# A phase transition model for the helium supercooling 

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#### Abstract

We build a predictive theory for the evolution of mixture of helium and supercooled helium at low temperature. The absolute temperature $\theta$ and the volume fraction $\beta$ of helium, which is dominant at temperature larger than the phase change temperature, are the state quantities. The predictive theory accounts for local interactions at the microscopic level, involving the gradient of $\beta$. The nonlinear heat flux in the supercooled phase results from a Norton-Hoff potential. We prove that the resulting set of partial differential equations has solutions within a convenient analytical frame.


Key words: Supercooled helium, phase change, predictive theory, existence theorem

## 1 Introduction

We investigate a phase transition model describing the thermal behaviour of helium. The model is based on some experimental observations at low temperature between $\theta>0 K$ and $\theta \simeq 5 K$, as described by E. Senger in [7]. Indeed, phase change occurs at $\theta_{c}=2.17 K$ (pressure being the atmospheric pressure): the helium, He II, is a superfluid at temperature lower than $\theta_{c}$, the
helium, He I, is a fluid at temperature larger than $\theta_{c}$. The Gorter-Mellink model assumes a nonlinear Fourier law based on a Norton-Hoff potential, 3. In our approach this model is upgraded by considering microscopic motions involved in the phase change, [7], [9]. Thus, the volume fraction $\beta$ of the He I is introduced, as well as the gradient of the volume fraction, $\nabla \beta$, to account for local interactions, [4]. A description of physics together with numerical results are given in [7].
Within the small perturbation we derive a predictive theory and prove that there exist solutions of the resulting set of partial differential equations in a convenient analytical frame. The main novelty of the resulting system is represented by the nonlinear structure of the diffusive term for the temperature (parabolic) equation. Indeed, we have the contribution of two degenerating terms, characterizing different diffusive behaviour in the two phases. Hence, the evolution equation governing the phase transition contains a multivalued operator leading internal constraints on the phase variable.

In Section 2 we detail the model derivation, leading to an initial and boundary value problem, whose analytical formulation is made precise in Section 3. Existence of a solution is proved in Section 4 by a fixed point argument, mainly combined with lower semicontinuity results.

## 2 The model

In this paper, we aim to model the phenomenon of helium supercooling by use of the phase transitions theory. In particular, our two phases are given by the helium at its normal state and at the supercooled state.

We consider helium located in a smooth bounded domain $\Omega \subset \mathbf{R}^{n}, n \leq 3$. At a first step, we study the evolution of the supercooling process during a finite time interval $(0, T)$ and assume that no macroscopic deformations act during the phenomenon. We use the notation $Q:=\Omega \times(0, T)$.

First, we introduce the state variables of the model $(\theta, \beta): \theta$ stands for the absolute temperature; the phase parameter $\beta$, denoting the volume fraction of helium at its normal state, (HE I), is

$$
\begin{equation*}
\beta \in[0,1] . \tag{2.1}
\end{equation*}
$$

Assuming that the two phases can coexist at each point, with suitable proportions, and that no voids nor overlapping can occur between the two phases, we let $1-\beta$ be the volume fraction of supercooled helium, (HE II).

The free energy is stated as follows

$$
\begin{equation*}
\Psi(\theta, \beta, \nabla \beta)=-c_{s} \theta \log \theta-\frac{l}{\theta_{c}}\left(\theta-\theta_{c}\right) \beta+k|\nabla \beta|^{2}+I_{[0,1]}(\beta), \tag{2.2}
\end{equation*}
$$

where $\theta_{c}$ is the phase transition temperature between helium and supercooled helium, $l$ the phase change latent heat, $c_{s}$ the heat capacity, and $k$ an interaction coefficient between the two phases holding at a microscopic level. The indicator function $I_{[0,1]}(\beta)$ accounts for (2.1), as it is $I_{[0,1]}(\beta):=0$ if $\beta \in[0,1]$ and $I_{[0,1]}(\beta):=+\infty$, otherwise, [6]. Then, we introduce the dissipative variables of the model $\beta_{t}$ and $\nabla \theta$, accounting for the thermodynamical evolution of the system, and make precise the pseudo-potential of dissipation, 4]

$$
\begin{equation*}
\Phi\left(\beta_{t}, \nabla \theta, \theta, \beta\right)=\frac{\mu}{2}\left|\beta_{t}\right|^{2}+\frac{d}{\theta}\left(\frac{|\nabla \theta|^{2}}{2}+(1-\beta) \frac{|\nabla \theta|^{p}}{p}\right), \tag{2.3}
\end{equation*}
$$

where $1<p<2$ and $\mu, d$ are positive constants. For the sake of simplicity, in the sequel we let let $\mu=d=l=k=c_{s}=1$.

The balance equations are recovered from the classical laws of continuum thermo-mechanics. More precisely, we exploit a generalized version of the principle of virtual power including also the effects of microscopic motions which are responsible for the phase transition, 4]. Thus, we can recover the equation governing the evolution of the phase parameter $\beta$ as a balance equation for microscopic movements. It is ( $\mathbf{n}$ is the normal vector to the boundary)

$$
\begin{equation*}
B-\operatorname{div} \mathbf{H}=0 \text { in } \Omega, \quad \mathbf{H} \cdot \mathbf{n}=0 \text { on } \partial \Omega, \tag{2.4}
\end{equation*}
$$

where $B$ and $\mathbf{H}$ are new interior forces for which we are specifying in a moment the constitutive relations. Note that we are not considering applied volume or surface forces acting on the microscopic level. The second equation is given by the energy balance, mainly governing the evolution of the temperature

$$
\begin{equation*}
e_{t}+\operatorname{div} \mathbf{q}=r+B \beta_{t}+\mathbf{H} \cdot \nabla \beta_{t} \text { in } \Omega, \quad \mathbf{q} \cdot \mathbf{n}=0 \text { on } \partial \Omega, \tag{2.5}
\end{equation*}
$$

$e$ denoting the internal energy, $\mathbf{q}$ the heat flux, $r$ an exterior heat source, and $B \beta_{t}+\mathbf{H} \cdot \nabla \beta_{t}$ microscopic mechanically induced heat sources.

Now, let us specify the constitutive relations for the above involved physical quantities. As usual, the internal energy $e$ is specified by

$$
\begin{equation*}
e=\Psi+\theta s \tag{2.6}
\end{equation*}
$$

where the entropy $s$ is

$$
\begin{equation*}
s=-\frac{\partial \Psi}{\partial \theta} \tag{2.7}
\end{equation*}
$$

Then, $B$ is given by the sum of a non-dissipative contribution (derived by the free energy $\Psi$ ) and a dissipative one (related to the pseudo-potential $\Phi$ ), i.e.

$$
\begin{equation*}
B=B^{n d}+B^{d}=\frac{\partial \Psi}{\partial \beta}+\frac{\partial \Phi}{\partial \beta_{t}}, \tag{2.8}
\end{equation*}
$$

while $\mathbf{H}=\mathbf{H}^{\text {nd }}$ is taken only as a non-dissipative vector defined by

$$
\begin{equation*}
\mathbf{H}^{n d}=\frac{\partial \Psi}{\nabla \beta} . \tag{2.9}
\end{equation*}
$$

Finally, the heat flux is recovered by the pseudo-potential of dissipation through the following relation

$$
\begin{equation*}
\mathbf{q}=-\theta \frac{\partial \Phi}{\nabla \theta} . \tag{2.10}
\end{equation*}
$$

Let us point out that, due to the explicit form of $\Phi$, we eventually get

$$
\begin{equation*}
\mathbf{q}=-\beta \nabla \theta-(1-\beta)|\nabla \theta|^{p-2} \nabla \theta \tag{2.11}
\end{equation*}
$$

In particular, we find the classical Fourier heat flux law for the normal state helium $(\beta=1)$, while the supercooled helium $(\beta=0)$ is characterized by a lower order diffusion term for the temperature $(p<2)$.

To derive our PDE system, we substitute (2.7)-(2.10) (cf. (2.11)) in (2.4)(2.5). On account of (2.2) and (2.3), assuming the small perturbations assumption (so that some higher order dissipative nonlinearities are neglected) and letting the model derived close to the phase transition temperature $\theta_{c}$, we recover the following equations, with $1<p<2$, (cf. Remark 2.1)

$$
\begin{align*}
& \theta_{t}+\beta_{t}-\operatorname{div}\left(\beta \nabla \theta+(1-\beta)|\nabla \theta|^{p-2} \nabla \theta\right)=r  \tag{2.12}\\
& \beta_{t}-\Delta \beta+\partial I_{[0,1]}(\beta) \ni \frac{1}{\theta_{c}}\left(\theta-\theta_{c}\right) . \tag{2.13}
\end{align*}
$$

Then, we have to combine (2.13) with suitable initial and boundary conditions. In particular, Cauchy conditions hold in $\Omega$

$$
\begin{equation*}
\theta(0)=\theta_{0}, \quad \beta(0)=\beta_{0}, \tag{2.14}
\end{equation*}
$$

and on $\partial \Omega, \partial_{n}$ being the normal derivative operator (cf. (2.5) and (2.4))

$$
\begin{equation*}
\partial_{n} \chi=0, \quad\left(\beta \nabla \theta+(1-\beta)|\nabla \theta|^{p-2} \nabla \theta\right) \cdot \mathbf{n}=0 \tag{2.15}
\end{equation*}
$$

Remark 2.1. Note that actually (2.12) is obtained regularizing the right hand side of the following equation

$$
\begin{equation*}
\theta_{t}+\frac{\theta}{\theta_{c}} \beta_{t}-\operatorname{div}\left(\beta \nabla \theta+(1-\beta)|\nabla \theta|^{p-2} \nabla \theta\right)=\left|\beta_{t}\right|^{2}+r \tag{2.16}
\end{equation*}
$$

neglecting the dissipative higher order nonlinearity (within the small perturbations assumption) and letting $\theta / \theta_{c} \sim 1$

## 3 Analytical formulation

In this section, we make precise the abstract version of the analytical problem we are dealing with (2.12)-(2.15), and state the main existence result (obtained under suitable assumptions on the data).

Concerning the regularity of initial data, we let

$$
\begin{align*}
& \theta_{0} \in L^{2}(\Omega),  \tag{3.17}\\
& \beta_{0} \in H^{1}(\Omega), \quad \beta_{0} \in[0,1] \text { a.e. in } \Omega . \tag{3.18}
\end{align*}
$$

Hence, we take in (2.12)

$$
\begin{equation*}
r \in L^{2}\left(0, T ; L^{2}(\Omega)\right) \tag{3.19}
\end{equation*}
$$

Now, we can state the main existence result, which actually refers to an abstract version of the system (2.13), (2.12) combined with (2.15), (2.14).

We first clarify the abstract setting we need as well as some related notation. The Hilbert triplet

$$
\begin{equation*}
V:=H^{1}(\Omega) \hookrightarrow H:=L^{2}(\Omega) \hookrightarrow V^{\prime}, \tag{3.20}
\end{equation*}
$$

is introduced $H$ being identified with its dual space, as usual. Then, by $\|\cdot\|_{X}$ we denote the norm in a Banach space $X$ and in any power of it. By $\langle\cdot, \cdot\rangle$ we denote the duality pairing between $V^{\prime}, V$.

Then, we make precise the notion of solution for the above problem. We are looking for

$$
\begin{align*}
& \theta \in H^{1}\left(0, T ; V^{\prime}\right) \cap L^{2}(0, T ; H) \cap L^{p}\left(0, T ; W^{1, p}(\Omega)\right),  \tag{3.21}\\
& \beta \in H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right),  \tag{3.22}\\
& \xi \in L^{2}(0, T ; H), \tag{3.23}
\end{align*}
$$

with

$$
\begin{align*}
& \xi \in \partial I_{[0,1]}(\beta) \quad \text { a.e. in } Q,  \tag{3.24}\\
& \beta^{1 / 2} \nabla \theta \in L^{2}(Q) \tag{3.25}
\end{align*}
$$

fulfilling a.e. in $(0, T)$ in $V^{\prime}$

$$
\begin{align*}
& \theta_{t}+\beta_{t}-\operatorname{div}\left(\beta \nabla \theta+(1-\beta)|\nabla \theta|^{p-2} \nabla \theta\right)=r  \tag{3.26}\\
& \beta_{t}-\Delta \beta+\xi=\frac{1}{\theta_{c}}\left(\theta-\theta_{c}\right) . \tag{3.27}
\end{align*}
$$

Here - div $(H)^{3} \rightarrow V^{\prime}$ stands for the abstract operator

$$
\langle-\operatorname{div} \mathbf{v}, \phi\rangle=\int_{\Omega} \mathbf{v} \cdot \nabla \phi,
$$

and $-\Delta: V \rightarrow V^{\prime}$ for

$$
\langle-\Delta v, \phi\rangle=\int_{\Omega} \nabla v \cdot \nabla \phi
$$

The following theorem holds.
Theorem 3.1. Let (3.17)-(3.19) hold. Then, there exists a solution $(\theta, \beta, \xi)$ to (3.26), (3.27), (3.24), (2.14) with regularity (3.21)-(3.23), (3.25). In particular, (3.27) is actually solved a.e. in $Q$.

## 4 Existence result

In order to prove the existence result stated by Theorem 3.1 we apply a fixed point argument on a regularized version of our problem, showing that it admits, at least locally in time, existence of a solution. Hence, exploiting an a priori estimates-passage to the limit procedure we pass to the limit showing that the limit problem admits a solution, at least in a weak sense. We point out that, actually, the limit problem is solved globally in time, due to suitable a priori estimates on the solutions, allowing to extend them on the whole time interval.

### 4.1 The regularized problem

First, letting $\varepsilon>0$, we regularize (3.26) as follows

$$
\begin{equation*}
\theta_{t}+\beta_{t}-\varepsilon \Delta \theta-\operatorname{div}\left(\beta \nabla \theta+(1-\beta)|\nabla \theta|^{p-2} \nabla \theta\right)=r \tag{4.28}
\end{equation*}
$$

and combine it with (3.27) and (2.14). Then, we construct an operator $\mathcal{T}$ fulfilling the assumption of the Schauder theorem and such that its fixed points are eventually solutions to our system (4.28), (3.27), (3.24) with (2.14) . To this aim, we introduce

$$
\begin{equation*}
\mathcal{B}:=\left\{v \in L^{2}(0, \widehat{T} ; H):\|v\|_{L^{2}(0, \widehat{T} ; H)} \leq R\right\} \tag{4.29}
\end{equation*}
$$

here $R>0$ is fixed and $\widehat{T} \in(0, T]$ will be chosen later. Note that, from now on, we denote by $c$ possibly different positive constants not depending
on the solutions, but just on the data of the problem, and by $c(R)$ positive constants depending in particular on $R$.

First, we take $\bar{\theta} \in \mathcal{B}$ and substitute $\theta$ in (3.27). Fairly standard results in the theory of evolution equations associated with maximal monotone operators ensure the existence and uniqueness of

$$
\beta:=\mathcal{T}_{1}(\bar{\theta}) \in H^{1}(0, \widehat{T} ; H) \cap L^{2}\left(0, \widehat{T} ; H^{2}(\Omega)\right)
$$

solving the resulting equation (actually a.e.), with (2.14) and (2.15) (note that the right hand side of (3.27) belongs to $L^{2}(\Omega \times(0, \widehat{T}))$ and the regularity of the initial datum is given by (3.18)). Then, we can exploit the following a priori estimate on the solution $\beta$. We test (3.27) by $\beta_{t}-\Delta \beta$ and integrate over $(0, t), t \in(0, \widehat{T})$. Integrating by parts, using a generalization of the chain rule for subdifferential operators and exploiting the monotonicity of the subdifferential $\partial I_{[0,1]}(\beta)$ (cf. [2] and [1]), as well aas the positivity of the indicator function $I_{[0,1]}(\cdot)$, we get (cf. also (3.18))

$$
\begin{align*}
& \left\|\beta_{t}\right\|_{L^{2}(0, t ; H)}^{2}+\|\nabla \beta(t)\|_{H}^{2}+\|\Delta \beta\|_{L^{2}(0, t ; H)}^{2}+\int_{\Omega} I(\beta(t))  \tag{4.30}\\
& \leq\left\|\beta_{0}\right\|_{V}^{2}+c \int_{0}^{t}\left\|\bar{\theta}-\theta_{c}\right\|_{H}\left(\left\|\beta_{t}\right\|_{H}+\|\Delta \beta\|_{H}\right) \\
& \leq c\left(1+\|\bar{\theta}\|_{L^{2}(0, \widehat{T} ; H)}^{2}\right)+\frac{1}{2}\left\|\beta_{t}\right\|_{L^{2}(0, t ; H)}^{2}+\frac{1}{2}\|\Delta \beta\|_{L^{2}(0, t ; H)}^{2},
\end{align*}
$$

where we have used the Young inequality. By (4.30), using the Gronwall lemma, we eventually deduce

$$
\begin{equation*}
\|\beta\|_{H^{1}(0, \widehat{T} ; H) \cap L^{\infty}(0, \widehat{T} ; V) \cap L^{2}\left(0, \widehat{T} ; H^{2}(\Omega)\right)} \leq c(R) . \tag{4.31}
\end{equation*}
$$

Moreover, we can infer that $\beta \in[0,1]$ a.e.. By a comparison in (3.27) $\xi \in$ $\partial I(\beta)$ satisfies

$$
\begin{equation*}
\|\xi\|_{L^{2}(0, \widehat{T} ; H)} \leq c(R) \tag{4.32}
\end{equation*}
$$

Now, as a second step, we take $\beta=\mathcal{T}_{1}(\bar{\theta})$ in (4.28) and look for a corresponding solution $\theta=\mathcal{T}_{2}(\beta)$ (actually depending on $\varepsilon>0$ ). To this aim, we apply the theory of parabolic nonlinear evolution equations (recall that $\beta \in[0,1]$ ), and get existence and uniqueness of a solution (see [5])

$$
\theta \in H^{1}\left(0, \widehat{T} ; V^{\prime}\right) \cap L^{2}(0, \widehat{T} ; V) .
$$

Then, let us proceed detailing some a priori estimates on this solution. For the sake of simplicity, if it is allowed, we directly perform estimates in which the positive bound $c$ does not depend on the approximating parameter $\varepsilon$, as
we are interested, in a second step, to pass to the limit on $\varepsilon \searrow 0$ on the whole $(0, T)$. We test (4.28) by $\theta$ and integrate over $(0, t), t \in[0, T]$. After integrating by parts in time and exploiting the Hölder inequality, we get

$$
\begin{align*}
& \frac{1}{2}\|\theta(t)\|_{H}^{2}-\frac{1}{2}\left\|\theta_{0}\right\|_{H}+\varepsilon \int_{0}^{t}\|\nabla \theta\|_{H}^{2}+\int_{0}^{t} \int_{\Omega} \beta|\nabla \theta|^{2}+\int_{0}^{t} \int_{\Omega}|\nabla \theta|^{p}  \tag{4.33}\\
& \leq \int_{0}^{t}\|r\|_{H}\|\theta\|_{H}+\int_{0}^{t}\left\|\beta_{t}\right\|_{H}\|\theta\|_{H}+\int_{0}^{t} \int_{\Omega} \beta|\nabla \theta|^{p} .
\end{align*}
$$

Now, let us handle the last term on the right hand side of (4.33) exploiting the Young inequality

$$
\begin{align*}
& \int_{0}^{t} \int_{\Omega} \beta|\nabla \theta|^{p}=\int_{0}^{t} \int_{\Omega} \beta^{1-p / 2} \beta^{p / 2}|\nabla \theta|^{p}  \tag{4.34}\\
& \leq \delta \int_{0}^{t} \int_{\Omega}\left(\beta^{p / 2}|\nabla \theta|^{p}\right)^{2 / p}+C_{\delta} \int_{0}^{t} \int_{\Omega}\left|\beta^{1-p / 2}\right|^{\frac{2}{2-p}} \\
& \leq \delta \int_{0}^{t} \int_{\Omega}\left(\beta|\nabla \theta|^{2}\right)+c\left(1+\|\beta\|_{L^{2}(0, \widehat{T} ; H)}^{2}\right) .
\end{align*}
$$

Thus, letting, e.g., $\delta=1 / 2$, we combine (4.34) with (4.33), and the Gronwall lemma implies

$$
\begin{equation*}
\|\theta\|_{L^{\infty}(0, \widehat{T} ; H)}+\left\|\beta^{1 / 2} \nabla \theta\right\|_{L^{2}(\Omega \times(0, \widehat{T}))}+\|\nabla \theta\|_{L^{p}(\Omega \times(0, \widehat{T}))}+\varepsilon^{1 / 2}\|\nabla \theta\|_{L^{2}(\Omega \times(0, \widehat{T}))} \leq c(R) . \tag{4.35}
\end{equation*}
$$

Then, a comparison in (3.26) yields (at least)

$$
\begin{equation*}
\|\theta\|_{H^{1}\left(0, \widehat{T} ; V^{\prime}\right)} \leq c \tag{4.36}
\end{equation*}
$$

Now, we define

$$
\mathcal{T}(\bar{\theta})=\mathcal{T}_{2}\left(\mathcal{T}_{1}(\bar{\theta})\right)
$$

It results that $\mathcal{T}: \mathcal{B} \rightarrow \mathcal{B}$ is well-defined, at least for some small $\widehat{T}$. Indeed, by virtue of (4.35) we can infer that

$$
\|\theta\|_{L^{2}(0, \widehat{T} ; H)}=\left(\int_{0}^{\widehat{T}}\|\theta\|_{H}^{2}\right)^{1 / 2} \leq\|\theta\|_{L^{\infty}(0, \widehat{T} ; H)} \frac{\widehat{T}^{1 / 2}}{2} \leq \frac{c(R)}{2} \widehat{T}^{1 / 2}
$$

and the right hand side is less than $R$ if $\widehat{T}$ is sufficiently small. However, as the above estimates do not depend on $\widehat{T}$. Then, as $\theta \in C^{0}([0, \widehat{T}] ; H)$, $\beta \in C^{0}([0, \widehat{T}] ; V)$, due to (3.17)-(3.18), we will be able to extend the result on the whole time interval $(0, T)$. Thus, for the sake of simplicity, in the
sequel we directly refer to the interval $(0, T)$. Hence, it is clear that any fixed point of $\mathcal{T}$ is a solution $\left.\theta, \mathcal{T}_{1}(\theta)\right)$ to (4.28), (3.27), (2.14).

Now, to prove that $\mathcal{T}$ admits a fixed point exploiting the Schauder theorem, we have to show that it is compact and continuous w.r.t. to the topology of $L^{2}(0, T ; H)$. Compactness easily follows by (4.35) and (4.36) (see [8]), as $\varepsilon>0$ (we have that $L^{2}(0, T ; H)$ is compact in $H^{1}\left(0, T ; V^{\prime}\right) \cap L^{2}(0, T ; V)$.

Now, we aim to prove (strong) continuity. To this aim we take a sequence

$$
\begin{equation*}
\bar{\theta}_{n} \rightarrow \bar{\theta} \quad \text { in } L^{2}(0, T ; H) \tag{4.37}
\end{equation*}
$$

and show that the corresponding $\theta_{n}$ strongly converges to

$$
\begin{equation*}
\theta_{n}=\mathcal{T}\left(\bar{\theta}_{n}\right) \rightarrow \theta=\mathcal{T}(\bar{\theta}) \tag{4.38}
\end{equation*}
$$

strongly in $L^{2}(0, T ; H)$.
First, we observe that (4.31), (4.35), and (4.361) hold for $\beta_{n}=\mathcal{T}_{1}\left(\bar{\theta}_{n}\right)$ and $\theta_{n}$ for constants $c$ independent of $n$. Thus, by weak and weak star compactness results, we get, at least for some suitable subsequences,

$$
\begin{equation*}
\beta_{n} \rightharpoonup^{*} \beta \quad \text { in } H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right) . \tag{4.39}
\end{equation*}
$$

Hence, we use strong compactness theorems yielding (at least for subsequences)

$$
\begin{equation*}
\beta_{n} \rightarrow \beta \quad \text { in } C^{0}([0, T] ; H) \cap L^{2}\left(0, T ; H^{2-\delta}(\Omega)\right), \quad \delta>0 . \tag{4.40}
\end{equation*}
$$

As $\beta_{n} \in[0,1]$ a.e. and, for a subsequence, $\beta_{n} \rightarrow \beta$ a.e. (cf. (4.40)), the Lebesgue theorem ensures that

$$
\begin{equation*}
\beta_{n} \rightarrow \beta \quad \text { in } L^{q}(Q), \quad \forall q<+\infty . \tag{4.41}
\end{equation*}
$$

Analogously, due to (4.32), we can infer that

$$
\begin{equation*}
\xi_{n} \rightharpoonup \xi \quad \text { in } L^{2}(0, T ; H) \tag{4.42}
\end{equation*}
$$

so that (4.40) and (4.42) lead to the identification (cf. [2])

$$
\begin{equation*}
\xi \in \partial I(\beta) \tag{4.43}
\end{equation*}
$$

a.e. in $Q$. The above convergences (4.39)-(4.43) allow us to pass to the limit in the equation (3.27) as $n \rightarrow+\infty$, actually in $L^{2}(0, T ; H)$ and thus to identify the limit equation a.e. in $Q$. By the uniqueness of the limit equation, once $\bar{\theta}$ is fixed, we identify $\beta=\mathcal{T}_{1}(\bar{\theta})$ and extend (4.39)-(4.43) to the whole sequences.

Now, we deal with (3.26) where $\beta_{n}$ is fixed and look for suitable convergence results of the corresponding solutions $\theta_{n}=\mathcal{T}_{2}\left(\beta_{n}\right)$, as $n$ tends to $+\infty$. We first point out that (4.35) and (4.36), due to weak and weak star compactness results, imply that the following convergence holds, up to the extraction of some suitable subsequences,

$$
\begin{align*}
& \theta_{n} \rightharpoonup^{*} \theta \quad \text { in } H^{1}\left(0, T ; V^{\prime}\right) \cap L^{\infty}(0, T ; H) \cap L^{p}\left(0, T ; W^{1, p}(\Omega)\right),  \tag{4.44}\\
& \beta_{n}^{1 / 2} \nabla \theta_{n} \rightharpoonup \eta \quad \text { in } L^{2}(0, T ; H),  \tag{4.45}\\
& \varepsilon^{1 / 2} \theta_{n} \rightharpoonup \varepsilon^{1 / 2} \theta \quad \text { in } L^{2}(0, T ; V) . \tag{4.46}
\end{align*}
$$

As $\varepsilon>0$, (4.35) implies that, at least for a subsequence,

$$
\begin{equation*}
\theta_{n} \rightarrow \theta \quad \text { in } L^{2}(0, T ; H) . \tag{4.47}
\end{equation*}
$$

Then, we can easily identify $\eta=\beta^{1 / 2} \nabla \theta$ in (4.45) due to (4.41) and (4.44). Indeed, $\beta_{n}$ strongly converges to $\beta$, and thus it converges almost everywhere. As a consequences $\beta_{n}^{1 / 2} \rightarrow \beta^{1 / 2}$ a.e. and $\beta_{n}^{1 / 2} \in[0,1]$ a.e. (i.e. they are uniformly bounded). The Lebesgue theorem ensures that $\beta_{n}^{1 / 2} \rightarrow \beta^{1 / 2}$ strongly in $L^{r}(Q)$ for any $1 \leq r<+\infty$. Finally, note that, due to (4.46), $\nabla \theta_{n}$ converges weakly in $L^{2}(Q)$.

Then, let us point out that (4.35) yields

$$
\begin{equation*}
\left\|\left|\nabla \theta_{n}{ }^{p-2} \nabla \theta_{n}\right|\right\|_{L^{p^{\prime}}(Q)} \leq c, \quad \frac{1}{p}+\frac{1}{p^{\prime}}=1, \tag{4.48}
\end{equation*}
$$

so that (4.41), combined with (4.48), yields

$$
\begin{equation*}
\left(1-\beta_{n}\right)\left|\nabla \theta_{n}{ }^{p-2}\right| \nabla \theta_{n} \rightharpoonup(1-\beta) \psi \quad \text { in } L^{p^{\prime}}(Q), \tag{4.49}
\end{equation*}
$$

(and consequently in $L^{2}(Q)$ ) where

$$
\begin{equation*}
\left|\nabla \theta_{n}{ }^{p-2}\right| \nabla \theta_{n} \rightharpoonup \psi \quad \text { in } L^{p^{\prime}}(Q) \tag{4.50}
\end{equation*}
$$

By the previous convergences we are allowed to pass to the limit as $n \rightarrow$ $+\infty$ in (4.28) and get

$$
\begin{equation*}
\theta_{t}-\varepsilon \Delta \theta-\operatorname{div}(\beta \nabla \theta+(1-\beta) \psi)+\beta_{t}=r, \tag{4.51}
\end{equation*}
$$

in $V^{\prime}$, for a.e. $t$. Now, our aim is to identify $\psi$ in (4.51) with $\left|\nabla \theta^{p-2}\right| \nabla \theta$.
Once $\beta$ is fixed we introduce the function

$$
\begin{align*}
& \widehat{J}_{\beta}: L^{p}(\Omega) \rightarrow(-\infty,+\infty], \widehat{J}_{\beta}(\nabla \theta)=\int_{\Omega} \frac{(1-\beta)}{p}|\nabla \theta|^{p}  \tag{4.52}\\
& \text { if }|\nabla \theta|^{p} \in L^{1}(\Omega), \widehat{J}_{\beta}(\nabla \theta)=+\infty \text { otherwise. }
\end{align*}
$$

Then, the subdifferential of $\widehat{J}_{\beta}$ can be standardly computed as

$$
\begin{equation*}
\partial \widehat{J}_{\beta}=(1-\beta)|\nabla \theta|^{p-2} \nabla \theta . \tag{4.53}
\end{equation*}
$$

Our goal is to prove that $(1-\beta) \psi \in \partial \widehat{J}_{\beta}(\nabla \theta)$ a.e. in $Q$, i.e. (by definition of the subdifferential)

$$
\begin{equation*}
\int_{Q}(1-\beta) \psi(\nabla w-\nabla \theta) \leq \int_{Q} \frac{(1-\beta)}{p}|\nabla w|^{p}-\int_{Q} \frac{(1-\beta)}{p}|\nabla \theta|^{p}, \tag{4.54}
\end{equation*}
$$

for any $w \in L^{p}\left(0, T ; W^{1, p}(\Omega)\right)$. To this aim we first prove that

$$
\begin{equation*}
\limsup _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right)\left|\nabla \theta_{n}\right|^{p-2} \nabla \theta_{n} \leq \int_{Q}(1-\beta) \psi \nabla \theta \text {. } \tag{4.55}
\end{equation*}
$$

Indeed, testing (3.26) written for $n$ by $\theta_{n}$ and integrating over $(0, T)$ leads to

$$
\begin{align*}
& \limsup _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right)\left|\nabla \theta_{n}\right|^{\mid p-2} \nabla \theta_{n}=\limsup _{n \rightarrow+\infty}-\frac{1}{2} \int_{\Omega}\left|\theta_{n}(t)\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\theta_{0}\right|^{2}(4  \tag{4.56}\\
& -\varepsilon \int_{Q}\left|\nabla \theta_{n}\right|^{2}-\int_{Q} \beta_{n}\left|\nabla \theta_{n}\right|^{2}-\int_{Q} \theta_{n} \beta_{n t}+\int_{Q} r \theta_{n} \\
& \leq-\frac{1}{2} \int_{\Omega}|\theta(t)|^{2}+\frac{1}{2} \int_{\Omega}\left|\theta_{0}\right|^{2}-\varepsilon \int_{Q}|\nabla \theta|^{2}-\int_{Q} \beta|\nabla \theta|^{2}-\int_{Q} \theta \beta_{t}+\int_{Q} r \theta \\
& =+\int_{Q}(1-\beta) \psi \nabla \theta .
\end{align*}
$$

To prove (4.56) we have used (weak) lower semicontinuity of norms, convergences (4.44)-(4.46), (4.49), and (4.47). Hence the right-hand side is identified by virtue of (4.51). As a consequence, we immediately have that

$$
\begin{equation*}
\int_{Q}(1-\beta) \psi(\nabla w-\nabla \theta) \leq \liminf _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right)\left|\nabla \theta_{n}\right|^{p-2} \nabla \theta_{n}\left(\nabla w-\nabla \theta_{n}\right) \tag{4.57}
\end{equation*}
$$

for any $w \in L^{p}\left(0, T ; W^{1, p}(\Omega)\right)$. Thus, it is now a standard matter to prove
the following chain of inequalities

$$
\begin{align*}
& \int_{Q}(1-\beta) \psi(\nabla w-\nabla \theta) \leq \liminf _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right)\left|\nabla \theta_{n}\right|^{p-2} \nabla \theta_{n}\left(\nabla w-\nabla \theta_{n}\right)  \tag{4.58}\\
& \leq \liminf _{n \rightarrow+\infty} \int_{Q} \frac{\left(1-\beta_{n}\right)}{p}\left(|\nabla w|^{p}-\left|\nabla \theta_{n}\right|^{p}\right) \\
& \leq \limsup _{n \rightarrow+\infty} \int_{Q} \frac{\left(1-\beta_{n}\right)}{p}\left(|\nabla w|^{p}-\left|\nabla \theta_{n}\right|^{p}\right) \\
& \leq \int_{Q}(1-\beta) \frac{|\nabla w|^{p}}{p}-\liminf _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right) \frac{\left|\nabla \theta_{n}\right|^{p}}{p} \\
& \leq \int_{Q}(1-\beta) \frac{|\nabla w|^{p}}{p}-\int_{Q}(1-\beta) \frac{|\nabla \theta|^{p}}{p}
\end{align*}
$$

which concludes our proof. Indeed, note in particular that, as (4.44) and (4.41) hold, we can infer that

$$
\begin{equation*}
\left(1-\beta_{n}\right)^{1 / p} \nabla \theta_{n} \rightharpoonup(1-\beta)^{1 / p} \nabla \theta \quad \text { in } L^{p}(Q) \tag{4.59}
\end{equation*}
$$

so that, by weak lower semicontinuity of norms,

$$
\liminf _{n \rightarrow+\infty} \int_{Q}\left(1-\beta_{n}\right) \frac{\left|\nabla \theta_{n}\right|^{p}}{p} \geq \int_{Q}(1-\beta) \frac{|\nabla \theta|^{p}}{p} .
$$

### 4.2 The existence result: the limit as $\varepsilon \searrow 0$

We are now in the position of proving Theorem 3.1 by passing to the limit in (4.28) as $\varepsilon \searrow 0$. To this aim, after denoting by $\left(\theta_{\varepsilon}, \beta_{\varepsilon}, \xi_{\varepsilon}\right)$ a solution to the system (4.28), (3.27), (3.24), with $\varepsilon>0$, we perform the analogous estimate as (4.30) and (4.33)-(4.34), i.e. we test (3.27) by $\partial_{t} \beta_{\varepsilon}-\Delta \beta_{\varepsilon}$ and (4.28) by
$\theta_{\varepsilon}$. After integrating over $(0, t)$ we get

$$
\begin{align*}
& \left.\left\|\left.\partial_{t} \beta_{\varepsilon}\right|_{L^{2}(0, t ; H)} ^{2}+\right\| \nabla \beta_{\varepsilon}(t)\left\|_{H}^{2}+\right\| \Delta \beta_{\varepsilon}\right|_{L^{2}(0, t ; H)} ^{2}+\int_{\Omega} I\left(\beta_{\varepsilon}(t)\right)  \tag{4.60}\\
& +\frac{1}{2}\left\|\theta_{\varepsilon}(t)\right\|_{H}^{2}+\varepsilon\left\|\nabla \theta_{\varepsilon}\right\|_{L^{2}(0, t ; H)}^{2}+\iint_{Q} \beta_{\varepsilon}\left|\nabla \theta_{\varepsilon}\right|^{2}+\iint_{Q}\left|\nabla \theta_{\varepsilon}\right|^{2} \\
& \leq \frac{1}{2}\left\|\left.\partial_{t} \beta_{\varepsilon}\right|_{L^{2}(0, t ; H)} ^{2}+\frac{1}{2}\right\| \Delta \theta_{\varepsilon} \|_{L^{2}(0, t ; H)}^{2} \\
& +c\left(1+\left\|\theta_{\varepsilon}\right\|_{L^{2}(0, t ; H)}^{2}+\int_{0}^{t}\|r\|_{H}\|\theta\|_{H}\right)+\iint_{Q} \beta\left|\nabla \theta_{\varepsilon}\right|^{p} \\
& \leq \frac{1}{2}\left\|\left.\partial_{t} \beta_{\varepsilon}\right|_{L^{2}(0, t ; H)} ^{2}+\frac{1}{2}\right\| \Delta \theta_{\varepsilon} \|_{L^{2}(0, t ; H)}^{2}+\frac{1}{2} \iint_{Q} \beta_{\varepsilon}\left|\nabla \theta_{\varepsilon}\right|^{2} \\
& +c\left(1+\left\|\theta_{\varepsilon}\right\|_{L^{2}(0, t ; H)}^{2}+\int_{0}^{t}\|r\|_{H}\|\theta\|_{H}+\left\|\beta_{\varepsilon}\right\|_{L^{2}(0, t ; H)}^{2}\right)
\end{align*}
$$

Then by using the Gronwall lemma, we get the analogous of (4.31), (4.32), (4.35), (4.36), now independently of $\varepsilon$, i.e.

$$
\begin{align*}
& \left\|\beta_{\varepsilon}\right\|_{H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right)} \leq c  \tag{4.61}\\
& \left\|\theta_{\varepsilon}\right\|_{H^{1}\left(0, T ; V^{\prime}\right) \cap L^{\infty}(0, T ; H) \cap L^{p}\left(0, T ; W^{1, p}(\Omega)\right)} \leq c  \tag{4.62}\\
& \left\|\xi_{\varepsilon}\right\|_{L^{2}(0, T ; H)} \leq c  \tag{4.63}\\
& \left\|\beta_{\varepsilon}^{1 / 2} \nabla \theta_{\varepsilon}\right\|_{L^{2}(0, T ; H)} \leq c  \tag{4.64}\\
& \varepsilon^{1 / 2}\left\|\theta_{\varepsilon}\right\|_{L^{2}(0, T ; H)} \leq c . \tag{4.65}
\end{align*}
$$

As in the previous section (cf. (4.39)-(4.42)), by compactness, we can deduce that, at least for some subsequences,

$$
\begin{align*}
& \beta_{\varepsilon} \rightharpoonup^{*} \beta \quad \text { in } H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right)  \tag{4.66}\\
& \beta_{\varepsilon} \rightarrow \beta \quad \text { in } C^{0}([0, T] ; H) \cap L^{2}\left(0, T ; H^{2-\delta}(\Omega)\right), \quad \delta>0  \tag{4.67}\\
& \beta_{n} \rightarrow \beta \quad \text { in } L^{q}(Q), \quad \forall q<+\infty .  \tag{4.68}\\
& \xi_{n} \rightharpoonup \xi \quad \text { in } L^{2}(0, T ; H) \tag{4.69}
\end{align*}
$$

and (at least)

$$
\begin{align*}
& \theta_{\varepsilon} \rightharpoonup^{*} \theta \quad \text { in } H^{1}\left(0, T ; V^{\prime}\right) \cap L^{\infty}(0, T ; H) \cap L^{p}\left(0, T ; W^{1, p}(\Omega)\right)  \tag{4.70}\\
& \theta_{\varepsilon} \rightarrow \theta \quad \text { in } L^{\infty}\left(0, T ; L^{q}(\Omega)\right), \tag{4.71}
\end{align*}
$$

where $q$ is such that $L^{q}$ is compact embedded in $W^{1, p}$, i.e. $p>\frac{n q}{q+n}(n$ standing for the dimension of $\Omega$ ). Note in particular that (4.47) still holds for $n=2$ for any $p \in(1,2)$, while for $n=3$ it is ensured just for $p>6 / 5$.

Then, by virtue of (4.66)-(4.69) and (4.70) we can pass to the limit in (3.27) (written for $\varepsilon>0$ ) as $\varepsilon$ tends to zero, identifying $\xi \in \partial I_{0,1}(\beta)$. Hence, we are interested in pass to the limit in (4.28). We make use of convergences (4.67), (4.66), (4.70), observing in particular that

$$
-\varepsilon \Delta \theta_{\varepsilon} \rightarrow 0 \quad \text { in } L^{2}\left(0, T ; V^{\prime}\right)
$$

Arguing as in the previous section, we can infer that

$$
\begin{equation*}
\beta_{\varepsilon}^{1 / 2} \nabla \theta_{\varepsilon} \rightharpoonup \beta^{1 / 2} \nabla \theta \quad \text { in } L^{2}(0, T ; H), \tag{4.72}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1-\beta_{\varepsilon}\right)\left|\nabla \theta_{\varepsilon}\right|^{p-2} \nabla \theta_{\varepsilon} \rightharpoonup(1-\beta) \psi \quad \text { in } L^{p^{\prime}}(Q) . \tag{4.73}
\end{equation*}
$$

We aim to identify $\psi=|\nabla \theta|^{p-2} \nabla \theta$. To this aim we proceed as in the previous section, thus we do not detail inequalities. We just make some comment on the equivalent version of (4.56). Indeed, in the general case we cannot infer that $\int_{Q} \theta_{\varepsilon} \partial_{t} \beta_{\varepsilon} \rightarrow \int_{Q} \theta \partial_{t} \beta$. Thus, to prove that

$$
\limsup _{\varepsilon \searrow 0}-\int_{Q} \theta_{\varepsilon} \partial_{t} \beta_{\varepsilon} \leq-\int_{Q} \theta \partial_{t} \beta
$$

we use semicontinuity for the equation (3.27) (written for $\varepsilon>0$ ) formally tested by $\theta_{c} \partial_{t} \beta_{\varepsilon}$. There holds

$$
\begin{aligned}
& \limsup -\int_{\varepsilon \searrow 0} \theta_{\varepsilon} \partial_{t} \beta_{\varepsilon} \\
& =\underset{\varepsilon \searrow 0}{\limsup }-\theta_{c}\left(\int_{Q}\left|\partial_{t} \beta_{\varepsilon}\right|^{2}-\frac{1}{2} \int_{\Omega}\left|\nabla \beta_{\varepsilon}(t)\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla \beta_{0}\right|^{2}-\int_{Q} \xi_{\varepsilon} \partial_{t} \beta_{\varepsilon}+\int_{Q} \partial_{t} \beta_{\varepsilon}\right) \\
& =-\liminf _{\varepsilon \searrow 0} \theta_{c}\left(\int_{Q}\left|\partial_{t} \beta_{\varepsilon}^{2}\right|+\frac{1}{2} \int_{\Omega}\left|\nabla \beta_{\varepsilon}(t)\right|^{2}-\frac{1}{2} \int_{\Omega}\left|\nabla \beta_{0}\right|^{2}+\int_{Q} \xi_{\varepsilon} \partial_{t} \beta_{\varepsilon}-\int_{Q} \partial_{t} \beta_{\varepsilon}\right) \\
& \leq-\theta_{c}\left(\int_{Q} \partial_{t} \beta^{2}-\frac{1}{2} \int_{\Omega}|\nabla \beta(t)|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla \beta_{0}\right|^{2}-\int_{Q} \xi \partial_{t} \beta+\int_{Q} \partial_{t} \beta\right) \\
& =-\int_{Q} \theta \partial_{t} \beta,
\end{aligned}
$$

by the fact that we have already identified the limit of (3.27) (written for $\varepsilon>0$, as $\varepsilon \searrow 0$. Note that we have exploited weak lower semicontinuity of norms with (4.66)-(4.67), and the lower semicontinuity of the indicator function. Thus, we are in the position of proving the analogous of (4.56) and thus (4.57)-(4.58) easily follow, concluding our proof.

## References

[1] E. Bonetti Global solution to a Frémond model for shape memory alloys with thermal memory Nonlinear Anal. 46 (2001), 535-565
[2] H. Brézis Opérateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert North-Holland Math. Stud. 5, NorthHolland, Amsterdam, 1973
[3] A. Friaa, La loi de Norton-Hoff généralisée en plasticité et en viscoplasticité, Thèse, Université Pierre et Marie Curie, Paris, 1979.
[4] M. Frémond, Non-smooth thermomechanics, Springer-Verlag, Berlin, 2002.
[5] J.L. Lions Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires Dunod, Gauthier-Villars Paris, 1969
[6] J.J. Moreau, Fonctionelles convexes, Collége de France (1966) and Dipartimento di Ingegneria Civile Università di Roma Tor Vergata, ISBN 978-88-6296-001-4 (2003).
[7] E. Senger, 2008, Modelagem Matematica e Métodos Numéricos para Simulacão da Conducão do Calor no Hélio Liquido, Tese de Doutorado, Laboratorio Nacional de Computação Cientifica, Petropolis.
[8] J. Simon Compact sets in the space $L^{p}(0, T ; B)$ Ann. Mat. Pura Appl. (4) 146 (1987), 65-96
[9] E. Senger, A.F.D. Loula, M.A. Raupp, 2004, Mathematical Modelling and Numerical Simulation in Liquid Helium, Proceeding of XXV Iberian Latin American Congress on Computational Methods in Engineering (CILAMCE).

