# RESONANCE WIDTHS IN A CASE OF MULTIDIMENSIONAL PHASE SPACE TUNNELING 

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#### Abstract

We consider a semiclassical $2 \times 2$ matrix Schrödinger operator of the form $P=-h^{2} \Delta \mathbf{I}_{2}+\operatorname{diag}\left(x_{n}-\mu, \tau V_{2}(x)\right)+h R\left(x, h D_{x}\right)$, where $\mu$ and $\tau$ are two small positive constants, $V_{2}$ is real-analytic and admits a non degenerate minimum at 0 , and $R=\left(r_{j, k}\left(x, h D_{x}\right)\right)_{1 \leq j, k \leq 2}$ is a symmetric off-diagonal $2 \times 2$ matrix of first-order differential operators with analytic coefficients. Then, denoting by $e_{1}$ the first eigenvalue of $-\Delta+\left\langle\tau V_{2}^{\prime \prime}(0) x, x\right\rangle / 2$, and under some ellipticity condition on $r_{1,2}=r_{2,1}^{*}$, we show that, for any $\mu$ sufficiently small, and for $0<\tau \leq \tau(\mu)$ with some $\tau(\mu)>0$, the unique resonance $\rho$ of $P$ such that $\rho=\tau V_{2}(0)+\left(e_{1}+r_{2,2}(0,0)\right) h+\mathcal{O}\left(h^{2}\right)\left(\right.$ as $\left.h \rightarrow 0_{+}\right)$satisfies, $$
\operatorname{Im} \rho=-h^{\frac{3}{2}} f\left(h, \ln \frac{1}{h}\right) e^{-2 S / h},
$$ where $f\left(h, \ln \frac{1}{h}\right) \sim \sum_{0<m<\ell} f_{\ell, m} h^{\ell}\left(\ln \frac{1}{h}\right)^{m}$ is a symbol with $f_{0,0}>0$, and $S$ is the imaginary part of the complex action along some convenient closed path containing $(0,0)$ and consisting of a union of complex nulbicharacteristics of $p_{1}:=\xi^{2}-x_{n}-\mu$ and $p_{2}:=\xi^{2}+\tau V_{2}(x)$ (broken instanton). This broken instanton is described in terms of the outgoing and incoming complex Lagrangian manifolds associated with $p_{2}$ at the point $(0,0)$, and their intersections with the characteristic set $p_{1}^{-1}(0)$ of $p_{1}$.


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## 1. Introduction

We consider the semiclassical $2 \times 2$ matrix Schrödinger operator,

$$
P=\left(\begin{array}{cc}
P_{1} & 0 \\
0 & P_{2}
\end{array}\right)+h R\left(x, h D_{x}\right)
$$

with,

$$
\begin{aligned}
& P_{1}:=-h^{2} \Delta+x_{n}-\mu ; \\
& P_{2}:=-h^{2} \Delta+\tau V_{2}(x),
\end{aligned}
$$

where $x=\left(x^{\prime}, x_{n}\right)=\left(x_{1}, \ldots, x_{n}\right)$ is the current variable in $\mathbb{R}^{n}(n \geq 2), h>0$ denotes the semiclassical parameter, $R\left(x, h D_{x}\right)=\left(r_{j, k}\left(x, h D_{x}\right)\right)_{1 \leq j, k \leq 2}$ is a formally self-adjoint off-diagonal $2 \times 2$ matrix of first-order semiclassical differential operators, and $\mu, \tau$ are two positive extra-parameters that will be taken sufficiently small later on. Moreover, we assume that the smooth potential $V_{2}$ is bounded on $\mathbb{R}^{n}$ and satisfies,

$$
\begin{equation*}
V_{2} \geq 0 ; V_{2}^{-1}(0)=\{0\} ; \operatorname{Hess}_{2}(0)>0 ; \liminf _{|x| \rightarrow \infty} V_{2}>0 \tag{1.1}
\end{equation*}
$$

In particular, the classically allowed region at energy 0 associated with $P_{2}$ (that is, $\left\{V_{2}(x) \leq 0\right\}=\{0\}$ ) is included into that associated with $P_{1}$ (that coincides with $\left.\mathbb{R}^{n-1} \times(-\infty, \mu]\right)$. For this reason, the possible tunneling between the two operators cannot be reduced to a tunneling in the $x$-variables (as it usually happens, for instance, in scalar multiple-wells problems - see, e.g., HeSj1, Ma1 - or for shape resonances - see, e.g., HeSj2), and actually, such a situation persists under any linear symplectic change of coordinates. Instead, one has to consider the corresponding energy shells which, in our case, are given by $\left\{(x, \xi) \in \mathbb{R}^{2 n} ; \xi^{2}+x_{n}=\mu\right\}$ for $P_{1}$, and by $\left\{(x, \xi) \in \mathbb{R}^{2 n} ; \xi^{2}+V_{2}(x)=0\right\}=\{(0,0)\}$ for $P_{2}$. Since they are disjoint, one expects some exponentially small tunneling between them. Indeed, some previous works already exists on such situations (see, e.g., Ba, Ma2, Ma3, Ma4, Ma5, MaNaSo, NaSo), but, except in the one dimensional case ( $\mathrm{Ba}, \mathrm{Na}$ ), no optimal bounds are obtained. Here, we plan to give such an optimal bound in terms of the width of the lower resonance appearing in our model.

Since we plan to study the resonances of $P$ near 0 , we also assume that $V_{2}$ and the coefficients of $R$ are globally analytic on $\mathbb{R}^{n}$, and extend to bounded holomorphic functions in a complex strip of the form $\mathcal{S}_{\delta}:=\{|\operatorname{Im} x|<\delta\}$ for
some $\delta>0$ (actually, in the sequel, we will need a little bit more). Then, the resonances of $P$ near 0 are defined, e.g., as the eigenvalues close to 0 of the complex translated operator $P_{\theta}(0<\theta<\delta)$ obtained by taking $x_{n}$ on $\mathbb{R}-i \theta$.

As in Ma2], one can construct formal WKB eigenfunctions of $P_{\theta}$ with energy close to 0 and, using the ellipticity of $P_{1, \theta}$, one can justify these formal constructions as in HeSj1, and show that the eigenvalues $\rho_{j}$ of $P_{\theta}$ in a complex domain of the form $[0, C h]-i[0, \varepsilon](\varepsilon>0$ small enough, $C>0$ arbitrarily large) admit asymptotic expansions as $h \rightarrow 0_{+}$, of the form,

$$
\rho_{j} \sim h e_{j}+h \sum_{k \geq 1} \rho_{j, k} h^{k / 2}
$$

where $\rho_{j, k} \in \mathbb{R}$ and $e_{j}$ is the $j$-th eigenvalue of the harmonic oscillator $-\Delta+\left\langle V_{2}^{\prime \prime}(0) x, x\right\rangle / 2$. Moreover, still as in Ma2, one can show that the width of $\rho_{j}$ is exponentially small, that is, there exists $S_{j}>0$ such that,

$$
\left|\operatorname{Im} \rho_{j}\right|=\mathcal{O}\left(e^{-S_{j} / h}\right)
$$

uniformly as $h \rightarrow 0$. However, the techniques used in Ma2 do not permit to have a reasonably good estimate on the best value that one can take for $S_{j}$, that is, on the quantity

$$
\mathbf{S}_{j}:=-\lim _{h \rightarrow 0} h \ln \left|\operatorname{Im} \rho_{j}\right|
$$

Better estimates can be found in Ma4, but they are not optimal, as can be deduced from the works $\mathrm{Ba}, \mathrm{Na}$ where optimal results are obtained in the one dimensional case.

The object of this paper is to compute the exact value of $\mathbf{S}_{1}$ (when $\mu$ is taken sufficiently small and $\tau$ is sufficiently small with respect to $\mu$ ), and show that $\mathbf{S}_{1}=S$ where $S$ is the imaginary part of some action integral along the complex bicharacteristics of $p_{1}(x, \xi):=\xi^{2}+x_{n}-\mu$ and $p_{2}(x, \xi):=\xi^{2}+\tau V_{2}(x)$ (see (2.2) below). Actually, our result is even more precise, since it gives a complete asymptotic expansion of $e^{S / h} \operatorname{Im} \rho_{1}$ (see Theorem 3.1). As a byproduct of the techniques, we also obtain the estimate $\mathbf{S}_{j} \geq S$ for all $j \geq 2$ (see Remark 3.2).

Our strategy consists in first making some (global) symplectic change of variables, in order to recover the geometrical situation of GrMa (where tunneling was in the position variables only), and then to adapt the argument used in GrMa. However, since this symplectic change is quadratic
in the dual variable $\xi$, a great amount of new technical difficulties appears, due to the bad behavior of the symbols as $|\xi| \rightarrow \infty$. In order to overcome them, it has been necessary for us to considerably improve the estimates of GrMa and to introduce some kind of exotic pseudodifferential calculus.

In the next section (Section2), we describe the geometric objects that enter in the statement of our main result (Section 3). Section 4 is devoted to the reduction of the problem to a geometric situation close to that of [GrMa], and to the construction of a WKB solution. In order to compare this asymptotic solution with the actual resonant state, in Section 5 we develop a generalization of Agmon estimates Ag , adapted to the peculiarities of the reduced operator (in particular, its somehow exotic aspects from a pseudodifferential point of view). In Section 6, the comparison between asymptotic and actual solution is studied, in the different regions of the $x$-space. Finally, the proof is completed in Section 7. Besides, various annex results and technical proofs are given in the six appendices.

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## 2. Geometrical Preliminaries

This section is devoted to the construction of the broken instanton that is used to express $\mathbf{S}_{1}$ as an action integral.
To start with, we work with the principal symbol $p_{2}(x, \xi)=\xi^{2}+\tau V_{2}(x)$ of $P_{2}$ near $(0,0)$. Since $V_{2}$ is analytic, we can consider the Hamilton field $H_{p_{2}}=\left(2 \xi,-\tau \nabla V_{2}(x)\right)$ in a complex neighborhood of $(0,0)$ in $\mathbb{C}^{2 n}$, and, due to the assumptions on $V_{2},(0,0)$ is a fix point for $H_{p_{2}}$. Moreover, following HeSj1], near $(0,0)$ we can define the two following (outgoing, respectively incoming) complex Lagragian submanifolds of $p_{2}^{-1}(0)$,

$$
\begin{equation*}
\Sigma_{ \pm}=\left\{\left(x, \pm i \nabla \varphi_{2}(x)\right) ; x \in \mathcal{V}_{0}\right\} \tag{2.1}
\end{equation*}
$$

where $\varphi_{2}(x)=d_{2}(x, 0), d_{2}$ is the analytic extension of the Agmon distance associated to the degenerate metric $\tau V_{2}(x) d x^{2}$ on $\mathbb{R}^{n}$, and $\mathcal{V}_{0}$ is a complex neighborhood of 0 in $\mathbb{C}^{n}$, sufficiently small so that $\varphi_{2}(x)$ remains holomorpic on it.

Next, we set $p_{1}(x, \xi)=\xi^{2}+x_{n}-\mu$ and,

$$
\Gamma_{ \pm}=\Sigma_{ \pm} \cap p_{1}^{-1}(0)
$$

and we say that a pair of points $\left(\rho_{+}, \rho_{-}\right) \in \Gamma_{+} \times \Gamma_{-}$are in $p_{1}$-correspondence, if there exits $t \in \mathbb{C}$ such that $\rho_{-}=\exp t H_{p_{1}}\left(\rho_{+}\right)$(that is, if $\rho_{+}$and $\rho_{-}$are on the same complex integral curve of $H_{p_{1}}$ ).
We have,

Lemma 2.1. Under Assumption (1.1), there exists a complex neighborhood of $(0,0)$ that, for each $\mu$ sufficiently small and for $\tau \in(0,1]$, contains a unique pair $\left(\rho_{+}, \rho_{-}\right) \in \Gamma_{+} \times \Gamma_{-}$in $p_{1}$-correspondence.

Proof. By a local change of symplectic coordinates centered at $(0,0)$ (of the form $\left.(x, \xi) \mapsto(x, \xi)+\mathcal{O}\left(\xi^{2}\right)\right)$, we can reduce to $p_{1}(x, \xi)=x_{n}-\mu$ and $p_{2}(x, \xi)=\xi^{2}+\tau V_{2}(x)+\mathcal{O}\left(|x| \xi^{2}+|\xi|^{4}\right)$, both even in $\xi$. In particular, the image of $\Sigma_{ \pm}$keeps the same form (2.1) with $\operatorname{Hess} \varphi_{2}(x)>0$ near 0 on the real. In these coordinates, $H_{p_{1}}$ just becomes $-\partial_{\xi_{n}}$, and thus the points $\rho_{ \pm}$ are necessarily given by $\rho_{ \pm}=\left(\left(x_{\mu}^{\prime}, \mu\right) ; \pm i \nabla \varphi_{2}\left(x_{\mu}^{\prime}, \mu\right)\right)$, where $x_{\mu}^{\prime} \in \mathbb{R}^{n-1}$ is the unique solution (near 0 ) of $\partial_{x^{\prime}} \varphi_{2}\left(x_{\mu}^{\prime}, \mu\right)=0$.

Now, let us denote by $\Gamma_{2}^{+}$the complex integral curve of $H_{p_{2}}$ containing $\rho_{+}$, $\Gamma_{2}^{-}$the complex integral curve of $H_{p_{2}}$ containing $\rho_{-}$, and $\Gamma_{1}$ the complex integral curve of $H_{p_{1}}$ containing $\rho_{-}$and $\rho_{+}$. In particular, $(0,0) \in \overline{\Gamma_{2}^{ \pm}}$, and we can choose a simple oriented loop $\gamma=\gamma_{2}^{+} \cup \gamma_{1} \cup \gamma_{2}^{-}$, such that,

$$
\begin{aligned}
& \gamma_{2}^{+} \text {is a path from }(0,0) \text { to } \rho_{+} \text {in } \Gamma_{2}^{+} \\
& \gamma_{1} \text { is a path from } \rho_{+} \text {to } \rho_{-} \text {in } \Gamma_{1} \\
& \gamma_{2}^{-} \text {is a path from } \rho_{-} \text {to }(0,0) \text { in } \Gamma_{2}^{-} .
\end{aligned}
$$

Therefore, $\gamma$ is included in the union of integral curves of $H_{p_{1}}$ and $H_{p_{2}}$, and for this reason we call it a broken instanton. Moreover, since the two Hamilton flows define canonical transformations, the action defined by

$$
I:=\int_{\gamma} \xi d x
$$

does not depend on the particular choice of $\gamma$, and we set

$$
\begin{equation*}
S:=\operatorname{Im} \int_{\gamma} \xi d x \tag{2.2}
\end{equation*}
$$

## 3. Main Result

Now, we can state our main result. In addition to the previous assumptions, we also assume that $V_{2}(x)$ and the coefficients $c(x)$ of $R$ extend to bounded holomorphic function near a complex domain of the form,

$$
\widetilde{\mathcal{S}}_{\delta}:=\left\{x \in \mathbb{C}^{n} ;|\operatorname{Im} x| \leq \delta\left\langle\operatorname{Re} x_{n}\right\rangle^{1 / 2}\right\}
$$

for some $\delta>0$. In particular, by Cauchy formula, for any multi-index $\alpha$, they satisfy,

$$
\begin{equation*}
\left|\partial^{\alpha} V_{2}(x)\right|+\left|\partial^{\alpha} c(x)\right|=\mathcal{O}\left(\left\langle\operatorname{Re} x_{n}\right\rangle^{-|\alpha| / 2}\right) \tag{3.1}
\end{equation*}
$$

uniformly on $\widetilde{\mathcal{S}}_{\delta}$. Moreover, we assume that the coefficient of interaction $r_{1,2}$ satisfies,

$$
\begin{equation*}
r_{1,2}\left(\rho_{+}\right) \neq 0 \tag{3.2}
\end{equation*}
$$

where $\rho_{+}$is defined in Lemma 2.1. Then, our main result is,

Theorem 3.1. Assume (1.1), (3.1) and (3.2). Then, for any $\mu>0$ small enough, there exists $\tau(\mu)>0$ such that, for $\tau \in(0, \tau(\mu)]$, the lowest resonance $\rho_{1}=e_{1} h+\mathcal{O}\left(h^{2}\right)$ of $P$ is such that,

$$
\operatorname{Im} \rho_{1}=-h^{\frac{3}{2}} f\left(h, \ln \frac{1}{h}\right) e^{-2 S / h}
$$

where $f\left(h, \ln \frac{1}{h}\right)$ admits an asymptotic expansion of the form,

$$
f\left(h, \ln \frac{1}{h}\right) \sim \sum_{0 \leq m \leq \ell} f_{\ell, m} h^{\ell}\left(\ln \frac{1}{h}\right)^{m}, \quad(h \rightarrow 0)
$$

with $f_{0,0}>0$, and $S=\operatorname{Im} \int_{\gamma} \xi d x>0$ is defined in (2.2).

Remark 3.2. As in GrMa, the same techniques also give upper bounds on the widths of the other resonances $\rho_{j}, j \geq 2$, of the type: $\left|\operatorname{Im} \rho_{j}\right|=$ $\mathcal{O}\left(h^{-\beta_{j}} e^{-2 S / h}\right)$, for some $\beta_{j}>0$. In particular, one has $\mathbf{S}_{j} \geq S$ for all $j$.

Remark 3.3. The assumption that the matrix $R\left(x, h D_{x}\right)$ is off-diagonal has been made for the sake of simplicity only. Our proof can be adapted without any problem to the case of a general (symmetric) matrix, with the difference that the lowest resonance is now given by: $\rho_{1}=\left(e_{1}+r_{2,2}(0,0)\right) h+\mathcal{O}\left(h^{2}\right)$.

## 4. Reduction and WKB Constructions

This section is devoted to the construction of asymptotics solutions of the equation $P u=\rho u$. In the next sections, these approximate solutions will be shown to be close enough to the actual ones, so that they can be used to find the asymptotics of $\operatorname{Im} \rho$.
First of all, we conjugate our operator $P$ by the unitary Fourier-integral operator,

$$
\begin{equation*}
\mathcal{U}:=\mathcal{F}^{-1} \exp \left[-\frac{2 i}{h}\left(\frac{1}{3} \xi_{n}^{3}+\sum_{k=1}^{n-1} \xi_{k}^{2} \xi_{n}\right)\right] \mathcal{F} \tag{4.1}
\end{equation*}
$$

where $\mathcal{F}$ stands for the usual $h$-dependent Fourier transform. The canonical transformation $\kappa:(x, \xi) \mapsto(y, \eta)$ associated with $\mathcal{U}$ is given by,

$$
\begin{align*}
& y_{k}=x_{k}+4 \xi_{k} \xi_{n} \quad(k \leq n-1) \\
& y_{n}=x_{n}+2 \xi^{2}  \tag{4.2}\\
& \eta=\xi
\end{align*}
$$

and thus, under the action of $\mathcal{U}$, the operator $P\left(x, h D_{x}\right)$ is transformed into $\widetilde{P}\left(y, h D_{y}\right)=\operatorname{diag}\left(\widetilde{P}_{1}, \widetilde{P}_{2}\right)+h \widetilde{R}\left(y, h D_{y}\right)$, with diagonal principal symbol $\widetilde{p}(y, \eta)=\operatorname{diag}\left(\widetilde{p}_{1}(y, \eta), \widetilde{p}_{2}(y, \eta)\right)$, given by,

$$
\widetilde{p}_{j}(y, \eta):=p_{j}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}-2 \eta^{2} ; \eta\right)
$$

In particular,

$$
\begin{align*}
& \widetilde{p}_{1}(y, \eta)=-\left(\eta^{2}-y_{n}+\mu\right) \\
& \widetilde{p}_{2}(y, \eta)=p_{2}(y, \eta)+\mathcal{O}\left(|(y, \eta)|^{3}\right)  \tag{4.3}\\
& \widetilde{p}_{2}(y,-\eta)=\widetilde{p}_{2}(y, \eta)
\end{align*}
$$

Moreover, thanks to (3.1), we observe that, for any compact set $K \subset \subset \mathbb{R}^{n}$, the full symbol of $\widetilde{P}$ extends, for $x \in K$, to an holomorphic function with respect to $\eta$ in a complex strip around $\mathbb{R}^{n}$, and that its derivatives of any order remain $\mathcal{O}\left(\langle\eta\rangle^{2}\right)$ on this strip.
Then, except from the minus sign in front of $\widetilde{p}_{1}$, we see on (4.3) that we are in a situation very similar to that studied by in GrMa] (see also [Pe]), namely, the transversal crossing between two potentials such that one of them admits a non degenerate point-well, and the two corresponding classically allowed regions are disjoints.

As in GrMa, the starting point of the construction consists in the WKB asymptotics given near the well $y=0$ by the method of Helffer and Sjöstrand HeSj1]. More precisely, because of the matricial nature of the operator and the fact that $\widetilde{p}_{1}$ is elliptic above $y=0$, one finds a formal solution $w_{1}$ of $\widetilde{P} w_{1}=\rho w_{1}$ of the form,

$$
\begin{equation*}
w_{1}(y ; h)=\binom{h a_{1}(y, h)}{a_{2}(y, h)} e^{-\widetilde{\varphi}_{2}(y) / h} \tag{4.4}
\end{equation*}
$$

where $\widetilde{\varphi}_{2}(y)$ is the (real and non negative on the real domain) solution of $\widetilde{p}_{2}\left(y, i \nabla \widetilde{\varphi}_{2}(y)\right)=0, \widetilde{\varphi}_{2}(0)=0$, defined by,

$$
\kappa\left(\Gamma_{+}\right)=\left\{\left(y, i \nabla \widetilde{\varphi}_{2}(y)\right) ; y \text { close to } 0\right\} .
$$

(In particular, we also have $\nabla \widetilde{\varphi}_{2}(0)=0$.) Moreover, the two $a_{j}$ 's $(j=1,2)$ are classical symbols of order 0 in $h$, that is, they are formal series in $h$ of the form,

$$
\begin{equation*}
a_{j}(y, h)=\sum_{k=0}^{\infty} h^{k} a_{j, k}(y) \tag{4.5}
\end{equation*}
$$

with $a_{j, k}$ smooth near 0 , and $a_{2}$ is elliptic in the sense that $a_{2,0}$ never vanishes.

It is easy to see that the above constructions can be continued along the integral curves of the (real) vector field $\partial_{\eta} \widetilde{p}_{2}\left(y, i \nabla \varphi_{2}\right) D_{y}=\left(2 \nabla \varphi_{2}(y)+\right.$ $\left.\mathcal{O}\left(y^{2}\right)\right) . \nabla_{y}$, as long as $\widetilde{p}_{1}\left(y, i \nabla \widetilde{\varphi}_{2}(y)\right)$ does not vanish. We set,

$$
\begin{align*}
& \Sigma_{2}^{+}=\left\{\left(y, i \nabla \widetilde{\varphi}_{2}(y)\right) ; y \in \mathcal{V}\right\} \\
& \Gamma=\Gamma_{\mu}=\Sigma_{2}^{+} \cap \widetilde{p}_{1}^{-1}(0) \tag{4.6}
\end{align*}
$$

where $\mathcal{V}$ is a fixed ( $\mu$-independent) sufficiently small neighbourhood of 0 in $\mathbb{C}^{n}$. Then $\Gamma$ is a complex $\mathbb{C}$-isotropic submanifold of $\mathbb{C}^{2 n}$ (that is, with respect to the complex symplectic form $\sigma=\sum d \eta_{k} \wedge d y_{k}$ ) of dimension $n-1$, and standard arguments of symplectic geometry show that if $\mathcal{T}$ is a sufficiently small $\mu$-independent neighborhood of 0 in $\mathbb{C}$, then the set:

$$
\begin{equation*}
\Sigma_{1}=\left\{\exp t H_{\tilde{p}_{1}}(y, \eta) ;(y, \eta) \in \Gamma, t \in \mathcal{T}\right\} \tag{4.7}
\end{equation*}
$$

is a complex Lagrangian manifold. Since the projection $\Pi_{y} \Gamma$ of $\Gamma$ on the base $\{\eta=0\}$ has an equation of the form $y_{n}=f\left(y^{\prime}\right)=\mu-\delta \mu^{2}-f_{1}\left(y^{\prime}\right)+\mathcal{O}\left(\left|y^{\prime}\right|^{3}+\right.$ $\mu^{3}$ ) with $f_{1}(0)=0, f_{1}\left(y^{\prime}\right) \geq \delta^{\prime}\left|y^{\prime}\right|^{2}\left(\right.$ where $y^{\prime}=\left(y_{1}, \ldots, y_{n-1}\right)$ and $\delta, \delta^{\prime}>0$ do not depend on $\mu$ ), an easy computation shows that, for $\mu>0$ small enough, $\Pi_{y} H_{\widetilde{p}_{1}}$ keeps transversal to $\Pi_{y} \Gamma$, as long as $\left|y^{\prime}\right| \ll 1$. Moreover, the only points of $\Sigma_{1}$ where $H_{\widetilde{p}_{1}}$ becomes vertical are above $\left\{y_{n}=\mu\right\}$,
and $\Pi_{y} \Gamma \cap \mathbb{R}^{n} \subset\left\{y_{n}<\mu\right\}$. As a consequence, there exists a complex neighborhood $\mathcal{W}_{\mu}$ of $\Gamma$ such that $\Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}$ is of the form,

$$
\Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}=\left\{-\delta_{0}<y_{n}<g\left(y^{\prime}\right),\left|y^{\prime}\right|<\delta_{0}\right\}
$$

with $\delta_{0}>0$ independent of $\mu$, and $g\left(y^{\prime}\right)=\mu+\mathcal{O}\left(\left|y^{\prime}-y_{\mu}^{\prime}\right|^{2}\right.$ ) (for some $\left.y_{\mu}^{\prime}=\mathcal{O}\left(\mu^{2}\right) \in \mathbb{R}^{n}\right)$, and such that $\Sigma_{1} \cap \mathcal{W}_{\mu}$ projects bijectively on the base. This implies the existence of an analytic function $\widetilde{\varphi}_{1}$ defined on $\Pi_{y} \mathcal{W}_{\mu}$, such that,

$$
\begin{equation*}
\Sigma_{1} \cap \mathcal{W}_{\mu}=\left\{\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right) ; y \in \Pi_{y} \mathcal{W}_{\mu}\right\} \tag{4.8}
\end{equation*}
$$

Moreover, $\widetilde{\varphi}_{1}$ can be normalized by requiring that $\widetilde{\varphi}_{1}=\widetilde{\varphi}_{2}$ on $\Pi_{y} \Gamma$, so that we then have,

$$
\begin{equation*}
\widetilde{\varphi}_{1}=\widetilde{\varphi}_{2} \quad \text { and } \quad \nabla \widetilde{\varphi}_{1}=\nabla \widetilde{\varphi}_{2} \quad \text { on } \quad \Pi_{y} \Gamma . \tag{4.9}
\end{equation*}
$$

Using that $\widetilde{\varphi}_{2}$ is real on the real, and $\widetilde{p}_{1}(y, \eta)$ is even with respect to $\eta$, we also see that,

$$
\begin{equation*}
\widetilde{\varphi}_{1} \text { is real on the real. } \tag{4.10}
\end{equation*}
$$

Finally, one can easily check that the quantity,

$$
S(\mu):=\widetilde{\varphi}_{1}\left(y_{\mu}^{\prime}, \mu\right),
$$

is $\mathcal{O}\left(\mu^{2}\right)$, and that, on $\Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}$, one has $\widetilde{\varphi}_{1} \leq \widetilde{\varphi}_{2}$. Moreover, since $\widetilde{\varphi}_{2}(y) \geq|y|^{2} / C$ with some constant $C>0$, for $\mu$ small enough, we have

$$
\widetilde{\varphi}_{2}(y) \sim 1 \gg S(\mu) \text { on } \partial\left(\Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}\right) \cap\left\{y_{n}<g\left(y^{\prime}\right)\right\} .
$$

Then, one can deduce,

$$
\widetilde{\varphi}_{1}(y)>S(\mu) \text { on } \partial\left(\Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}\right) \cap\left\{f\left(y^{\prime}\right) \leq y_{n}<g\left(y^{\prime}\right)\right\} .
$$

On the other hand, one can see as in HeSj2, Lemma 10.2, that, for $y^{\prime} \neq y_{\mu}^{\prime}$, one has,

$$
\widetilde{\varphi}_{1}\left(y^{\prime}, g\left(y^{\prime}\right)\right)>S(\mu) .
$$

In addition, a local study of $H_{p_{1}}$ shows that $\left(\Pi_{y} \Sigma_{1}\right) \cap \mathbb{R}^{n}$ has a contact of order exactly 2 with $\left\{y_{n}=\mu\right\} \cap \mathbb{R}^{n}$ at ( $y_{\mu}^{\prime}, \mu$ ), and one can deduce the existence of a constant $C=C(\mu)>0$ such that $C^{-1}\left|y^{\prime}-y_{\mu}^{\prime}\right|^{2} \leq \mu-g\left(y^{\prime}\right) \leq$ $C\left|y^{\prime}-y_{\mu}^{\prime}\right|^{2}$.

Remark 4.1. Let us observe that, by standard symplectic arguments, the quantity $S(\mu)$ is nothing but the constant $S$ defined in (2.2).

As in GrMa, the extension of the WKB constructions (4.4) accross $\Gamma$ are obtained by taking a formal ansatz of the form,
$w_{2}(y ; h)=\sum_{k \geq 0} h^{k}\left(\alpha_{k}(y, h) Y_{k, 0}\left(\frac{z(y)}{\sqrt{h}}\right)+\sqrt{h} \beta_{k}(y, h) Y_{k, 1}\left(\frac{z(y)}{\sqrt{h}}\right)\right) e^{-\psi(y) / h}$,
where,

$$
\begin{equation*}
\alpha_{k}(y, h)=\binom{h \alpha_{k, 1}(y, h)}{\alpha_{k, 2}(y, h)} \quad ; \quad \beta_{k}(y, h)=\binom{\beta_{k, 1}(y, h)}{h \beta_{k, 2}(y, h)}, \tag{4.12}
\end{equation*}
$$

$\alpha_{k, j}$ and $\beta_{k, j}$ are formal symbols of the form,

$$
\begin{equation*}
\sum_{l \geq 0} \sum_{m=0}^{l} h^{l}(\ln h)^{m} \gamma^{l, m}(y) \tag{4.13}
\end{equation*}
$$

(with $\gamma^{l, m}$ analytic near $\Gamma$ ), the function $\psi$ is defined as,

$$
\begin{equation*}
\psi(y):=\frac{1}{2}\left(\widetilde{\varphi}_{2}(y)+\widetilde{\varphi}_{1}(y)\right), \tag{4.14}
\end{equation*}
$$

the function $z(y)$ is the only analytic function defined near $\Gamma$ such that,

$$
\begin{align*}
& z(y)^{2}=2\left(\widetilde{\varphi}_{2}(y)-\widetilde{\varphi}_{1}(y)\right) \\
& z(y)<0 \quad \text { on } \quad \Gamma_{-}:=\left\{y_{n}<f\left(y^{\prime}\right) ; y \in \Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}\right\}, \tag{4.15}
\end{align*}
$$

and for any $k \geq 0$ and $\varepsilon \in \mathbb{C}$, the function $Y_{k, \varepsilon}$ is the so-called Weber function, defined by,

$$
\begin{equation*}
Y_{k, \varepsilon}(z)=\partial_{\varepsilon}^{k} Y_{0, \varepsilon}(z) \tag{4.16}
\end{equation*}
$$

where $Y_{0, \varepsilon}$ is the unique entire function with respect to $\varepsilon$ and $z$, solution of the Weber equation,

$$
\begin{equation*}
Y_{0, \varepsilon}^{\prime \prime}+\left(\frac{1}{2}-\varepsilon-\frac{z^{2}}{4}\right) Y_{0, \varepsilon}=0 \tag{4.17}
\end{equation*}
$$

such that, for $\varepsilon>0$, one has,

$$
\begin{equation*}
Y_{0, \varepsilon}(z) \sim \frac{\sqrt{2 \pi}}{\Gamma(\varepsilon)} e^{z^{2} / 4} z^{\varepsilon-1} \quad(z \rightarrow+\infty) \tag{4.18}
\end{equation*}
$$

Then, a resummation of (4.11) is possible up to an error of order $\mathcal{O}\left(h^{\infty} e^{-\phi / h}\right)$, with,

$$
\begin{align*}
& \phi=\widetilde{\varphi}_{2} \text { in } \Gamma_{-} ; \\
& \phi=\widetilde{\varphi}_{1}=\widetilde{\varphi}_{2} \quad \text { on } \Gamma  \tag{4.19}\\
& \phi=\widetilde{\varphi}_{1} \quad \text { in } \Gamma_{+}:=\left\{y_{n}>f\left(y^{\prime}\right) ; y \in \Pi_{y} \mathcal{W}_{\mu} \cap \mathbb{R}^{n}\right\} .
\end{align*}
$$

Actually, looking a little bit more carefully at the results of Pe (in particular Proposition A. 2 and the proof of Lemma 3.3), we see that the resummation
can actually be done in a small enough complex neighborhood $\Omega$ of $\Gamma$, under the condition that $\Omega \subset\left\{y \in \mathbb{C}^{n} ;|\operatorname{Im} z(y)| \leq C \sqrt{h}\right\}$ (where $C>0$ is an arbitrary constant). In that case, the resummation is valid up to $\mathcal{O}\left(h^{\infty} e^{-\operatorname{Re} \phi / h}\right)$ error-terms. In what follows, we will use the notations,

$$
\psi_{\delta}(y):=\psi(y+i \delta) \quad ; \quad \phi_{\delta}(y):=\phi(y+i \delta),
$$

where $\delta=\delta(h)=\mathcal{O}(\sqrt{h}) \in \mathbb{R}^{n}$.
For $\nu_{0}>0$ and $g \in C^{\infty}\left(\mathbb{R}^{2 n} ; \mathbb{R}_{+}\right)$, we denote by $S_{\nu_{0}}(g(x, \xi))$ the set of (possibly $h$-dependent) functions $p \in C^{\infty}\left(\mathbb{R}^{2 n}\right)$ that extend to holomorphic functions with respect to $\xi$ in the strip,

$$
\mathcal{A}_{\nu_{0}}:=\left\{(x, \xi) \in \mathbb{R}^{n} \times \mathbb{C}^{n} ;|\operatorname{Im} \xi|<\nu_{0}\right\},
$$

and such that, for all $\alpha \in \mathbb{N}^{2 n}$, one has,

$$
\begin{equation*}
\partial^{\alpha} p(x, \xi)=\mathcal{O}(g(x, \operatorname{Re} \xi)), \tag{4.20}
\end{equation*}
$$

uniformly with respect to $(x, \xi) \in \mathcal{A}_{\nu_{0}}$ and $h>0$ small enough. We also denote by $S_{0}(g)$ the analogous space of smooth symbols obtained by substituting $\mathbb{R}^{2 n}$ to $\mathcal{A}_{\nu_{0}}$.

If $a$ and $b$ are (scalar) formal symbols of the type (4.13), and $k \in \mathbb{N}$, we set,

$$
\begin{equation*}
I_{k}(a, b)(y ; h)=a(y ; h) Y_{k, 0}\left(\frac{z(y)}{\sqrt{h}}\right)+b(y ; h) Y_{k, 1}\left(\frac{z(y)}{\sqrt{h}}\right) . \tag{4.21}
\end{equation*}
$$

For $M \in \mathbb{Z}$ and $\Omega \subset \mathbb{R}^{n}$ open, we also consider the space of sequences of formal symbols,

$$
\begin{array}{r}
S^{M}(\Omega):=\left\{a=\left(a_{k}\right)_{k \in \mathbb{N}} ; a_{k}(y, h)=\sum_{l=-M}^{\infty} \sum_{m=0}^{l} h^{l}(\ln h)^{m} \gamma_{k}^{l, m}(y) ;\right. \\
\left.\gamma_{k}^{l, m} \in C^{\infty}(\Omega)\right\} .
\end{array}
$$

and, for $a, b \in S^{M}\left(\Pi_{y} \mathcal{W}_{\mu}\right)$, we set,

$$
\begin{equation*}
I(a, b) e^{-\psi_{\delta} / h}:=\sum_{k \geq 0} h^{k} I_{k}\left(a_{k}, \sqrt{h} b_{k}\right) e^{-\psi_{\delta} / h} . \tag{4.22}
\end{equation*}
$$

Then, the arguments of GrMa, Section4, give a natural way to make act $\mathrm{Op}_{h}^{W}(p)$ on expressions of the type,

$$
\begin{equation*}
w=\binom{I\left(h \alpha_{1}, \beta_{1}\right)}{I\left(\alpha_{2}, h \beta_{2}\right)} e^{-\psi_{\delta} / h}, \tag{4.23}
\end{equation*}
$$

where $\alpha_{j}=\left(\alpha_{j, k}\right)_{k \geq 0}$ and $\beta_{j}=\left(\beta_{j, k}\right)_{k \geq 0}$ are in $S^{0}\left(\Pi_{y} \mathcal{W}_{\mu}\right) \quad(j=1,2)$. Moreover, this action is well defined up to error terms that are exponentially smaller than $e^{\operatorname{Re} \phi_{\delta} / h}$.

Now, denote by $p$ the full symbol of $P$, and recall the definition $\widetilde{P}:=\mathcal{U}^{P} \mathcal{U}^{-1}$, where $\mathcal{U}$ is given in (4.1). In order to be able to make act $\widetilde{P}$, too, on such expressions, we first show,

Lemma 4.2. Let $c=c(x)$ satisfying (3.1). Then, for any $\chi=\chi\left(y_{n}\right) \in$ $C_{0}^{\infty}(\mathbb{R})$, there exists $\nu_{0}>0$ such that the operator $\chi \mathcal{U ~}^{c} \mathcal{U}^{-1}$ can be written as,

$$
\chi \mathcal{U} c \mathcal{U}^{-1}=\mathrm{Op}_{h}^{W}(\check{c}+r)
$$

with $\check{c} \in S_{\nu_{0}}(1), r \in S_{0}(1)$, and,

$$
r=\mathcal{O}\left(e^{-\nu_{0} / h}\right) \text { in } S_{0}(1)
$$

Moreover, the symbol č satisfies,

$$
\check{c}(y, \eta)=\chi\left(y_{n}\right) c\left(y^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}-2 \eta^{2}\right)+\mathcal{O}(h) \text { in } S_{\nu_{0}}(1) .
$$

Proof. A straightforward computation leads to,

$$
\begin{equation*}
\chi \mathcal{U} c \mathcal{U}^{-1}=\mathrm{Op}_{h}^{L}(a) \tag{4.24}
\end{equation*}
$$

with,

$$
\begin{aligned}
a(y, \eta)= & \frac{1}{(2 \pi h)^{n}} \int \chi\left(y_{n}\right) e^{i(y-x)(\xi-\eta) / h} \times \\
& \times c\left(x^{\prime}-2 \xi_{n}\left(\xi^{\prime}+\eta^{\prime}\right), x_{n}-\frac{2}{3}\left(\eta_{n}^{2}+\eta_{n} \xi_{n}+\xi_{n}^{2}\right)-2\left|\eta^{\prime}\right|^{2}\right) d x d \xi
\end{aligned}
$$

In this integral, we can perform the change of contour of integration,

$$
\mathbb{R}^{n} \ni x \mapsto x-i \delta \frac{\xi-\eta}{\langle\xi-\eta\rangle} \in \mathbb{C}^{n}
$$

where $\delta>0$ is a small enough constant. Setting $f=\left(f^{\prime}, f_{n}\right):=\delta(\xi-\eta) /\langle\xi-$ $\eta\rangle$ and,
$F(x, \eta, \xi):=\left(x^{\prime}-i f^{\prime}-2 \xi_{n}\left(\xi^{\prime}+\eta^{\prime}\right), x_{n}-i f_{n}-\frac{2}{3}\left(\eta_{n}^{2}+\eta_{n} \xi_{n}+\xi_{n}^{2}\right)-2\left|\eta^{\prime}\right|^{2}\right)$,
we obtain,

$$
\begin{aligned}
a(y, \eta) & =\frac{1}{(2 \pi h)^{n}} \int \chi\left(y_{n}\right) e^{i(y-x)(\xi-\eta) / h-\frac{\delta}{h}|\xi-\eta|^{2} /\langle\xi-\eta\rangle} c(F(x, \eta, \xi)) d x d \xi \\
& =\frac{1}{(2 \pi h)^{n}} \int \chi\left(y_{n}\right) e^{i(y-x) \xi / h-\frac{\delta}{h}|\xi|^{2} /\langle\xi\rangle} c(F(x, \eta, \xi+\eta)) d x d \xi
\end{aligned}
$$

and thus, for any fix $\chi_{0} \in C_{0}^{\infty}\left(\mathbb{R}^{n} ;[0,1]\right)$ such that $\chi_{0}=1$ near 0 ,

$$
\begin{array}{r}
a(y, \eta)=\frac{1}{(2 \pi h)^{n}} \int \chi\left(y_{n}\right) \chi_{0}(\xi) e^{i(y-x) \xi / h-\frac{\delta}{h}|\xi|^{2} /\langle\xi\rangle} c(F(x, \eta, \xi+\eta)) d x d \xi \\
+r_{1}(y, \eta)
\end{array}
$$

where, by means of integrations by parts in $\xi$, the symbol $r_{1}$ can be seen to satisfy,

$$
\partial^{\alpha} r_{1}=\mathcal{O}\left(e^{-\delta^{\prime} / h}\right) \text { in } S_{0}(1)
$$

for some $\delta^{\prime}>0$ independent of $\mu$ and $h$.
Then, we split the previous integral into,

$$
I_{1}+I_{2}+I_{3}:=\int_{\left|x_{n}\right| \leq 2 C}+\int_{2 C \leq\left|x_{n}\right| \leq 2 C+\langle\eta\rangle^{2}}+\int_{\left|x_{n}\right| \geq 2 C+\langle\eta\rangle^{2}}
$$

where $C>0$ is chosen sufficiently large in order to have $\left|y_{n}-x_{n}\right| \geq\left|x_{n}\right| / 2$ when $\left|x_{n}\right| \geq 2 C$ and $y_{n} \in \operatorname{Supp} \chi$.
On $\left\{\left|x_{n}\right| \leq 2 C+\langle\eta\rangle^{2}\right\} \cap \operatorname{Supp} \chi_{0}$, we observe that $\left|\operatorname{Re} F_{n}\right| \sim\langle\eta\rangle^{2}$. Therefore, if $\chi_{1} \in C_{0}^{\infty}\left(\mathbb{R}^{n} ;[0,1]\right)$ is such that $\chi_{1}=1$ near 0 and $\operatorname{Supp} \chi_{1} \subset\left\{\chi_{0}=1\right\}$, in this region we can perform the change of contour of integration,

$$
\mathbb{R}^{n} \ni \xi \mapsto \widetilde{\xi}:=\xi+i \delta^{\prime} \chi_{1}(\xi) \frac{y-x}{\langle y-x\rangle}
$$

that gives us,

$$
\begin{array}{r}
I_{1}=\chi\left(y_{n}\right) \int_{\left|x_{n}\right| \leq 2 C} \mathcal{O}\left(\chi_{0}(\xi) h^{-n} e^{-\frac{\delta^{\prime} \chi_{1}(\xi)|y-x|^{2}}{2 h\langle y-x\rangle}-\frac{\delta^{\prime}|\xi|^{2}}{h\langle\xi\rangle}}\right) c(F(x, \eta, \widetilde{\xi}+\eta)) d x d \xi ; \\
I_{2}=\chi\left(y_{n}\right) \int_{2 C \leq\left|x_{n}\right| \leq 2 C+\langle\eta\rangle^{2}} \mathcal{O}\left(\chi_{0}(\xi) h^{-n} e^{-\frac{\delta^{\prime} \chi_{1}(\xi)|y-x|^{2}}{2 h\langle y-x\rangle}-\frac{\delta^{\prime} \mid\left\langle\left.\right|^{2}\right.}{h\langle\xi\rangle}}\right) \\
\times c(F(x, \eta, \widetilde{\xi}+\eta)) d x d \xi
\end{array}
$$

and thus, using Assumption (3.1), we see that, for $\nu_{0}>0$ sufficiently small, we have $I_{1} \in S_{\nu_{0}}(1)$. Moreover, a stationary-phase expansion shows that,

$$
I_{1}=\chi\left(y_{n}\right) c(F(y, \eta, \eta))+\mathcal{O}(h)=\chi\left(y_{n}\right) c\left(y^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}-2 \eta^{2}\right)+\mathcal{O}(h)
$$

where the estimates $\mathcal{O}(h)$ hold in the space $S_{\nu_{0}}(1)$.
On the other hand, since $|x-y| \geq\left|x_{n}-y_{n}\right| \geq\left|x_{n}\right| / 2$ for $\left|x_{n}\right| \geq 2 C$ and $y_{n} \in \operatorname{Supp} \chi$, we see that,

$$
I_{2}=\mathcal{O}\left(e^{-\nu_{1} / h}\right) \text { in } S_{0}(1)
$$

for some $\nu_{1}>0$.
Finally, in the expression of $I_{3}$, we perform the change of contour of integration,

$$
\mathbb{R}^{n} \ni \xi \mapsto \hat{\xi}:=\xi+i \delta \chi_{1}(\xi) \frac{y-x}{\langle y-x\rangle\langle\eta\rangle},
$$

and since, in the relevant region, $|y-x| \geq\left|x_{n}-y_{n}\right| \geq\left|x_{n}\right| / 2 \geq C+\langle\eta\rangle^{2} / 2$, this leads to,

$$
\begin{aligned}
& I_{3}=\chi\left(y_{n}\right) \int_{\left|x_{n}\right| \geq 2 C+\langle\eta\rangle^{2}} \mathcal{O}\left(\chi_{0}(\xi) h^{-n} e^{-\frac{\delta^{\prime} x_{1}|x|}{h\langle\eta\rangle}-\frac{\delta^{\prime} \chi_{1}\langle\eta\rangle}{h}-\frac{\delta^{\prime}|\xi|^{2}}{h\langle\xi\rangle}}\right) \\
& \times V_{2}(F(x, \eta, \hat{\xi}+\eta)) d x d \xi
\end{aligned}
$$

and thus,

$$
I_{3}=\mathcal{O}\left(e^{-\delta^{\prime} / h}\right) \text { in } S_{0}(1)
$$

Therefore, we have proved the result of the lemma, except from the fact that we have worked with the left-quantization instead of the Weyl-one. But the passage from one quantization to the other one can be done in a standard way (see, e.g., Ma6), and this completes the proof of the lemma.

As a corollary, and since $\mathcal{U} h D_{x} \mathcal{U}^{-1}=h D_{y}$, we immediately obtain (denoting by $\mathcal{M}_{2}(E)$ the space of $2 \times 2$ matrices with coefficients in $E$ ),

Proposition 4.3. Assume (3.1), and denote by $T_{\delta}$ the complex translation given by $T_{\delta} u(y):=u(y+i \delta)$. Then, for any $\chi=\chi\left(y_{n}\right) \in C_{0}^{\infty}(\mathbb{R})$, there exists $\nu_{0}>0$ (independent of $\mu$ ) such that the operator $\chi \widetilde{P}^{\delta}:=\chi T_{\delta} \widetilde{P} T_{\delta}^{-1}$ can be written as,

$$
\chi \widetilde{P}^{\delta}=\mathrm{Op}_{h}^{W}\left(\check{p}_{\delta}+r_{0}\right),
$$

with $\check{p}_{\delta} \in \mathcal{M}_{2}\left(S_{\nu_{0}}\left(\langle\eta\rangle^{2}\right)\right), r_{0} \in \mathcal{M}_{2}\left(S_{0}\left(\langle\eta\rangle^{2}\right)\right)$, and,

$$
r_{0}=\mathcal{O}\left(e^{-\nu_{0} / h}\right) \text { in } \mathcal{M}_{2}\left(S_{0}\left(\langle\eta\rangle^{2}\right)\right)
$$

Moreover, the symbol $\check{p}_{\delta}$ satisfies,
$\check{p}_{\delta}(y, \eta)=\chi\left(y_{n}\right) p\left(y^{\prime}+i \delta^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}+i \delta_{n}-2 \eta^{2} ; \eta\right)+\mathcal{O}(h)$ in $\mathcal{M}_{2}\left(S_{\nu_{0}}\left(\langle\eta\rangle^{2}\right)\right)$.
Consequently, for $\mu$ sufficienly small, and up to error terms that are exponentially smaller that $e^{-\operatorname{Re} \phi_{\delta} / h}$, the operator $\widetilde{P}^{\delta}$ naturally acts on expressions of the type (4.23).

Then, proceeding as in GrMa (but in a quite simpler geometric situation, here), we finally obtain the existence of a $\mu$-independent neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^{n}$, such that, if one sets,

$$
\begin{align*}
& \mathcal{V}_{\mu}^{-}=\mathcal{V} \cap\left\{y_{n}<f\left(y^{\prime}\right)\right\} \\
& \mathcal{V}_{\mu}^{+}=\mathcal{V} \cap\left\{f\left(y^{\prime}\right)<y_{n}<g\left(y^{\prime}\right)\right\} \tag{4.25}
\end{align*}
$$

then, we have,

Proposition 4.4. There exists a neighborhood $\Omega_{\mu}$ of $\Pi_{y} \Gamma$ and symbols,

$$
\begin{align*}
& a_{1}, a_{2} \in S^{0}\left(\mathcal{V}_{\mu}^{-}\right), \quad a_{2} \text { elliptic } \\
& \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in S^{0}\left(\Omega_{\mu}\right), \quad \alpha_{2} \text { and } \beta_{1} \text { elliptic } ;  \tag{4.26}\\
& b_{1}, b_{2} \in S^{0}\left(\mathcal{V}_{\mu}^{+}\right), \quad b_{1} \text { elliptic, }
\end{align*}
$$

such that, if one sets,

$$
\begin{aligned}
& w_{1}=\binom{h a_{1}}{a_{2}} e^{-\widetilde{\varphi}_{2} / h} ; w_{2}=\binom{I\left(h \alpha_{1}, \beta_{1}\right)}{I\left(\alpha_{2}, h \beta_{2}\right)} e^{-\psi / h} ; \\
& w_{3}=\sqrt{2 \pi h}\binom{b_{1}}{h b_{2}} e^{-\widetilde{\varphi}_{1} / h},
\end{aligned}
$$

then, one formally has,

$$
\widetilde{P} w_{1}=\rho w_{1} \quad ; \quad \widetilde{P} w_{2}=\rho w_{2} \quad ; \quad \widetilde{P} w_{3}=\rho w_{3} .
$$

Moreover, if $\widetilde{w}_{1}, \widetilde{w}_{2}$ and $\widetilde{w}_{3}$ denote resummations of $w_{1}, w_{2}$ and $w_{3}$ respectively, one has,

$$
\begin{array}{ll}
\widetilde{w}_{1}-\widetilde{w}_{2}=\mathcal{O}\left(h^{\infty} e^{-\widetilde{\varphi}_{2} / h}\right) & \text { in } \mathcal{V}_{\mu}^{-} \cap \Omega_{\mu} \\
\widetilde{w}_{2}-\widetilde{w}_{3}=\mathcal{O}\left(h^{\infty} e^{-\widetilde{\varphi}_{1} / h}\right) & \text { in } \mathcal{V}_{\mu}^{+} \cap \Omega_{\mu} . \tag{4.28}
\end{array}
$$

Now, let $\left(\chi_{1}, \chi_{2}, \chi_{3}\right)$ be a partition of unity adapted to $\left(\mathcal{V}_{\mu}^{-}, \Omega_{\mu}, \widetilde{\mathcal{V}}_{\mu}^{+}\right)$with $\widetilde{\mathcal{V}}_{\mu}^{+} \subset \subset \mathcal{V}_{\mu}^{+}$and $\chi_{3} \in C_{0}^{\infty}\left(\mathcal{V}_{\mu}^{+}\right)$, and set,

$$
\begin{equation*}
\widetilde{U}=\sum_{j=1}^{3} \chi_{j} \widetilde{w}_{j} \in C_{0}^{\infty}\left(\mathcal{V} \cap\left\{y_{n}<g\left(y^{\prime}\right)\right\}\right) . \tag{4.29}
\end{equation*}
$$

Then, the arguments in GrMa show that,

$$
\begin{equation*}
\widetilde{P} \widetilde{U}=\rho \widetilde{U}+\mathcal{O}\left(h^{\infty} e^{-\phi / h}\right) \quad \text { uniformly in }\left\{\chi_{1}+\chi_{2}+\chi_{3}=1\right\} . \tag{4.30}
\end{equation*}
$$

Finally, the caustic set $\mathcal{C}:=\left\{y_{n}=g\left(y^{\prime}\right)\right\}$ can be crossed over, too, by using, as in HeSj2 Section 10, an Airy representation near $\mathcal{C}$, and by using the stationary phase method in $\mathcal{V} \cap\left\{y_{n}>g\left(y^{\prime}\right)\right\}$ in order to recover there an expression of the type $\sqrt{2 \pi h}\binom{c_{1}}{h c_{2}} e^{\widetilde{\varphi} / h}$ where the symbol $c_{1}$ is elliptic and $\widetilde{\varphi}$ is a complex-valued analytic function that satisfies $(\nabla \widetilde{\varphi})^{2}=\mu-$ $y_{n}, \operatorname{Re} \widetilde{\varphi}=C t e=S(\mu)$ on $\gamma_{\mu}:=\left\{\exp t H_{\widetilde{p}_{1}}\left(y_{\mu}^{\prime}, \mu\right) ; t \in \mathbb{R}\right.$ small enough $\}$, $\operatorname{Re} \widetilde{\varphi}(y) \geq S(\mu)+\delta_{1} \operatorname{dist}\left(y, \gamma_{\mu}\right)^{2}$ for some $\delta_{1}>0$. In particular, one also has,

$$
\begin{equation*}
\left|\partial_{y_{n}} \operatorname{Im} \widetilde{\varphi}\left(y_{\mu}^{\prime}, y_{n}\right)\right|=\sqrt{y_{n}-\mu} . \tag{4.31}
\end{equation*}
$$

By using an additional cut-off function $\chi_{4}$ near $(0, \mu)$, and by shrinking a little bit (but still in a $\mu$-independent way) $\mathcal{V}$ around 0 , we finally obtain a
function,

$$
\mathbf{w}=\sum_{j=1}^{4} \chi_{j} \widetilde{w}_{j} \in C_{0}^{\infty}\left(\mathcal{V} \cap\left\{y_{n}<g\left(y^{\prime}\right)+2 \varepsilon\right\}\right.
$$

$(\varepsilon=\varepsilon(\mu)>0)$, such that, setting $\widetilde{\mathcal{V}}_{1}:=\left\{\chi_{1}+\chi_{2}+\chi_{3}+\chi_{4}=1\right\}$ and $\widetilde{\mathcal{V}}_{2}:=\bigcup_{1 \leq j \leq 4} \operatorname{Supp}\left(\chi_{j}\right)$, one has,

$$
\begin{align*}
& \text { (4.32) } \quad \widetilde{P} \mathbf{w}=\rho \mathbf{w}+\mathcal{O}\left(h^{\infty} e^{-\varphi / h}\right) \quad \text { on } \quad \widetilde{\mathcal{V}}_{1} \cap\left\{y_{n}<g\left(y^{\prime}\right)+\varepsilon\right\}  \tag{4.32}\\
& (4.33) \quad \widetilde{P} \mathbf{w}=\rho \mathbf{w}+\mathcal{O}\left(h^{-N_{0}} e^{-S(\mu) / h}\right) \quad \text { on } \widetilde{\mathcal{V}}_{2} \cap\left\{y_{n} \geq g\left(y^{\prime}\right)\right\}, \\
& \text { (4.34) } \widetilde{P} \mathbf{w}=\rho \mathbf{w}+\mathcal{O}\left(h^{-N_{0}} e^{-\varphi / h}\right) \quad \text { on } \quad \widetilde{\mathcal{V}}_{2} \backslash \widetilde{\mathcal{V}}_{1} \\
& \text { with } \varphi=\widetilde{\varphi}_{2} \text { in } \widetilde{\mathcal{V}}_{2} \cap\left\{y_{n} \leq f\left(y^{\prime}\right)\right\}, \varphi=\widetilde{\varphi}_{1} \text { in } \widetilde{\mathcal{V}}_{2} \cap\left\{f\left(y^{\prime}\right) \leq y_{n} \leq g\left(y^{\prime}\right)\right\}, \\
& \varphi=\operatorname{Re} \widetilde{\varphi} \text { in } \widetilde{\mathcal{V}}_{2} \cap\left\{g\left(y^{\prime}\right) \leq y_{n} \leq g\left(y^{\prime}\right)+\varepsilon\right\}, \text { and } N_{0} \geq 0
\end{align*}
$$

Observe that, by construction, $\varphi>S(\mu)$ on $\left(\widetilde{\mathcal{V}}_{2} \backslash \widetilde{\mathcal{V}}_{1}\right) \cap\left\{y_{n} \leq g\left(y^{\prime}\right)\right\}$.

## 5. Agmon estimates

In order to estimate the width of the resonance, we need to prove that the asymptotic solution $\mathbf{w}$ constructed in the previous section gives a sufficiently good approximation of the true resonant state $\mathbf{u}$, specially in the region $\left\{y_{n}>\mu\right\}$.

For this, we would like to adopt the usual strategy of Agmon estimates (see Ag ), but, since we work with operators that are not differential (even at the level of principal symbols), we have to be more careful. In particular, our weight functions need to be smooth (and not only Lipschitz like in the case of a Schrödinger operator). Moreover, these weight functions need to be very close (at least up to $O(h)$ ) to the $C^{1,1}$ weight $\phi$ introduced in the previous section. This lead to estimates of the type $\partial^{\alpha} \widetilde{\phi}=O\left(h^{-(|\alpha|-2)_{+}}\right)$for such functions, and also estimates of the type $\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)=O\left(h^{-(|\alpha|-1)_{+}}\right)$ for the related symbols.

Thus, we first have to establish some preliminary results for such objects.
5.1. Preliminaries. In the sequel, we denote by $\lambda_{h}$ a positive function of $h$ such that,

$$
\lambda_{h} \geq 1 \quad ; \quad h \lambda_{h}=\mathcal{O}(1) \text { as } h \rightarrow 0_{+}
$$

For $\Omega \subset \mathbb{R}^{n}$, we also denote by $\mathbf{1}_{\Omega}$ the characteristic function of $\Omega$.

Proposition 5.1. Let $\nu_{0}>0, m \geq 0, a=a(y, \eta) \in S_{\nu_{0}}\left(\langle\eta\rangle^{2 m}\right)$, and $\Omega \subset \mathbb{R}^{n}$. For $h>0$ small enough, let also $\Phi_{h} \in C^{\infty}\left(\mathbb{R}^{n}\right)$ real valued, such that,

$$
\begin{equation*}
\sup \left|\nabla \Phi_{h}\right|<\nu_{0} \tag{5.1}
\end{equation*}
$$

and, for any multi-index $\alpha \in \mathbb{N}^{n}$ with $|\alpha| \geq 2$,

$$
\begin{equation*}
\partial^{\alpha} \Phi_{h}(y)=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\Omega}(y)\right) h^{2-|\alpha|}\right) \tag{5.2}
\end{equation*}
$$

uniformly for $y \in \mathbb{R}^{n}$ and $h>0$ small enough. Then, for any $\widetilde{\Omega} \subset \mathbb{R}^{n}$ with $\operatorname{dist}\left(\Omega, \mathbb{R}^{n} \backslash \widetilde{\Omega}\right)>0$, the operator $e^{\Phi_{h} / h} A e^{-\Phi_{h} / h}:=e^{\Phi_{h} / h} \mathrm{Op}_{h}^{W}(a) e^{-\Phi_{h} / h}$ can be split into ,

$$
\begin{equation*}
e^{\Phi_{h} / h} A e^{-\Phi_{h} / h}=\mathrm{Op}_{h}^{W}\left(a_{0}\left(y, \eta+i \nabla \Phi_{h}(y)\right)+h B+R\right. \tag{5.3}
\end{equation*}
$$

where $a_{0}$ is the principal symbol of $A=\mathrm{Op}_{h}^{W}(a)$, and the two operators $B$ and $R$ satisfy,

$$
\begin{align*}
& |\langle B u, u\rangle| \leq C_{1}\left(\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|^{2}+\lambda_{h}\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|_{L^{2}(\widetilde{\Omega})}^{2}\right)  \tag{5.4}\\
& |\langle R u, u\rangle| \leq C_{2} e^{\left(2 \sup |\Phi|-\varepsilon_{0}\right) / h}\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|^{2} \tag{5.5}
\end{align*}
$$

where the two positive constants $C_{2}$ and $\varepsilon_{0}$ do not depend on $\Phi$, and where the estimate holds for all $h>0$ small enough and $u \in H^{m}\left(\mathbb{R}^{n}\right)$.

Proof. For $u \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$, we write,
$e^{\Phi / h} A e^{-\Phi / h} u(y)=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(y-y^{\prime}\right) \eta / h+\left(\Phi(y)-\Phi\left(y^{\prime}\right)\right) / h} a\left(\frac{y+y^{\prime}}{2}, \eta\right) u\left(y^{\prime}\right) d y^{\prime} d \eta$,
and we first make the change of contour of integration given by,

$$
\mathbb{R}^{n} \ni \eta \mapsto \eta+i \delta \frac{y-y^{\prime}}{\left\langle y-y^{\prime}\right\rangle} \in \mathbb{C}^{n}
$$

where $\delta>0$ is fixed small enough (independently of $\Phi$ ).
Then, we fix a cut-off $\chi_{1} \in C_{0}^{\infty}\left(\mathbb{R}^{n} ;[0,1]\right)$ such that $\chi_{0}=1$ near 0 . Using this cut-off to split the integral, and setting $f:=\delta \frac{y-y^{\prime}}{\left\langle y-y^{\prime}\right\rangle}$ and $g:=\delta \frac{\left|y-y^{\prime}\right|^{2}}{\left\langle y-y^{\prime}\right\rangle}$, we obtain,

$$
e^{\Phi / h} A e^{-\Phi / h} u(y)=I_{1}+I_{2}
$$

with,

$$
\begin{array}{r}
I_{1}:=\frac{1}{(2 \pi h)^{n}} \int \chi_{0}\left(y-y^{\prime}\right) e^{i\left(y-y^{\prime}\right) \eta / h+\left(\Phi(y)-\Phi\left({ }^{\prime} y\right)\right) / h-g / h} \\
\times a\left(\frac{y+y^{\prime}}{2}, \eta+i f\right) u\left(y^{\prime}\right) d y^{\prime} d \eta
\end{array}
$$

$$
\begin{aligned}
& I_{2}:=\frac{1}{(2 \pi h)^{n}} \int\left(1-\chi_{0}\left(y-y^{\prime}\right)\right) e^{i\left(y-y^{\prime}\right) \eta / h+\left(\Phi(y)-\Phi\left(y^{\prime}\right)\right) / h-g / h} \\
& \times a\left(\frac{y+y^{\prime}}{2}, \eta+i f\right) u\left(y^{\prime}\right) d y^{\prime} d \eta
\end{aligned}
$$

Now, on the support of $1-\chi_{0}\left(y-y^{\prime}\right)$, we have $g \geq 2 \varepsilon_{0}$ for some $\varepsilon_{0}>0$ that does not depend on $\Phi$, and thus, by the Calderón-Vaillancourt theorem (see, e.g., Ma6), we immediately obtain,

$$
\begin{equation*}
\left|\left\langle I_{2}, u\right\rangle\right| \leq C e^{\left(2 \sup |\Phi|-\varepsilon_{0}\right) / h}\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|^{2} \tag{5.6}
\end{equation*}
$$

where the constant $C>0$ does not depend on $\Phi$.
On the other hand, if $\delta$ has been chosen sufficiently small, the property (5.1) shows that, in $I_{1}$, we can make the change of contour of integration given by,

$$
\mathbb{R}^{n} \ni \eta \mapsto \eta+i \Psi\left(y, y^{\prime}\right)
$$

where the smooth vectorial function $\Psi\left(y, y^{\prime}\right)$ is defined by the identity: $\Phi(y)-\Phi\left(y^{\prime}\right)=\left(y-y^{\prime}\right) \Psi\left(y, y^{\prime}\right)$. Then, denoting by $\mathrm{Op}_{h}$ the semiclassical quantization of symbols depending on $3 n$ variables (see e.g. Ma6] Section 2.5), we obtain,

$$
I_{1}=\mathrm{Op}_{h}\left(e^{-g / h} \chi_{0}\left(y-y^{\prime}\right) a\left(\frac{y+y^{\prime}}{2}, \eta+i \Psi\left(y, y^{\prime}\right)+i f\right)\right) u(y)
$$

Next, passing to symbols depending on $2 n$ variables (see e.g. Ma6] Theorem 2.7.1), this gives,

$$
I_{1}=\mathrm{Op}_{h}^{W}(b) u(y)
$$

with $b$ given by the oscillatory integral, $b(y, \eta)=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h-g_{1} / h} \chi_{0}(\theta) a\left(y, \eta^{\prime}+i \Psi\left(y+\frac{\theta}{2}, y-\frac{\theta}{2}\right)+i f_{1}\right) d \eta^{\prime} d \theta$, where we have used the notations: $g_{1}:=\delta|\theta|^{2} /\langle\theta\rangle ; f_{1}:=\delta \theta /\langle\theta\rangle$. Then, we need,

Lemma 5.2. For any $\delta>0$, the symbol $b$ can be split into,

$$
b(y, \eta)=a(y, \eta+i \nabla \Phi(y))+\widetilde{b}(y, \eta)
$$

where, for any $\alpha, \beta \in \mathbb{N}^{n}, \widetilde{b}$ satisfies,

$$
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} \widetilde{b}(y, \eta)=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\Omega_{\delta}}(y)\right) h^{1-|\alpha|}\langle\eta\rangle^{2 m}\right)
$$

uniformly for $h>0$ small enough and $(y, \eta) \in \mathbb{R}^{n} \times \mathbb{C}^{n},|\operatorname{Im} \eta|$ small enough. Here, we have set,

$$
\Omega_{\delta}:=\left\{x \in \mathbb{R}^{n} ; \operatorname{dist}(x, \Omega) \leq \delta\right\}
$$

Proof. see Appendix A.
This means that $h^{-1}\langle\xi\rangle^{-2 m} \widetilde{b}$ belongs to the space of symbols $\widetilde{S}_{0}\left(\lambda_{h}, \Omega_{\delta}\right)$ introduced in Appendix B. Therefore, applying Proposition B. 1 (iv), we deduce that, for any $\delta>0$, one has,

$$
\begin{equation*}
\left|\left\langle\mathrm{Op}_{h}^{W}(\widetilde{b}) u, u\right\rangle\right|=\mathcal{O}\left(h \lambda_{h}\right)\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|_{L^{2}\left(\Omega_{\delta}\right)}^{2}+\mathcal{O}(h)\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|^{2} \tag{5.7}
\end{equation*}
$$

uniformly as $h \rightarrow 0$ (see e.g. Ma6 Exercise 2.10.15). Moreover, one also has,

$$
a\left(y, \eta+i \nabla \Phi_{h}(y)\right)=a_{0}\left(y, \eta+i \nabla \Phi_{h}(y)\right)+h a_{1}\left(y, \eta+i \nabla \Phi_{h}(y)\right)
$$

and,

$$
\begin{aligned}
\partial_{y}^{\alpha} \partial_{\eta}^{\beta}\left[a_{1}\left(y, \eta+i \nabla \Phi_{h}(y)\right)\right] & =\mathcal{O}\left(\left(1+\sup _{1 \leq|\gamma| \leq|\alpha|+1}\left|\partial^{\gamma} \Phi_{h}\right|\right)\langle\eta\rangle^{2 m}\right) \\
& =\mathcal{O}\left(\left(1+\lambda_{h} h^{1-|\alpha|}\right)\langle\eta\rangle^{2 m}\right) \\
& =\mathcal{O}\left(h^{-|\alpha|}\langle\eta\rangle^{2 m}\right) .
\end{aligned}
$$

Thus, $\left|\left\langle\mathrm{Op}{ }_{h}^{W}\left(a_{1}\right) u, u\right\rangle\right|=\mathcal{O}(1)\left\|\left\langle h D_{y}\right\rangle^{m} u\right\|^{2}$, and the result follows.

Thanks to the previous lemma, only the principal symbols of our operators will be involved in the estimates. However, the study of $\widetilde{P}_{2}$ remains delicate since its principal symbol is not a polynome in $\eta$. However, due to the parity of $\widetilde{p}_{2}(y, \eta)$ with respect to $\eta$, we have for any real-valued Lipschitz function $\Phi(y)$ with $|\nabla \Phi|$ small enough,
$\operatorname{Re} \widetilde{p}_{2}(y, \eta+i \nabla \Phi(y))=\widetilde{p}_{2}(y, i \nabla \Phi(y))+\frac{1}{2}\left\langle\operatorname{Hess}_{\eta} \widetilde{p}_{2}(y, i \nabla \Phi(y)) \eta, \eta\right\rangle+\mathcal{O}\left(|\eta|^{3}\right)$ and therefore, since $\operatorname{Hess}_{\eta} \widetilde{p}_{2}(0,0)$ is positive definite,

$$
\begin{equation*}
\operatorname{Re} \widetilde{p}_{2}(y, \eta+i \nabla \Phi(y)) \geq \widetilde{p}_{2}(y, i \nabla \Phi(y))+\delta_{0}|\eta|^{2} \tag{5.9}
\end{equation*}
$$

for some constant $\delta_{0}>0$ and $(y, \eta)$ small enough. Actually, we have a more complete result on $\widetilde{P}_{2}:=\mathcal{U} P_{2} \mathcal{U}^{-1}$, as stated in the following proposition:

Proposition 5.3. Let $K \subset \mathbb{R}$ be a fix compact set, and let $\Phi=\Phi_{h, \mu} \in$ $C^{\infty}\left(\mathbb{R}^{n} ; \mathbb{R}\right)$ be such that,

$$
\begin{equation*}
\operatorname{Supp} \nabla \Phi \subset\left\{y_{n} \in K\right\} \tag{5.10}
\end{equation*}
$$

and,

$$
\begin{equation*}
|\Phi|+|\nabla \Phi|=\mathcal{O}(\mu) \tag{5.11}
\end{equation*}
$$

uniformly with respect to $h, \mu>0$ both small enough. Moreover, assume that for some $\Omega \subset \mathbb{R}^{n}$, for any multi-index $\alpha \in \mathbb{N}^{n}$ with $|\alpha| \geq 2$, and for any fix $\mu>0$ small enough, one has,

$$
\begin{equation*}
\partial^{\alpha} \Phi=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\Omega}\right) h^{2-|\alpha|}\right) \tag{5.12}
\end{equation*}
$$

uniformly with respect to $h>0$ small enough.
Then, for $\mu$ sufficiently small, there exist three positive constants $\tau(\mu), \delta=$ $\delta(\mu)$ and $C_{\mu}$, such that, for all $\tau \in(0, \tau(\mu)]$ and for all $h>0$ small enough, one has,

$$
\operatorname{Re} e^{\Phi / h} \widetilde{P}_{2} e^{-\Phi / h} \geq \frac{1}{C_{\mu}}\left(-\frac{h^{2}}{2} \Delta+\widetilde{p}_{2}(y, i \nabla \Phi(y))\right)-C_{\mu} h\left(1+\lambda_{h} \mathbf{1}_{\Omega_{\delta}}\right)
$$

Proof. Let $\chi \in C_{0}^{\infty}(\mathbb{R})$ be such that $\chi=1$ in a n neighborhood of $K \cup\{0\}$. Since $P_{2}=-h^{2} \Delta+\tau V_{2}$ with $V_{2}$ satisfying (3.1), by Lemma 4.2 there exists $\nu_{1}>0$, such that,

$$
\chi\left(y_{n}\right) \widetilde{P}_{2}=\chi\left(y_{n}\right)\left(-h^{2} \Delta_{y}\right)+\tau \mathrm{Op}_{h}^{W}\left(\check{V}_{2}\right)+\mathcal{O}\left(e^{-\nu_{1} / h}\right)
$$

with $\check{V}_{2} \in S_{\nu_{1}}(1)$ satisfying,

$$
\check{V}_{2}(y, \eta)=\chi\left(y_{n}\right) V_{2}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}-2 \eta^{2}\right)+\mathcal{O}(h) \text { in } S_{\nu_{1}}(1) .
$$

Moreover, by assumption $|\Phi|+|\nabla \Phi(y)| \leq C \mu$ for some constant $C>0$, and thus, for $\mu$ sufficiently small, we can apply Proposition 5.1 with $m=0$, and we obtain,

$$
e^{\Phi / h} \operatorname{Op}_{h}^{W}\left(\check{V}_{2}\right) e^{-\Phi / h}=\operatorname{Op}_{h}^{W}\left(\chi\left(y_{n}\right) \widetilde{V}_{2}(y, \eta+i \nabla \Phi)\right)+h B+R,
$$

where we have set,

$$
\begin{equation*}
\widetilde{V}_{2}(y, \eta):=V_{2}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}, y_{n}-2 \eta^{2}\right) \tag{5.13}
\end{equation*}
$$

and with,

$$
|\langle B u, u\rangle| \leq C_{1}\left(\|u\|^{2}+\lambda_{h}\|u\|_{L^{2}\left(\Omega_{\delta}\right)}^{2}\right) ;|\langle R u, u\rangle| \leq C_{2} e^{\left(2 C \mu-\varepsilon_{0}\right) / h}\|u\|^{2} .
$$

Here, $\varepsilon_{0}>0$ does not depend on $\mu$, and thus, for $\mu$ small enough, this gives, $\operatorname{Re} e^{\Phi / h} \operatorname{Op}_{h}^{W}\left(\check{V}_{2}\right) e^{-\Phi / h} \geq \operatorname{Re} \operatorname{Op}_{h}^{W}\left(\chi\left(y_{n}\right) \widetilde{V}_{2}(y, \eta+i \nabla \Phi)\right)-C h\left(1+\lambda_{h} \mathbf{1}_{\Omega_{\delta}}\right)$.

Since we also have,

$$
\operatorname{Re} e^{\Phi / h} \chi\left(y_{n}\right)\left(-h^{2} \Delta\right) e^{-\Phi / h}=\operatorname{Re} \chi\left(y_{n}\right)\left(-h^{2} \Delta-|\nabla \Phi|^{2}\right)+\mathcal{O}(h),
$$

we obtain,

$$
\begin{align*}
& \operatorname{Re} e^{\Phi / h} \chi \widetilde{P}_{2} e^{-\Phi / h}  \tag{5.14}\\
& \geq \operatorname{Re} \operatorname{Op}_{h}^{W}\left(\chi\left(y_{n}\right)\left(\eta^{2}-|\nabla \Phi|^{2}+\tau \widetilde{V}_{2}(y, \eta+i \nabla \Phi)\right)\right) \\
& -C h\left(1+\lambda_{h} \mathbf{1}_{\Omega_{\delta}}\right)
\end{align*}
$$

with some new constant $C>0$.
On the other hand, using (5.13) and making a Taylor expansion, we see,

$$
\operatorname{Re} \widetilde{V}_{2}(y, \eta+i \nabla \Phi)=V_{2}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}+4\left(\partial_{y_{n}} \Phi\right) \nabla_{y^{\prime}} \Phi, y_{n}-2 \eta^{2}+2(\nabla \Phi)^{2}\right)
$$

$$
-\langle Y, A Y\rangle
$$

where we have set,

$$
Y=\left(Y^{\prime}, Y_{n}\right):=\left(4 \eta_{n} \nabla_{y^{\prime}} \Phi+\eta^{\prime}\left(\partial_{y_{n}} \Phi\right), 4 \eta \cdot \nabla \Phi\right) \in \mathbb{R}^{n}
$$

and,
$A:=\int_{0}^{1} \operatorname{Re} V_{2}^{\prime \prime}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}+4\left(\partial_{y_{n}} \Phi\right) \nabla_{y^{\prime}} \Phi-i t Y^{\prime}, y_{n}-2 \eta^{2}+2(\nabla \Phi)^{2}-i t Y_{n}\right) d t$.
In particular, by (5.11)-(5.12), we see that, for $\mu$ sufficienly small,

$$
\chi\left(y_{n}\right)\langle Y, A Y\rangle \in \mathcal{S}_{0}\left(\lambda_{h}, \Omega\right)
$$

where $\mathcal{S}_{0}\left(\lambda_{h}, \Omega\right)$ is the exotic class of symbols introduced in Appendix B Actually, both $Y$ and $\chi A$ are in similar (vectorial) classes of symbols (with weight $\langle\eta\rangle$ for $Y$, and, for any $u \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$, the results of Appendix B permit us to write,

$$
\begin{aligned}
\left\langle\mathrm{Op}_{h}^{W}\left(\chi\left(y_{n}\right)\langle Y, A Y\rangle\right) u, u\right\rangle= & \left\langle\mathrm{Op}_{h}^{W}(\chi A) \mathrm{Op}_{h}^{W}(Y) u, \mathrm{Op}_{h}^{W}(Y) u\right\rangle \\
& +\mathcal{O}_{\mu}\left(h \lambda_{h}\right)\left\|\left\langle h D_{x}\right\rangle u\right\|_{\Omega_{\delta}}^{2}+\mathcal{O}_{\mu}(h)\left\|\left\langle h D_{x}\right\rangle u\right\|^{2} \\
= & \mathcal{O}_{\mu}\left(\left\|\mathrm{Op}_{h}^{W}(Y) u\right\|^{2}\right. \\
& +\mathcal{O}_{\mu}\left(h \lambda_{h}\right)\left\|\left\langle h D_{x}\right\rangle u\right\|_{\Omega_{\delta}}^{2}+\mathcal{O}_{\mu}(h)\left\|\left\langle h D_{x}\right\rangle u\right\|^{2} \\
= & \mathcal{O}_{\mu}\left(\|h \nabla u\|^{2}+h \lambda_{h}\|u\|_{\Omega_{\delta}}^{2}+h\|u\|^{2}\right) .
\end{aligned}
$$

Here, $\mathcal{O}_{\mu}$ means that the estimate is not necessarily uniform with respect to $\mu$ (and this is precisely the reason why we need the small parameter $\tau$ in our problem).
In the same way, we also have,

$$
\begin{aligned}
\mathrm{Op}_{h}^{W} & \left(\chi\left(y_{n}\right) V_{2}\left(y^{\prime}-4 \eta_{n} \eta^{\prime}+4\left(\partial_{x_{n}} \Phi\right) \nabla_{x^{\prime}} \Phi, y_{n}-2 \eta^{2}+2(\nabla \Phi)^{2}\right)\right) \\
& =\operatorname{Op}_{h}^{W}\left(\chi\left(y_{n}\right) V_{2}\left(y^{\prime}+4\left(\partial_{x_{n}} \Phi\right) \nabla_{x^{\prime}} \Phi, y_{n}+2(\nabla \Phi)^{2}\right)\right)+B_{1}
\end{aligned}
$$

with,

$$
\left\langle B_{1} u, u\right\rangle=\mathcal{O}_{\mu}\left(\|h \nabla u\|^{2}+h \lambda_{h}\|u\|_{\Omega_{\delta}}^{2}+h\|u\|^{2}\right) .
$$

Therefore, coming back to (5.14), we deduce,

$$
e^{\Phi / h} \chi\left(y_{n}\right) \widetilde{P}_{2} e^{-\Phi / h} \geq \operatorname{Op}_{h}^{W}\left(\left(\chi\left(y_{n}\right)-C_{\mu} \tau\right) \eta^{2}+\chi\left(y_{n}\right) \widetilde{p}_{2}(y, i \nabla \Phi(y))\right)
$$

$$
\begin{equation*}
-C_{\mu} h \lambda_{h} \mathbf{1}_{\Omega_{\delta}}-C_{\mu} h, \tag{5.15}
\end{equation*}
$$

where the constant $C_{\mu}>0$ may depend on $\mu$. Since $e^{\Phi / h}(1-\chi) \widetilde{P}_{2} \chi e^{-\Phi / h}=$ $\left(e^{-\Phi / h} \chi \widetilde{P}_{2}(1-\chi) e^{\Phi / h}\right)^{*}$, we also have,

$$
\operatorname{Re} e^{\Phi / h}(1-\chi) \widetilde{P}_{2} \chi e^{-\Phi / h} \geq \operatorname{Op}_{h}^{W}\left(\left(1-\chi\left(y_{n}\right) \chi\left(y_{n}\right)\left(\eta^{2}+\widetilde{p}_{2}(y, i \nabla \Phi(y))\right)\right.\right.
$$

$$
\begin{equation*}
-C_{\mu} \tau\left(h D_{y}\right)^{2}-C_{\mu} h \lambda_{h} \mathbf{1}_{\Omega_{\delta}}-C_{\mu} h \tag{5.16}
\end{equation*}
$$

Finally, since $\Phi$ is constant on the support of $(1-\chi)$, we can write,

$$
\begin{align*}
& \operatorname{Re}\left\langle e^{\Phi / h}(1-\chi) \widetilde{P}_{2}(1-\chi) e^{-\Phi / h} u, u\right\rangle \\
& =\left\langle(1-\chi) \widetilde{P}_{2}(1-\chi) u, u\right\rangle  \tag{5.17}\\
& =\left\langle\mathrm{Op}_{h}^{W}\left((1-\chi)^{2} \eta^{2}\right) u, u\right\rangle+\tau\left\langle(1-\chi) \mathcal{U} V_{2} \mathcal{U}^{-1}(1-\chi) u, u\right\rangle .
\end{align*}
$$

Here, one must be aware of the fact that the pseudodifferential operator $(1-\chi) \mathcal{U} V_{2} \mathcal{U}^{-1}(1-\chi)$, though being uniformly bounded on $L^{2}$, does not have its symbol in the class $S_{0}\left(1 ; \mathbb{R}^{n}\right)$ (there are directions where the repeated derivatives with respect to $\eta$ increase more and more as $|\eta| \rightarrow \infty)$. However, we have,

Lemma 5.4. The operator $(1-\chi) \mathcal{U} V_{2} \mathcal{U}^{-1}(1-\chi)$ satisfies,

$$
\left\langle(1-\chi) \mathcal{U} V_{2} \mathcal{U}^{-1}(1-\chi) \geq \frac{(1-\chi)^{2}}{C}-C\left(h D_{y}\right)^{2}-C h^{2}\right.
$$

where $C>0$ is a constant.
Proof. Let $\chi_{0} \in C_{0}^{\infty}\left(\mathbb{R}^{n} ; \mathbb{R}\right)$ be such that Supp $\chi_{0} \subset\left\{\chi\left(y_{n}\right)=1\right\}$ and $\chi_{0}(0) \neq 0$. Then, the assumptions on $V_{2}$ imply the existence of a positive constant $C$ such that,

$$
V_{2}+\chi_{0}^{2} \geq \frac{1}{C}
$$

As a consequence,

$$
\begin{equation*}
(1-\chi) \mathcal{U} V_{2} \mathcal{U}^{-1}(1-\chi) \geq \frac{(1-\chi)^{2}}{C}-(1-\chi) \mathcal{U} \chi_{0}^{2} \mathcal{U}^{-1}(1-\chi) \tag{5.18}
\end{equation*}
$$

and thus, it is enough to estimate $(1-\chi) \mathcal{U} \chi_{0}^{2} \mathcal{U}^{-1}(1-\chi)$. We write it as,

$$
(1-\chi) \mathcal{U} \chi_{0}^{2} \mathcal{U}^{-1}(1-\chi)=\mathcal{U} A \chi_{0}^{2} A \mathcal{U}^{-1}
$$

with,

$$
A:=\mathcal{U}^{-1}(1-\chi) \mathcal{U}=1-\mathcal{U}^{-1} \chi \mathcal{U} .
$$

By a straightforward computation (similar to that of (4.24)), we have,

$$
A \chi_{0}=\chi_{0}-\mathrm{Op}_{h}^{R}(a)
$$

with,

$$
a(y, \eta):=\frac{1}{2 \pi h} \int e^{i\left(x_{n}-y_{n}\right)\left(\xi_{n}-\eta_{n}\right) / h} \chi\left(x_{n}+\theta\left(\xi_{n}, \eta\right)\right) \chi_{0}(y) d x_{n} d \xi_{n}
$$

where we have set: $\theta(\xi, \eta)=2\left|\eta^{\prime}\right|^{2}+2\left(\eta_{n}^{2}+\eta_{n} \xi_{n}+\xi_{n}^{2}\right) / 3$. Therefore, writing $\chi\left(x_{n}+\theta\right)=\chi\left(x_{n}\right)+\theta \int_{0}^{1} \chi^{\prime}\left(x_{n}+t \theta\right) d t$ and using the fact that $\chi \chi_{0}=\chi_{0}$, we find,

$$
\chi_{0}(y)-a(y, \eta)=\mathrm{Op}_{h}^{R}(b)
$$

with,
$b(y, \eta):=\frac{1}{2 \pi h} \int_{0}^{1} \int_{\mathbb{R}^{2}} e^{i\left(x_{n}-y_{n}\right)\left(\xi_{n}-\eta_{n}\right) / h} \theta\left(\xi_{n}, \eta\right) \chi^{\prime}\left(x_{n}+t \theta(\xi, \eta)\right) \chi_{0}(y) d x d \xi d t$.
Then, writing $\theta\left(\xi_{n}, \eta\right)=2 \eta^{2}+2 \eta_{n}\left(\xi_{n}-\eta_{n}\right)+\frac{2}{3}\left(\xi_{n}-\eta_{n}\right)^{2}$, and making integrations by parts with respect to $x_{n}$ and $\xi_{n}$, for any $M, N \geq 1$, we find,

$$
b(y, \eta)=2 \eta^{2} A_{1}+2 h \eta_{n} A_{2}+h^{2} A_{3}
$$

where, for $j=1,2,3$,

$$
A_{j}=\int_{0}^{1} \int_{\mathcal{E}_{t}} \mathcal{O}\left(h^{-1} \frac{\left(1+t\left|\xi_{n}\right|+t|\eta|\right)^{M}}{\left\langle h^{-1}\left(\xi_{n}-\eta_{n}\right)\right\rangle^{N}\left\langle x_{n}-y_{n}\right\rangle^{M}}\right) d x_{n} d \xi_{n} d t
$$

Here, $\mathcal{E}_{t}$ stands for the support of $\chi\left(x_{n}+t \theta\left(\xi_{n}, \eta\right)\right) \chi_{0}(y)$. In particular, on this set we have,

$$
t\left|\xi_{n}\right|+t|\eta|=\mathcal{O}\left(\sqrt{t\left|\xi_{n}\right|^{2}+t|\eta|^{2}}\right)=\mathcal{O}\left(\left\langle x_{n}\right\rangle^{1 / 2}+\left\langle\xi_{n}-\eta_{n}\right\rangle\right) \quad ; \quad|y|=\mathcal{O}(1)
$$

and thus, for $N \geq 2 M$, this gives,

$$
h^{-1} \frac{\left(1+t\left|\xi_{n}\right|+t|\eta|\right)^{M}}{\left\langle h^{-1}\left(\xi_{n}-\eta_{n}\right)\right\rangle^{N}\left\langle x_{n}-y_{n}\right\rangle^{M}}=\mathcal{O}\left(\frac{h^{-1}}{\left\langle h^{-1}\left(\xi_{n}-\eta_{n}\right)\right\rangle^{M}\left\langle x_{n}\right\rangle^{M / 2}}\right) .
$$

Therefore, $A_{j}=\mathcal{O}(1)$ uniformly, and the same is true for all its derivatives. The same procedure also shows that $a \in S_{0}(1)$, and we can write,

$$
\begin{aligned}
\left\langle A \chi_{0}^{2} A u, u\right\rangle=\left\langle\mathrm{Op}_{h}^{R}(b) \chi_{0} A u, u\right\rangle & =\mathcal{O}\left(\left\|h D_{y} \chi_{0} A u\right\| \cdot\left\|h D_{y} u\right\|\right. \\
& \left.+h\left\|\chi_{0} A u\right\| \cdot\left\|h D_{y} u\right\|+h^{2}\left\|\chi_{0} A u\right\| \cdot\|u\|\right)
\end{aligned}
$$

that is, since $\chi_{0} A=\mathrm{Op}_{h}^{L}(a)$ and $\left[D_{y}, \chi_{0} A\right]$ are uniformly bounded,

$$
\left\langle A \chi_{0}^{2} A u, u\right\rangle=\mathcal{O}\left(\left\|h D_{y} u\right\|^{2}+h^{2}\|u\|^{2}\right)
$$

Then, by (5.18) (and the fact that $D_{y}$ and $\mathcal{U}$ commute), the result follows.

Going back to (5.17), we deduce,

$$
\begin{aligned}
& \operatorname{Re} e^{\Phi / h}(1-\chi) \widetilde{P}_{2}(1-\chi) e^{-\Phi / h} \\
& \geq \mathrm{Op}_{h}^{W}\left(\left((1-\chi)^{2}-C \tau\right) \eta^{2}+\frac{\tau}{C}(1-\chi)^{2}-C h^{2}\right) .
\end{aligned}
$$

Summing up with (5.15)-(5.16), we have proved,

$$
\begin{aligned}
\operatorname{Re} e^{\Phi / h} \widetilde{P}_{2} e^{-\Phi / h} \geq \operatorname{Op}_{h}^{W} & \left((1-C \tau) \eta^{2}+\left(1-(1-\chi)^{2}\right) \widetilde{p}_{2}(y, i \nabla \Phi(y))\right) \\
& +\frac{\tau}{C} \operatorname{Op}_{h}^{W}\left((1-\chi)^{2}\right)-C h \lambda_{h} \mathbf{1}_{\Omega_{\delta}}-C h .
\end{aligned}
$$

On the other hand, on the support of $(1-\chi)$, we have $\widetilde{p}_{2}(y, i \nabla \Phi(y))=$ $\widetilde{p}_{2}(y, 0)=\tau V_{2}(y) \geq 0$, and thus $\left(1-(1-\chi)^{2}\right) \widetilde{p}_{2}(y, i \nabla \Phi(y))+\frac{\tau}{C} \geq \frac{\tau}{C} \geq$ $C_{1}^{-1} \tau V_{2}(y)=C_{1}^{-1} \widetilde{p}_{2}(y, i \nabla \Phi(y))$, for some positive constant $C_{1}$. As a consequence, for $\tau>0$ sufficiently small, we deduce from (5.19),

$$
\operatorname{Re} e^{\Phi / h} \widetilde{P}_{2} e^{-\Phi / h} \geq \frac{1}{C} \mathrm{Op}_{h}^{W}\left(\eta^{2}+\widetilde{p}_{2}(y, i \nabla \Phi(y))\right)-C h \lambda_{h} \mathbf{1}_{\Omega_{\delta}}-C h,
$$

where the (new) constant $C>0$ may depend on $\mu$. Thus, Proposition 5.3 is proved.
5.2. Agmon Estimates. Now, we can specify what we call Agmon estimates in our case. In the rest of the paper, we set,

$$
k:=h \ln h^{-1},
$$

and, for $t \in \mathbb{R}$, we consider the translation $e^{i t D_{x_{n}}}$ given by,

$$
e^{i t D_{x_{n}}} f(y)=f\left(y^{\prime}, y_{n}+t\right) .
$$

We also set,

$$
\widetilde{P}^{t}:=e^{i t D_{y_{n}}} \widetilde{P} e^{-i t D_{y_{n}}} .
$$

Since $e^{i t D_{y_{n}}}$ and $\mathcal{U}$ commute, we have,

$$
\widetilde{P}^{t}=\mathcal{U} P^{t} \mathcal{U}^{-1}
$$

where,

$$
P^{t}:=e^{i t D_{x_{n}}} P e^{-i t D_{x_{n}}}=\left(\begin{array}{cc}
P_{1}^{t} & 0 \\
0 & P_{2}^{t}
\end{array}\right)+h R\left(x^{\prime}, x_{n}+t ; h D_{x}\right),
$$

with,

$$
P_{1}^{t}:=-h^{2} \Delta+x_{n}+t-\mu \quad ; \quad P_{2}^{t}:=-h^{2} \Delta+\tau V_{2}\left(x^{\prime}, x_{n}+t\right) .
$$

In particular, by Assumption (3.1), we see that $\widetilde{P}^{t}$ extends analytically to complex values of $t,|t|$ small enough, and we set,

$$
\begin{equation*}
Q:=\widetilde{P}^{-i k} . \tag{5.20}
\end{equation*}
$$

Then, the essential spectrum of $Q$ is $\mathbb{R}-i k$, and the resonances of $P$ (or $\widetilde{P}$ ) close enough to 0 are discrete eigenvalues of $Q$. In particular, the resonant state $\widetilde{\mathbf{u}}=\mathcal{U} \mathbf{u}$ of $\widetilde{P}$ associated to the resonance $\rho$ (where $\mathbf{u}$ is the corresponding resonant state of $P$ ), is such that $\mathbf{v}:=e^{k D_{y_{n}}} \widetilde{\mathbf{u}}$ is well defined, belongs to $L^{2}\left(\mathbb{R}^{n}\right) \oplus L^{2}\left(\mathbb{R}^{n}\right)$, and satisfies,

$$
(Q-\rho) \mathbf{v}=0 .
$$

Moreover, $\mathbf{v}$ can be normalized by requiring: $\|\mathbf{v}\|=1$.
In the sequel, we set,

$$
Q_{j}=\widetilde{P}_{j}^{-i k}:=\mathcal{U} P_{j}^{-i k} \mathcal{U}^{-1} .
$$

Actually, since $\mathcal{F} x_{n} \mathcal{F}^{-1}=i h \partial_{\xi_{n}}$, the operator $Q_{1}$ can be explicitly computed, and one finds,

$$
Q_{1}=h^{2} \Delta+y_{n}-i k-\mu .
$$

Moreover, setting $\widetilde{R}_{k}:=\mathcal{U} R\left(x^{\prime}, x_{n}-i k ; h D_{x}\right) \mathcal{U}^{-1}$, we have,

$$
Q=\left(\begin{array}{cc}
Q_{1} & 0  \tag{5.21}\\
0 & Q_{2}
\end{array}\right)+h \widetilde{R}_{k},
$$

Now, we fix an $h$-dependent cut-off function $\chi_{h} \in C^{\infty}(\mathbb{R} ;[0,1])$, such that,

$$
\chi_{h}=1 \text { on }\left(-\infty,-2 k^{2 / 3}\right] ; \chi_{h}=0 \text { on }\left[-k^{2 / 3},+\infty\right) ; \partial^{\alpha} \chi_{h}=\mathcal{O}\left(k^{-2|\alpha| / 3}\right),
$$

and we set,

$$
\begin{align*}
& \widetilde{V}_{1}\left(y_{n}\right):=\left(\mu-y_{n}\right) \chi_{n}\left(y_{n}-\mu\right)+k^{2 / 3}\left\langle y_{n}\right\rangle\left(1-\chi_{h}\left(y_{n}-\mu\right)\right) ; \\
& \widetilde{Q}_{1}:=h^{2} \Delta-i k-\widetilde{V}_{1}\left(y_{n}\right) ; \\
& \widetilde{Q}=\left(\begin{array}{cc}
\widetilde{Q}_{1} & 0 \\
0 & Q_{2}
\end{array}\right)+h \widetilde{R}_{k} . \tag{5.22}
\end{align*}
$$

We also denote by $\mathcal{D}\left(\widetilde{Q}_{1}\right)$ the (natural) domain of $\widetilde{Q}_{1}$, and we prove,
Proposition 5.5. Let $\Phi$ be as in Proposition 5.3] Then, for any $\delta>0$, there exists a positive constant $C_{2}=C_{2}(\mu)$ such that, for any $v=\left(v_{1}, v_{2}\right) \in$
$\mathcal{D}\left(\widetilde{Q}_{1}\right) \times H^{2}\left(\mathbb{R}^{n}\right)$, and setting $\check{v}:=\left(-v_{1}, v_{2}\right)$, one has,

$$
\begin{aligned}
\operatorname{Re}\left\langle e^{\Phi / h} \widetilde{Q} v, e^{\Phi / h} \check{v}\right\rangle \geq & \frac{1}{C_{2}}\left\|h \nabla\left(e^{\Phi / h} v_{1}\right)\right\|^{2}+\left\|h \nabla\left(e^{\Phi / h} v_{2}\right)\right\|^{2} \\
& +\frac{1}{C_{2}}\left\langle\widetilde{p}_{2}(y, i \nabla \Phi(y)) e^{\Phi / h} v_{2}, e^{\Phi / h} v_{2}\right\rangle \\
& \left.+\left\langle\left(\widetilde{V}_{1}\left(y_{n}\right)-(\nabla \Phi)^{2}\right)\right) e^{\Phi / h} v_{1}, e^{\Phi / h} v_{1}\right\rangle \\
& -C_{2} h\left\|e^{\Phi / h} v\right\|^{2}-C_{2} h \lambda_{h}\left\|e^{\Phi / h} v_{2}\right\|_{\Omega_{\delta}}^{2},
\end{aligned}
$$

uniformly for $h>0$ small enough.
Proof. We first prove,
Lemma 5.6. Let $c=c(x)$ satisfying (3.1). Then, with $\Phi$ as in Proposition 5.3, the operator $e^{\Phi / h} \mathcal{U} \mathcal{U}^{-1} e^{-\Phi / h}$ is uniformly bounded on $L^{2}$ as $h \rightarrow 0_{+}$.

Proof. Using Lemma 4.2, and Propositions 5.1 and B.1, we see that is is enough to study the operator $(1-\chi) e^{\Phi / h} \mathcal{U C}^{-1} e^{-\Phi / h}(1-\chi)$, where $\chi \in$ $C_{0}^{\infty}(\mathbb{R})$ is such that $\chi\left(y_{n}\right)=1$ on $\operatorname{Supp} \nabla \Phi$. But then, this operator coincides with $(1-\chi) \mathcal{U} \mathcal{U}^{-1}(1-\chi)$, and the result is obvious.

Now, if $c$ satisfies (3.1), so does $c\left(x^{\prime}, x_{n}-i k\right)$, and thus, we deduce from the previous lemma that we have,

$$
\begin{equation*}
\left\|e^{\Phi / h} h \widetilde{R}_{k} v\right\| \leq C h\left\|\left\langle h D_{y}\right\rangle e^{\Phi / h} v\right\| . \tag{5.23}
\end{equation*}
$$

As a consequence, by (5.22), we deduce,

$$
\begin{align*}
&\left\langle e^{\Phi / h} \widetilde{Q} v, e^{\Phi / h} \check{v}\right\rangle=-\left\langle e^{\Phi / h} \widetilde{Q}_{1} v_{1}, e^{\Phi / h} v_{1}\right\rangle+\left\langle e^{\Phi / h} Q_{2} v_{2}, e^{\Phi / h} v_{2}\right\rangle \\
&+\mathcal{O}\left(h\left\|\left\langle h D_{y}\right\rangle e^{\Phi / h} v\right\|^{2}\right) . \tag{5.24}
\end{align*}
$$

We also compute,

$$
-e^{\Phi / h} \widetilde{Q}_{1} e^{-\Phi / h}=-h^{2} \Delta+\widetilde{V}_{1}\left(y_{n}\right)-|\nabla \Phi|^{2}+i k-i h\left(D_{y} \cdot \nabla \Phi+\nabla \Phi \cdot D_{y}\right),
$$

and thus,

$$
-\operatorname{Re}\left\langle e^{\Phi / h} \widetilde{Q}_{1} v_{1}, e^{\Phi / h} v_{1}\right\rangle=\|h \nabla u\|^{2}+\left\langle\left(\widetilde{V}_{1}\left(y_{n}\right)-|\nabla \Phi|^{2}\right) e^{\Phi / h} v, e^{\Phi / h} v\right\rangle .
$$

Then, the result follows from (5.24) by applying Proposition 5.3 and by taking $h$ small enough.

Remark 5.7. It results from the proof that the dependence with respect to $\Phi$ of the constant $C_{2}$ is performed only through the estimates (5.11)-(5.12).

As a consequence, if $\Phi$ depends on some extra-parameter and satisfies (5.11)(5.12) uniformly with respect to this parameter, then the constant $C_{2}$ can be taken independent of it.

## 6. Comparison Between Formal and True Solution

6.1. On $\{\phi<S(\mu)\}$. Now, we start by studying the region $\{\phi<S(\mu)\}$. The idea is to use the previous Agmon estimates in order to have informations on the decay of the first eigenfunction $\widetilde{\mathbf{v}}$ of $\widetilde{Q}$, and then to transfer these informations on $\mathbf{v}$ by comparing $\widetilde{\mathbf{v}}$ and $\mathbf{v}$.

Note that, by construction, we have $\widetilde{V}_{1} \geq \delta k^{2 / 3}\left\langle y_{n}\right\rangle$ everywhere (with some $\delta>0$ constant). As a consequence, we see that the operator $\widetilde{Q}_{1}-z$ is invertible for any complex number $z$ with $z=\mathcal{O}(h)$, and the norm of its inverse is $\mathcal{O}\left(k^{-2 / 3}\right)$. Therefore, the $2 \times 2$ system of equations: $Q u=z u$ (with $u=\left(u_{1}, u_{2}\right)$ ), can be reduced to a scalar equation on $u_{2}$, of the form,

$$
\left(Q_{2}+h^{2} k^{-2 / 3} B(z)\right) u_{2}=z u_{2},
$$

with $\|B(z)\|=\mathcal{O}(1)$ uniformly. Since $h^{2} k^{-2 / 3}=h^{4 / 3}|\ln h|^{-2 / 3} \ll h$, one can perform the standard regular-perturbation procedure, and conclude (also using Proposition C. 1 in Appendix C) that there exists a unique discrete eigenvalue $\widetilde{\rho}$ of $\widetilde{Q}$ near $\rho$, that coincides with $\rho$ up to $\mathcal{O}\left(h^{4 / 3}|\ln h|^{-2 / 3}\right)$. Finally, using, e.g., the WKB constructions of Section 4 we easily conclude that $\widetilde{\rho}$ coincides with $\rho$ up to $\mathcal{O}\left(h^{\infty}\right)$.

We have the following result:
Proposition 6.1. For any multi-index $\alpha$, the (conveniently normalized) eigenfunction $\widetilde{\mathbf{v}}$ of $\widetilde{Q}$, associated with the eigenvalue $\widetilde{\rho}$, satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\widetilde{\mathbf{v}}-\mathbf{w})=\mathcal{O}\left(h^{\infty} e^{-\phi / h}\right) \tag{6.1}
\end{equation*}
$$

locally uniformly in $\{\phi<S(\mu)\}$. Moreover, for any $\varepsilon>0$

$$
\begin{equation*}
\partial^{\alpha} \widetilde{\mathbf{v}}=\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right), \tag{6.2}
\end{equation*}
$$

locally uniformly on $\mathbb{R}^{n} \backslash\{\phi<S(\mu)\}$.

Proof. At first, we observe that, by arguments similar to those of HeSjl (that is, using the WKB solution w constructed in Section 4 , and comparing it with $\widetilde{\Pi} \mathbf{w}$, where $\widetilde{\Pi}$ is the spectral projector of $\widetilde{Q}$ onto $\operatorname{Ker}(\widetilde{Q}-\widetilde{\rho})$ ), for
any multi-index $\alpha$, one has,

$$
\begin{equation*}
\partial^{\alpha}(\widetilde{\mathbf{v}}-\mathbf{w})=\mathcal{O}\left(h^{\infty}\right) \tag{6.3}
\end{equation*}
$$

uniformly in some ( $\mu$-dependent) neighborhood of 0 .
Then, we consider the regularization $\widetilde{\phi}$ of $\phi$ obtained by modifying it near $\Gamma$ in the following way:

$$
\widetilde{\phi}=\widetilde{\varphi}_{2}+\chi_{0}(z / h)\left(\widetilde{\varphi}_{1}-\widetilde{\varphi}_{2}\right)
$$

where $z=z(y)$ is defined in (4.15), and $\chi_{0} \in C^{\infty}(\mathbb{R} ;[0,1]), \chi_{0}(t)=0$ for $t \leq-1, \chi_{0}(t)=1$ for $t \geq 1$, and $0 \leq \chi^{\prime} \leq 3 / 2$. In particular, $\widetilde{\phi}$ is well defined on $\{\phi<S(\mu)\}$, and, by (4.15), we see that $\tilde{\phi}$ satisfies (5.11)-(5.12) locally uniformly, with $\lambda_{h}=1$. Moreover, one also has,

$$
\widetilde{\phi}-\phi=\mathcal{O}\left(h^{2}\right) \quad ; \quad \nabla \widetilde{\phi}-\nabla \phi=\mathcal{O}(h) \quad \text { a.e. }
$$

and, for some $\delta=\delta(\mu)>0$ small, $\partial^{\alpha} \widetilde{\phi}=\mathcal{O}(1)$ on $\{\widetilde{\phi}<\delta\}$.
Now, for $C>0$ fixed large enough, we consider the function

$$
\begin{equation*}
\Phi_{0}=\widetilde{\phi}-C h \ln \left(1+\left(\frac{\widetilde{\phi}}{C h}-1\right) \chi(\widetilde{\phi} /(C h))\right) \tag{6.4}
\end{equation*}
$$

where $\chi \in C^{\infty}(\mathbb{R}), \chi(t)=0$ for $t \leq 1, \chi(t)=1$ for $t \geq 2,0 \leq \chi^{\prime} \leq 2$.
On the support of $\chi^{\prime}(\widetilde{\phi} /(C h))$, we have $\widetilde{\Phi}=\widetilde{\varphi}_{2}$ e $\nabla \widetilde{\phi}=\mathcal{O}(\sqrt{C h})$. Therefore, on this set, we see that $\Phi_{0}$ satisfies,

$$
\partial^{\alpha} \Phi_{0}=\mathcal{O}\left(1+h^{1-|\alpha| / 2}\right)
$$

uniformly with respect to $C \geq 1$ and $h$ small enough. On the other hand, away from $\operatorname{Supp} \chi^{\prime}(\widetilde{\phi} /(C h))$, we have either $\Phi_{0}=\widetilde{\phi}$, or $\Phi_{0}=\widetilde{\phi}-C h \ln (\widetilde{\phi} / C h)$ $=\widetilde{\phi}-C h \ln \tilde{\phi}+C h \ln (C h)$. In any case, we conclude that $\Phi_{0}$ satisfies (5.11)(5.12) with $\lambda_{h}=1$, locally uniformly on $\{\phi<S(\mu)\}$, and uniformly with respect to the pair $(C, h)$ such that $C h \leq 1$.

We also have,

$$
\widetilde{p}_{2}\left(y, i \nabla \Phi_{0}(y)\right)=\widetilde{p}_{2}(y, i(1-F(\widetilde{\phi} /(C h))) \nabla \widetilde{\phi}(y))
$$

with $F(t)=\left((t-1) \chi^{\prime}(t)+\chi(t)\right) /(1+(t-1) \chi(t))$, and one can easily check that $|F(t)| \leq 4 / t$. Therefore, $F(\widetilde{\phi} /(C h)) \nabla \widetilde{\phi}(y) \leq 4 h|\nabla \widetilde{\phi}| / \widetilde{\phi}$, and thus, by a Taylor expansion, we find,

$$
\begin{aligned}
\widetilde{p}_{2}\left(y, i \nabla \Phi_{0}(y)\right)=\widetilde{p}_{2}(y, i \nabla \widetilde{\phi}(y))-i F(\widetilde{\phi} /(C h)) \nabla \widetilde{\phi}(y) \nabla_{\eta} \widetilde{p}_{2}(y, i \nabla \widetilde{\phi}(y)) \\
+\mathcal{O}\left((C h|\nabla \widetilde{\phi}| / \widetilde{\phi})^{2}\right) .
\end{aligned}
$$

Since $\widetilde{p}_{2}\left(y, i \nabla \widetilde{\varphi}_{2}(y)\right)=0$ and $\widetilde{p}_{2}\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right)>0$ on $\Gamma_{+}$(see Appendix (D), and $-i \nabla \widetilde{\phi}(y) \nabla_{\eta} \widetilde{p}_{2}(y, i \nabla \widetilde{\phi}(y)) \geq \delta_{0}|\nabla \widetilde{\phi}(y)|^{2}$ in some fixed neighborhood of 0 and for some uniform constant $\delta_{0}>0$, we conclude,
$\widetilde{p}_{2}\left(y, i \nabla \Phi_{0}(y)\right) \geq \delta_{0} F(\widetilde{\phi} /(C h))|\nabla \widetilde{\phi}(y)|^{2}+\mathcal{O}\left((C h|\nabla \widetilde{\phi}| / \widetilde{\phi})^{2}\right)+\mathcal{O}\left(h^{2}\right)+\mathcal{O}_{\Gamma}(h)$
where the last $\mathcal{O}_{\Gamma}(h)$ is uniform with respect to $C$ and supported near $\Gamma$. In particular, on $\left\{\widetilde{\phi} \geq C^{2} h\right\}$, and if $C$ is large enough,

$$
\widetilde{p}_{2}\left(y, i \nabla \Phi_{0}(y)\right) \geq C \frac{h}{2 \widetilde{\phi}}|\nabla \widetilde{\phi}(y)|^{2}+\mathcal{O}\left(h^{2}\right)+\mathcal{O}_{\Gamma}(h)
$$

Since, near $\Gamma,|\nabla \widetilde{\phi}|$ stays uniformly away from 0 , we finally obtain, by taking $C$ large enough,

$$
\begin{equation*}
\widetilde{p}_{2}\left(y, i \nabla \Phi_{0}(y)\right) \geq C \frac{h}{4 \widetilde{\phi}}|\nabla \widetilde{\phi}(y)|^{2} \quad \text { on } \quad\left\{\widetilde{\phi} \geq C^{2} h\right\} \tag{6.5}
\end{equation*}
$$

uniformly for $h$ small enough.
Now, for a given $\varepsilon>0$ arbitrarily small, we set,

$$
\Phi(y)=\chi_{\varepsilon}\left(\Phi_{0}(y)\right)
$$

where $\chi_{\varepsilon} \in C^{\infty}\left(\mathbb{R}_{+}\right)$is such that $\chi_{\varepsilon}(t)=t$ if $0 \leq t \leq S(\mu)-3 \varepsilon, \chi_{\varepsilon}(t)=$ $S(\mu)-2 \varepsilon$ if $t \geq S(\mu)-\varepsilon, 0 \leq \chi_{\varepsilon}^{\prime}(t) \leq 1$ everywhere. Then, the function $\Phi$ is smooth on $\mathbb{R}^{n}$ and it satisfies,

$$
\begin{aligned}
& \Phi=\Phi_{0}(y) \quad \text { on } \quad\{\phi \leq S(\mu)-4 \varepsilon\} \\
& \Phi=S(\mu)-2 \varepsilon \quad \text { on } \quad\left\{\phi \geq S(\mu)-\frac{\varepsilon}{2}\right\} \\
& |\nabla \Phi| \leq\left|\nabla \Phi_{0}\right| \quad \text { everywhere; } \\
& \partial^{\alpha} \Phi=\mathcal{O}\left(h^{-(2-|\alpha|)_{+}+}\right) \text {uniformly on } \mathbb{R}^{n} .
\end{aligned}
$$

Moreover, the last estimate is also uniform with respect to the pair $(C, h)$ as long as $C h \leq 1$.

Then, thanks to (6.5), and the fact that $|\nabla \phi|^{2} \geq \delta \phi$ for some positive constant $\delta$ on $\{\phi \leq S(\mu)-\varepsilon\}$, we see that, (denoting by $\mathbf{1}_{A}$ the characteristic function of a set $A$ ),

$$
\widetilde{p}_{2}(y, i \nabla \Phi(y)) \geq C^{\prime} h+\delta_{3} \mathbf{1}_{\{\phi(y) \geq S(\mu)-\varepsilon / 2\}}
$$

on $\left\{\widetilde{\phi} \geq C^{2} h\right\}$, with $\delta_{3}>0$ and $C^{\prime}=C^{\prime}(C)>o$ arbitrarily large, depending on the choice of $C$ (here, we also use the fact that $\widetilde{p}_{2}\left(y, i s \nabla \widetilde{\varphi}_{1}(y)\right)>0$ for $y \in \Gamma_{+}$and $0 \leq s \leq 1$ : See Appendix (D). On the other hand, on the support
of $\nabla \Phi$, we have $|\nabla \Phi|^{2} \leq\left|\nabla \Phi_{0}\right|^{2} \leq(1-C h / \widetilde{\phi})|\nabla \widetilde{\phi}|^{2}$, and we also obtain on this set,

$$
\widetilde{V}_{1}(y)-(\nabla \Phi)^{2}=\mu-y_{n}-(\nabla \Phi)^{2} \geq C^{\prime} h\left\langle y_{n}\right\rangle
$$

with $C^{\prime}>0$ arbitrarily large, while, away from $\operatorname{Supp} \nabla \Phi$, we have $\widetilde{V}_{1}(y)-$ $(\nabla \Phi)^{2}=\widetilde{V}_{1}(y) \geq \delta k^{2 / 3}\left\langle y_{n}\right\rangle \gg h\left\langle y_{n}\right\rangle$.
Finally, on $\left\{\phi \leq C^{2} h\right\}$, one has $\Phi=\phi$, and thus $e^{\Phi / h}=\mathcal{O}(1)$.
Now, we apply the Agmon estimates of Proposition 5.5 with $v:=\widetilde{\mathbf{v}}-\mathbf{w}$ (and $\lambda_{h}=1$ ). Then, thanks also to (4.32)-(4.34) and Remark (5.7) we obtain,

$$
\begin{aligned}
& \frac{1}{C_{2}}\left\|h \nabla\left(e^{\Phi / h} v\right)\right\|^{2}+\left(C^{\prime}-C_{2}\right) h\left\|e^{\Phi / h} v_{2}\right\|_{\phi \geq C^{2} h}^{2}+\left(C^{\prime}-C_{2}\right) h\left\|\sqrt{\left\langle y_{n}\right\rangle} e^{\Phi / h} v_{1}\right\|^{2} \\
&=\mathcal{O}\left(\|v\|_{\phi \leq C^{2} h}\right)+\mathcal{O}\left(h^{\infty}\right),
\end{aligned}
$$

where the constant $C_{2}$ does not depend on the choice of $C$ in (6.4), and where $C^{\prime}=C^{\prime}(C)$ tends to $\infty$ as $C \rightarrow+\infty$. Therefore, choosing $C$ sufficiently large, and using (6.3), we conclude,

$$
\left\|h \nabla\left(e^{\Phi / h} v\right)\right\|+\left\|\sqrt{\left\langle y_{n}\right\rangle} e^{\Phi / h} v_{1}\right\|+\left\|e^{\Phi / h} v_{2}\right\|=\mathcal{O}\left(h^{\infty}\right)
$$

Finally, using repeatedly the partial differential equations satisfied by $\widetilde{\mathbf{v}}$ and $\mathbf{w}$, and taking advantage of the classical ellipticity of $\widetilde{Q}_{1}$ and $Q_{2}$, together with the positivity of $\widetilde{V}_{1}$ for $\left|y_{n}\right|$ large, we deduce in a standard way that, for all $s \in \mathbb{R}$,

$$
\begin{equation*}
\left\|\left\langle y_{n}\right\rangle^{s} e^{\Phi / h} v_{1}\right\|_{H^{s}\left(\mathbb{R}^{n}\right)}+\left\|e^{\Phi / h} v_{2}\right\|_{H^{s}\left(\mathbb{R}^{n}\right)}=\mathcal{O}\left(h^{\infty}\right) \tag{6.6}
\end{equation*}
$$

and thus, for any $\alpha \in \mathbb{N}^{n}$, by standard Sobolev estimates,

$$
\partial^{\alpha} v=\mathcal{O}\left(h^{\infty} e^{-\Phi / h}\right)
$$

uniformly. Then, the results follows from the fact that, on $\{\phi \leq S(\mu)-4 \varepsilon\}$, one has $\Phi=\Phi_{0} \geq \phi-C h \ln (S(\mu) / C h)$, while, on $\mathbb{R}^{n} \backslash\{\phi \leq S(\mu)-4 \varepsilon\}$, one has $\Phi \geq S(\mu)-5 \varepsilon$.

Now, we are able to compare the (complex-translated) resonant state $\mathbf{v}$ with w.

Proposition 6.2. For any multi-index $\alpha$, the first (conveniently normalized) eigenfunction $\mathbf{v}$ of $Q$ satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\mathbf{v}-\mathbf{w})=\mathcal{O}\left(h^{\infty} e^{-\phi / h}\right) \tag{6.7}
\end{equation*}
$$

locally uniformly in $\{\phi<S(\mu)\}$. Moreover, for any $\varepsilon>0$

$$
\begin{equation*}
\partial^{\alpha} \mathbf{v}=\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right) \tag{6.8}
\end{equation*}
$$

locally uniformly on $\mathbb{R}^{n} \backslash\{\phi<S(\mu)\}$.
Proof. Let $\gamma$ be the oriented circle centered at $\rho$ with radius $\delta h$, where $\delta>0$ is a small enough constant. Then, $\gamma$ does not meet the spectrum of $Q$, and we can write the spectral projector $\Pi$ of $Q$ associated with $\rho$, as,

$$
\Pi=\frac{1}{2 i \pi} \oint_{\gamma}(z-Q)^{-1} d z .
$$

We set,

$$
\mathbf{v}_{0}:=\Pi \widetilde{\mathbf{v}} .
$$

Then, $\mathbf{v}_{0}$ is solution of $(Q-\rho) \mathbf{v}_{0}=0$ and is colinear with $\mathbf{v}$. Therefore, the result will be an easy consequence of Proposition 6.1 and the following lemma:

Lemma 6.3. For any multi-index $\alpha$ and for any $\varepsilon>0$, one has,

$$
\begin{equation*}
\partial^{\alpha}\left(\mathbf{v}_{0}-\widetilde{\mathbf{v}}\right)=\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right) \tag{6.9}
\end{equation*}
$$

locally uniformly in $\mathbb{R}^{n}$.
Proof. Since $\widetilde{Q} \widetilde{\mathbf{v}}=\widetilde{E} \widetilde{\mathbf{v}}$, for any $z \in \gamma$, we have,

$$
\begin{equation*}
(Q-z) \widetilde{\mathbf{v}}=(\widetilde{E}-z) \widetilde{\mathbf{v}}+r, \tag{6.10}
\end{equation*}
$$

with $r:=(Q-\widetilde{Q}) \widetilde{\mathbf{v}}=\left(\left(\left\langle y_{n}\right\rangle+y_{n}-\mu\right)\left(1-\chi_{h}\left(y_{n}-\mu\right) \widetilde{\mathbf{v}}_{1}, 0\right)\right.$ (where we have set $\left.\widetilde{\mathbf{v}}=:\left(\widetilde{\mathbf{v}}_{1}, \widetilde{\mathbf{v}}_{2}\right)\right)$. In particular, by (6.6), for any $s \geq 0$ and any $\varepsilon>0$, we see,

$$
\begin{equation*}
\|r\|_{H^{s}\left(\mathbb{R}^{n}\right)}=\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right) \tag{6.11}
\end{equation*}
$$

Moreover, we deduce from (6.10),

$$
(z-Q)^{-1} \widetilde{\mathbf{v}}=(z-\widetilde{E})^{-1} \widetilde{\mathbf{v}}+(z-\widetilde{E})^{-1}(z-Q)^{-1} r,
$$

and thus,

$$
\begin{equation*}
\mathbf{v}_{0}=\widetilde{\mathbf{v}}+\frac{1}{2 i \pi} \int_{\gamma}(z-\widetilde{E})^{-1}(z-Q)^{-1} r d z . \tag{6.12}
\end{equation*}
$$

Now, along $\gamma$, we have $(z-\widetilde{E})^{-1}=\mathcal{O}\left(h^{-1}\right),\left(z-Q_{1}\right)^{-1}=\mathcal{O}\left(k^{-1}\right)$, and $\left(z-Q_{2}\right)^{-1}=\mathcal{O}\left(h^{-1}\right)$ (this last estimate comes from Appendix C and the fact that $\left.Q_{2}=\mathcal{U}\left(-h^{2} \Delta+V_{2}\left(x^{\prime}, x_{n}-i k\right)\right) \mathcal{U}^{-1}\right)$. Therefore,

$$
\left\|h R_{2,1}\left(Q_{1}-z\right)^{-1} h R_{1,2}\left(Q_{2}-z\right)^{-1}\right\|=\mathcal{O}\left(h k^{-1}\right)=\mathcal{O}\left(\frac{1}{|\ln h|}\right) .
$$

In particular, for $h$ small enough, the operator $I+h R_{2,1}\left(Q_{1}-z\right)^{-1} h R_{1,2}\left(Q_{2}-\right.$ $z)^{-1}$ is invertible with uniformly bounded inverse, and one easily deduces
that $\left\|(z-Q)^{-1}\right\|=\mathcal{O}\left(h^{-1}\right)$. Since one also has $(z-\widetilde{E})^{-1}=\mathcal{O}\left(h^{-1}\right)$, we learn from (6.11)-(6.12),

$$
\left\|\mathbf{v}_{0}-\widetilde{\mathbf{v}}\right\|_{L^{2}}=\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right)
$$

The same estimate for the $H^{s}$-norms is proved in the same way, by applying any arbitrary power of $Q$ to (6.12), and the result follows by Sobolev estimates.

Now, to complete the proof of the proposition, we just observe that, by construction, $\mathbf{v}$ and $\mathbf{v}_{0}$ are co-linear, and $\left\|\mathbf{v}_{0}\right\|=1+\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right)$. therefore, we can take $\mathbf{v}=\lambda \mathbf{v}_{0}$ with $\lambda=1+\mathcal{O}\left(e^{-(S(\mu)-\varepsilon) / h}\right)$, and the result follows from Lemma 6.3 and Proposition 6.1.
6.2. Near $\left\{y_{n}=\mu\right\}$.
6.2.1. A priori estimates. We first prove the following a priori estimates:

Proposition 6.4. For any $\alpha \in \mathbb{N}^{n}$, there exists $N_{\alpha} \geq 0$ such that, for any $N \geq 1$, the eigenfunction $\widetilde{\mathbf{v}}$ satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\widetilde{\mathbf{v}}-\mathbf{w})=\mathcal{O}\left(h^{-N_{\alpha}} e^{-S(\mu) / h}\right) \tag{6.13}
\end{equation*}
$$

locally uniformly in $\left\{y_{n} \geq \mu-(N k)^{2 / 3}\right\}$ (where $k:=h \ln h^{-1}$ ).
Remark 6.5. An important feature of the last estimate is that the number $N_{\alpha}$ does not depend on the choice of $N$.

Proof. Again, the proof is based on Agmon estimates, but we have to be more precise on the construction of the weight function near $\left\{y_{n}=\mu\right\}$.
Fix $\varepsilon>0$ small enough, and let $\chi \in C^{\infty}(\mathbb{R} ;[0,1])$ such that $\chi=1$ on $[\mu-\varepsilon,+\infty)$, and $\chi=0$ on $(-\infty, \mu-2 \varepsilon]$. We set,

$$
\begin{equation*}
\widetilde{\phi}_{0}\left(y_{n}\right)=\chi\left(y_{n}\right)\left(S(\mu)-\frac{2}{3}\left(\mu-y_{n}\right)^{3 / 2}\right) \tag{6.14}
\end{equation*}
$$

and, for $N \geq 1$ large enough and $y_{n} \in[0, \mu)$, we define,

$$
\begin{equation*}
\Phi_{N}\left(y_{n}\right)=\widetilde{\chi}\left(\widetilde{\phi}_{0}\left(y_{n}\right)+3 k\left(S(\mu)-\widetilde{\phi}_{0}\left(y_{n}\right)\right)^{1 / 3}+3 N k\right) \tag{6.15}
\end{equation*}
$$

where $\widetilde{\chi} \in C^{\infty}\left(\mathbb{R}_{+}\right)$depends on $N k$ and is such that $\widetilde{\chi}(s)=s$ for $s \leq S(\mu)$, $\widetilde{\chi}(s)=S(\mu)+N k$ for $s \geq S(\mu)+2 N k, 0 \leq \widetilde{\chi}^{\prime} \leq 1$ everywhere, and, for all $\ell \geq 0, \widetilde{\chi}^{(\ell)}=\mathcal{O}\left((N k)^{-(\ell-1)+}\right)$ uniformly.
Then, $\Phi_{N}$ is well defined and smooth on $(-\infty, \mu)$, and we have $\Phi_{N}=S(\mu)+$ $N k$ when $\widetilde{\phi}_{0}+3 k\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1 / 3} \geq S(\mu)-N k$. The latter condition is
implied by $S(\mu)-\widetilde{\phi}_{0} \leq N k$, and thus, if we extend $\widetilde{\Phi}_{N}$ by the constant value $S(\mu)+N k$ on $\left\{\widetilde{\phi}_{0} \geq S(\mu)-N k\right\}$, we obtain a smooth function on $\mathbb{R}$, that satisfies,

$$
\begin{aligned}
\Phi_{N}^{\prime} & =\widetilde{\chi}^{\prime}\left(\widetilde{\phi}_{0}+3 k\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1 / 3}+3 N k\right)\left(1-\frac{k}{\left(S(\mu)-\widetilde{\phi}_{0}\right)^{2 / 3}}\right) \widetilde{\phi}_{0}^{\prime} \\
(6.16) & =\mathcal{O}(1) .
\end{aligned}
$$

Moreover, one can check that, for $\ell \geq 1$, one has,

$$
\begin{equation*}
\tilde{\phi}_{0}^{(\ell)}=\mathcal{O}\left(\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1-2 \ell / 3}\right) \text { in }(\mu-2 \varepsilon, \mu) . \tag{6.17}
\end{equation*}
$$

In particular, $\widetilde{\phi}_{0}^{\prime}=\mathcal{O}\left(\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1 / 3}\right)$, and, since $S(\mu)-\widetilde{\phi}_{0} \in[N k, 3 N k]$ on the support of $\widetilde{\chi}^{\prime \prime}\left(\widetilde{\phi}_{0}+3 k\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1 / 3}+3 N k\right)$, and $\widetilde{\chi}^{(\ell)}=\mathcal{O}\left((N k)^{-(\ell-1)_{+}}\right)$, we deduce that, for all $\ell \geq 2$,

$$
\Phi_{N}^{(\ell)}=\mathcal{O}\left((N k)^{1-2 \ell / 3}\right)
$$

uniformly in $(\mu-2 \varepsilon, \mu)$. Moreover, by construction, $\Phi_{N}$ is constant on both $(-\infty, \mu-2 \varepsilon]$ and $[\mu,+\infty)$, and since $S(\mu)-\widetilde{\phi}_{0}$ vanishes at $y_{n}=\mu$ only, we also have $\partial^{\alpha} \Phi_{N}=\mathcal{O}(1)$ uniformly on $[\mu-2 \varepsilon, \mu-\varepsilon]$. Therefore, $\Phi_{N}$ satisfies (5.12) with $\lambda(h)=k^{-1 / 3}$ and $\Omega=\widetilde{\Omega}:=\left\{y \in \mathbb{R}^{n} ; \mu-\varepsilon \leq y_{n} \leq \mu\right\}$, and we can apply Proposition 5.5 with $\Phi=\Phi_{N}, \lambda_{h}=k^{-1 / 3}$, and $v=\widetilde{\mathbf{v}}-\mathbf{w}$. Then, taking into account (4.32)-(4.34) and the fact that $\widetilde{\rho}-\rho=\mathcal{O}\left(h^{\infty}\right)$, for any $\delta>0$, we obtain,

$$
\begin{gathered}
\frac{1}{C_{2}}\left\|h \nabla\left(e^{\Phi_{N} / h} v\right)\right\|^{2}+\frac{1}{C_{2}}\left\langle\widetilde{p}_{2}\left(y, i \nabla \Phi_{N}(y)\right) e^{\Phi_{N} / h} v_{2}, e^{\Phi_{N} / h} v_{2}\right\rangle \\
\left.+\left\langle\left(\widetilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2}\right)\right) e^{\Phi_{N} / h} v_{1}, e^{\Phi_{N} / h} v_{1}\right\rangle \\
\leq C_{2} h\left\|e^{\Phi_{N} / h} v_{1}\right\|^{2}+C_{2} h k^{-1 / 3}\left\|e^{\Phi_{N} / h} v_{2}\right\|_{\tilde{\Omega}_{\delta}}^{2}+C_{2} h\left\|e^{\Phi_{N} / h} v_{2}\right\|^{2} \\
+\varepsilon(h) e^{2 c_{1} / h}+C_{2} h^{-2 N_{0}} e^{2 c_{2} / h}+C_{2} h^{-2 N_{0}} e^{2 c_{3} / h},
\end{gathered}
$$

where $C_{2}>0$ is a constant, the quantities $N_{0}, \varphi$ are those appearing in (4.32)-(4.34), $\varepsilon(h)=\mathcal{O}\left(h^{\infty}\right)$, and where we have set,

$$
\begin{aligned}
& c_{1}:=\sup _{\widetilde{\mathcal{V}}_{2}}\left(\Phi_{N}-\varphi\right) ; \\
& c_{2}:=\sup _{\tilde{\mathcal{V}}_{2} \cap\left\{y_{n} \geq g\left(y^{\prime}\right)\right\}}\left(\Phi_{N}-S(\mu)\right) ; \\
& c_{3}:=\sup _{\tilde{\mathcal{V}}_{2} \backslash \widetilde{\mathcal{V}}_{1}}\left(\Phi_{N}-\varphi\right) .
\end{aligned}
$$

Now, using Lemma E.1. we see,

$$
\Phi_{N}-\varphi \leq 3 k\left(S(\mu)-\widetilde{\phi}_{0}\right)^{1 / 3}+3 N k \leq 3\left(N+S(\mu)^{1 / 3}\right) k,
$$

and thus, for $h$ small enough,

$$
e^{2 c_{1} / h} \leq h^{-6\left(N+S(\mu)^{1 / 3}\right)} .
$$

Moreover, since $\Phi_{N} \leq S(\mu)+N k$ everywhere, we have,

$$
e^{2 c_{2} / h}=h^{-2 N} .
$$

Finally, since $\varphi>S(\mu)$ on $\widetilde{\mathcal{V}}_{2} \backslash \widetilde{\mathcal{V}}_{1}$, we have $c_{3}<0$ and thus $e^{2 c_{3} / h}=\mathcal{O}\left(h^{\infty}\right)$. Summing up, for $N$ large enough, we have proved,

$$
\begin{aligned}
\frac{1}{C_{2}}\left\|h \nabla\left(e^{\Phi_{N} / h} v\right)\right\|^{2}+\frac{1}{C_{2}}\left\langle\widetilde{p}_{2}(y,\right. & \left.\left.i \nabla \Phi_{N}(y)\right) e^{\Phi_{N} / h} v_{2}, e^{\Phi_{N} / h} v_{2}\right\rangle \\
& \left.+\left\langle\left(\widetilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2}\right)\right) e^{\Phi_{N} / h} v_{1}, e^{\Phi_{N} / h} v_{1}\right\rangle
\end{aligned}
$$

(6.18) $C_{2} h\left\|e^{\Phi_{N} / h} v_{1}\right\|^{2}+C_{2} h k^{-1 / 3}\left\|e^{\Phi_{N} / h} v_{2}\right\|_{\tilde{\Omega}_{\delta}}^{2}$

$$
+C_{2} h\left\|e^{\Phi_{N} / h} v_{2}\right\|^{2}+C_{2} h^{-2\left(N+N_{0}\right)}+C_{N},
$$

where $C_{N}>0$ is an $h$-dependent constant. Now, for $N$ sufficiently large, on $\operatorname{Supp} \nabla \Phi_{N}$, we have $\widetilde{V}_{1}\left(y_{n}\right)=\mu-y_{n}$. Therefore, by (6.16), on $\widetilde{\Omega} \cap$ $\operatorname{Supp}\left(\nabla \Phi_{N}\right)$, we have,

$$
\begin{aligned}
\widetilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2} & \geq \mu-y_{n}-\left(1-k /\left(S-\widetilde{\phi}_{0}\right)^{2 / 3}\right)^{2}\left(\nabla \widetilde{\phi}_{0}\right)^{2} \\
& \geq\left(\mu-y_{n}\right)\left(\frac{2 k}{\left(S-\widetilde{\phi}_{0}\right)^{2 / 3}}-\frac{k^{2}}{\left(S-\widetilde{\phi}_{0}\right)^{4 / 3}}\right) .
\end{aligned}
$$

Since, on this set, we also have $k\left(S-\widetilde{\phi}_{0}\right)^{-2 / 3} \leq k^{1 / 3}$ and $S-\widetilde{\phi}_{0}=\mathcal{O}((\mu-$ $\left.y_{n}\right)^{3 / 2}$, we conclude that, for $h$ small enough,

$$
\widetilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2} \geq \frac{k}{C_{3}} \quad \text { on } \widetilde{\Omega} \cap \operatorname{Supp}\left(\nabla \Phi_{N}\right),
$$

where $C_{3}>0$ is a large enough constant. On the other hand, away from $\operatorname{Supp}\left(\nabla \Phi_{N}\right)$, we have $\widetilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2}=\widetilde{V}_{1}\left(y_{n}\right) \geq k^{2 / 3}\left\langle y_{n}\right\rangle / C_{3}$, and thus, in any case,

$$
\begin{equation*}
\tilde{V}_{1}\left(y_{n}\right)-\left(\nabla \Phi_{N}\right)^{2} \geq \frac{k\left\langle y_{n}\right\rangle}{C_{3}} \quad \text { on }\left\{y_{n} \geq \mu-\varepsilon\right\} . \tag{6.19}
\end{equation*}
$$

Moreover, since $\nabla \widetilde{\phi}_{0}=0$ on $\left\{y_{n}=\mu\right\}$, while $V_{2}>0$ there, one easily checks that, if $\varepsilon$ and $\delta$ are chosen sufficiently small, we have,

$$
\begin{equation*}
\widetilde{p}_{2}\left(y, i \nabla \Phi_{N}(y)\right) \geq c_{\mu} \quad \text { on } \quad\left\{y_{n} \geq \mu-\varepsilon-\delta\right\}, \tag{6.20}
\end{equation*}
$$

where $c_{\mu}>0$ is a constant.
On the other hand, on $\left\{y_{n} \leq \mu-\varepsilon\right\}$, using Lemma E.1, we see that,

$$
\Phi_{N} \leq \operatorname{Min}\left\{S(\mu)-\delta_{\mu}, \widetilde{\phi}+4 N k\right\},
$$

for some $\delta_{\mu}>0$ independent of $h$. Therefore, by using Proposition 6.1, we obtain,

$$
\begin{equation*}
\left\|h \nabla\left(e^{\Phi_{N} / h} v\right)\right\|_{\left\{y_{n} \leq \mu-\varepsilon\right\}}+\left\|e^{\Phi_{N} / h} v\right\|_{\left\{y_{n} \leq \mu-\varepsilon\right\}}=\mathcal{O}(1) \tag{6.21}
\end{equation*}
$$

Going back to (6.18), and using (6.19)-(6.21) (plus the fact that $\widetilde{V}_{1}=\left|y_{n}\right|+\mu$ for $y_{n} \leq 0$ ), we finally obtain,

$$
\begin{aligned}
& \left.\left\|h \nabla\left(e^{\Phi_{N} / h} v\right)\right\|^{2}+\left\|e^{\Phi_{N} / h} v_{2}\right\|^{2}+k \| \sqrt{\left\langle y_{n}\right.}\right\rangle e^{\Phi_{N} / h} v_{1} \|^{2} \\
& \leq C h\left\|e^{\Phi_{N} / h} v_{1}\right\|^{2}+C h k^{-1 / 3}\left\|e^{\Phi_{N} / h} v_{2}\right\|_{\tilde{\Omega}_{\delta}}^{2} \\
& \quad+C h\left\|e^{\Phi_{N} / h} v_{2}\right\|^{2}+C h^{-2\left(N+N_{0}\right)}+C,
\end{aligned}
$$

where $C>0$ is a constant (that may depend on $N$ and $\mu$ ). Since $k \gg h$ and $h k^{-1 / 3} \rightarrow 0$ as $h \rightarrow 0_{+}$, we conclude,
(6.22) $\left\|h \nabla\left(e^{\Phi_{N} / h} v\right)\right\|^{2}+\left\|e^{\Phi_{N} / h} v_{2}\right\|^{2}+k\left\|\sqrt{\left\langle y_{n}\right\rangle} e^{\Phi_{N} / h} v_{1}\right\|^{2}=\mathcal{O}\left(h^{-2\left(N+N_{0}\right)}\right)$,
uniformly for $h>0$ small enough. As before, using iteratively the equation, this permits us to obtain,

$$
\begin{equation*}
\left\|e^{\Phi_{N} / h} v_{2}\right\|_{H^{s}}+\left\|\left\langle y_{n}\right\rangle^{s} e^{\Phi_{N} / h} v_{1}\right\|_{H^{s}}=\mathcal{O}\left(h^{-\left(N+N_{s}\right)}\right) \tag{6.23}
\end{equation*}
$$

for any $s \in \mathbb{R}$, where $N_{s} \geq 0$ does not depend on N .
Now, by LemmaE.1, for any $M \geq 1$, on $\left[\mu-(M k)^{2 / 3}, \mu\right)$ we have $\widetilde{\phi}_{0} \geq S(\mu)-$ $\frac{2}{3} M k$. Therefore, taking $M=N^{\prime}:=3 N / 2$, we obtain $\widetilde{\phi}_{0} \geq S(\mu)-N k$, and thus (still by construction),

$$
\Phi_{N}=S(\mu)+N k \text { for } y_{n} \geq \mu-\left(N^{\prime} k\right)^{2 / 3}
$$

In particular, $e^{\Phi_{N} / h} \geq h^{-N} e^{S(\mu) / h}$ there, and thus, we deduce from (6.23),

$$
\begin{array}{r}
\left\|v_{2}\right\|_{H^{s}\left(y_{n} \geq \mu-\left(N^{\prime} k\right)^{2 / 3}\right)}+\left\|\left\langle y_{n}\right\rangle^{s} v_{1}\right\|_{H^{s}\left(y_{n} \geq \mu-\left(N^{\prime} k\right)^{2 / 3}\right)} \\
=\mathcal{O}\left(h^{-N_{s}} e^{-S(\mu) / h}\right), \tag{6.24}
\end{array}
$$

Since $N^{\prime} \rightarrow+\infty$ as $N \rightarrow+\infty$, the result of Proposition 6.4 follows by standard Sobolev estimates.

Proposition 6.6. For any multi-index $\alpha$, there exists $N_{\alpha} \geq 0$, such that, for all $N \geq 1$, the first eigenfunction $\mathbf{v}$ of $Q$ satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\mathbf{v}-\mathbf{w})=\mathcal{O}\left(h^{-N_{\alpha}} e^{-S(\mu) / h}\right) \tag{6.25}
\end{equation*}
$$

locally uniformly in $\left\{y_{n} \geq \mu-(N k)^{2 / 3}\right\}$.

Proof. The proof is similar to that of Proposition 6.2.
With the same notations, we see that $(Q-z) \widetilde{\mathbf{v}}=(\widetilde{E}-z) \widetilde{\mathbf{v}}+r$, with $r$ supported in $\left\{y_{n} \geq \mu-k^{2 / 3}\right\}$. Then, using (6.24), we find that, for any $s \geq 0,\|r\|_{H^{s}}=\mathcal{O}\left(h^{-N_{s}} e^{-S(\mu) / h}\right)$ for some $N_{s} \geq 0$, and we deduce that $\left\|\mathbf{v}_{0}-\widetilde{\mathbf{v}}\right\|=\mathcal{O}\left(h^{-N_{0}^{\prime}} e^{-S(\mu) / h}\right)$ for some $N_{0}^{\prime} \geq 0$. In particular, $\left\|\mathbf{v}_{0}\right\|=$ $1+\mathcal{O}\left(h^{-N_{0}^{\prime}} e^{-S(\mu) / h}\right)$, thus $\mathbf{v}=\lambda \mathbf{v}_{0}$ with $\lambda=1+\mathcal{O}\left(h^{-N_{0}^{\prime}} e^{-S(\mu) / h}\right)$, and the result follows by applying Proposition 6.4,
6.2.2. Propagation. Now, we are able to prove,

Proposition 6.7. For any multi-index $\alpha$, there exists $\delta_{\alpha}>0$, such that, for all $N \geq 1$, the first eigenfunction $\mathbf{v}$ of $Q$ satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\mathbf{v}-\mathbf{w})=\mathcal{O}\left(h^{\delta_{\alpha} N} e^{-S(\mu) / h}\right) \tag{6.26}
\end{equation*}
$$

uniformly with respect to $h>0$ small enough, and locally uniformly with respect to $y$ in $\left\{\mu-(N k)^{2 / 3} \leq y_{n} \leq \mu+(N k)^{2 / 3}\right\}$.

Proof. We proceed in a way very similar to that of FuLaMa (but in a simpler context, here). At first, for $\nu>0$, we introduce the Bargmann transform $T_{\nu}$, defined by the formula,

$$
T_{\nu} u(y, \eta ; h):=\int_{\mathbb{R}^{n}} e^{i(y-z) \eta / h-\left(y^{\prime}-z^{\prime}\right)^{2} / 2 h-\nu\left(y_{n}-z_{n}\right)^{2} / 2 h} u(z) d z
$$

We also set $q_{1}(y, \eta):=\eta^{2}+\mu-y_{n}$ and, for $y^{\prime} \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$,

$$
\gamma\left(t, y^{\prime}\right):=\exp t H_{q_{1}}\left(y^{\prime}, \mu ; 0\right)=\left(y^{\prime}, \mu+t^{2} ; 0,-t\right)
$$

By the same arguments as at the beginning of Section 6.1, we see that the estimate (6.3) remains valid in any complex neighborhood of 0 of the form: $\left\{y \in \mathbb{C}^{n} ;|\operatorname{Re} y| \leq 1 / C ;|\operatorname{Im} y| \leq C \sqrt{h}\right\}$, with a large enough constant $C>0$. Moreover, because of the quadratic behaviour of $\widetilde{\varphi}_{2}(y)$ near 0 , we also have $\|\mathbf{w}(y+i \delta)\|_{L_{l o c}^{2}}=\mathcal{O}(1)$ for any $\delta=\mathcal{O}(\sqrt{h})$. As a consequence, the same is true for $\widetilde{\mathbf{v}}$, and we see that all the previous arguments can be extended to $\operatorname{Im} y_{n}=-\delta(h)$, if $\delta(h)$ satisfies,

$$
0 \leq \delta(h) \leq C \sqrt{h}
$$

In particular, we obtain,

$$
\begin{equation*}
\partial^{\alpha} \mathbf{v}(y)=\mathcal{O}\left(h^{-N_{\alpha}} e^{-S(\mu) / h}\right) \tag{6.27}
\end{equation*}
$$

locally uniformly in $\left\{\operatorname{Re} y_{n} \geq \mu,-C \sqrt{h} \leq \operatorname{Im} y_{n} \leq 0\right\}$. (Observe that, since $\widetilde{\varphi}_{1}$ and $\widetilde{\varphi}_{2}$ are real on the real, one has $\operatorname{Re} \widetilde{\varphi}_{j}\left(y^{\prime}, y_{n}-i \delta(h)\right)=\widetilde{\varphi}_{j}(y)+$
$\left.\mathcal{O}\left(\delta(h)^{2}\right).\right)$ Therefore, taking $\nu=1$ and making a complex change of contour in the expression of $T_{1} \mathbf{v}$, it is easy to see that, if $(y, \eta) \in \mathbb{R}^{2 n}$ is fixed (independently of $\mu$ ), with $\eta_{n}>0$ and $y_{n}>\mu$, then,

$$
T_{1} \mathbf{v}=\mathcal{O}\left(e^{-1 / \sqrt{h}} e^{-S(\mu) / h}\right)=\mathcal{O}\left(h^{\infty}\right) e^{-S(\mu) / h}
$$

uniformly near $(y, \eta)$. In particular, for any $t \leq-1$ and $y^{\prime} \in \mathbb{R}^{n-1}$, we have,

$$
\gamma\left(t, y^{\prime}\right) \notin F S\left(e^{S(\mu) / h} \mathbf{v}\right),
$$

where $F S\left(e^{S(\mu) / h} \mathbf{v}\right.$ ) stands for the frequency set of $e^{S(\mu) / h} \mathbf{v}$ (see, e.g., GuSt, Ma6]). By the standard result of propagation of the frequency set, we deduce that $\gamma\left(t, y^{\prime}\right)$ stays away from $F S\left(e^{S(\mu) / h} \mathbf{v}\right)$ as long as the a priori estimate (6.27) remains valid in some fix neighborhood of $\pi_{y} \gamma\left(t, y^{\prime}\right):=\left(y^{\prime}, \mu+t^{2}\right)$, that is, as long as $t$ remains (strictly) negative. In other words, for any fixed $t<0$ (arbitrarily close to 0 ), one has $\gamma\left(t, y^{\prime}\right) \notin F S\left(e^{S(\mu) / h} \mathbf{v}\right)$.

On the other hand, if one has chosen the correct critical point in the definition of w on $\left\{y_{n} \geq \mu\right\}$, one also has $\gamma\left(t, y^{\prime}\right) \notin F S\left(e^{S(\mu) / h} \mathbf{w}\right)$ for any $t<0$ close enough to 0 . Therefore, $\gamma\left(t, y^{\prime}\right) \notin F S\left(e^{S(\mu) / h}(\mathbf{v}-\mathbf{w})\right.$, and thanks to Proposition 6.6, we can apply the same argument of propagation used in FuLaMa] Section 6 (see also GrMa]), and conclude to the existence of a constant $\delta>0$, such that, for all $N \geq 1$,

$$
\begin{equation*}
T_{1} \widetilde{v}_{N}\left(\widetilde{y}, \widetilde{\eta} ; \widetilde{h}_{N}\right)=\mathcal{O}\left(e^{-\delta / \widetilde{h}_{N}}\right) \text { uniformly in } \mathcal{V}_{N}(\delta), \tag{6.28}
\end{equation*}
$$

where we have set,

$$
\begin{aligned}
& \widetilde{v}_{N}(\widetilde{y}):=e^{S(\mu) / h}(\mathbf{v}-\mathbf{w})\left(y_{0}^{\prime}+(N k)^{\frac{1}{2}} \widetilde{y}^{\prime}, \mu+(N k)^{\frac{2}{3}} \widetilde{y}_{n}\right) \\
& \widetilde{h}_{N}:=(N k)^{-1} h ; \\
& \mathcal{V}_{N}(\delta):=\left\{(\widetilde{y}, \widetilde{\eta}) \in \mathbb{R}^{2 n} ;|\widetilde{y}| \leq \delta,(N k)^{\frac{1}{6}}\left|\widetilde{\eta}^{\prime}\right|+\left|\widetilde{\eta}_{n}\right| \leq \delta\right\} .
\end{aligned}
$$

Here, $y_{0}^{\prime} \in \mathbb{R}^{n-1}$ is fixed arbitrarily, and the estimate is uniform with respect to $h>0$ small enough and locally uniform with respect to $y_{0}^{\prime}$ (see FuLaMa, Proposition 6.8).
Note that, respect to FuLaMa, here the situation is slightly simpler, because the asymptotic solution $\mathbf{w}$ exists in a whole $h$-independent neighborhood of $\left(y^{\prime}, \mu\right)$. This is the reason why the estimate (6.28) is valid for all $N$ large enough, while that of FuLaMa, Proposition 6.8 was valid for special values of $N$ only.

As in FuLaMa], the main point of the proof consists in observing that the function $\widetilde{v}_{N}$ is solution of an equation of the form,

$$
\begin{equation*}
\widetilde{\mathbf{Q}}_{N}\left(\widetilde{y}, \widetilde{h}_{N} D_{\widetilde{y}} ; \widetilde{h}_{N}\right) \widetilde{v}_{N}=0 \tag{6.29}
\end{equation*}
$$

where the symbol $\widetilde{\mathbf{q}}_{N}$ of $\widetilde{\mathbf{Q}}_{N}$ satisfies,

$$
\begin{array}{r}
\widetilde{\mathbf{q}}_{N}\left(\widetilde{y}, \widetilde{\eta} ; \widetilde{h}_{N}\right)=\left((N k)^{1 / 3}\left|\widetilde{\eta}^{\prime}\right|^{2}+\widetilde{\eta}_{n}^{2}\right)\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
\widetilde{y}_{n} & 0 \\
0 & \check{V}_{N}(\widetilde{y}, \widetilde{\eta})
\end{array}\right) \\
+\mathcal{O}\left(\widetilde{h}_{N}(N k)^{1 / 3}\right)
\end{array}
$$

with $\operatorname{Re} \check{V}_{N}>0$ near $\mathcal{V}_{N}(\delta)$.
Using this equation, and in particular the fact that $\operatorname{det} \widetilde{\mathbf{q}}_{N}\left(\widetilde{y}, \widetilde{\eta} ; \widetilde{h}_{N}\right) \neq 0$ in $\left\{|\widetilde{y}| \leq \delta^{\prime} ;(N k)^{1 / 3}\left|\widetilde{\eta}^{\prime}\right|^{2}+\widetilde{\eta}_{n}^{2} \geq \delta\right\}$ if $\delta^{\prime}>0$ is small enough ( $\delta>0$ fixed), we deduce as in [FuLaMa] that, for all $m \geq 0$, one has,

$$
\begin{equation*}
\left\|\widetilde{v}_{N}\right\|_{H^{m}\left(|\tilde{y}| \leq \frac{1}{2} \delta^{\prime}\right)}=\mathcal{O}\left(e^{-\delta_{1} / \widetilde{h}_{N}}\right)=\mathcal{O}\left(h^{\delta_{1} N}\right), \tag{6.30}
\end{equation*}
$$

for some constant $\delta_{1}>0$. Then, the result follows from Sobolev estimates and from the fact that the previous estimate is locally uniform with respect to $y_{0}^{\prime} \in \mathbb{R}^{n-1}$.

Now, recall from Section 5.2 that the resonant state $\widetilde{\mathbf{u}}$, solution of $\widetilde{P} \widetilde{\mathbf{u}}=\rho \widetilde{\mathbf{u}}$, is related to $\mathbf{v}$ by,

$$
\widetilde{\mathbf{u}}(y)=\mathbf{v}\left(y^{\prime}, y_{n}+i k\right)
$$

We have,

Proposition 6.8. For any $\varepsilon>0$ small enough, set,

$$
\phi_{\varepsilon}:=\operatorname{Max}\left(\widetilde{\phi}_{0}, \min (\phi, S(\mu)-\varepsilon)\right)
$$

where $\widetilde{\phi}_{0}(y):=\chi_{0}\left(y_{n}\right)\left(S(\mu)-\frac{2}{3}\left(\mu-y_{n}\right)_{+}^{3 / 2}\right)$, with $\chi_{0} \in C^{\infty}(\mathbb{R} ;[0,1])$, $\chi_{0}=$ 1 on $\left[\mu-\varepsilon_{0}+\infty\right)$, $\chi_{0}=0$ on $\left(-\infty, \mu-2 \varepsilon_{0}\right]$, $\varepsilon_{0}>0$ fixed sufficiently small. Then, for any $\alpha \in \mathbb{N}^{n}$, there exists $N_{\alpha} \geq 0$, such that, for any $\varepsilon>0$,

$$
\partial^{\alpha} \widetilde{\mathbf{u}}(y)=\mathcal{O}\left(h^{-N_{\alpha}} e^{-\phi_{\varepsilon}(y) / h}\right)
$$

locally uniformly on $\mathbb{R}^{n}$.

Proof. We first observe that, by (6.6) and (6.23), the result is true for the function $\widetilde{\mathbf{v}}-\mathbf{w}$ (instead of $\widetilde{\mathbf{u}})$. Moreover, by construction and the proof of Lemma E.1, we see that it is also true for $\mathbf{w}$. Therefore, it is true for $\widetilde{\mathbf{v}}$, and the proofs of Propositions 6.2 and 6.6 permit us to deduce that it is also true for $\mathbf{v}$.

Then, we use the elementary fact that,

$$
\begin{equation*}
\left(T_{1} \widetilde{\mathbf{u}}\right)(y, \eta)=\left(T_{1} \mathbf{v}\right)\left(y^{\prime}, y_{n}+i k ; \eta\right)=e^{k^{2} / 2 h+k \eta_{n} / h}\left(T_{1} \mathbf{v}\right)\left(y ; \eta^{\prime}, \eta_{n}+k\right) \tag{6.31}
\end{equation*}
$$ and, since $\|\mathbf{v}\|_{L^{2}}=1$, we also know that $\left\|T_{1} \mathbf{v}\right\|_{L^{2}}$ is uniformly bounded (actually, it is $\left.\mathcal{O}\left(h^{\frac{3 n}{4}}\right)\right)$. Therefore, $e^{-k \eta_{n} / h} T_{1} \widetilde{\mathbf{u}} \in L^{2}\left(\mathbb{R}^{2 n}\right)$, and,

$$
\begin{equation*}
\left\|e^{-k \eta_{n} / h} T_{1} \widetilde{\mathbf{u}}\right\|_{L^{2}}=\mathcal{O}(1) \tag{6.32}
\end{equation*}
$$

uniformly as $h \rightarrow 0_{+}$. In particular, for any $\varepsilon>0, e^{-\varepsilon k^{2 / 3}\left\langle\eta_{n}\right\rangle / h} T_{1} \widetilde{\mathbf{u}} \in$ $L^{2}\left(\mathbb{R}^{2 n}\right)$.
Then, using the classical ellipticity (that is, as $|\eta| \rightarrow+\infty$ ) of the symbol of $\widetilde{P}$, together with the analyticity of $V_{2}$, one can show,

Lemma 6.9. For any $\chi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and for any $M \geq 1$, there exists a constant $C>0$ independent of $\mu$ such that,

$$
\sup _{|\eta| \geq C}\left|\langle\eta\rangle^{M} \chi(y) T_{1} \widetilde{\mathbf{u}}\right|=\mathcal{O}\left(e^{-1 / C h}\right)
$$

uniformly for $h>0$ small enough.
Proof. See Appendix F
Now, for $y$ in some fixed arbitrary compact set, we write,

$$
\begin{aligned}
\widetilde{\mathbf{u}}(y) & =\frac{1}{(2 \pi h)^{n}} \int_{\mathbb{R}^{2 n}} e^{i(y-z) \eta / h-(y-z)^{2} / 2 h} \widetilde{\mathbf{u}}(z) d z d \eta \\
& =\frac{1}{(2 \pi h)^{n}} \int_{\mathbb{R}^{n}} T_{1} \widetilde{\mathbf{u}}(y, \eta) d \eta,
\end{aligned}
$$

and thus, by Lemma 6.9 and (6.31),

$$
\begin{align*}
& \widetilde{\mathbf{u}}(y) \\
& =\frac{1}{(2 \pi h)^{n}} \int_{|\eta| \leq C} e^{k^{2} / 2 h+k \eta_{n} / h} T_{1} \mathbf{v}\left(y ; \eta^{\prime}, \eta_{n}+k\right) d \eta+\mathcal{O}\left(e^{-1 / C h}\right) \\
& =\frac{1}{(2 \pi h)^{n}} \int_{|\eta| \leq C} e^{k^{2} / 2 h+k \eta_{n} / h+i(y-z) \eta / h+i k\left(y_{n}-z_{n}\right) / h-(y-z)^{2} / 2 h} \mathbf{v}(z) d z d \eta \\
& +\mathcal{O}\left(e^{-1 / C h}\right) \tag{6.33}
\end{align*}
$$

In this integral, we perform the (singular) complex change of contour of integration given by,

$$
\begin{equation*}
\mathbb{R}^{n} \ni \eta \mapsto \eta+i \delta \chi(|\eta|) \frac{y-z}{|y-z|} \in \mathbb{C}^{n}, \tag{6.34}
\end{equation*}
$$

with $\delta>0$ small enough independent of $\mu$, and where $\chi \in C_{0}^{\infty}([0, C) ;[0.1])$, $\chi=1$ on $[0, C / 2]$. We obtain,

$$
\widetilde{\mathbf{u}}(y)=\int_{|\eta| \leq C} \mathcal{O}\left(h^{-n} e^{k \eta_{n} / h-\delta \chi(|\eta|)\left(|y-z| / h-(y-z)^{2} / 2 h\right.}\right) \mathbf{v}(z) d z d \eta+\mathcal{O}\left(e^{-1 / C h}\right) .
$$

Then, taking advantage that we are in dimension $n \geq 2$, we use the polar coordinates in $\eta: \eta=r \omega\left(r>0, \omega \in S^{n-1}\right)$, and since the integrated function is analytic with respect to $\omega$, we can perform the (singular) complex change of contour of integration given by,

$$
\begin{equation*}
S^{n-1} \ni \omega \mapsto \omega+i \delta \frac{y-z}{|y-z|} \in \mathbb{C}^{n} \tag{6.35}
\end{equation*}
$$

This gives,
$\widetilde{\mathbf{u}}(y)=\int_{|\eta| \leq C} \mathcal{O}\left(h^{-n} e^{k \eta_{n} / h-\delta\left(\chi_{1}(|\eta|)\right)|y-z| / h-(y-z)^{2} / 2 h}\right) \mathbf{v}(z) d z d \eta+\mathcal{O}\left(e^{-1 / C h}\right)$, with $\chi_{1}(|\eta|):=|\eta|+\chi(|\eta|) \geq 1$, and thus,

$$
\begin{equation*}
\widetilde{\mathbf{u}}(y)=\int \mathcal{O}\left(h^{-n-C} e^{-\delta|y-z| / h}\right) \mathbf{v}(z) d z+\mathcal{O}\left(e^{-1 / C h}\right) \tag{6.36}
\end{equation*}
$$

Finally, we observe that, if $\varepsilon$ is sufficiently enough, the Lipshitz function $\phi_{\varepsilon}$ is such that $\left\|\nabla \phi_{\varepsilon}\right\|_{L^{\infty}}=\mathcal{O}(\sqrt{\mu})$. Therefore, there exists a $\mu$-independent constant $C>0$ such that,

$$
-\delta^{\prime}|y-z|+\phi_{\varepsilon}(y)-\phi_{\varepsilon}(z) \leq-\left(\delta^{\prime}-C \sqrt{\mu}\right)|y-z|
$$

and thus, since $\delta^{\prime}>0$ can be taken independently of $\mu$, we deduce from (6.36) and from the estimate on $\mathbf{v}$ that, for $\mu$ sufficiently small, we have,

$$
\widetilde{\mathbf{u}}=\mathcal{O}\left(h^{-N_{0}} e^{-\phi_{\varepsilon} / h}\right)
$$

uniformly for $h>0$ small enough, and with some constant $N_{0} \geq 0$.
The estimates on the derivatives of $\widetilde{\mathbf{u}}$ can be done in the same way.

Now, we denote by $\widetilde{\mathbf{w}}$ the asymptotic solution constructed in Section 4 on the real domain. Then, we claim,

Proposition 6.10. For any $\alpha \in \mathbb{N}^{n}$, there exists $\delta_{\alpha}>0$, such that, for all $N \geq 1$, the resonant state $\widetilde{\mathbf{u}}$ of $\widetilde{P}$ satisfies,

$$
\begin{equation*}
\partial^{\alpha}(\widetilde{\mathbf{u}}-\widetilde{\mathbf{w}})=\mathcal{O}\left(h^{\delta_{\alpha} N} e^{-S(\mu) / h}\right) \tag{6.37}
\end{equation*}
$$

uniformly with respect to $h>0$ small enough, and locally uniformly with respect to $y$ in $\left\{\left|y_{n}-\mu\right| \leq(N k)^{2 / 3}\right\}$. Moreover, for any $\alpha \in \mathbb{N}^{n}$, there exists $N_{\alpha} \geq 0$ such that, for all $N \geq 1$, one has,

$$
\partial^{\alpha} \widetilde{\mathbf{u}}=\mathcal{O}\left(h^{-N_{\alpha}} e^{-S(\mu) / h}\right)
$$

locally uniformly in $\left\{y_{n} \geq \mu\right\}$.

Proof. We use Proposition 6.6 and the representation of $\widetilde{\mathbf{u}}$ given in 6.33). After the changes of contour (6.34)-(6.35), we obtain,

$$
\begin{array}{r}
\widetilde{\mathbf{u}}(y)=\frac{e^{k^{2} / 2 h}}{(2 \pi h)^{n}} \int_{|\eta| \leq C} e^{i \theta / h+k \eta_{n} / h-\delta \chi_{1}(|\eta|)|y-z|-(y-z)^{2} / 2 h} \mathbf{v}(z) J(y, z, \eta) d z d \eta \\
+\mathcal{O}\left(e^{-1 / h}\right)
\end{array}
$$

with $\theta:=\delta k \chi_{1}(|\eta|) \frac{y_{n}-z_{n}}{|y-z|}+(y-z) \eta+k\left(y_{n}-z_{n}\right)$, and $J(y, z, \eta)=\mathcal{O}(1)$. On $\left\{z_{n} \leq \mu-\varepsilon\right\}$ (with $\varepsilon>0$ fixed small enough), the estimates on $\mathbf{v}$ show that, for any $\varepsilon^{\prime}>0$, one has,

$$
e^{-\delta|y-z| / h} \mathbf{v}(z)=\mathcal{O}\left(h^{-N_{0}} e^{-\delta|y-z| / h-\min \left(\phi, S-\varepsilon^{\prime}\right) / h}\right.
$$

and thus, if in addition $y_{n} \geq \mu-(N k)^{2 / 3}$ (and using the fact that $\|\nabla \phi\|_{L^{\infty}}=$ $\mathcal{O}(\mu))$, for $\varepsilon^{\prime}$ small enough, on this set we obtain,

$$
e^{-\delta|y-z| / h} \mathbf{v}(z)=\mathcal{O}\left(h^{-N_{0}} e^{-\left(S / h+\delta \varepsilon / 2 h-\varepsilon^{\prime} / h\right)}\right)=\mathcal{O}\left(e^{-(S+\delta \varepsilon / 4) / h)}\right)
$$

Moreover, on $\left\{\mu-\varepsilon \leq z_{n} \leq \mu-2(N k)^{2 / 3}\right\}$, we have,

$$
e^{-\delta|y-z| / h} \mathbf{v}(z)=\mathcal{O}\left(h^{-N_{0}} e^{-\delta|y-z| / h-S / h+2\left(\mu-z_{n}\right)^{3 / 2} / h}\right),
$$

and thus, for $y_{n} \geq \mu-(N k)^{2 / 3}$,

$$
e^{-\delta|y-z| / h} \mathbf{v}(z)=\mathcal{O}\left(h^{-N_{0}} e^{-\delta(N k)^{2 / 3} / 2 h-S / h}\right)=\mathcal{O}\left(h^{\infty} e^{-S / h}\right)
$$

As a consequence, for $\widetilde{\mathbf{u}}(y)$ with $y_{n} \geq \mu-(N k)^{2 / 3}$, we obtain,
$\widetilde{\mathbf{u}}(y)$

$$
\begin{array}{r}
\frac{e^{k^{2} / 2 h}}{(2 \pi h)^{n}} \int_{\substack{|\eta| \leq C}} e^{i \theta / h+k \eta_{n} / h-\delta \chi_{1}(|\eta|)|y-z|-(y-z)^{2} / 2 h} \mathbf{v}(z) J(y, z, \eta) d z d \eta \\
+\mathcal{O}\left(h^{\infty} e^{-S / h}\right)
\end{array}
$$

In particular, for $y_{n} \geq \mu$, by Proposition 6.6 we deduce,

$$
\widetilde{\mathbf{u}}(y)=\mathcal{O}\left(h^{-N_{0}} e^{-S(\mu) / h}\right.
$$

Moreover, if $\left|y_{n}-\mu\right| \leq(N k)^{2 / 3}$, the previous arguments lead to,
$\widetilde{\mathbf{u}}(y)$
$=\frac{e^{k^{2} / 2 h}}{(2 \pi h)^{n}} \int_{\substack{|\eta| \leq C \\\left|z_{n}-\mu\right| \leq 2(N k)^{2 / 3}}} e^{i \theta / h+k \eta_{n} / h-\delta \chi_{1}(|\eta|)|y-z|-(y-z)^{2} / 2 h} \mathbf{v}(z) J(y, z, \eta) d z d \eta$ $+\mathcal{O}\left(h^{\infty} e^{-S / h}\right)$.

Then, using Proposition 6.7, we conclude,
$\widetilde{\mathbf{u}}(y)$
$=\frac{e^{k^{2} / 2 h}}{(2 \pi h)^{n}} \int_{\substack{|\eta| \leq C \\\left|z_{n}-\mu\right| \leq 2(N k)^{2 / 3}}} e^{i \theta / h+k \eta_{n} / h-\delta \chi_{1}(|\eta|)|y-z|-(y-z)^{2} / 2 h} \mathbf{w}(z) J(y, z, \eta) d z d \eta$

$$
+\mathcal{O}\left(h^{\delta N} e^{-S / h}\right)
$$

for some $\delta>0$ independent of $N$. Finally, taking $y^{\prime}$ close to some $y_{0}^{\prime} \in$ $\mathbb{R}^{n-1}$, either $\mathbf{w}$ is analytic around $\left(y_{0}^{\prime}, y_{n}\right)$, or it is exponentially smaller than $e^{-S(\mu) / h}$. In the second case, we obtain $\widetilde{\mathbf{u}}(y)=\mathcal{O}\left(h^{\delta N} e^{-S / h}\right)$, while, in the first case, a change of contour of the type $z_{n} \mapsto z_{n}+i k \chi\left(\left|z_{n}-\mu\right|(N k)^{-2 / 3}\right)$ gives, by the same arguments as before, $\widetilde{\mathbf{u}}(y)=\widetilde{\mathbf{w}}(y)+\mathcal{O}\left(h^{\delta N} e^{-S / h}\right)$. The same estimates holds for the derivatives, and this proves the proposition.
6.3. All around $\{\phi=S(\mu)\}$. We fix $\nu_{0}>0$ sufficiently small, but independent of $\mu$, and, for $\nu \in\left(0, \nu_{0} / 4\right]$, we consider the following spherical domain of $\mathbb{R}^{n}$,

$$
\mathcal{A}_{\nu}:=\left\{\nu_{0}-\nu<|y|<\nu_{0}+\nu\right\} .
$$

In this section, we plan to estimate $\widetilde{\mathbf{u}}$ on $\mathcal{A}_{\nu}$. By construction, $\widetilde{\mathbf{w}}=0$ in $\mathcal{A}_{2 \nu}$ if $\mu \ll \nu_{0}$, and then we also have $\mathcal{A}_{2 \nu} \subset \mathbb{R}^{n} \backslash\{\phi<S(\mu)\}$. Therefore, Propositions 6.8 and 6.10 already tell us that,

- $\partial^{\alpha} \widetilde{u}=\mathcal{O}\left(h^{-N_{\alpha}} e^{-S(\mu) / h}\right)$ on $\mathcal{A}_{2 \nu} \cap\left\{y_{n} \geq \mu\right\} ;$
- $\partial^{\alpha} \widetilde{u}=\mathcal{O}\left(h^{\delta N} e^{-S(\mu) / h}\right)$ on $\mathcal{A}_{2 \nu} \cap\left\{\left|y_{n}-\mu\right| \leq 4(N k)^{2 / 3}\right\}$;
- For any $\varepsilon>0, \partial^{\alpha} \widetilde{u}=\mathcal{O}\left(h^{-N_{\alpha}} e^{-\max \left(\widetilde{\phi}_{0}, S(\mu)-\varepsilon\right) / h}\right)$ on $\mathcal{A}_{2 \nu} \cap\left\{y_{n} \leq\right.$ $\left.\mu-(N k)^{2 / 3}\right\}$,
where $N \geq 1$ is arbitrary, and the positive constants $\delta, N_{\alpha}$ do not depend on $N$ (note that the same estimates also hold for $\mathbf{v}$ ). Then, recording that $S(\mu)=\mathcal{O}\left(\mu^{2}\right)$, we consider a weight function $\psi \in C^{\infty}\left(\mathbb{R}^{n} ; \mathbb{R}_{+}\right)$such that,
- $\psi$ is supported in $\mathcal{A}_{2 \nu}$;
- $\psi \leq S(\mu)$ on $\left\{y_{n} \geq \mu+4(N k)^{2 / 3}\right\}$;
- $\psi \leq S(\mu)+\delta N k$ on $\left\{y_{n} \leq \mu+4(N k)^{2 / 3}\right\}$;
- $\psi=S(\mu)$ on $\left\{y_{n} \geq \mu+4(N k)^{2 / 3}\right\} \cap \mathcal{A}_{\nu}$;
- $\psi=S(\mu)+\delta N k$ on $\left\{y_{n} \leq \mu+2(N k)^{2 / 3}\right\} \cap \mathcal{A}_{\nu}$;
- $|\nabla \psi| \leq \mu$ on $\left\{y_{n} \leq \mu+2(N k)^{2 / 3}\right\} \cup\left\{y_{n} \geq \mu+4(N k)^{2 / 3}\right\}$;
- $\partial^{\alpha} \psi=\mathcal{O}(1)$ on $\left\{y_{n} \leq \mu+2(N k)^{2 / 3}\right\} \cup\left\{y_{n} \geq \mu+4(N k)^{2 / 3}\right\},(|\alpha| \geq 2)$;
- $\partial^{\alpha} \psi=\mathcal{O}\left((N k)^{1-2|\alpha| / 3}\right)$ on $\left\{\mu+2(N k)^{2 / 3} \leq y_{n} \leq \mu+4(N k)^{2 / 3}\right\}$, $(|\alpha| \geq 1)$.

Then, as before, the Agmon estimates applied with this weight function $\psi$ lead to $\partial^{\alpha} \widetilde{u}=\mathcal{O}\left(h^{-N_{\alpha}} e^{-\psi / h}\right)$, and thus, in particular, for any $N \geq 1$, one has,

$$
\begin{equation*}
\partial^{\alpha} \widetilde{u}=\mathcal{O}\left(h^{\delta N} e^{-S(\mu) / h}\right) \tag{6.38}
\end{equation*}
$$

uniformly on $\left\{y_{n} \leq \mu+2(N k)^{2 / 3}\right\} \cap \mathcal{A}_{\nu}$.

## 7. Completion of the Proof

We fix $\nu \in\left(0, \nu_{0} / 4\right]$ small enough, independent of $\mu$, and $\chi_{\nu} \in C^{\infty}(\mathbb{R} ;[0,1])$, such that $\chi_{\nu}=1$ on $\left(-\infty, \nu_{0}-\nu\right]$, $\chi_{\nu}=0$ on $\left[\nu_{0}+\nu,+\infty\right)$. We also fix $\chi_{0} \in C^{\infty}(\mathbb{R}:[0,1])$ such that $\chi_{0}=1$ on $(-\infty, 1], \chi_{0}=0$ on $[2,+\infty)$. Then, for $y=\left(y^{\prime}, y_{n}\right) \in \mathbb{R}^{n}$ and $N \geq 1$, we set,

$$
\chi_{N}(y):=\chi_{\nu}(|y|) \chi_{0}\left(\frac{y_{n}-\mu}{(N k)^{2 / 3}}\right)
$$

In particular, the support of $\nabla \chi_{N}$ is included in the set.

$$
\begin{aligned}
\mathcal{B}_{N}:= & \left(\mathcal{A}_{\nu} \cap\left\{y_{n}-\mu \leq 2(N k)^{2 / 3}\right\}\right) \\
& \cup\left(\left\{(N k)^{2 / 3} \leq y_{n}-\mu \leq 2(N k)^{2 / 3}\right\} \cap\left\{|y| \leq \nu_{0}+\nu\right\}\right)
\end{aligned}
$$

Then, we write,

$$
(\operatorname{Im} \rho)\left\|\chi_{N} \widetilde{\mathbf{u}}\right\|^{2}=\operatorname{Im}\left\langle\chi_{N} \widetilde{P} \widetilde{\mathbf{u}}, \chi_{N} \widetilde{\mathbf{u}}\right\rangle=-\operatorname{Im}\left\langle\chi_{N}\left[\widetilde{P}, \chi_{N}\right] \widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}\right\rangle
$$

and, at this point, we can readily follow the arguments of GrMa, Section 8 (with, in our case, $n_{0}=1$ and $n_{\Gamma}=0$ ), and Theorem 3.1 follows.

## APPENDIX

## Appendix A. Proof of Lemma 5.2

Setting $F(\theta, y):=\Psi\left(y+\frac{\theta}{2}, y-\frac{\theta}{2}\right)$, we write,

$$
\begin{equation*}
a\left(y, \eta^{\prime}+i F(\theta, y)+i f_{1}\right)=a\left(y, \eta^{\prime}+i \nabla \Phi(y)+i f_{1}\right)+\widetilde{a}\left(y, \eta^{\prime}, \theta\right) \tag{A.1}
\end{equation*}
$$

where,
(A.2) $\quad \widetilde{a}\left(y, \eta^{\prime}, \theta\right)=i \theta \int_{0}^{1}\left(\nabla_{\theta} F\right)(t \theta, y) \cdot\left(\nabla_{\eta} a\right)\left(y, \eta^{\prime}+i F(t \theta, y)+i f_{1}\right) d t$.

Now, by assumption (5.2), $F$ satisfies,

$$
\begin{aligned}
& |F| \leq \nu_{0} \\
& \partial^{\alpha} F=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\substack{y \in \Omega_{\delta} \\
|\theta|<\delta}}\right) h^{1-|\alpha|}+\lambda_{h} h^{2-|\alpha|}\right) \quad(|\alpha| \geq 1)
\end{aligned}
$$

where $\delta>0$ is arbitrarily small.
In the following, for any set $\mathcal{E}$, we set,

$$
\Lambda_{h}(\mathcal{E}):=1+\lambda_{h} \mathbf{1}_{\mathcal{E}},
$$

where, as before, $\mathbf{1}_{\mathcal{E}}$ stands for the characteristic function of $\mathcal{E}$.
Then, we see on (A.2) that, for any $\alpha, \beta, \gamma \in \mathbb{N}^{n}$, we have,

$$
\begin{align*}
\partial_{\theta}^{\alpha} \partial_{y}^{\beta} \partial_{\eta^{\prime}}^{\gamma} \widetilde{a}\left(y, \eta^{\prime}, \theta\right)= & \mathcal{O}\left(\left\langle\eta^{\prime}\right\rangle^{2 m}\left((|\theta|+h) \Lambda_{h}\left(|\theta|<\delta ; y \in \Omega_{\delta}\right) h^{-|\alpha|-|\beta|}\right)\right. \\
& +\mathcal{O}\left(\left\langle\eta^{\prime}\right\rangle^{2 m}\left((|\theta|+h) \lambda_{h} h^{1-|\alpha|-|\beta|}\right) .\right. \tag{A.3}
\end{align*}
$$

(Note that in the factor $(|\theta|+h)$, the term in $h$ appears if $\alpha \neq 0$ only, and that the validity of the estimate extends to $\left(y, \eta^{\prime}\right) \in \mathbb{R}^{n} \times \mathbb{C}^{n},\left|\operatorname{Im} \eta^{\prime}\right|$ small enough.)
Now, using (A.1), we can split $b(y, \eta)$ into,

$$
\begin{equation*}
b(y, \eta)=b_{0}(y, \eta)+b_{1}(y, \eta)+b_{2}(y, \eta), \tag{A.4}
\end{equation*}
$$

with,
$b_{0}(y, \eta):=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h-g_{1} / h} a\left(y, \eta^{\prime}+i \nabla \Phi(y)+i f_{1}\right) d \eta^{\prime} d \theta ;$
$b_{1}(y, \eta):=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h-g_{1} / h}\left(\chi_{0}(\theta)-1\right) a\left(y, \eta^{\prime}+i \nabla \Phi(y)+i f_{1}\right) d \eta^{\prime} d \theta ;$
$b_{2}(y, \eta):=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h-g_{1} / h} \chi_{0}(\theta) \widetilde{a}\left(y, \eta^{\prime}, \theta\right) d \eta^{\prime} d \theta$.
Making the change of contour of integration $\eta^{\prime} \mapsto \eta^{\prime}-i f_{1}$ in the expression of $b_{0}$, and using the fact that $\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h} d \theta=\delta_{\eta^{\prime}=\eta}$, we immediately obtain,

$$
\begin{equation*}
b_{0}(y, \eta)=a(y, \eta+i \nabla \Phi(y)) . \tag{A.5}
\end{equation*}
$$

On the other hand, if we set,

$$
L:=\left(1+\left|\eta^{\prime}-\eta\right|^{2}+|\theta|^{2}\right)^{-1}\left(1+\left(\eta^{\prime}-\eta\right) \cdot h D_{\theta}+\theta \cdot h D_{\eta^{\prime}}\right),
$$

then, by integrations by parts, for any $N \geq 1$, we have,

$$
\begin{align*}
& b_{1}(y, \eta)=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h} c_{N}\left(y, \eta, \eta^{\prime}, \theta\right) d \eta^{\prime} d \theta  \tag{A.6}\\
& b_{2}(y, \eta)=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\eta^{\prime}-\eta\right) \theta / h} \widetilde{c}_{N}\left(y, \eta, \eta^{\prime}, \theta\right) d \eta^{\prime} d \theta \tag{A.7}
\end{align*}
$$

with,

$$
\begin{aligned}
& c_{N}\left(y, \eta, \eta^{\prime}, \theta\right):=\left({ }^{t} L\right)^{N}\left(e^{-g_{1} / h}\left(\chi_{0}(\theta)-1\right) a\left(y, \eta^{\prime}+i \nabla \Phi(y)+i f_{1}\right)\right) ; \\
& \widetilde{c}_{N}\left(y, \eta, \eta^{\prime}, \theta\right):=\left({ }^{t} L\right)^{N}\left(e^{-g_{1} / h} \chi_{0}(\theta) \widetilde{a}\left(y, \eta^{\prime}, \theta\right)\right) .
\end{aligned}
$$

Therefore, using (5.2), (A.3), and the fact that $g \geq 2 \varepsilon_{0}$ on the support of $1-\chi_{0}\left(y-y^{\prime}\right)$, we see that, for any $\alpha, \beta, \gamma \in \mathbb{N}^{n}$, we have,

$$
\begin{array}{r}
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} \partial_{\eta^{\prime}}^{\gamma} c_{N}\left(y, \eta^{\prime}, \theta\right)=\mathcal{O}\left(e^{-\varepsilon_{0} / h}\left(1+\left|\eta^{\prime}-\eta\right|+|\theta|\right)^{-N}\right)\left\langle\eta^{\prime}\right\rangle^{2 m} \\
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} \partial_{\eta^{\prime}}^{\gamma} \widetilde{c}_{N}\left(y, \eta^{\prime}, \theta\right)=\mathcal{O}\left(\left(\Lambda_{h}\left(|\theta|<\delta ; y \in \Omega_{\delta}\right)+h \lambda_{h}\right)(|\theta|+h) h^{-|\alpha|}\right) \\
\left.\times\left(1+\left|\eta^{\prime}-\eta\right|+|\theta|\right)^{-N}\left\langle\eta^{\prime}\right\rangle^{2 m}\right)
\end{array}
$$

uniformly for $h>0$ small enough, $(\theta, \eta) \in \mathbb{R}^{2 n}$, and $\left(y, \eta^{\prime}\right) \in \mathbb{R}^{n} \times \mathbb{C}^{n}$, $\left|\operatorname{Im} \eta^{\prime}\right|$ small enough.
In particular, by choosing $N$ sufficiently large, we derive,

$$
\begin{equation*}
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} b_{1}(y, \eta)=\mathcal{O}\left(e^{-\varepsilon_{0} / h}\langle\eta\rangle^{2 m}\right) \tag{A.8}
\end{equation*}
$$

Concerning $b_{2}$, in (A.7) we make the (singular) change of contour of integration,

$$
\mathbb{R}^{n} \ni \eta^{\prime} \mapsto \eta^{\prime}+i \delta \frac{\theta}{|\theta|}
$$

where, as before, $\delta>0$ is taken sufficiently small. We obtain,

$$
\begin{array}{r}
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} b_{2}(y, \eta)=\frac{1}{(2 \pi h)^{n}} \int \mathcal{O}\left(e^{-\delta|\theta| / h}\left(\Lambda_{h}\left(|\theta|<\delta ; y \in \Omega_{\delta}\right)+h \lambda_{h}\right)(|\theta|+h)\right) \\
\times h^{-|\alpha|}\langle\theta\rangle^{-N}\langle\eta\rangle^{2 m} d \theta
\end{array}
$$

and thus,

$$
\begin{equation*}
\partial_{y}^{\alpha} \partial_{\eta}^{\beta} b_{2}(y, \eta)=\mathcal{O}\left(\Lambda_{h}\left(y \in \Omega_{\delta}\right) h^{1-|\alpha|}\langle\eta\rangle^{2 m}\right) \tag{A.9}
\end{equation*}
$$

Then, Lemma 5.2 follows from (A.4), (A.5), (A.8), and (A.9).

## Appendix B. An exotic pseudofifferential calculus

In this appendix we have gathered some useful results concerning the pseudodifferential operators $A=\mathrm{Op}_{h}^{W}(a)$ with $a=a(x, \xi ; h)$ smooth, such that $a$ extends to a holomorphic function of $\xi$ in a strip $\mathcal{A}_{\nu}:=\{(x, \xi) \in$ $\left.\mathbb{R}^{n} \times \mathbb{C}^{n} ;|\operatorname{Im} \xi|<\nu\right\}$ with $\nu>0$, and, for all $\alpha, \beta \in \mathbb{N}^{n}$, it satisfies,

$$
\begin{align*}
& \partial_{\xi}^{\beta} a(x, \xi)=\mathcal{O}\left(\langle\operatorname{Re} \xi\rangle^{m}\right)  \tag{B.1}\\
& \partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\Omega}(x)\right) h^{1-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m}\right) \quad(|\alpha| \geq 1) \tag{B.2}
\end{align*}
$$

uniformly for $(x, \xi) \in \mathcal{A}_{\nu}$ and $h>0$ small enough. Here, $m \in \mathbb{R}$ is some fix real number, $\Omega$ is some fix subset of $\mathbb{R}^{n}$, and $\lambda_{h} \geq 1$ is some given function of $h$ such that $h \lambda_{h}=\mathcal{O}(1)$ uniformly.

We denote by $\mathcal{S}_{m}\left(\lambda_{h}, \Omega\right)$ the space symbols $a$ such that the previous properties hold for some $\nu>0$. For $t \in[0,1]$, we also denote by $\operatorname{Op}_{h}^{t}(a)$ the $t$-quantization of $a$ (see, e.g., Ma6]).

In order to describe the remainders terms that will appear in the symbolic calculus, we need to introduce another class of symbols. We say that a smooth symbol $a=a(x, \xi ; h)$ is in $\widetilde{\mathcal{S}}_{m}\left(\lambda_{h}, \Omega\right)$ if there exists some $\nu>0$ such that $a$ extends to a holomorphic function of $\xi$ in $\mathcal{A}_{\nu}$, and, for all $\alpha, \beta \in \mathbb{N}^{n}$, it satisfies,

$$
\begin{equation*}
\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)=\mathcal{O}\left(\left(1+\lambda_{h} \mathbf{1}_{\Omega}(x)\right) h^{-|\alpha|}\langle\operatorname{Re} \xi\rangle^{2 m}\right) \tag{B.3}
\end{equation*}
$$

uniformly for $(x, \xi) \in \mathcal{A}_{\nu}$ and $h>0$ small enough.
In particular, one has,

$$
\mathcal{S}_{m}\left(\lambda_{h}, \Omega\right) \subset \widetilde{\mathcal{S}}_{m}\left(\lambda_{h}, \Omega\right) \subset h^{-1} \mathcal{S}_{m}\left(\lambda_{h}, \Omega\right)
$$

Proposition B.1. With the previous notations, one has,
(i) If $a \in \mathcal{S}_{0}\left(\lambda_{h}, \mathbb{R}^{n}\right)$, then $\mathrm{Op}_{h}^{W}(a)$ is uniformly bounded on $L^{2}\left(\mathbb{R}^{n}\right)$.
(ii) If $a \in \mathcal{S}_{m}\left(\lambda_{h}, \Omega\right)$, then, for any $t \in[0,1]$ any $\delta>0$, and any $u \in L^{2}$, one has,

$$
\left\|\left\langle h D_{x}\right\rangle^{-\frac{m}{2}}\left(\mathrm{Op}_{h}^{W}(a)-\mathrm{Op}_{h}^{t}(a)\right)\left\langle h D_{x}\right\rangle^{-\frac{m}{2}} u\right\|=\mathcal{O}\left(h\|u\|+h \lambda_{h}\|u\|_{L^{2}\left(\Omega_{\delta}\right)}\right)
$$

where, as before, $\Omega_{\delta}:=\left\{x \in \mathbb{R}^{n} ; \operatorname{dist}(x, \Omega) \leq \delta\right\}$, and where the estimate is uniform with respect to $h$ and $u$.
(iii) Let $m, m^{\prime} \in \mathbb{R}, a \in \mathcal{S}_{m}\left(\lambda_{h}, \Omega\right)$, and $b \in \mathcal{S}_{m^{\prime}}\left(\lambda_{h}, \Omega\right)$. Then, for any $\delta>0, a \#^{W} b \in \mathcal{S}_{m+m^{\prime}}\left(\lambda_{h}, \Omega_{\delta}\right)$, and one has,

$$
a \#^{W} b=a b+h c
$$

with $c \in \widetilde{\mathcal{S}}_{m+m^{\prime}}\left(\lambda_{h}, \Omega_{\delta}\right)$.
(iv) If $a \in \widetilde{\mathcal{S}}_{0}\left(\lambda_{h}, \Omega\right)$, then, for any $\delta>0, \mathrm{Op}_{h}^{W}(a)$ satisfies,

$$
\left\|\mathrm{Op}_{h}^{W}(a) u\right\|=\mathcal{O}\left(\|u\|+\lambda_{h}\|u\|_{L^{2}\left(\Omega_{\delta}\right)}\right)
$$ uniformly with respect to $u \in L^{2}\left(\mathbb{R}^{n}\right)$ and $h>0$ small enough.

(v) Let $m, m^{\prime} \in \mathbb{R}, a \in \widetilde{\mathcal{S}}_{m}\left(\lambda_{h}, \Omega\right)$, and $b \in \mathcal{S}_{m^{\prime}}\left(\lambda_{h}, \Omega\right)$. Then, for any $\delta>0, a \#^{W} b \in \widetilde{\mathcal{S}}_{m+m^{\prime}}\left(\lambda_{h}, \Omega_{\delta}\right)$.
(vi) If $a \in \mathcal{S}_{0}\left(\lambda_{h}, \mathbb{R}^{n}\right)$ takes it values in $\mathbb{R}_{+}$, then, there exists a constant $C>0$ such that,

$$
\operatorname{Op}_{h}^{W}(a) \geq-C \sqrt{h \lambda_{h}} .
$$

Remark B.2. No complete symbolic calculus is available for this exotic class of symbols, but just up to $\mathcal{O}\left(h \lambda_{h}\right)$.

Remark B.3. It results from (iii) and (v) that, if $a_{j} \in \mathcal{S}_{m_{j}}\left(\lambda_{h}, \Omega\right)(j=$ $1,2,3$ ), then, for any $\delta>0$, the symbol $a_{1} \#^{W} a_{2} \#^{W} a_{3}$ can be written as,

$$
\begin{equation*}
a_{1} \#^{W} a_{2} \#^{W} a_{3}=a_{1} a_{2} a_{3}+h b \tag{B.4}
\end{equation*}
$$

with $c \in \widetilde{\mathcal{S}}_{m_{1}+m_{2}+m_{3}}\left(\lambda_{h}, \Omega_{\delta}\right)$. Indeed, writing $a_{1} \#^{W} a_{2}=a_{1} a_{2}+h b_{1}$ with $b_{1} \in \mathcal{S}_{m_{1}+m_{2}}\left(\lambda_{h}, \Omega_{\delta / 2}\right)$, it comes out,

$$
a_{1} \#^{W} a_{2} \#^{W} a_{3}=\left(a_{1} a_{2}\right) \#^{W} a_{3}+h b_{1} \#^{W} a_{3},
$$

and since $a_{1} a_{2} \in \mathcal{S}_{m_{1}+m_{2}}\left(\lambda_{h}, \Omega\right)$ (see (B.5)-(B.6)), the result follows from a new application of (iii), and from (v). In particular, by (i) and (iv), this implies,

$$
\mathrm{Op}_{h}^{W}\left(a_{1}\right) \mathrm{Op}_{h}^{W}\left(a_{2}\right) \mathrm{Op}_{h}^{W}\left(a_{3}\right)=\mathrm{Op}_{h}^{W}\left(a_{1} a_{2} a_{3}\right)+h B,
$$

with,

$$
\left\|\left\langle h D_{x}\right\rangle^{-\frac{m_{1}+m_{2}+m_{3}}{2}} B\left\langle h D_{x}\right\rangle^{-\frac{m_{1}+m_{2}+m_{3}}{2}} u\right\|=\mathcal{O}\left(\|u\|+\lambda_{h}\|u\|_{L^{2}\left(\Omega_{\delta}\right)}\right) .
$$

Proof. (i) The assumptions imply that $\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a=\mathcal{O}\left(h^{-|\alpha|}\right)$. Then, this is just an easy consequence of the Calderón-Vaillancourt Theorem: see, e.g., Ma6, exercise 2.10.15.
(ii) It is well known that $\mathrm{Op}_{h}^{W}(a)=\mathrm{Op}_{h}^{t}\left(a_{t}\right)$, with (see, e.g., Ma6], Theorem 2.7.1),

$$
a_{t}(x, \xi):=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\xi^{\prime}-\xi\right) \theta / h} a\left(x+\left(t-\frac{1}{2}\right) \theta, \xi^{\prime}\right) d \xi^{\prime} d \theta
$$

and thus,
$a_{t}(x, \xi)-a(x, \xi):=\frac{1}{(2 \pi h)^{n}} \int e^{i\left(\xi^{\prime}-\xi\right) \theta / h}\left(a\left(x+\left(t-\frac{1}{2}\right) \theta, \xi^{\prime}\right)-a\left(x, \xi^{\prime}\right)\right) d \xi^{\prime} d \theta$. Then, first making $N$ integrations by parts by means of the operator $L:=$ $\frac{1}{1+\left(\xi^{\prime}-\xi\right)^{2}+\theta^{2}}\left(1+\left(\xi^{\prime}-\xi\right) \cdot h D_{\theta}+\theta \cdot h D_{\xi^{\prime}}\right)$, and next performing the change of contour of integration given by,

$$
\mathbb{R}^{n} \ni \xi^{\prime} \mapsto \xi^{\prime}+i \delta \frac{\theta}{|\theta|},
$$

(with $\delta>0$ small enough), we obtain as in Appendix A

$$
\begin{array}{r}
\partial_{x}^{\alpha} \partial_{\xi}^{\beta}\left(a_{t}(x, \xi)-a(x, \xi)\right)=\int \mathcal{O}\left(h^{-n} e^{-\delta|\theta| / h} \frac{\left(\Lambda_{h}\left(|\theta| \leq \delta ; x \in \Omega_{\delta}\right) h^{-|\alpha|}\right.}{\left(1+\left(\xi^{\prime}-\xi\right)^{2}+\theta^{2}\right)^{N}}\right) \\
\times(|\theta|+h)\left\langle\xi^{\prime}\right\rangle^{m} d \xi^{\prime} d \theta
\end{array}
$$

with the notation $\Lambda_{h}(\mathcal{E}):=1+\lambda_{h} \mathbf{1}_{\mathcal{E}}$. Thus, if $N$ has been taken sufficiently large, this implies,

$$
\partial_{x}^{\alpha} \partial_{\xi}^{\beta}\left(a_{t}(x, \xi)-a(x, \xi)\right)=\mathcal{O}\left(\Lambda_{h}\left(x \in \Omega_{\delta}\right) h^{1-|\alpha|}\langle\xi\rangle^{m}\right)
$$

Then, as in Appendix A the result follows by taking a partition of unity adapted to the pair ( $\Omega_{\delta}, \mathbb{R}^{n} \backslash \Omega_{\delta / 2}$ ).
(iii) For any $X=(x, \xi) \in \mathbb{R}^{2 n}$, one has,

$$
\begin{array}{r}
a \#^{W} b(X)=\frac{1}{(2 \pi h)^{4 n}} \int_{\mathbb{R}^{8 n}} e^{i\left[(X-Y) Y^{*}+(X-Z) Z^{*}+\sigma\left(Y^{*}, Z^{*}\right)\right] / h} a(Y) b(Z) \\
\times d\left(Y, Z, Y^{*}, Z^{*}\right),
\end{array}
$$

where $\sigma$ stands for the usual canonical simplectic form on $\mathbb{R}^{2 n}$. Therefore, integrating first with respect to $\left(y^{*}, z^{*}\right)$ (where we have set $Y^{*}=\left(y^{*}, \eta^{*}\right)$ and $\left.Z^{*}=\left(z^{*}, \zeta^{*}\right)\right)$, we obtain,

$$
\begin{array}{r}
a \#^{W} b(X)=\frac{1}{(2 \pi h)^{2 n}} \int_{\mathbb{R}^{4 n}} e^{i\left[(\xi-\eta) \eta^{*}+(\xi-\zeta) \zeta^{*}\right] / h} a\left(x-\zeta^{*}, \eta\right) b\left(x+\eta^{*}, \zeta\right) \\
\times d\left(\eta, \eta^{*}, \zeta, \zeta^{*}\right) .
\end{array}
$$

Here, we observe that, by Leibniz formula, for $x \in \Omega$, we have,

$$
\begin{equation*}
\nabla_{x}(a b)=a \nabla_{x} b+b \nabla_{x} a=\mathcal{O}\left(\lambda_{h}\langle\xi\rangle^{m+m^{\prime}}\right), \tag{B.5}
\end{equation*}
$$

and, for $|\alpha| \geq 2$,

$$
\begin{align*}
\partial_{x}^{\alpha}(a b) & =a \partial_{x}^{\alpha} b+b \partial_{x}^{\alpha} a+\sum_{\substack{\beta \leq \alpha \\
1<|\beta|<|\alpha|}} \mathcal{O}\left(\left|\partial^{\beta} a\right| \cdot\left|\partial^{\alpha-\beta} b\right|\right) \\
& =\mathcal{O}\left(\lambda_{h} h^{1-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}+\lambda_{h}^{2} h^{2-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right) \\
& =\mathcal{O}\left(\lambda_{h} h^{1-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right), \tag{B.6}
\end{align*}
$$

where the last estimate comes from the fact that $h \lambda_{h}=\mathcal{O}(1)$.
In particular $a b \in S_{m+m^{\prime}}\left(\lambda_{h}, \Omega\right)$ and, making integrations by parts as before, and performing the change of contour,

$$
\mathbb{R}^{2 n} \ni(\eta, \zeta) \mapsto(\eta, \zeta)-i \delta^{\prime}\left(\frac{\eta^{*}}{\left|\eta^{*}\right|}, \frac{\eta^{*}}{\left|\eta^{*}\right|}\right),
$$

with $\delta^{\prime}>0$ small enough, we see that, for any $\delta>0, a \#^{W} b$ belongs to $S_{m+m^{\prime}}\left(\lambda_{h}, \Omega_{\delta}\right)$. Moreover, by an argument similar to that of the proof of
(ii), this also gives,

$$
\begin{array}{r}
a \#^{W} b(X)=\frac{1}{(2 \pi h)^{2 n}} \int_{\mathbb{R}^{4 n}} e^{i\left[(\xi-\eta) \eta^{*}+(\xi-\zeta) \zeta^{*}\right] / h} a(x, \eta) b(x, \zeta) d\left(\eta, \eta^{*}, \zeta, \zeta^{*}\right) \\
+h c(X)
\end{array}
$$

with,

$$
\partial_{x}^{\alpha} \partial_{\xi}^{\beta} c(x, \xi)=\mathcal{O}\left(\Lambda_{h}\left(x \in \Omega_{\delta}\right) h^{-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right)
$$

Thus, $c \in \widetilde{\mathcal{S}}_{m+m^{\prime}}\left(\lambda_{h}, \Omega_{\delta}\right)$, and one finds,

$$
a \#^{W} b(X)=a(X) b(X)+h c(X)
$$

(iv) For any fix $\delta>0$, by a partition of unity we can write $a=a_{1}+a_{2}$ with $a_{1} \in \widetilde{\mathcal{S}}_{0}\left(1, \mathbb{R}^{n}\right)$, and $a_{2} \in \widetilde{\mathcal{S}}_{0}\left(\lambda_{h}, \Omega\right)$ supported in $\left\{x \in \Omega_{\delta / 4}\right\}$. Then, the Calderón-Vaillancourt theorem tells us that $\mathrm{Op}_{h}^{W}\left(a_{1}\right)$ is uniformly bounded on $L^{2}\left(\mathbb{R}^{n}\right)$, and it just remains to study $\mathrm{Op}_{h}^{W}\left(a_{2}\right)$. Let $\chi \in C^{\infty}\left(\mathbb{R}^{n} ;[0,1]\right)$ be such that $\operatorname{Supp} \chi \subset \Omega_{\delta}$ and $\chi=1$ near $\Omega_{\delta / 2}$. Then, we write,

$$
\mathrm{Op}_{h}^{W}\left(a_{2}\right) u=\mathrm{Op}_{h}^{W}\left(a_{2}\right)(\chi u)+\operatorname{Op}_{h}^{W}\left(a_{2}\right)((1-\chi) u)
$$

and, still by the Calderón-Vaillancourt theorem, we have,

$$
\left\|\mathrm{Op}_{h}^{W}\left(a_{2}\right)(\chi u)\right\|=\mathcal{O}\left(\lambda_{h}\|\chi u\|\right)=\mathcal{O}\left(\lambda_{h}\|u\|_{L^{2}\left(\Omega_{\delta}\right)}\right)
$$

On the other hand, since the function $a_{2}((x+y) / 2, \xi)(1-\chi(y))$ vanishes identically in $\{|x-y| \leq \delta / 4\}$, in the expression of $\mathrm{Op}_{h}^{W}\left(a_{2}\right)((1-\chi) u)(x)$ we can make integrations by means of the operator,

$$
L:=|x-y|^{-2}(x-y) \cdot h D_{\xi}
$$

Then, for any $N \geq 1$, we obtain,

$$
\mathrm{Op}_{h}^{W}\left(a_{2}\right)((1-\chi) u)=\operatorname{Op}_{h}^{W}\left(b_{N}\right) u
$$

where $b_{N}:=(-1)^{N} L^{N}\left(a_{2}((x+y) / 2, \xi)(1-\chi(y))\right)$ satisfies,

$$
\partial_{x}^{\alpha} \partial_{y}^{\beta} \partial_{\xi}^{\gamma} b_{N}=\mathcal{O}\left(h^{N} \lambda_{h} h^{-|\alpha|-|\beta|}\right)
$$

Therefore, $\left\|\mathrm{Op}_{h}^{W}\left(b_{N}\right)\right\|=\mathcal{O}\left(\lambda_{h} h^{N}\right)=\mathcal{O}\left(h^{N-1}\right)$, and the result follows by taking $N \geq 2$.
(v) For $a \in \widetilde{\mathcal{S}}_{m}\left(\lambda_{h}, \Omega\right)$ and $b \in \mathcal{S}_{m^{\prime}}\left(\lambda_{h}, \Omega\right)$, on $\Omega$ we have,
$\nabla_{x}(a b)=a \nabla_{x} b+b \nabla_{x} a=\mathcal{O}\left(\left(\lambda_{h}^{2}+\lambda_{h} h^{-1}\right)\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right)=\mathcal{O}\left(\lambda_{h} h^{-1}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right)$,
and, for $|\alpha| \geq 2$,

$$
\begin{aligned}
\partial_{x}^{\alpha}(a b) & =a \partial_{x}^{\alpha} b+b \partial_{x}^{\alpha} a+\sum_{\substack{\beta \leq \alpha \\
1<|\beta|<|\alpha|}} \mathcal{O}\left(\left|\partial^{\beta} a\right| \cdot\left|\partial^{\alpha-\beta} b\right|\right) \\
& =\mathcal{O}\left(\lambda_{h}^{2} h^{1-|\alpha|}+\lambda_{h} h^{-|\alpha|}+\lambda_{h}^{2} h^{1-|\alpha|}\right)\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}} \\
& =\mathcal{O}\left(\lambda_{h} h^{-|\alpha|}\langle\operatorname{Re} \xi\rangle^{m+m^{\prime}}\right) .
\end{aligned}
$$

Therefore, $a b$ is in $\widetilde{\mathcal{S}}_{m+m^{\prime}}\left(\lambda_{h}, \Omega\right)$, and the rest of the proof proceeds exactly as in (iii).
(vi) We introduce the modified anti-Wick quantization, given by,

$$
\widetilde{\mathrm{Op}}_{h}^{A W}(a):=\mathrm{Op}_{h}^{W}(\widetilde{a})
$$

where we have set,

$$
\widetilde{a}(x, \xi):=\frac{1}{(\pi h)^{n}} \int a(y, \eta) e^{-\lambda_{h}^{1 / 2}(x-y)^{2} / h^{3 / 2}-\lambda_{h}^{-1 / 2}(\xi-\eta)^{2} / h^{1 / 2}} d y d \eta .
$$

Then, by a second-order Taylor expansion of $a(y, \eta)$ at $(x, \xi)$, and using the fact that $\nabla_{x}^{2} a=\mathcal{O}\left(\lambda_{h} h^{-1}\right)$ and $\nabla_{\xi}^{2} a=\mathcal{O}(1)$, we see that,

$$
\widetilde{a}-a=\mathcal{O}\left(\sqrt{h \lambda_{h}}\right),
$$

uniformly. In the same way, using that $\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \widetilde{a}=\widetilde{\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a}$, we obtain,

$$
\partial_{x}^{\alpha} \partial_{\xi}^{\beta}(\widetilde{a}-a)=\mathcal{O}\left(\sqrt{h \lambda_{h}} h^{-|\alpha|}\right) .
$$

As a consequence, by the Calderón-Vaillancourt theorem,

$$
\begin{equation*}
\left\|\widetilde{\mathrm{Op}}_{h}^{A W}(a)-\mathrm{Op}_{h}^{W}(a)\right\|=\mathcal{O}\left(\sqrt{h \lambda_{h}}\right) \tag{B.7}
\end{equation*}
$$

On the other hand, for any $u \in L^{2}\left(\mathbb{R}^{n}\right)$, a straightforward computation gives,

$$
\left\langle\widetilde{\mathrm{Op}}_{h}^{A W}(a) u, u\right\rangle=\pi^{-n / 2} h^{-7 n / 4} \lambda_{h}^{n / 2} \int a(y, \eta)\left|H_{u}(y, \eta)\right|^{2} e^{-\lambda_{h}^{1 / 2} y^{2} / h^{3 / 2} / h} d y d \eta
$$

where $H_{u}$ is the (modified) Husimi function, defined by,

$$
H_{u}(y, \eta):=\int e^{i\left[x \eta / h-\lambda_{h}^{1 / 2} x^{2} /\left(2 h^{3 / 2}\right)-\lambda_{h}^{1 / 2} x y / h^{3 / 2}\right]} u(x) d x .
$$

In particular, $\left\langle\widetilde{\mathrm{Op}}_{h}^{A W}(a) u, u\right\rangle \geq 0$, and the result follows by using (B.7).

## Appendix C. An estimate on $P_{2}$

Here we prove,

Proposition C.1. Let $E_{1}<E_{2}$ be the first two eigenvalue of $P_{2}=-h^{2} \Delta+$ $V_{2}$, with $V_{2}$ satisfying (1.1), and let $k=h \ln \left(h^{-1}\right)$. Then, for any $z \in \mathbb{C}$ such that $\left|z-E_{1}\right|=\delta h$ with $0<\delta<\inf _{0<h \ll 1}\left(E_{2}-E_{1}\right) / h$, one has,

$$
\left\|\left(-h^{2} \Delta+V_{2}\left(x^{\prime}, x_{n}-i k\right)-z\right)^{-1}\right\|=\mathcal{O}\left(h^{-1}\right)
$$

Proof. By a Taylor expansion, we have,

$$
V\left(x^{\prime}, x_{n}-i k\right)-z=V_{2}(x)-i k \partial_{x_{n}} V_{2}(x)+k^{2} W
$$

with $W=\mathcal{O}(1)$, and thus, (C.1)
$-h^{2} \Delta+V_{2}\left(x^{\prime}, x_{n}-i k\right)-z=\left(I+\left(-i k \partial_{x_{n}} V_{2}(x)+k^{2} W\right)\left(P_{2}-z\right)^{-1}\right)\left(P_{2}-z\right)$.
Moreover, the assumptions on $V_{2}$ imply that $\left(\partial_{x_{n}} V_{2}\right)^{2}=\mathcal{O}\left(V_{2}\right)$. Therefore, for any $u \in L^{2}$, we have,

$$
\begin{aligned}
\left\|\left(\partial_{x_{n}} V_{2}\right)\left(P_{2}-z\right)^{-1} u\right\|^{2} & =\left\langle\left(\partial_{x_{n}} V_{2}\right)^{2}\left(P_{2}-z\right)^{-1} u,\left(P_{2}-z\right)^{-1} u\right\rangle \\
& \leq C\left\langle V_{2}\left(P_{2}-z\right)^{-1} u,\left(P_{2}-z\right)^{-1} u\right\rangle \\
& \leq C\left\langle P_{2}\left(P_{2}-z\right)^{-1} u,\left(P_{2}-z\right)^{-1} u\right\rangle \\
& \leq C\left\langle\left(1+z\left(P_{2}-z\right)^{-1}\right) u,\left(P_{2}-z\right)^{-1} u\right\rangle \\
& \leq C\|u\| \cdot\left\|\left(P_{2}-z\right)^{-1} u\right\|+C|z| \cdot\left\|\left(P_{2}-z\right)^{-1} u\right\|^{2}
\end{aligned}
$$

Since $|z|=\mathcal{O}(h)$ and we also know that $\left\|\left(P_{2}-z\right)^{-1}\right\|=\mathcal{O}\left(h^{-1}\right)$, we deduce,

$$
\left\|\left(\partial_{x_{n}} V_{2}\right)\left(P_{2}-z\right)^{-1} u\right\|^{2}=\mathcal{O}\left(h^{-1}\right)\|u\|^{2}
$$

and thus,

$$
\left\|\left(\partial_{x_{n}} V_{2}\right)\left(P_{2}-z\right)^{-1}\right\|=\mathcal{O}\left(h^{-1 / 2}\right)
$$

As a consequence,

$$
\left\|\left(-i k \partial_{x_{n}} V_{2}(x)+k^{2} W\right)\left(P_{2}-z\right)^{-1}\right\|=\mathcal{O}\left(k h^{-1 / 2}+k^{2} h^{-1}\right)=\mathcal{O}\left(\sqrt{h} \ln \frac{1}{h}\right)
$$

and the result follows by taking the inverse in (C.1).

## Appendix D. A geometric estimate

Here we prove the following result, that we use in Section 5:

Lemma D.1. The function $\widetilde{\varphi}_{1}$ defined in Section 4 is such that, for any $s \in[0,1]$ and for any $y \in \Gamma_{+}$,

$$
\widetilde{p}_{2}\left(y, i s \nabla \widetilde{\varphi}_{1}(y)\right)>0
$$

Proof. We first observe that

$$
\frac{\partial}{\partial s} \widetilde{p}_{2}\left(y, i s \nabla \widetilde{\varphi}_{1}(y)\right)=\partial_{\eta} \widetilde{p}_{2}\left(y, i s \nabla \widetilde{\varphi}_{1}(y)\right) \cdot i \nabla \widetilde{\varphi}_{1}(y) \leq-\frac{1}{C} s\left(\nabla \widetilde{\varphi}_{1}(y)\right)^{2} \leq 0
$$

and therefore, it is enough to prove the result for $s=1$. Moreover, since we already know that $\widetilde{p}_{2}\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right) \neq 0$ outside $\Pi_{y} \Gamma$, it is enough to prove it for $y \in \Gamma_{+}$close enough to $\Pi_{y} \Gamma$, that is, for $y \in \Gamma_{+}$such that

$$
\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right)=\exp t H_{\widetilde{p}_{1}}\left(\widetilde{y}, i \nabla \widetilde{\varphi}_{2}(\widetilde{y})\right)
$$

for some $t \in \mathbb{C} \backslash\{0\}$ small enough and $\widetilde{y} \in \Pi_{y} \Gamma \cap \mathbb{R}^{n}$. In particular, we have

$$
\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right)=\left(\widetilde{y}, i \nabla \widetilde{\varphi}_{2}(\widetilde{y})\right)-t\left(2 i \nabla \widetilde{\varphi}_{2}(\widetilde{y}),(0, \ldots, 1)\right)+\mathcal{O}\left(|t|^{2}\right)
$$

and thus, for $y$ to be real it is necessary that $t=i t^{\prime}$ for some real $t^{\prime}$. Moreover, in order that $y$ is in $\Gamma_{+}$, we need that $y_{n}>\widetilde{y}_{n}$ (at least up to $\left.\mathcal{O}\left(|t|^{2}\right)\right)$, that is,

$$
2 t^{\prime} \partial_{n} \widetilde{\varphi}_{2}(\widetilde{y})>0
$$

and thus: $t^{\prime}>0\left(\right.$ since $\left.\partial_{n} \widetilde{\varphi}_{2}(\widetilde{y})>0\right)$.
Now, since $\widetilde{p}_{2}\left(\widetilde{y}, i \nabla \widetilde{\varphi}_{2}(\widetilde{y})\right)=0$, we obtain by a Taylor expansion,

$$
\begin{aligned}
\widetilde{p}_{2}\left(y, i \nabla \widetilde{\varphi}_{1}(y)\right) & =2 t^{\prime}\left(\partial_{y} \widetilde{p}_{2}\right)\left(\widetilde{y}, i \nabla \widetilde{\varphi}_{2}(\widetilde{y})\right) \cdot \nabla \widetilde{\varphi}_{2}(\widetilde{y})-i t^{\prime}\left(\partial_{\eta_{n}} \widetilde{p}_{2}\right)\left(\widetilde{y}, i \nabla \widetilde{\varphi}_{2}(\widetilde{y})\right)+\mathcal{O}\left(|t|^{2}\right) \\
& \geq \frac{t^{\prime}}{C}\left(\widetilde{y} \cdot \nabla \widetilde{\varphi}_{2}(\widetilde{y})+\partial_{n} \widetilde{\varphi}_{2}(\widetilde{y})\right)>0
\end{aligned}
$$

This finishes the proof.

## Appendix E. An estimate on $\widetilde{\phi}_{0}$

We have,
Lemma E.1. Let $\widetilde{\phi}_{0}$ be the function defined in (6.14). Then, there exists $\varepsilon^{\prime}>0$ such that, for all $y \in \mathbb{R}^{n}$ with $y_{n} \leq \mu-\varepsilon$, one has,

$$
\widetilde{\phi}_{0}\left(y_{n}\right) \leq \operatorname{Min}\left\{\tilde{\phi}(y), S(\mu)-\varepsilon^{\prime}\right\} .
$$

Proof. By definition, for $\mu-2 \varepsilon \leq y_{n}<g\left(y^{\prime}\right)$ (where $g\left(y^{\prime}\right)$ is the function defining the caustic set of $\widetilde{\phi}$ ), we have $\left(\partial_{y_{n}} \widetilde{\phi}_{0}\left(y_{n}\right)\right)^{2}=|\nabla \widetilde{\phi}(y)|^{2}=\mu-y_{n}$, and $\partial_{y_{n}} \widetilde{\phi}_{0} \geq 0$. Therefore, $\left|\partial_{y_{n}} \widetilde{\phi}(y)\right| \leq \partial_{y_{n}} \widetilde{\phi}_{0}\left(y_{n}\right)$, and we also have $\widetilde{\phi}\left(y^{\prime}, g\left(y^{\prime}\right) \geq\right.$ $S(\mu) \geq \widetilde{\phi}_{0}\left(g\left(y^{\prime}\right)\right)$. Therefore, integrating from $y_{n}=g\left(y^{\prime}\right)-t$ to $y_{n}=g\left(y^{\prime}\right)$ (with $0 \leq t \leq g\left(y^{\prime}\right)-\mu+2 \varepsilon$ ), we deduce,

$$
\widetilde{\phi}_{0}\left(y_{n}\right) \leq \widetilde{\phi}(y) \text { if } \mu-2 \varepsilon \leq y_{n}<g\left(y^{\prime}\right) .
$$

In particular, $\widetilde{\phi}_{0}\left(y_{n}\right) \leq \widetilde{\phi}(y)$ on $\{\widetilde{\phi}(y)<S(\mu)\} \cap\left\{y_{n} \geq \mu-2 \varepsilon\right\}$. Since we also have $\sup _{y_{n} \leq \mu-\varepsilon} \widetilde{\phi}_{0}\left(y_{n}\right)<S(\mu)$, and $\widetilde{\phi}_{0}\left(y_{n}\right)=0$ for $y_{n} \leq \mu-2 \varepsilon$, the result follows.

## Appendix F. Proof of Lemma 6.9

If $c=c(x)$ satisfies (3.1), we have,

$$
\mathcal{U C O}^{-1}=\mathrm{Op}_{h}^{L}(\widetilde{c}),
$$

with,

$$
\begin{aligned}
\widetilde{c}(y, \eta)= & \frac{1}{(2 \pi h)^{n}} \int e^{i(y-x)(\xi-\eta) / h} \times \\
& \times c\left(x^{\prime}-2 \xi_{n}\left(\xi^{\prime}+\eta^{\prime}\right), x_{n}-\frac{2}{3}\left(\eta_{n}^{2}+\eta_{n} \xi_{n}+\xi_{n}^{2}\right)-2\left|\eta^{\prime}\right|^{2}\right) d x d \xi .
\end{aligned}
$$

Here, the oscillatory integral is well defined thanks to (3.1), and it defines a smooth function on $\mathbb{R}^{2 n}$, holomorphic with respect to $y$ in a complex strip around $\mathbb{R}^{n}$. Moreover, observing that,

$$
\left.|\eta| \cdot\left|\nabla\left\langle y_{n}\right\rangle^{1 / 2}\right| \leq|\eta|\left\langle y_{n}\right\rangle^{-1 / 2}=\mathcal{O}\left(\left.\left\langle y_{n}-2\right| \eta\right|^{2}\right\rangle^{1 / 2}\right),
$$

we see that, by (3.1), we can make a deformation of contour of integration in order to obtain,

$$
e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U C}^{-1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h}=\mathrm{Op}_{h}^{L}(\check{c}),
$$

where $\check{c}$ has properties similar to those of $\widetilde{c}$. Then, by standard properties of $T_{1}$ (see, e.g., Ma6), we have,

$$
T_{1} e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U}^{\prime} \mathcal{U}^{-1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h}=\operatorname{Op}_{h}^{L}\left(\check{c}\left(y-\eta^{*}, y^{*}\right)\right) T_{1},
$$

where $\left(y^{*}, \eta^{*}\right)$ stands for the dual variable of $(y, \eta)$. Next, using the fact that $\check{c}\left(y-\eta^{*}, y^{*}\right)$ is holomorphic with respect to $y$, for any $\varepsilon>0$, we find,

$$
\begin{aligned}
& e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U C} \mathcal{U}^{-1} e^{\left.4 k\left\langle y_{n}\right\rangle\right\rangle^{1 / 2} / h} \\
& \quad=\operatorname{Op}_{h}^{L}\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1},
\end{aligned}
$$

where $c_{\varepsilon}\left(y, \eta, \eta^{*}, y^{*}\right)$ can be computed explicitly, and has essentially the same properties as $\check{c}\left(y-\eta^{*}, y^{*}\right)$, with estimates uniform with respect to $\varepsilon>0$ sufficiently small.

Then, taking a cut-off function $\chi_{1}=\chi_{1}\left(\left|\eta^{*}\right|\right)$ that is equal to 1 near 0 , and using the fact that $T_{1}^{*} T_{1}=1$, we can write,
(F.1) $\mathrm{Op}_{h}^{L}\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1}$

$$
\begin{aligned}
& =\operatorname{Op}_{h}^{L}\left(\chi_{1}\left(\left|\eta^{*}\right|\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1}\right. \\
& \quad+\operatorname{Op}_{h}^{L}\left(\left(1-\chi_{1}\left(\left|\eta^{*}\right|\right)\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) K_{\varepsilon} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1},\right.
\end{aligned}
$$

with $K_{\varepsilon}:=e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} T_{1}^{*} e^{\varepsilon k^{2 / 3}\langle\eta\rangle / h}$.
A simple computation (see, e.g., Ma6, Section 3) shows that the distributional kernel of $K_{\varepsilon}$ is given by,

$$
K_{\varepsilon}(y, \eta ; z, \zeta):=(\pi h)^{\frac{n}{2}} e^{i(y-z)(\eta+\zeta) / 2 h-(y-z)^{2} / 4 h-(\eta-\zeta)^{2} / 4 h+\varepsilon k^{2 / 3}(\langle\zeta\rangle-\langle\eta\rangle) / h}
$$

Therefore, the distributional kernel of $\mathrm{Op}_{h}^{L}\left(\left(1-\chi_{1}\left(\left|\eta^{*}\right|\right)\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) K_{\varepsilon}\right.$ is,
$I_{1}(y, \eta ; z, \zeta):=\frac{1}{2^{n}(\pi h)^{\frac{3 n}{2}}} \int e^{\Phi_{1} / h}\left(1-\chi_{1}\left(\left|\eta^{*}\right|\right)\right) c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right) d y^{\prime} d \eta^{\prime} d y^{*} d \eta^{*}$, with,

$$
\begin{aligned}
\Phi_{1}:=i\left(y-y^{\prime}\right) y^{*}+i\left(\eta-\eta^{\prime}\right) \eta^{*} & +\frac{i}{2}\left(y^{\prime}-z\right)\left(\eta^{\prime}+\zeta\right)-\frac{1}{4}\left(y^{\prime}-z\right)^{2} \\
& -\frac{1}{4}\left(\eta^{\prime}-\zeta\right)^{2}+\varepsilon k^{2 / 3}(\langle\zeta\rangle-\langle\eta\rangle),
\end{aligned}
$$

while the distributional kernel of $\mathrm{Op}_{h}^{L}\left(\chi_{1}\left(\left|\eta^{*}\right|\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right)\right.$ is,

$$
I_{2}(y, \eta ; z, \zeta):=\frac{1}{(2 \pi h)^{n}} \int e^{i(y-z) y^{*} / h+i(\eta-\zeta) \eta^{*} / h} \chi_{1}\left(\left|\eta^{*}\right|\right) c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right) d y^{*} d \eta^{*}
$$

In $I_{1}$, we can perform the change of contour of integration,

$$
\eta^{\prime} \mapsto \eta^{\prime}-i \delta \eta^{*} /\left\langle\eta^{*}\right\rangle
$$

with $\delta>0$ small enough and independent of $\mu$. Since $k^{2 / 3}(\langle\zeta\rangle-\langle\eta\rangle) \leq$ $k^{4 / 3}+(\eta-\zeta)^{2}$ and $k^{4 / 3} / h \ll 1$, we see that the resulting phase factor can be estimated by,

$$
e^{-\delta\left|\eta^{*}\right|^{2} / 2 h\left\langle\eta^{*}\right\rangle-(y-z)^{2} / 8 h-(\eta-\zeta)^{2} / 8 h}
$$

As a consequence, since $\eta^{*}$ stays away from 0 on the support of $1-\chi_{1}\left(\left|\eta^{*}\right|\right)$, and since, for any $N \geq 1$ one has $\left.\left.\left|y^{*}\right|^{2 N}\left\langle\eta_{n}^{*}+\right| y^{*}\right|^{2}\right\rangle^{-N} e^{-\delta\left\langle\eta^{*}\right\rangle / 4 h}=\mathcal{O}(1)$, by the Calderón-Vaillancourt theorem we conclude that, for any $\chi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$, we have,

$$
\begin{gathered}
\| \chi(y) \operatorname{Op}_{h}^{L}\left(\left(1-\chi_{1}\left(\left|\eta^{*}\right|\right)\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) K_{\varepsilon} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}} \|\right. \\
= \\
=\mathcal{O}\left(e^{-\delta_{1} / h}\right)\left\|e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|
\end{gathered}
$$

where we have set,

$$
\check{\mathbf{u}}(y):=e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \widetilde{\mathbf{u}}(y),
$$

and where $\delta_{1}>0$ is independent of $\mu$. In particular, if $\psi \in C_{b}^{\infty}\left(\mathbb{R}^{2 n} ; \mathbb{R}_{+}\right)$is such that $\sup |\psi| \leq \delta_{1} / 2$, we obtain,

$$
\begin{gather*}
\| e^{\psi / h} \chi(y) \operatorname{Op}_{h}^{L}\left(\left(1-\chi_{1}\left(\left|\eta^{*}\right|\right)\right)\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) K_{\varepsilon} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}} \|\right. \\
=\mathcal{O}\left(e^{-\delta_{1} / 2 h}\right)\left\|e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|, \tag{F.2}
\end{gather*}
$$

On the other hand, for $y$ in some arbitrarily large fixed compact set $K$, in $I_{2}$ we can successively perform the two changes of contour,

$$
\begin{align*}
& \mathbb{R}^{n} \ni \eta^{*} \mapsto \eta^{*}+i \delta \chi_{2}\left(\left|\eta^{*}\right|\right)(\eta-\zeta) /|\eta-\zeta| \\
& S^{n-1} \ni \omega^{*}:=\eta^{*} /\left|\eta^{*}\right| \mapsto \omega^{*}+i \delta(\eta-\zeta) /|\eta-\zeta|, \tag{F.3}
\end{align*}
$$

where $\chi_{2} \in C_{0}^{\infty}$ is supported inside $\left\{\chi_{1}=1\right\}$, $\chi_{2}=1$ near 0 , and $\delta>0$ depends on $K$ only. This makes appear the exponential factor,

$$
e^{-\delta\left(\left|\eta^{*}\right|+\chi_{2}\left(\left|\eta^{*}\right|\right)\right)|\eta-\zeta| / h} .
$$

Moreover, since $\left|\eta^{*}\right|$ remains bounded on the support of $\chi_{1}\left(\left|\eta^{*}\right|\right)$, if $\psi \in$ $C_{b}^{\infty}\left(\mathbb{R}^{2 n} ; \mathbb{R}_{+}\right)$is such that sup $|\nabla \psi| \ll \delta$, writing: $\psi(y, \eta)=\psi(z, \zeta)+(y-$ $z) \phi_{1}+(\eta-\zeta) \phi_{2}$, then we see that we can also perform the change of contour,

$$
\mathbb{R}^{n} \ni y^{*} \mapsto y^{*}+i \phi_{1}(y, z, \eta, \zeta),
$$

and we easily conclude that,

$$
\begin{aligned}
&\left\|e^{\psi / h} \chi(y) \chi_{1}\left(\left|\eta^{*}\right|\right) \operatorname{Op}_{h}^{L}\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\| \\
&=\mathcal{O}\left(\left\|e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|\right) .
\end{aligned}
$$

Here, $\chi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ is an arbitrary cut-off function supported in $K$.
Summing up, we have proved that, for any $\chi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and for any $\psi \in$ $C_{b}^{\infty}\left(\mathbb{R}^{2 n} ; \mathbb{R}_{+}\right)$with $\sup |\psi|+\sup |\nabla \psi|$ small enough (independently of $\mu$ ), there exists a constant $C>0$ (independent of $\mu$ ) such that, for any $\varepsilon>0$, we have,

$$
\begin{equation*}
\left\|\chi(y) e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \mathcal{U} \mathcal{U}^{-1} \check{\mathbf{u}}\right\| \leq C\left\|e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\| . \tag{F.4}
\end{equation*}
$$

It is important to observe that, in all these computations, the estimates are uniform with respect to $\varepsilon>0$. Actually, the presence of $\varepsilon$ is required only to insure that the norms are finite.

Remark F.1. One can extract from the previous proof the following (maybe already well-know) result: If $a(x, \xi)$ is holomorphic with respect to $\xi$ is a
complex strip $\Gamma$ around $\mathbb{R}^{n}(n \geq 2)$, and is bounded together with all its derivatives in $\mathbb{R}^{n} \times \Gamma$, then, for any cut-off function $\chi \in C_{0}^{\infty}\left(\mathbb{R}_{+}\right)$such that $\chi=1$ near 0 , and for any $\varphi \in C_{b}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\sup (|\varphi|+|\nabla \varphi|)$ small enough, the operator $e^{\varphi / h} \mathrm{Op}_{h}^{W}(\chi(|\xi|) a(x, \xi)) e^{-\varphi / h}$ is uniformly bounded on $L^{2}\left(\mathbb{R}^{n}\right)$. (Indeed, this can be seen by first making integrations by parts with respect to $\xi$, and then by performing changes of contour similar to those of (F.3).)

In the sequel, we also need,
Lemma F.2. There exists a constant $C>0$ such that, for all $\varepsilon, h>0$ small enough, one has,

$$
\begin{aligned}
& \left\|e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U C \mathcal { U }}^{-1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \check{\mathbf{u}}\right\| \\
& \leq C\left\|\left(1+\frac{\langle\eta\rangle^{2 C}}{\left\langle y_{n}-2 \eta^{2}\right\rangle^{C}}\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|
\end{aligned}
$$

Proof. Using (3.1), one can check that the symbol $c_{\varepsilon}$ satisfies,

$$
\left.\partial_{y, \eta, \eta^{*}}^{\alpha} \partial_{y^{*}}^{\beta} c_{\varepsilon}=\mathcal{O}\left(1+\left.\left\langle y^{*}\right\rangle^{|\beta|}\left\langle y_{n}-\eta_{n}^{*}-2\right| y^{*}\right|^{2}\right\rangle^{-|\beta| / 2}\right),
$$

uniformly with respect to $\varepsilon$. Therefore, we learn from the Calderón-Vaillancourt Theorem the existence of some constant $C>0$ such that,

$$
\begin{aligned}
& \left\|\mathrm{Op}_{h}^{L}\left(c_{\varepsilon}\left(\eta, y-\eta^{*}, y^{*}\right)\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\| \\
& \leq C\left\|\left(1+\left\langle h D_{y}\right\rangle^{2 C}\left\langle y_{n}-h D_{\eta_{n}}-2\left(h D_{y}\right)^{2}\right\rangle^{-C}\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|
\end{aligned}
$$

Then, the result follows from standard properties of the transform $T_{1}$, that permit us to replace $h D_{y}$ by $\eta$, and $h D_{\eta}$ by 0 , in the previous estimate (see, e.g., Ma6]).

Now, we set,

$$
\check{P}:=e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \widetilde{P} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h}
$$

and we observe that, for any integer $N \geq 1$, the operator,

$$
(\check{P}-\rho)^{N}=e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U}(P-\rho)^{N} \mathcal{U}^{-1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h}
$$

is a sum of terms of the type: $a\left(y, h D_{y}\right) e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \mathcal{U} c \mathcal{U}^{-1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h}$, where $a\left(y, h D_{y}\right)$ is a $2 \times 2$ matrix of partial differential operator with polynomial coefficients, and $c=c(x)$ satisfies (3.1). Moreover, the leading term (that is, the one corresponding to the operator $a\left(y, h D_{y}\right)$ with higher degree in $\left.h D_{y}\right)$ is just $\left(-h^{2} \Delta\right)^{N}\left(\begin{array}{cc}(-1)^{N} & 0 \\ 0 & 1\end{array}\right)$. Therefore, combining (I) (F.4) with standard microlocal exponential weighted estimates (see [Ma6], Section 3.5), we see that the equation $(\check{P}-\rho)^{N} \check{\mathbf{u}}=0$ implies that, for any $\chi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$, for any
$\psi \in C_{b}^{\infty}\left(\mathbb{R}^{2 n} ; \mathbb{R}_{+}\right)$with $\sup |\psi|+\sup |\nabla \psi|$ small enough, and for any $\varepsilon>0$ small enough, one has,

$$
\begin{equation*}
\left\||\eta|^{2 N} \chi(y) e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\| \leq C\left\|\langle\eta\rangle^{2 N-1} e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\| \tag{F.5}
\end{equation*}
$$

where the constant $C$ does not depend on $\varepsilon$, and depends on $\psi$ through the quantities $\sup \left|\partial^{\alpha} \psi\right|$ only. In particular, we can take $\psi$ such that $\psi=0$ on $\{\chi(y) \neq 1\} \cup\left\{|\eta|^{2 N} \leq 2 C\langle\eta\rangle^{2 N-1}\right\}$, and $\psi \geq \delta_{0}$ on $K \times\left\{|\eta| \geq C_{0}\right\}$ with $\delta_{0}, C_{0}>0$ independent of $\varepsilon$ and $\mu$, and where $K$ is an arbitrary compact subset of the interior of $\{\chi=1\}$. In this case, we deduce from (F.5),

$$
\begin{equation*}
\left\||\eta|^{2 N} e^{\psi / h-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\operatorname{Supp} \psi} \leq 2 C\left\|\langle\eta\rangle^{2 N-1} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\mathbb{R}^{2 n} \backslash \operatorname{Supp} \psi} \tag{F.6}
\end{equation*}
$$

On the other hand, using Lemma F. 2 and the equation $(\check{P}-\rho)^{N} \check{\mathbf{u}}=0$, we also deduce,

$$
\begin{array}{r}
\left\|\left|y_{n}-\eta^{2}\right|^{N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}_{1}\right\|+\left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}_{2}\right\| \\
\leq C\left\|\left(\langle\eta\rangle^{2 N-1}+\left\langle y_{n}-\eta^{2}\right\rangle^{N-1}\right)\left(1+\frac{\langle\eta\rangle^{2 C}}{\left\langle y_{n}-2 \eta^{2}\right\rangle^{C}}\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|,
\end{array}
$$

with some other constant $C>0$, still independent of $\varepsilon$. Now, on $\left\{\eta^{2} \geq\right.$ $\left.2\left|y_{n}\right|\right\}$, we have $\eta^{2}-y_{n} \geq \frac{1}{2} \eta^{2}$, while $\langle\eta\rangle^{2 C}\left\langle y_{n}-2 \eta^{2}\right\rangle^{-C}$ is bounded. Therefore, we obtain,

$$
\begin{aligned}
& \left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq 2\left|y_{n}\right|} \\
& \begin{aligned}
\leq C \|\left(\langle\eta\rangle^{2 N-1}+\left\langle y_{n}-\eta^{2}\right\rangle^{N-1}\right) & \left(1+\frac{\langle\eta\rangle^{2 C}}{\left\langle y_{n}-2 \eta^{2}\right\rangle C}\right) e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}} \|_{\eta^{2} \leq 2\left|y_{n}\right|} \\
& +C\left\||\eta|^{2 N-1} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq 2\left|y_{n}\right|},
\end{aligned}
\end{aligned}
$$

and thus, if $C_{1}>0$ is sufficiently large (in order to have $|\eta|^{2 N} \geq\langle\eta\rangle^{2 N-1}$ when $|\eta|^{2} \geq C_{1}$ ),
(F.7) $\left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq \max \left(C_{1}, 2\left|y_{n}\right|\right)}$

$$
\leq C_{N}\left\|\left\langle y_{n}\right\rangle^{N+C} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \leq 2\left|y_{n}\right|}+C_{N}\left\|e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \leq C_{1}}
$$

with $C_{N}>0$ independent of $\varepsilon$. Setting $\theta_{1}=\theta_{1}\left(y_{n} ; h\right):=4 k\left\langle y_{n}\right\rangle^{1 / 2} / h$ and $\theta_{2}=\theta_{2}(\eta ; h):=k\langle\eta\rangle / h$, for $\eta^{2} \leq 2\left|y_{n}\right|$, we write,

$$
\begin{aligned}
\left\langle y_{n}\right\rangle^{N+C} T_{1} \check{\mathbf{u}} & =\left\langle y_{n}\right\rangle^{N+C} T_{1} e^{-\theta_{1}} \widetilde{\mathbf{u}} \\
& =\left\langle y_{n}\right\rangle^{N+C} e^{\theta_{2}-\theta_{1}}\left(e^{\theta_{1}-\theta_{2}} T_{1} e^{-\theta_{1}} T_{1}^{*} e^{\theta_{2}}\right) e^{-\theta_{2}} T_{1} \widetilde{\mathbf{u}}
\end{aligned}
$$

and since $\theta_{2}(\eta)-\theta_{1}\left(y_{n}\right) \leq-\frac{1}{2} \theta_{1}\left(y_{n}\right)$, and $\left\langle y_{n}\right\rangle^{N+C} e^{-\frac{1}{2} \theta_{1}\left(y_{n}\right)}=\mathcal{O}(1)$, we obtain,

$$
\begin{equation*}
\left\|\left\langle y_{n}\right\rangle^{N+C} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \leq 2\left|y_{n}\right|} \leq C_{N}^{\prime}\left\|\left(e^{\theta_{1}-\theta_{2}} T_{1} e^{-\theta_{1}} T_{1}^{*} e^{\theta_{2}}\right) e^{-\theta_{2}} T_{1} \widetilde{\mathbf{u}}\right\| \tag{F.8}
\end{equation*}
$$

with $C_{N}^{\prime}>0$ independent of $\varepsilon$. Then, we have,
Lemma F.3. The operator $J:=e^{\theta_{1}-\theta_{2}} T_{1} e^{-\theta_{1}} T_{1}^{*} e^{\theta_{2}}$ is bounded on $L^{2}\left(\mathbb{R}^{2 n}\right)$, uniformly with respect to $h>0$ small enough.

Proof. The distributional kernel of $J$ is given by,

$$
\begin{array}{r}
J(y, \eta ; z, \zeta)=\int_{\mathbb{R}^{n}} e^{i(y-x) \eta / h+i(x-z) \zeta / h-(y-x)^{2} / 2 h-(x-z)^{2} / 2 h} \\
\times e^{\theta_{2}(\zeta)-\theta_{2}(\eta)+\theta_{1}\left(y_{n}\right)-\theta_{1}\left(x_{n}\right)} d x .
\end{array}
$$

In this integral, we can perform the change of contour,

$$
\mathbb{R}^{n} \ni x \mapsto x+i k^{2 / 3} \frac{\zeta-\eta}{|\zeta-\eta|}
$$

and, since $\left|\nabla \theta_{1}\right| \leq \frac{1}{2} k^{2 / 3}$ and $\left|\nabla \theta_{2}\right| \leq \frac{1}{4} k^{2 / 3}$, we obtain,

$$
\begin{array}{r}
|J(y, \eta ; z, \zeta)| \leq \int_{\mathbb{R}^{n}} e^{-k^{2 / 3}|\zeta-\eta| / h-(y-x)^{2} / 2 h-(x-z)^{2} / 2 h+k^{4 / 3} / h} \\
\times e^{k^{2 / 3}|\zeta-\eta| / 4 h+k^{2 / 3}|y-x| / 2 h} d x
\end{array}
$$

Therefore, using the fact that $k^{2 / 3}|y-x| \leq 2 k^{4 / 3}+(y-x)^{2} / 2$ and $k^{4 / 3} / h \rightarrow 0$, for $h$ small enough, we find,

$$
|J(y, \eta ; z, \zeta)| \leq 2 \int_{\mathbb{R}^{n}} e^{-3 k^{2 / 3}|\zeta-\eta| / 4 h-(y-x)^{2} / 4 h-(x-z)^{2} / 2 h} d x
$$

and the result follows from the Schur Lemma (see, e.g., Ma6]).
We deduce from Lemma F. 3 and (F.7)-(F.8) that we have,

$$
\begin{aligned}
& \left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq \max \left(C_{1}, 2\left|y_{n}\right|\right)} \\
& \leq C_{N}\left\|e^{-k\langle\eta\rangle / h} T_{1} \widetilde{\mathbf{u}}\right\|+C_{N}\left\|e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \leq C_{1}}
\end{aligned}
$$

with $C_{1}, C_{N}$ independent of $\varepsilon$, and thus,

$$
\begin{aligned}
& \left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq \max \left(C_{1}, 2\left|y_{n}\right|\right)} \\
& \leq C_{N}\left\|e^{-k\langle\eta\rangle / h} T_{1} \widetilde{\mathbf{u}}\right\|+C_{N} h^{-C_{1}}\left\|e^{-k\langle\eta\rangle / h} T_{1} e^{-4 k\left\langle y_{n}\right\rangle^{2 / 3}} \widetilde{\mathbf{u}}\right\| .
\end{aligned}
$$

Now, one can see as in the proof of Lemma F. 3 that we have,

$$
\begin{equation*}
\left\|e^{-k\langle\eta\rangle / h} T_{1} e^{-4 k\left\langle y_{n}\right\rangle^{2 / 3}} \widetilde{\mathbf{u}}\right\| \leq C\left\|e^{-k\langle\eta\rangle / h} T_{1} \widetilde{\mathbf{u}}\right\|, \tag{F.9}
\end{equation*}
$$

and, in virtue of (6.32), we deduce,

$$
\left\||\eta|^{2 N} e^{-\varepsilon k^{2 / 3}\langle\eta\rangle / h} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq \max \left(C_{1}, 2\left|y_{n}\right|\right)}=\mathcal{O}\left(h^{-C_{1}}\right),
$$

uniformly with respect to $h$ and $\varepsilon$. Sending $\varepsilon$ to 0 in this estimate, we conclude,

$$
\begin{equation*}
\left\||\eta|^{2 N} T_{1} \check{\mathbf{u}}\right\|_{\eta^{2} \geq \max \left(C_{1}, 2\left|y_{n}\right|\right)}=\mathcal{O}\left(h^{-C_{1}}\right) . \tag{F.10}
\end{equation*}
$$

Gathering (F.8), (F.9), (F.10), and using Lemma F.3, we obtain,

$$
\begin{equation*}
\left\|\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right\|=\mathcal{O}\left(h^{-C}\right), \tag{F.11}
\end{equation*}
$$

and, inserting (F.11) into (F.6), and sending again $\varepsilon$ to 0 , this gives us,

$$
\begin{equation*}
\left\||\eta|^{2 N} e^{\psi / h} T_{1} \check{\mathbf{u}}\right\|_{\text {Supp } \psi}=\mathcal{O}\left(h^{-C}\right) \tag{F.12}
\end{equation*}
$$

Therefore, thanks to our choice of the function $\psi$, we finally obtain,

$$
\begin{equation*}
\left\|\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right\|_{K \times\left\{|\eta| \geq C_{0}\right\}}=\mathcal{O}\left(e^{-\delta_{0} / 2 h}\right) . \tag{F.13}
\end{equation*}
$$

where $K \subset \mathbb{R}^{n}$ is an arbitrary compact set, $C_{0}, N \geq 1$ are arbitraryily large and $\delta_{0}=\delta_{0}\left(K, C_{0}, N\right)$ is a positive constant that does not depend on $\mu$.

To complete the proof, we fix some arbitrary constant $C \geq 1$, we take $K^{\prime} \subset \subset K \cap\left\{\left\langle y_{n}\right\rangle \leq C^{2}\right\}$, and we write,

$$
\begin{aligned}
& \left\|\langle\eta\rangle^{2 N} T_{1} \widetilde{\mathbf{u}}\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}} \\
& \leq h^{-4 C}\left\|\langle\eta\rangle^{2 N} e^{-4 k\left\langle y_{n}\right)^{1 / 2} / h} T_{1} \widetilde{\mathbf{u}}\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}} \\
& \leq h^{-4 C}\left\|\langle\eta\rangle^{2 N} e^{-4 k\left\langle y_{n}\right)^{1 / 2} / h} T_{1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} \check{\mathbf{u}}\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}} \\
& \leq h^{-4 C}\left\|J_{1}\left(\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right)\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}},
\end{aligned}
$$

with,

$$
J_{1}:=\langle\eta\rangle^{2 N} e^{-4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} T_{1} e^{4 k\left\langle y_{n}\right\rangle^{1 / 2} / h} T_{1}^{*}\langle\eta\rangle^{-2 N} .
$$

Then, in a similar way as in Lemma.3, we see that the distributional kernel of $J_{1}$ satisfies,

$$
\left|J_{1}(y, \eta ; z, \zeta)\right| \leq 2 \int e^{-\delta|\zeta-\eta| / h-(y-x)^{2} / 4 h-(x-z)^{2} / 2 h+\delta^{2} / h} d x
$$

where $\delta>0$ can be taken arbitrarily small. We deduce,

$$
\begin{array}{r}
\left\|J_{1}\left(\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right)\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}} \leq 2 e^{\delta^{2} / h}\left\|\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right\|_{K \times\left\{|\eta| \geq C_{0}\right\}} \\
+e^{\delta^{2} / h-\delta_{1} / h}\left\|\langle\eta\rangle^{2 N} T_{1} \check{\mathbf{u}}\right\|,
\end{array}
$$

where the positive number $\delta_{1}$ only depends on the distance between $K^{\prime}$ and $\mathbb{R}^{n} \backslash K$. As a consequence, taking $\delta^{2} \leq \min \left(\frac{1}{2} \delta_{1}, \frac{1}{4} \delta_{0}\right)$, and using (F.12)(F.13) we conclude,

$$
\left\|\langle\eta\rangle^{2 N} T_{1} \widetilde{\mathbf{u}}\right\|_{K^{\prime} \times\left\{|\eta| \geq 2 C_{0}\right\}}=\mathcal{O}\left(h^{-4 C} e^{-\delta_{0} / 4 h}+h^{-5 C} e^{-\delta_{1} / 2 h}\right)
$$

Since $K$ was an arbitrary compact set, so is $K^{\prime}$, and thus Lemma 6.9 is proved.

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