d x \\
& -\int_{\mathbb{R}^{3}} R(n, p, x) \ln \left(\frac{n p}{\mu_{n} \mu_{p}}\right) d x
\end{aligned}
$$

$$
\text { where } N=D_{n} e^{-\psi(t)} \mu_{n}, \quad P=D_{p} e^{\psi(t)} \mu_{p} .
$$

Based on the entropy dissipation inequality, we can obtain uniform $L^{r}$ bounds $(r \in[1,+\infty)$ ) for $n(t)$ and $p(t)$ exactly as in [36, Lemma 4.1]. It only remains to show the uniform $L^{\infty}$ estimate. We note that $L^{\infty}$ bounds of solutions to a simplified drift-diffusion system (without self-consistent potential $\psi$ and with a recombination-generation rate of Shockley-Read-Hall type) have been obtained in [11] via a Nash-Moser type iteration method and the results could be extended to the case with self-consistent potential [12]. For the convenience of the readers, we sketch the proof for our present case with a more general recombination-generation rate.

For $r \geq 2$, using integration by parts and the nonnegativity of $n, p$, we get

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x+\frac{4 r}{r+1} \int_{\mathbb{R}^{3}}\left(\left|\nabla\left(n^{\frac{r+1}{2}}\right)\right|^{2}+\left|\nabla\left(p^{\frac{r+1}{2}}\right)\right|^{2}\right) d x \\
= & r \int_{\mathbb{R}^{3}} \Delta \psi\left(n^{r+1}-p^{r+1}\right)+r \int_{\mathbb{R}^{3}}\left(\Delta V_{n} n^{r+1}+\Delta V_{p} p^{r+1}\right) d x \\
& -(r+1) \int_{\mathbb{R}^{3}} R(n, p, x)\left(n^{r}+p^{r}\right) d x \\
= & -r \int_{\mathbb{R}^{3}}(n-p)\left(n^{r+1}-p^{r+1}\right) d x+r \int_{\mathbb{R}^{3}} D(x)\left(n^{r+1}-p^{r+1}\right) d x \\
& +r \int_{\mathbb{R}^{3}}\left(\Delta V_{n} n^{r+1}+\Delta V_{p} r^{r+1}\right) d x \\
& -(r+1) \int_{\mathbb{R}^{3}} F(n, p)\left(n^{r+1} p+n p^{r+1}\right) d x+(r+1) \int_{\mathbb{R}^{3}} \mu^{2} F(n, p)\left(n^{r}+p^{r}\right) d x \\
\leq & r\left(\|D\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta V_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{\mathbb{R}^{3}} n^{r+1} d x \\
& +r\left(\|D\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}+\left\|\Delta V_{p}\right\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}\right) \int_{\mathbb{R}^{3}} p^{r+1} d x \\
& +C(r+1) \int_{\mathbb{R}^{3}}(1+n+p)\left(n^{r}+p^{r}\right) d x
\end{aligned}
$$

$$
\begin{equation*}
\leq C(r+1) \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x+C(r+1) \int_{\mathbb{R}^{3}}\left(n^{r}+p^{r}\right) d x \tag{4.4}
\end{equation*}
$$

where we use the facts that

$$
(n-p)\left(n^{r+1}-p^{r+1}\right) \geq 0, \quad F(n, p)\left(n^{r+1} p+n p^{r+1}\right) \geq 0 .
$$

Since

$$
(r-1) r^{\frac{2-r}{r-1}}>0 \text { for } r \geq 2, \quad \lim _{r \rightarrow+\infty}(r-1) r^{\frac{2-r}{r-1}}=1,
$$

by the Young's inequality and the uniform $L^{1}$ estimate of $n, p$, we deduce that

$$
\begin{align*}
\int_{\mathbb{R}^{3}}\left(n^{r}+p^{r}\right) d x & \leq \frac{1}{r^{2}} \int_{\mathbb{R}^{3}}(n+p) d x+(r-1) r^{\frac{2-r}{r-1}} \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x \\
& \leq \frac{C}{r^{2}}+C^{\prime} \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x, \quad \forall r \geq 2, \tag{4.5}
\end{align*}
$$

where the constants $C, C^{\prime}$ are independent of $r$. Therefore, it follows from (4.4) and (4.5) that

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x+\frac{4 r}{r+1} \int_{\mathbb{R}^{3}}\left(\left|\nabla\left(n^{\frac{r+1}{2}}\right)\right|^{2}+\left|\nabla\left(p^{\frac{r+1}{2}}\right)\right|^{2}\right) d x \\
\leq & C(r+1) \int_{\mathbb{R}^{3}}\left(n^{r+1}+p^{r+1}\right) d x+\frac{C}{r}, \tag{4.6}
\end{align*}
$$

where $C$ is independent of $r$. Based on the differential inequality (4.6), we can argue as in [11, Supplement Lemma 5.1] to obtain the uniform $L^{\infty}$ bounds for $n(t)$ and $p(t)$. The proof is complete.

Proof of Theorem 1.2. Lemma 4.1 yields the uniform-in-time $L^{r}$ estimates (1.9). Then the conclusion of Theorem 1.2 follows from the same argument as in [36, Theorem 4.1]. The proof is complete.

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