Decay of solutions to the three-dimensional generalized Navier-Stokes equations

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Abstract

In this paper, we first obtain the temporal decay estimates for weak solutions to the three dimensional generalized Navier-Stokes equations. Then, with these estimates at disposal, we obtain the temporal decay estimates for higher order derivatives of the smooth solution with small initial data. The decay rates are optimal in the sense that they coincides with ones of the corresponding generalized heat equation. These results improve the previous known results to the classical Navier-Stokes equations.

Keywords: Generalized Navier-Stokes equations; decay; Fourier-splitting method

1 Introduction

The incompressible Navier-Stokes equations can be written as

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u = -\nabla p, \\ \text{div } u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n,\ n \geq 2,\ t > 0$, the vector field $u = u(x,t)$ denotes the velocity of the fluid, $p = p(x,t)$ is the pressure of the fluid and the positive ν is the viscosity coefficient.

Whether or not weak solutions of (1.1) decay to zero in L^2 as time tends to infinity was posed by Leray in his pioneering paper [10, 11]. Kato [7] gave the first affirmative answer to the strong solutions with small data to system (1.1). Algebraic decay rates for

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weak solutions to system (1.1) were first obtained by Schonbek [16], in which the Fourier splitting method was introduced to prove that there exists a Leray-Hopf weak solution of (1.1) in three space dimension with arbitrary data in $L^1 \cap L^2$, satisfying

$$\|u(t)\|_2 \leq C(t+1)^{-\frac{1}{4}}$$

where the constant C depends only on the L^1 and L^2 norms of the initial data. Later the method in [16] was extended by Schonbek [17] (see also Kajikiya and Miyakawa [6], Wiegner [22] for the case $\mathbb{R}^n$ (n=2,3,4)) and it was proved that the decay rate for Leray-Hopf solutions of (1.1) in three space dimension with large data in $L^p \cap L^2$ with $1 \leq p < 2$ is same as those for the solution of the heat equation. That is,

$$\|u(t)\|_2 \leq C(t+1)^{-\frac{3}{4}(\frac{2}{p}-1)},$$

where the constant C only depends on the L^p and L^2 norms of the initial data. On the decay of solutions to the Navier-Stokes equations, it is also referred to [2, 3, 5, 9, 13, 21] and references therein.

In this paper, we consider the large-time behavior of solutions to the following Cauchy problem for the incompressible generalized Navier-Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)u + \nu \Lambda^{2\alpha} u = -\nabla p, \\ \text{div } u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^n,\ n \geq 2,\ t > 0$, $\Lambda^{2\alpha}$ is defined through Fourier transform (see [19])

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi), \quad \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$

It is known that if $(u(x,t), p(x,t))$ is a solution to the three-dimensional generalized Navier-Stokes equations, then for any $\lambda > 0$, the scalings $(u_\lambda(x,t), p_\lambda(x,t)) = (\lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \lambda^{4\alpha-2} p(\lambda x, \lambda^{2\alpha} t))$ also solves the generalized Navier-Stokes equations. The corresponding energy is

$$E(u_\lambda) = \sup_t \int_{\mathbb{R}^3} |u_\lambda|^2 dx + \int_0^\infty \int_{\mathbb{R}^3} |\Lambda^\alpha u_\lambda|^2 dx dt = \lambda^{4\alpha-5} E(u).$$

It follows that $E(u_\lambda) \to \infty$ as $\lambda \to \infty$ when $\alpha < \frac{5}{4}$. In this sense, we say that the three-dimensional generalized Navier-Stokes equations (1.2) is supercritical if $\alpha < \frac{5}{4}$, critical for $\alpha = \frac{5}{4}$ and subcritical with $\alpha > \frac{5}{4}$. It has been proved that when $\alpha \geq \frac{5}{4}$, the three-dimensional generalized Navier-Stokes equations admits a global and unique regular solution (see [12], [23] for instance).

In this paper, we are concerned with the asymptotic behavior of solution of (1.2) in the supercritical case $\alpha < \frac{5}{4}$. Motivated by [16]-[18], we will show that the weak solutions to (1.2) subject to large initial data decay in L^2 at a uniform algebraic rate. The decay estimates for the higher order derivatives of the smooth solution with small initial data will also be established in L^2. To prove our main results, the Fourier splitting method due to Schonbek [16] with appropriate modification will be applied. It should be noted that the decay rates obtained in this paper are optimal in the sense that they coincide with ones of the corresponding generalized heat equation $v_t + \nu\Lambda^{2\alpha}v = 0$ with the same initial data u_0 (see Lemma 3.1 in [15]). Therefore, our results improve ones obtained in [17] in which the classical Navier-Stokes equations ($\alpha = 1$ in (1.2)) are investigated. For completeness, the proof of existence of weak solutions will be sketched in Appendix in the end of the paper.

Throughout the rest of the paper the L^p- norm of a function f is denoted by $\|f\|_p$ and the H^s- norm by $\|f\|_{H^s}$. We will also set $\nu = 1$ for simplicity.

Our main results are listed as follows.

Theorem 1.1. Let $0 < \alpha \leq 1$. Then for divergence-free vector-field $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $1 \leq p < 2$, the system (1.2) admits a weak solution such that

$$\|u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}, \tag{1.3}$$

where the constant C depends on α, the L^p and L^2 norms of the initial data.

Theorem 1.2. Let $1 \leq \alpha < \frac{5}{4}$. Then for divergence-free vector-field $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $\frac{1}{3-2\alpha} \leq p < 2$, the system (1.2) admits a weak solution such that

$$\|u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}, \tag{1.4}$$

where the constant C depends on α, the L^p and L^2 norms of the initial data.

The following are decay estimates for the higher order derivatives of the smooth solution, of which global-in-time existence for sufficiently small initial data is guaranteed in [24].

Theorem 1.3. Let $0 < \alpha \leq 1$ and $u_0 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with div $u_0 = 0$. Then, for $m \in \mathbb{N}$ (the set of positive integers), there exist $T_0 > 0$ and $C > 0$ such that the small global-in-time solution satisfies

$$\|D^m u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}-\frac{m}{\alpha}}$$

for all $t > T_0$, where the constant C depends on m, α and $\|u_0\|_{L^2 \cap L^1}$.

Remark 1.1. The following cases can be dealt with in a similar fashion:

(1) If $0 < \alpha \leq \frac{1}{2}$ and $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $1 \leq p \leq \frac{6}{4\alpha+3}$, one has

$$\|D^m u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)-\frac{m}{\alpha}}.$$

To prove this result, we just modify the estimate (3.14) as

$$\|\nabla u\|_\infty \leq C(t+1)^{-\frac{3}{4\alpha}(\frac{2}{p}-1)}.$$

(2) If $\frac{1}{2} < \alpha \leq 1$ and $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $1 \leq p \leq \frac{6}{4\alpha+1}$, one has

$$\|D^m u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)-\frac{m}{\alpha}}.$$

To prove this result, we just modify the estimate (3.10) as

$$\|u\|_\infty \leq C(t+1)^{-\frac{3}{4\alpha}(\frac{2}{p}-1)}.$$

Theorem 1.4. Let $1 \leq \alpha < \frac{5}{4}$ and $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with div $u_0 = 0$ and $\frac{1}{3-2\alpha} \leq p < 2$. Then, for $m \in \mathbb{N}$ (the set of positive integers), there exist $T_0 > 0$ and $C > 0$ such that the small global-in-time solution satisfies

$$\|D^m u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)-\frac{m}{\alpha}}$$

for all $t > T_0$, where the constant C depends on m, α and $\|u_0\|_{L^2 \cap L^p}$.

Remark 1.2. The decay rates for higher order of derivatives of the solutions was studied in [4] for the classical Navier-Stokes equations and in [18] for the Hall-magnetohydrodynamic equations.

The paper unfolds as follows: Section 2 is devoted to the proof of Theorem 1.1 and Theorem 1.2 whereas Section 3 deals with the proof of Theorem 1.3 and Theorem 1.4. The existence of weak solutions is given in the Appendix in the end of the paper.

2 Proof of Theorem 1.1 and Theorem 1.2

In this section, Theorem 1.1 and Theorem 1.2 will be proved. We start with two key lemmas.

Lemma 2.1. *Let u be a smooth solution to system* (1.2) *with initial data $u_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $1 \leq p < 2$. Then there exists a constant $C > 0$ depending only on $\|u_0\|_2$ such that*

$$|\widehat{u}(\xi,t)| \leq C(|\widehat{u_0}(\xi)| + \frac{1}{|\xi|^{2\alpha-1}}). \tag{2.1}$$

Proof. Taking the Fourier transform of the first equation of (1.2) yields

$$\widehat{u}_t(\xi,t) + |\xi|^{2\alpha}\widehat{u}(\xi,t) = H(\xi,t), \tag{2.2}$$

where

$$H(\xi,t) = -\widehat{u\cdot\nabla u}(\xi,t) - \widehat{\nabla p}(\xi,t).$$

Multiplying (2.2) by $e^{|\xi|^{2\alpha}t}$ gives

$$\frac{d}{dt}(e^{|\xi|^{2\alpha}t}\widehat{u}(\xi,t)) = e^{|\xi|^{2\alpha}t}H(\xi,t).$$

Integrating with respect to time from 0 to t, we have

$$\widehat{u}(\xi,t) = e^{-|\xi|^{2\alpha}t}\widehat{u_0}(\xi) + \int_0^t e^{-|\xi|^{2\alpha}(t-s)}H(\xi,s)ds. \tag{2.3}$$

Hence

$$|\widehat{u}(\xi,t)| \leq |\widehat{u_0}(\xi)| + \int_0^t e^{-|\xi|^{2\alpha}(t-s)}|H(\xi,s)|ds. \tag{2.4}$$

To complete the proof we need to establish an estimate for $H(\xi,s)$. Taking the divergence operator on the first equation of (1.2) yields

$$-\Delta p = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u^i u^j).$$

Since the Fourier transform is a bounded map from L^1 into L^∞, it follows that

$$\begin{aligned}|\widehat{\nabla p}(\xi,t)| \leq &|\xi||\widehat{p}(\xi,t)|\\ \leq &\sum_{i,j=1}^{3}\frac{|\xi_i\xi_j|}{|\xi|}|\widehat{u^iu^j}(\xi,t)|\\ \leq &C|\xi|\|u(t)u(t)\|_1\\ \leq &C|\xi|\|u(t)\|_2^2.\end{aligned}$$

Similarly, for the convection term, using the divergence free condition, we have

$$\begin{aligned}|\widehat{u\cdot\nabla u}(\xi,t)| \leq &\sum_{i}^{3}|\xi||\widehat{u^iu}(\xi,t)|\\ \leq &C|\xi|\|u(t)u(t)\|_1\\ \leq &C|\xi|\|u(t)\|_2^2.\end{aligned}$$

Combing the above two estimates, we obtain

$$|H(\xi,t)| \leq C|\xi|\|u(t)\|_2^2. \tag{2.5}$$

Inserting (2.5) into (2.4) and using the boundedness of the L^2 norm of the solution lead to

$$\begin{aligned}|\widehat{u}(\xi,t)| \leq &|\widehat{u_0}(\xi)| + \frac{C}{|\xi|^{2\alpha-1}}\|u_0\|_2^2(1-e^{-|\xi|^{2\alpha}t})\\ \leq &C(|\widehat{u_0}(\xi)| + \frac{1}{|\xi|^{2\alpha-1}}).\end{aligned}$$

The proof of the lemma is finished. $\square$

Lemma 2.2. *Let $u_0 \in L^p(\mathbb{R}^3)$ with $1 \leq p \leq 2$. Then*

$$\int_{S(t)} |\widehat{u_0}(\xi)|^2 d\xi \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}, \tag{2.6}$$

where

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \leq g(t)\}, \quad g(t) = (\frac{\gamma}{t+1})^{\frac{1}{2\alpha}}, \tag{2.7}$$

the constant C depends on γ and the L^p norm of u_0.

Proof. Denote $\mathcal{F}$ the Fourier transform. By Riesz theorem, if $1 \leq p \leq 2$, the Fourier transform $\mathcal{F} : L^p \to L^q$ is bounded, and

$$\|\mathcal{F}u_0\|_q \leq C\|u_0\|_p, \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, one has

$$\int_{S(t)} |\widehat{u_0}|^2 d\xi \leq (\int_{S(t)} |\widehat{u_0}|^q d\xi)^{\frac{2}{q}}(\int_{S(t)} d\xi)^{1-\frac{2}{q}}. \tag{2.9}$$

Thanks to (2.8) and noting that the volume $|S(t)| = Cg^3(t)$, we get

$$\int_{S(t)} |\widehat{u_0}(\xi)|^2 d\xi \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

The proof of the lemma is finished.

$\square$

In the rest of this section, we first present a formal argument by the Fourier splitting method (see [16]).

Proof of Theorem 1.1. By taking L^2-inner product on both sides of the first equation of (1.2) with u, we get

$$\frac{d}{dt}\|u(t)\|_2^2 = -2\|\Lambda^\alpha u(t)\|_2^2.$$

Applying the Plancherel theorem, one has

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\widehat{u}(\xi)|^2 d\xi = -2\int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi.$$

Let

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le g(t)\}, \quad g(t) = (\frac{\gamma}{t+1})^{\frac{1}{2\alpha}}, \tag{2.10}$$

where γ is a constant to be determined. Then

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} |\widehat{u}(\xi)|^2 d\xi \le &- g^{2\alpha}(t)\int_{|\xi|\ge g(t)} |\widehat{u}(\xi)|^2 d\xi - \int_{|\xi|\le g(t)} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi \\ \le &- g^{2\alpha}(t)\int_{\mathbb{R}^3} |\widehat{u}(\xi)|^2 d\xi + g^{2\alpha}(t)\int_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 d\xi.\end{aligned} \tag{2.11}$$

Multiplying (2.11) by $G(t) = e^{\int_0^t g^{2\alpha}(\tau)d\tau}$ yields

$$\frac{d}{dt}(G(t)\|u(t)\|_2^2) \le g^{2\alpha}(t)G(t)\int_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 d\xi.$$

Note that $G(t) = (t+1)^\gamma$ by (2.10). It follows that

$$\frac{d}{dt}((t+1)^\gamma\|u(t)\|_2^2) \le \gamma(t+1)^{\gamma-1}\int_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 d\xi. \tag{2.12}$$

To complete the proof we will use Lemma 2.1 and 2.2 to estimate the right hand of (2.12). Indeed, by plugging (2.1) into the right hand of (2.12) and using (2.6), we have

$$\begin{aligned}\frac{d}{dt}((t+1)^\gamma\|u(t)\|_2^2) \le & C(t+1)^{\gamma-1}\int_{|\xi|\le g(t)} |\widehat{u_0}(\xi)|^2 d\xi + C(t+1)^{\gamma-1}\int_{|\xi|\le g(t)} \frac{1}{|\xi|^{2(2\alpha-1)}} d\xi \\ \le & C(t+1)^{\gamma-1-\frac{3}{2\alpha}(\frac{2}{p}-1)} + C(t+1)^{\gamma-1-\frac{5-4\alpha}{2\alpha}}.\end{aligned}$$

Integrating in time from 0 to t yields

$$\|u(t)\|_2^2 \le C((t+1)^{-\gamma} + (t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)} + (t+1)^{-\frac{5-4\alpha}{2\alpha}}). \tag{2.13}$$

When $0 < \alpha \le \frac{1}{2}$ and $p \ge 1 \ge \frac{3}{4-2\alpha}$, we have $\frac{3}{2\alpha}(\frac{2}{p}-1) \le \frac{5-4\alpha}{2\alpha}$. Hence, by choosing $\gamma = \frac{3}{2\alpha}$, we obtain

$$\|u(t)\|_2^2 \le C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

When $\frac{1}{2} < \alpha \le 1$, two cases will be considered respectively. In case of $1 \le p < \frac{3}{4-2\alpha}$, one has $\frac{3}{2\alpha}(\frac{2}{p}-1) > \frac{5-4\alpha}{2\alpha}$. Hence, by choosing $\gamma = 3$, we have

$$\|u(t)\|_2^2 \le C(t+1)^{-\frac{5-4\alpha}{2\alpha}}. \tag{2.14}$$

In case of $\frac{3}{4-2\alpha} \le p < 2$, one has $\frac{3}{2\alpha}(\frac{2}{p}-1) \le \frac{5-4\alpha}{2\alpha}$. Hence, by choosing $\gamma = \frac{3}{2\alpha}$, we have

$$\|u(t)\|_2^2 \le C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}. \tag{2.15}$$

Now we improve the decay rate in (2.14). We will use (2.14) to show that

$$|\widehat{u}(\xi,t)| \le |\widehat{u_0}(\xi)| + C \text{ for } \xi \in S(t).$$

Then a bootstrap-type argument will lead to a better decay rate. Using (2.5) and (2.14), for $\frac{1}{2} < \alpha \le 1$ and $\alpha \ne \frac{5}{6}$, we have

$$\begin{aligned} \int_0^t e^{-|\xi|^{2\alpha}(t-s)}|H(\xi,s)|ds \le & C|\xi| \int_0^t (s+1)^{-\frac{5-4\alpha}{2\alpha}} ds \\ \le & C\frac{2\alpha}{6\alpha-5}|\xi|((t+1)^{\frac{6\alpha-5}{2\alpha}}-1) \\ \le & C\frac{2\alpha}{6\alpha-5}(t+1)^{-\frac{1}{2\alpha}}((t+1)^{\frac{6\alpha-5}{2\alpha}}-1) \\ \le & C. \end{aligned} \tag{2.16}$$

If $\alpha = \frac{5}{6}$, we have

$$\begin{aligned} \int_0^t e^{-|\xi|^{2\alpha}(t-s)}|H(\xi,s)|ds \le & C|\xi| \int_0^t (s+1)^{-1} ds \\ \le & C(t+1)^{-\frac{3}{5}} \ln(t+1) \\ \le & C. \end{aligned} \tag{2.17}$$

Hence by (2.3), (2.16) and (2.17)

$$|\widehat{u}(\xi,t)| \le |\widehat{u_0}(\xi)| + C, \text{ for } \xi \in S(t).$$

This, combined with (2.12), yields

$$\begin{aligned} \frac{d}{dt}((t+1)^\gamma \|u(t)\|_2^2) \le & C(t+1)^{\gamma-1} \int_{|\xi|\le g(t)} |\widehat{u_0}(\xi)|^2 d\xi + C(t+1)^{\gamma-1} \int_{|\xi|\le g(t)} d\xi \\ \le & C(t+1)^{\gamma-1-\frac{3}{2\alpha}(\frac{2}{p}-1)} + C(t+1)^{\gamma-1-\frac{3}{2\alpha}}. \end{aligned}$$

Integrating with respect to time yields

$$\begin{aligned} \|u(t)\|_2^2 \le & C((t+1)^{-\gamma} + (t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)} + (t+1)^{-\frac{3}{2\alpha}}) \\ \le & C((t+1)^{-\gamma} + (t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}). \end{aligned}$$

By choosing γ suitably large, we have

$$\|u(t)\|_2^2 \le C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

$\square$

Proof of Theorem 1.2. Two cases will be considered respectively.

Case I. When $1 \leq \alpha < \frac{5}{4}$ and $\frac{3}{4-2\alpha} \leq p < 2$, similar to the proof of (2.12), (2.13) and (2.15), one has

$$\|u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

Case II. When $1 \leq \alpha < \frac{5}{4}$ and $1 \leq \frac{1}{3-2\alpha} \leq p < \frac{3}{4-2\alpha}$, similar to the proof of (2.12), (2.13) and (2.14), one has

$$\|u(t)\|_2^2 \leq C(t+1)^{-\frac{5-4\alpha}{2\alpha}}. \tag{2.18}$$

It follows from (2.5) and (2.18) that

$$\begin{aligned}\int_0^t e^{-|\xi|^{2\alpha}(t-s)}|H(\xi,s)|ds \leq & C|\xi|\int_0^t (s+1)^{-\frac{5-4\alpha}{2\alpha}}ds \\ \leq & C\frac{2\alpha}{6\alpha-5}|\xi|((t+1)^{\frac{6\alpha-5}{2\alpha}}-1) \\ \leq & C\frac{2\alpha}{6\alpha-5}(t+1)^{-\frac{1}{2\alpha}}((t+1)^{\frac{6\alpha-5}{2\alpha}}-1) \\ \leq & C(t+1)^{\frac{3\alpha-3}{\alpha}}.\end{aligned} \tag{2.19}$$

Thanks to (2.3), we have $|\widehat{u}(\xi,t)| \leq C(|\widehat{u_0}(\xi)| + (t+1)^{\frac{3\alpha-3}{\alpha}})$. Applying (2.12) again leads to

$$\frac{d}{dt}((t+1)^\gamma\|u(t)\|_2^2) \leq C(t+1)^{\gamma-1-\frac{3}{2\alpha}(\frac{2}{p}-1)} + C(t+1)^{\gamma-1-\frac{3}{2\alpha}+\frac{6\alpha-6}{\alpha}}.$$

Integrating with respect to time and choosing γ suitably yield

$$\|u(t)\|_2^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

The proof of Theorem 1.2 is finished.

$\square$

Remark 2.1. The proof of Theorems 1.1 and 1.2 is formal and we assume that all the calculus in the proof make sense. To make it more rigorous, we apply the a prior estimates to the approximate solutions constructed in the Appendix. Let us recall that u_N is a solution of the approximate equation

$$\begin{cases}\partial_t u_N + PJ_N(u_N\cdot\nabla u_N) + \Lambda^{2\alpha}u_N = 0, \\ \text{div } u_N = 0, \\ u_N(x,0) = J_N u_0,\end{cases}$$

where J_N is the spectral cutoff defined by

$$\widehat{J_N f}(\xi) = 1_{[0,N]}(|\xi|)\widehat{f}(\xi)$$

and P is the Leray projector over divergence-free vector-fields.

It is shown that the u_N converges strongly in $L^2(0,T;L^2_{loc}(\mathbb{R}^3))$ to a weak solution of the generalized three-dimensional Navier-Stokes equation (1.2) in the Appendix. Hence the L^2 decay of u_N will imply the L^2 decay of the weak solution of (1.2).

3 Proof of Theorem 1.3 and Theorem 1.4

In this section, we will give the proof of Theorem 1.3 and Theorem 1.4. Before that, we recall the following result established in [24].

Theorem 3.1. Let $s \geq \frac{5}{2} - 2\alpha$ with $0 < \alpha < \frac{5}{4}$. Suppose that $u_0 \in H^s(\mathbb{R}^3)$ with div $u_0 = 0$ and there exists a constant ϵ such that $\|u_0\|_{H^s} \leq \epsilon$. Then there exists a unique solution $u \in L^\infty(0, +\infty; H^s)$ satisfying

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq -\|\Lambda^\alpha u\|_{H^s}^2. \tag{3.1}$$

Lemma 3.2. *Let $1 \leq p < 2$. Suppose that $u_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with* div $u_0 = 0$. *Then, for any $|\xi| \leq 1$ and $j \geq 0$, we have*

$$|\widehat{\Lambda^j u}(\xi, t)| \leq C(|\widehat{u_0}(\xi)| + \frac{1}{|\xi|^{2\alpha-1}}), \tag{3.2}$$

where C depends only on $\|u_0\|_{L^p \cap L^2}$.

Proof. Since $|\xi| \leq 1$, we have

$$|\widehat{\Lambda^j u}(\xi, t)| \leq |\xi|^j |\widehat{u}(\xi, t)| \leq |\widehat{u}(\xi, t)|.$$

Using Lemma 2.1 leads to the desired (3.2). $\square$

The following are decay estimates for high order derivatives of the smooth solution.

Theorem 3.3. Let $0 < \alpha \leq 1$. Suppose that $u_0 \in L^p(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ with $1 \leq p < 2$ and $s \geq \frac{5}{2} - 2\alpha$, satisfying div $u_0 = 0$. Then, there exists a $T_0 > 0$ such that for any $t > T_0$ the global-in-time solution established in Theorem 3.1 satisfies

$$\|u(t)\|_{H^s}^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}, \tag{3.3}$$

where C depends on α and $\|u_0\|_{H^s \cap L^p}$.

Theorem 3.4. Let $1 \leq \alpha < \frac{5}{4}$. Suppose that $u_0 \in L^p(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ with $s \geq \frac{5}{2} - 2\alpha$ and $\frac{1}{3-2\alpha} \leq p < 2$, satisfying div $u_0 = 0$. Then, there exists a $T_0 > 0$ such that for any $t > T_0$ the global-in-time solution established in Theorem 3.1 satisfies

$$\|u(t)\|_{H^s}^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}, \tag{3.4}$$

where C depends on α and $\|u_0\|_{H^s \cap L^p}$.

Proof of Theorem 3.3 and 3.4. We adopt to the Fourier splitting method again. It follows from (3.1) that

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\widehat{u}(\xi, t)|^2 + |\widehat{\Lambda^s u}(\xi, t)|^2 d\xi \leq -\int_{\mathbb{R}^3} |\xi|^{2\alpha}(|\widehat{u}(\xi, t)|^2 + |\widehat{\Lambda^s u}(\xi, t)|^2) d\xi.$$

Similar to the proof of Theorem 1.1 and Theorem 1.2, we have

$$\frac{d}{dt}((t+1)^\gamma \int_{\mathbb{R}^3} |\widehat{u}(\xi, t)|^2 + |\widehat{\Lambda^s u}(\xi, t)|^2 d\xi) \leq \gamma(t+1)^{\gamma-1} \int_{|\xi| \leq g(t)} |\widehat{u}(\xi, t)|^2 + |\widehat{\Lambda^s u}(\xi, t)|^2 d\xi.$$

Similar to (1.3) and (1.4), using Lemma 3.2, we get

$$\|u(t)\|_{H^s}^2 \leq C(t+1)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}$$

for any $t > T_0$. The proof of Theorem 3.3 and 3.4 are finished. $\square$

To prove Theorem 1.3 and 1.4, we first present the following commutator estimate.

Lemma 3.5. *Let $s > 0$ and $1 < p < \infty$. Then*

$$\|\Lambda^s(fg)\|_p \leq C\|\Lambda^s f\|_{p_1}\|g\|_{p_2} + \|\Lambda^s g\|_{q_1}\|f\|_{q_2}, \tag{3.5}$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \leq C\|\Lambda^s f\|_{p_1}\|g\|_{p_2} + \|\nabla f\|_{q_1}\|\Lambda^{s-1} g\|_{q_2}, \tag{3.6}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

The proof is referred to [8] and the details are omitted here .

Now we give the proof of Theorem 1.3 and 1.4.

Proof of Theorems 1.3. For any $m \in \mathbb{N}$, applying Λ^m on both sides of the first equation of (1.2), multiplying the resulting equation by $\Lambda^m u$ and integrating by parts, we obtain

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 = -\int_{\mathbb{R}^3} \Lambda^m u \cdot \Lambda^m(u \cdot \nabla u)dx. \tag{3.7}$$

By (3.5), we have

$$\begin{aligned} \frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 \leq & C\|\Lambda^{m+\alpha} u\|_2\|\Lambda^{m-\alpha+1}(u \otimes u)\|_2 \\ \leq & \|\Lambda^{m+\alpha} u\|_2\|\Lambda^{m-\alpha+1} u\|_2\|u\|_\infty. \end{aligned} \tag{3.8}$$

Since

$$\|\Lambda^{m-\alpha+1} u\|_2 \leq C\|\Lambda^{m+\alpha} u\|_2^{\frac{1}{\alpha}-1}\|\Lambda^m u\|_2^{2-\frac{1}{\alpha}}, \quad \frac{1}{2} \leq \alpha \leq 1,$$

one has

$$\begin{aligned} \frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 \leq & \|\Lambda^{m+\alpha} u\|_2^{\frac{1}{\alpha}}\|\Lambda^m u\|_2^{2-\frac{1}{\alpha}}\|u\|_\infty \\ \leq & \frac{1}{4}\|\Lambda^{m+\alpha} u\|_2^2 + C\|u\|_\infty^{\frac{2\alpha}{2\alpha-1}}\|\Lambda^m u\|_2^2. \end{aligned} \tag{3.9}$$

Using Theorem 1.1 and Theorem 3.3 yields

$$\|u\|_\infty \leq C\|u\|_2^{\frac{1}{2}}\|\Lambda^3 u\|_2^{\frac{1}{2}} \leq C(t+1)^{-\frac{3}{4\alpha}} \tag{3.10}$$

for any $t > T_0$ and $\frac{1}{2} \leq \alpha \leq 1$. Putting (3.10) into (3.9), one has

$$\begin{aligned} \frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 \leq & C(t+1)^{-\frac{3}{4\alpha-2}}\|\Lambda^m u\|_2^2 \\ \leq & C(t+1)^{-1}\|\Lambda^m u\|_2^2 \end{aligned} \tag{3.11}$$

for $\frac{1}{2} < \alpha \le 1$. In the case of $0 < \alpha \le \frac{1}{2}$, we can also establish the similar estimate as in (3.11). Indeed, by divergence free condition, (3.7) can be rewritten as

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 = -\int_{\mathbb{R}^3} \Lambda^m u \cdot (\Lambda^m(u\cdot\nabla u) - u\cdot\nabla\Lambda^m u)dx. \tag{3.12}$$

Use the commutator estimate (3.6) to get

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 \le C\|\nabla u\|_\infty \|\Lambda^m u\|_2^2. \tag{3.13}$$

It follows from Theorem 1.1 and Theorem 3.3 that, for any $t > T_0$ and $0 < \alpha \le \frac{1}{2}$,

$$\begin{aligned}\|\nabla u\|_\infty \le& C\|u\|_2^{\frac{1}{6}} \|\Lambda^3 u\|_2^{\frac{5}{6}} \le C(t+1)^{-\frac{3}{4\alpha}} \\ \le& C(t+1)^{-1}.\end{aligned} \tag{3.14}$$

Hence, we obtain that, for $0 < \alpha \le 1$,

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha} u\|_2^2 \le C(t+1)^{-1}\|\Lambda^m u\|_2^2.$$

Let

$$D_i(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le f_i(t)\},\ l > \frac{3}{2\alpha} + \frac{m}{\alpha},\ f_i(t) = (\frac{l+i}{t+1})^{\frac{1}{2\alpha}}.\ i = 0, 1.$$

Then

$$\begin{aligned}\|\Lambda^{m+\alpha} u\|_2^2 =& \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\widehat{\Lambda^m u}(\xi,t)|^2 d\xi \\ \ge& \int_{|\xi|\ge f_i(t)} |\xi|^{2\alpha} |\widehat{\Lambda^m u}(\xi,t)|^2 d\xi \\ \ge& f_i^{2\alpha}(t)\|\Lambda^m u\|_2^2 - f_i^{2\alpha+2}(t) \int_{|\xi|\le f_i(t)} |\widehat{\Lambda^{m-1} u}(\xi,t)|^2 d\xi \\ \ge& f_i^{2\alpha}(t)\|\Lambda^m u\|_2^2 - f_i^{2\alpha+2}(t) \int_{\mathbb{R}^3} |\widehat{\Lambda^{m-1} u}(\xi,t)|^2 d\xi,\end{aligned} \tag{3.15}$$

where $i = 0, 1$. Inserting (3.15) with $i = 1$ into (3.11), we get

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \frac{l}{t+1}\|\Lambda^m u\|_2^2 \le C(\frac{l+1}{t+1})^{\frac{2\alpha+2}{2\alpha}} \|\Lambda^{m-1} u\|_2^2. \tag{3.16}$$

To complete the proof, we use the inductions for m. The case $m = 0$ has been proved in Theorem 1.1. Assume that

$$\|\Lambda^{m-1} u\|_2^2 \le C_{m-1}(t+1)^{-\rho_{m-1}},\quad \rho_{m-1} = \frac{3}{2\alpha} + \frac{m-1}{\alpha}.$$

Then, thanks to (3.16), we have

$$\frac{d}{dt}((t+1)^l \|\Lambda^m u\|_2^2) \le C_{m-1}(t+1)^{l-\rho_{m-1}-\frac{\alpha+1}{\alpha}}. \tag{3.17}$$

Integrating (3.17) in time from T_0 to t yields

$$(t+1)^l \|\Lambda^m u(t)\|_2^2 \le (T_0+1)^l \|\Lambda^m u(T_0)\|_2^2 + C_{m-1}(t+1)^{l-\rho_{m-1}-\frac{1}{\alpha}},$$

which implies

$$\begin{aligned}\|\Lambda^m u(t)\|_2^2 \leq & C_m(t+1)^{-\rho_{m-1}-\frac{1}{\alpha}} \\ \leq & C_m(t+1)^{-\frac{3}{2\alpha}-\frac{m}{\alpha}}.\end{aligned}$$

The proof of Theorems 1.3 is finished. □

Proof of Theorems 1.4. In case of $1 \leq \alpha < \frac{5}{4}$, since $\frac{5}{2} - 2\alpha \in (0, \frac{1}{2})$, we obtain that, for any $m \in \mathbb{N}$, $m \geq \frac{5}{2} - 2\alpha$. Therefore, Theorem 3.1 implies

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 \leq -\|\Lambda^{m+\alpha} u\|_2^2. \tag{3.18}$$

Inserting (3.15) with $i = 0$ into (3.18) yields

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \frac{l}{t+1}\|\Lambda^m u\|_2^2 \leq C(\frac{l}{t+1})^{\frac{2\alpha+2}{2\alpha}}\|\Lambda^{m-1} u\|_2^2. \tag{3.19}$$

Adopting to similar procedure in the proof of Theorem 1.3, we finish the proof of Theorem 1.4. □

A Existence of weak solutions

In this section we show that the generalized Navier-Stokes equations with $\alpha > 0$ have a global weak solution corresponding to any prescribed L^2 initial data.

We start with a definition of weak solutions for (1.2) with L^2 initial data u_0. Let $T > 0$ be arbitrarily fixed.

Definition D.1. *The function pair $(u(x,t), p(x,t))$ is called a weak solution of the problem* (1.2) *if the following conditions are satisfied:*

(1) $u \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^\alpha(\mathbb{R}^3))$,

(2) for any $\Phi \in C_0^\infty([0,T) \times \mathbb{R}^3)$ *with* $\Phi(\cdot, T) = 0$, *we have*

$$\int_0^T \langle u, \Phi_t\rangle - \langle \Lambda^\alpha u, \Lambda^\alpha \Phi\rangle - \langle u \cdot \nabla u, \Phi\rangle dt = -\langle u(0), \Phi(0)\rangle,$$

(3) $\operatorname{div}\, u(x,t) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^3 \times [0,T)$.

The following theorem states that there exists global-in-time weak solutions of (1.2).

Theorem D.1. Let $T > 0$ be fixed and $\alpha > 0$. Assume that $u_0 \in L^2(\mathbb{R}^3)$. Then the system (1.2) possess a weak solution obeying Definition D.1 over $[0,T]$.

We will use the Friedrichs method to prove Theorem (D.1). Before that, let us recall the following Picard theorem [14] and Bernstein inequality [1].

Theorem D.2. (Picard Theorem on a Banach Space). Let $O \subseteq B$ be an open subset of a Banach Space B and let $F : O \to B$ be a mapping that satisfies the following parameters:

(i) $F(X)$ maps O to B.

(ii) F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood U_X of X such that

$$\|F(\widetilde{X}) - F(\widehat{X})\|_B \leq L\|\widetilde{X} - \widehat{X}\|_B, \quad \text{for all} \quad \widetilde{X}, \widehat{X} \in U_X.$$

Then, for any $X_0 \in O$, there exists a time T such that the ODE

$$\frac{dX}{dt} = F(X), X|_{t=0} = X_0 \in O$$

has a unique (local) solution $X \in C^1[(-T, T); O]$. In addition, the unique solution $X(t)$ either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set O as $t \nearrow T$.

Proposition D.3. *(Bernstein inequality). Let B be a ball of $\mathbb{R}^d$. Then, there exists a positive constant C such that for all integer $k \geq 0$, all $b \geq a \geq 1$ and $u \in L^a$, the following estimates are satisfied:*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_b \leq C^{k+1}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_a, \quad supp\hat{u} \subset \lambda B.$$

Proof of Theorems D.1. For $N \geq 1$, let J_N be the spectral cutoff defined by

$$\widehat{J_N f}(\xi) = 1_{[0,N]}(|\xi|)\widehat{f}(\xi).$$

Let P denote the Leray projector over divergence-free vector-fields. Consider the following ODE in the space $L^2_N = \{f \in L^2(\mathbb{R}^3) : \ supp\widehat{f}(\xi) \subset B(0, N)\}$,

$$\begin{cases} u_t + PJ_N(PJ_N u \cdot PJ_N \nabla u) + PJ_N\Lambda^{2\alpha}u = 0, \\ u(x, 0) = J_N u_0. \end{cases} \tag{4.1}$$

We shall apply Picard Theorem to show the existence (local) and uniqueness of solution to (4.1). We write

$$\frac{du}{dt} = -PJ_N(PJ_N u \cdot PJ_N \nabla u) - PJ_N\Lambda^{2\alpha}u \triangleq F(u).$$

Then F satisfies the local Lipschitz condition. In fact, for any $u, v \in L^2_N$, by the Hölder inequality and the Bernstein inequality, we get

$$\begin{aligned}
&\|PJ_N(PJ_N u \cdot PJ_N \nabla u) - PJ_N(PJ_N v \cdot PJ_N \nabla v)\|_2 \\
\leq&\|PJ_N(PJ_N(u - v) \cdot \nabla PJ_N u)\|_2 + \|PJ_N(PJ_N v \cdot \nabla PJ_N(u - v))\|_2 \\
\leq&\|PJ_N(u - v)\|_2\|\nabla PJ_N u\|_\infty + \|\nabla PJ_N(u - v))\|_2\|PJ_N v\|_\infty \\
\leq&N^{\frac{5}{2}}(\|u\|_2 + \|v\|_2)\|u - v\|_2.
\end{aligned}$$

By the Bernstein inequality, it follows that

$$\begin{aligned}
&\|PJ_N\Lambda^{2\alpha}u - PJ_N\Lambda^{2\alpha}v\|_2 \\
\leq&\|J_N\Lambda^{2\alpha}(u - v)\|_2 \\
\leq&N^{2\alpha}\|u - v\|_2.
\end{aligned}$$

Consequently,

$$\|F(u)-F(v)\|_2 \le (N^{\frac{5}{2}}(\|u\|_2+\|v\|_2)+N^{2\alpha})\|u-v\|_2.$$

Picard Theorem implies that (4.1) has a unique local (in time) solution $u_N \in C^1([0,T_N);L^2_N)$. Recall that $P^2=P$, $J_N^2=J_N$ and $PJ_N=J_NP$, it is easy to check that Pu_N and J_Nu_N are also solutions of (4.1). By the uniqueness, $Pu_N=u_N$ (i.e. $\mathrm{div}u_N=0$) and $J_Nu_N=u_N$. Then (4.1) can be simplified as

$$\begin{cases} \partial_t u_N + PJ_N(u_N\cdot\nabla u_N)+\Lambda^{2\alpha}u_N=0, \\ \mathrm{div}\ u_N=0, \\ u_N(x,0)=J_Nu_0. \end{cases} \tag{4.2}$$

Multiplying the first equation of(4.2) by u_N and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_2^2+\|\Lambda^\alpha u_N(t)\|_2^2=0,$$

which implies that

$$\|u_N(t)\|_2^2+2\int_0^t\|\Lambda^\alpha u_N(s)\|_2^2\,ds=\|u_N(0)\|_2^2\le\|u_0\|_2^2. \tag{4.3}$$

This implies that u_N remains bounded in L^2_N for finite time, whence $T_N=T$.

Next, we will use Aubin-Lions lemma [20] to prove the strong convergence of u_N (or its subsequence) in $L^2(0,T;L^2(\Omega))$ for any $\Omega\subset\mathbb{R}^3$. In fact, for any $h\in L^2(0,T;H^3(\mathbb{R}^3))$ and $\alpha\le\frac{5}{2}$, we obtain

$$\begin{aligned} \int_0^T\langle PJ_N(u_N(s)\cdot\nabla u_N(s)),\,h(s)\rangle\,ds &\le \int_0^T\|u_N(s)\|_2\|u_N(s)\|_{\frac{6}{3-2\alpha}}\|\nabla h(s)\|_{\frac{3}{\alpha}}\,ds \\ &\le C\int_0^T\|u_N(s)\|_2\|\Lambda^\alpha u_N(s)\|_2\|\nabla^3h(s)\|_2^{\frac{5-2\alpha}{6}}\|h(s)\|_2^{\frac{1+2\alpha}{6}}\,ds \\ &\le C\|u_N\|_{L^\infty(0,T;\,L^2(\mathbb{R}^3))}\|\Lambda^\alpha u_N\|_{L^2(0,T;\,L^2(\mathbb{R}^3))}\|h\|_{L^2(0,T;\,H^3(\mathbb{R}^3))} \\ &\le C\|u_0\|_2^2\|h\|_{L^2(0,T;\,H^3(\mathbb{R}^3))}, \end{aligned} \tag{4.4}$$

where the Hölder inequality and the Gagliardo-Nirenberg inequality have been used. The Hölder inequality and Sobolev embedding $H^3(\mathbb{R}^3)\hookrightarrow H^\alpha(\mathbb{R}^3),\ \alpha\le\frac{5}{2}$ yield that

$$\begin{aligned} \int_0^T\langle\Lambda^{2\alpha}u_N(s),\,h(s)\rangle\,ds &\le \int_0^T\|\Lambda^\alpha u_N(s)\|_2\|\Lambda^\alpha h(s)\|_2\,ds \\ &\le \Big(\int_0^T\|\Lambda^\alpha u_N(s)\|_2^2\,ds\Big)^{1/2}\Big(\int_0^T\|\Lambda^\alpha h(s)\|_2^2\,ds\Big)^{1/2} \\ &\le \|u_0\|_2\|h\|_{L^2(0,T;\,H^3(\mathbb{R}^3))}. \end{aligned}$$

Combining these estimates with the first equation of (4.2), we obtain

$$\partial_t u_N\in L^2(0,T;\,H^{-3}(\mathbb{R}^3)), \tag{4.5}$$

which together with (4.3) yields that

$$u_N\to u \quad\text{in}\quad L^2(0,T;L^2(\Omega)) \quad\text{for any}\quad \Omega\subset\mathbb{R}^3.$$

We choose $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset ...$ with smooth boundary satisfying $\cup_{i=1}^{\infty}\Omega_i = \mathbb{R}^3$. For any fixed $i = 1, 2, ...$, we obtain that there exists a subsequence of $\{u_N\}_{N=1}^{\infty}$ still denote by itself, such that u_N strongly converges u in $L^2(0,T;L^2(\Omega_i))$. By the diagonal principle, there exists a subsequence $\{u_{N_j}\}_{j=1}^{\infty}$ of $\{u_N\}_{N=1}^{\infty}$ such that u_{N_j} strongly converges u in $L^2(0,T;L^2(\Omega_i))$ for any $i = 1, 2, ...$ and hence in $L^2(0,T;L^2_{loc}(\mathbb{R}^3))$. These convergence guarantee that $u(x,t)$ is a weak solution of (1.2).

When $\alpha > \frac{5}{2}$, it can be proved in a similar way that system (1.2) possess a weak solution obeying Definition D.1. The proof of the Theorem is finished.

$\square$

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