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Abstract. – In this article, we derive a viscous Boussinesq system for surface water waves from Navier-Stokes equations. So, we use neither the irrotationality assumption, nor the Zakharov-Craig-Sulem formulation. During the derivation, we find the bottom shear stress, and also the decay rate for shallow (and not deep) water. In order to justify our derivation, we check it by deriving the viscous Korteweg-de Vries equation from our viscous Boussinesq system. We also extend the system to the 3-D case.

Key words: water waves, shallow water, Boussinesq system, viscosity, KdV equation.

AMS Classification Codes: 76N20, 74J15, 76M45

1 Introduction

1.1 Motivation

The propagation of water waves over a fluid is a long run issue both of mathematics, fluid mechanics, hydrogeology, coastal engineering, ... In case of an inviscid fluid, the topic stemmed many research and even broadened with time. Various equations have been proposed to model this propagation of water waves. The goal is to find reduced (in size) models on simplified domains with as little fields as possible, should they be valid only in an asymptotic regime.
This article is a step forward in the direction of a rigorous derivation of an asymptotic system for surface water waves in the so-called Boussinesq regime, taking into account the

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viscosity. While viscous effects can be neglected for most oceanic situations, they cannot for surface waves in relatively shallow channels.
In the inviscid potential case, the complete and rigorous justification of most asymptotic models for water waves has been recently carried out (see the book [15] and the bibliography therein). This includes the proof of the consistency and stability of the models, the proof of the existence of solutions of the water waves systems and of the asymptotic models on the relevant time scales and the proof of "optimal" error estimates between the two solutions. The curlfree assumption allows to use the Zakharov-Craig-Sulem formulation of the water waves system and facilitates the rigorous derivation of the models, by expanding the Dirichlet to Neumann operator with respect to suitable small parameters.
Things are more delicate when viscosity is taken into account and a complete justification of the asymptotic models is still lacking. The main difficulties, for not only a derivation but a rigorous proof, arise from the matching between the boundary layer coming from the bottom and the "Euler" regime in the upper part of the flow.
In this article, we derive an asymptotic system (Boussinesq system) for the viscous flow in a flat channel of water waves in the Boussinesq regime, that is in the long wave, small amplitude regime with an *ad hoc* balance between the two effects.

1.2 The literature

When deriving models of water waves in a channel, taking viscosity into account, numerous things must be done in order to be rigorous.
Since there are various dimensionless parameters, a linear study must be done so as to determine the most interesting regime between the parameters. One must also either assume linearized Navier-Stokes Equation (NSE), or justify that the nonlinear terms can be dropped. This is not so obvious because numerous authors extend the inviscid theory by assuming the velocity to be the sum of an inviscid velocity and a viscous one. Then they force only one condition (for instance the vanishing velocity on the bottom) to be satisfied by the total velocity, once the inviscid velocity is taken unchanged by viscosity. This deserves to be jutified or assumed.
At a certain level, a heat-like equation arises. Most people solve it with a Fourier transform while the only physical problem is a Cauchy one, so with an initial condition. The only possibility is to use either Laplace (in time) transform or a sine-transform (in the vertical dimension) with a complete treatment of the initial condition.
One must also derive the bottom shear stress because it is meaningful for the physicist who deals with sediment transport.
Last the order up to which the expansion is done must be consistent throughout the article.
To the best of our knowledge, no article does all this. Yet various articles have been written on this topic. Let us review those that retained our attention and interest.

The very first article taking viscosity into account is from Boussinesq in 1895 [2]. Lamb [14] derived the decay rate of the linear wave amplitude by a dissipation calculation (paragraph 348 of the sixth edition and done also in [2]) and by a direct calculation based on the linearized NSE (paragraph 349 of the sixth edition).
Both of them use linearized NSE on deep-water, and compute the dispersion relation. The imaginary part of the phase velocity gives the decay rate:

$$\frac{\partial A}{\partial t} = -2\nu k^2 A, \tag{1}$$

where A is the amplitude of the wave, ν the kinematic viscosity and k the wavenumber.
In an article [23], Ott and Sudan make a formal derivation (in nine lines) of a dissipative KdV equation (different from ours). They use the linear damping of shallow water waves already given by Landau-Lifschitz. This drives them to an additional term to KdV which looks like a half integral. They also find back the damping in time of a solitary wave over a finite depth as $(1+T)^{-4}$ (already found by [12], and later by [11], [21], [10] (p. 374)).
J. Byatt-Smith studied the effect of a laminar viscosity (in the boundary layer where a laminar flow takes place) on the solution of an undular bore [3]. He found the (almost exact) Boussinesq system of evolution with a half derivative but with no treatment of the initial condition. He did an error when providing the solution to the heat equation: his convolution in time is over $(0, +\infty)$.
In 1975, Kakutani and Matsuuchi [11] find a minor error in the computation of [23]. They start from the NSE and perform a clean boundary layer analysis. First, they make a linear analysis that gives the dispersion relation and, with some assumptions, the phase velocity as a function of both the wavenumber of the wave and the Reynold's number Re. They distinguish various regimes of Re as a function of the classical small parameter of any Boussinesq study. Then, they derive the corresponding viscous KdV equation. We want to stress that, at the level of the heat equation, they use a Fourier (in time) transform. As a consequence, they may not have any initial condition.
In [20], one of the authors of the last article tries to validate the equation they were led to. He shows that their model does not modify the number of crest found by an experimental study and by a (non-viscous) KdV simulation, but induces a shift in phase. Yet the numerical treatment seems light because the space step is between some percent and 10 %, the numerical relaxation is not very efficient, and there is no numerical validation of the full algorithm. The author concludes that their "modified K-dV equation can describe the observed wave behaviours". Yet, very fair, he adds that "the phase shift obtained by the calculations is not confirmed by [the] experiments". Indeed, the regime is not the Boussinesq one (dispersion's coefficient is about 0.002 and the viscous coefficient is 0.03). Moreover, the phase shift numerically measured has three digits while the space step is some percent.
In an article of 1987, Khabakhpashev [13] extends the derivation of the viscous KdV evolution equation to the derivation of a Boussinesq system, studies the dispersion relation

and predicts a reverse flow in the bottom in case of the propagation of a soliton wave. Yet, the equations are not made dimensionless, so the right regime is not discussed and a numerical method very inefficient is used (Taylor series expansion is replaced in the convolution term). The existence of solitary waves to the damped KdV equation is claimed, but not justified. He uses a Laplace transform (instead of Fourier as [11] did) with vanishing initial conditions since he assumes starting from rest. He pays attention to the justification of this assumption and stresses that “the time required for the boundary layer to develop over the entire thickness of the fluid [is] much greater than the characteristic time of the wave process”.
In the book [10] (part 5 pp. 356–391), Johnson finds the same dispersion relation as [11], studies the attenuation of the solitary wave by a multi-scales derivation, reaches a heat equation, but solves it only with vanishing initial condition. He exhibits a convolution with a square root integrated on $(0, +\infty)$ (like Byatt-Smith [3]). Some numerical simulations (already partialy done by [3]) enable him to recover the mecanism of undular bore slightly damped.

Later, Liu and Orfila wrote a founding article [19] (LO hereafter) in which they study water waves in an infinite channel (so without meniscus). They derive a Boussinesq system with an additionnal half integration (seen as a convolution), and an initial condition assumed to be vanishing, but implicitely added to the system when numerical simulation must be done.
More precisely, the authors take a linearized Navier-Stokes fluid, use the Helmholtz-Leray decomposition and define the parameters (index LO denotes their parameters):

$$\alpha_{LO}^2 = \nu/\left(l\sqrt{gh_0}\right),$$
$$\varepsilon_{LO} = A/h_0,$$
$$\mu_{LO} = h_0/l,$$

where the following notations will be used throughout the present article: A is the characteristic amplitude of the wave, h_0 is the mean height of the channel, g is the gravitational acceleration, l is the characteristic wavelength of the wave. They say they make expansions up to order α_{LO} which square is a kind of a Reynold's number inverse. They use the classical Boussinesq approximation: $\varepsilon_{LO} \simeq \mu_{LO}^2$, but they set also the link between the viscosity and ε_{LO} by requiring $O(\alpha_{LO}) \simeq O(\varepsilon_{LO}^2) \simeq O(\mu_{LO}^4)$ without further justification. Although “the boundary layer thickness is of $O(\alpha_{LO})$”, they stretch the coordinates by a larger factor $\alpha_{LO}/\mu_{LO} \simeq \mu_{LO}^3$ (see their (2.9)). More important, and maybe linked, they keep the $\alpha_{LO}\mu_{LO}$ terms (in their (2.8) or (2.21) for instance) and yet drop $o(\alpha_{LO})$ terms ! This can explain why their final system (3.10-3.11) has a $\alpha_{LO}/\mu_{LO} = O(\varepsilon_{LO}^{3/2})$ term before the half integration, while we will justify an $O(\varepsilon_{LO})$ term for our system.
Let us stress that assuming $\alpha_{LO}^2 = \varepsilon_{LO}^4$ as do [19] amounts to Re $= \varepsilon_{LO}^{-7/2}$ with our (further redefined) Reynold's number: Re$= \nu/(h_0\sqrt{gh_0})$, while we prove below that the regime at

which gravity and viscosity are both relevant is $\mathrm{Re} = \varepsilon_{LO}^{-5/2}$. Our regime was also exhibited by [11], [3], [10]. So, [19] look at a regime different from ours.
Last, they claim the shear stress at the bottom is:

$$\tau_{bottom} = -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{u(x,T)}{\sqrt{(t-T)^3}} \mathrm{d}T,$$

where $u(x,T)$ is the depth averaged horizontal velocity. Indeed this integral is infinite as they acknowledge in a later corrigendum where they claim the right formula to be:

$$\tau_{bottom} = \frac{1}{\sqrt{\pi}} \frac{u(x,0)}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{u_{,T}}{\sqrt{t-T}} \mathrm{d}T,$$

but do not provide a justification. Moreover, their solution (2.15) to the heat equation, computed in [22] (pp. 153–159), assumes zero initial condition. So the treatment of the initial condition is not done. One of our goal in the present article is precisely to provide a better treatment of this initial condition.
In this article, they also raise the question of the eligible boundary condition. Indeed, they remind us that for a laminar boundary layer, the phase shift between the bottom shear stress and the free stream velocities being $\pi/4$. So it prohibits any bottom condition of the Navier type $\tau_{xy} = -ku_{bottom}$ as is usually assumed (and not derived).
Although we presented some critics, we acknowledge the modeling, derivation and explanations of this article are clever and, last but not least, very well written. Yet our critics apply to all subsequent articles of the same vein.
In [17], Liu *et al.* validate experimentaly LO's equations in the particular case of a solitary wave over a boundary layer. By Particle Image Velocimetry (PIV), they measure the horizontal velocity in the boundary layer over which the solitary wave runs and confirm the theory.
In [18], Liu *et al.* extend the derivation of the viscous Boussinesq system of [19] to the case of an unflat bottom. They compare the viscous damping and shoaling of a solitary wave propagating in a wave tank from the experimental and numerical point of view. They provide a condition on the slope of the bottom and pay attention to the (line) meniscus on the sidewall of the rectangular cross section.
In [16], Liu and Chan use the same process to study the flow of an inviscid fluid over a mud bed modeled by a *very* viscous fluid. They also study the damping rate of progressive linear waves and solitary waves. In [24], Park *et al.* validate this model with experiments. They also study the influence of the ratio of the "mud bed thickness and the wave-induced boundary-layer thickness in the mud bed".

In a very separate way, Wang and Joseph [25] find back the Boussinesq-Lamb decay rate of free gravity waves of a viscous fluid over an infinite depth. They take linearized NSE and use the Leray-Helmholtz decomposition. They determine a (viscous) pressure correction

so as to balance the normal stress. Oddly, their viscous velocity is curl-free. Since they have only a new pressure, they cannot satisfy the full NSE. Such a modeling is mainly motivated by satisfying some equations, yet, it gives good results since the authors can reproduce the decay rate of Boussinesq-Lamb over an infinite depth flow.
In 2008, Dias *et al.* [6] take the (linearized) NSE of a deep water flow with a free boundary and use the Leray-Helmholtz decomposition. Both Bernoulli's equation (through an irrotationnal pressure) and the kinematic boundary are modified. Then, they make an *ad hoc* modeling for the nonlinear term. Starting from such a model, they provide the evolution equation for the enveloppe A of a Stokes wavetrain which, in case of an inviscid fluid is Non-Linear Schrödinger (NLS). The obtained equation happens to be a commonly used dissipative generalization of NLS.
Although it was published earlier (2007), [8] is a further development of [6] to a finite-depth flow. In this article, the authors still linearize NSE and generalize by including additional nonlinearities.
In a later article [9], D. Dutykh linearizes NSE and works on dimensionned equations, considering the viscosity ν to be small (in an absolute meaning). The author "generalizes" by "including nonlinear terms" and reaches a viscous Boussinesq system (his (11-12)). Making this system dimensionless triggered very odd terms and its order zero was no more the wave equation. He further derives a KdV equation by making a change of variable in space (but not the associated change in time $\tau = \varepsilon t$). He also makes a study of the dispersion relation by plugging $e^{i(kx-\omega t)}$ function but then he freezes the half derivative term. Indeed as is well known, plugging these exponentials amounts to making a Fourier or Laplace transform. Here, the Fourier/Laplace transform of the half derivative is very simple :$\mid \xi \mid^{1/2}$ and could have been used instead of freezing this half derivative term.

In [4], Chen *et al.* investigate the well-posedness and decay rate of solutions to a viscous KdV equation which has a nonlocal term that is the same as Liu and Orfila's [19] and [9] but not the same as [11] nor the same as ours. The theoretical proofs are made with no dispersive term (u_{xxx}), but with a dissipative term (u_{xx}). The tools are either theoretical by finding the kernel and study its decay rate, or numerical. In the numerical study, they take the dispersive term into account. As expected, they notice that the "local dissipative term produces a bigger decay rate when compared with the nonlocal dissipative term".
In [5], the authors prove the global existence of solution to the viscous KdV derived by [11] (with the dispersive term) and investigate numerically the decay rate for various norms.

In the following, we first make a linear study of NSE in our geometry (Section 2). We will get the dispersion relation and state various asymptotics that give different phase velocities, and so the decay rate (in finite depth). In Section 3, we make the formal derivation of the viscous Boussinesq system by splitting the upper domain and the bottom one (the boundary layer). The explicit shear stress at the bottom is computed. The equations obtained are cleaned of any dependence on z. In Section 4, we give the 2-D system, and cross-check we get the right viscous KdV equation.

2 The linear theory

In order to make a linear theory, we need first to get dimensionless equations. It is done in the next subsection. Then we investigate three asymptotics in the following subsections.

2.1 Dimensionless equations

Let us denote $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{w})$ the velocity of a fluid in a 2-D domain $\tilde{\Omega} = \{(\tilde{x}, \tilde{z}) \ / \ \tilde{x} \in \mathbb{R},\ \tilde{z} \in [-h, \tilde{\eta}(\tilde{x}, \tilde{t})]\}$. So we assume the bottom is flat and the free surface is characterized by $\tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t})$ with $\tilde{\eta}(\tilde{x}, \tilde{t}) > -h$ (the bottom does not get dry). The dimensionless Figure is drawn in Figure 1. Let $\tilde{p}$ denote the pressure and $\tilde{\mathbf{D}}[\tilde{\mathbf{u}}]$ the symmetric part of the velocity gradient.

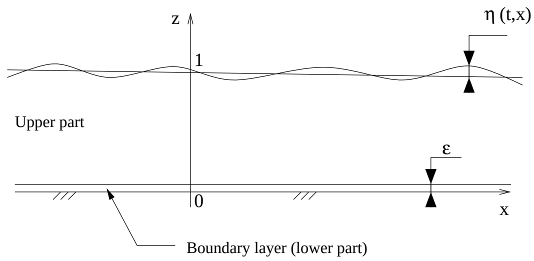

Figure 1: The dimensionless domain

We also denote ρ the density of the fluid, ν the viscosity of the fluid, g the gravity constant, $\mathbf{k}$ the unit vertical vector, $\mathbf{n}$ the outward unit normal to the upper frontier of $\tilde{\Omega}$, $\tilde{p}_{atm}$ the atmospheric pressure. The original "simplified" system reads:

$$\left\{ \begin{array}{ll} \rho\left(\dfrac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}}.\tilde{\nabla}\tilde{\mathbf{u}}\right) - \nu\tilde{\Delta}\tilde{\mathbf{u}} + \tilde{\nabla}\tilde{p} = -\rho g \mathbf{k} & \text{in } \tilde{\Omega} \\ \widetilde{\text{div}}\ \tilde{\mathbf{u}} = 0 & \text{in } \tilde{\Omega} \\ \left(-\tilde{p}\mathbf{I} + 2\nu\tilde{\mathbf{D}}[\tilde{\mathbf{u}}]\right).\mathbf{n} = -\tilde{p}_{atm}\mathbf{n} & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}) \\ \tilde{\eta}_{\tilde{t}} + \tilde{u}\tilde{\eta}_{\tilde{x}} - \tilde{w} = 0 & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}) \\ \tilde{\mathbf{u}} = 0 & \text{on } \tilde{z} = -h, \end{array} \right. \qquad (2)$$

where we write with bold letters both the second order tensors and the vectors. Of course, we need to add an initial condition and conditions at infinity.

So as to get dimensionless fields and variables, we need to choose a characteristic horizontal length l which is the wavelength (roughly the inverse of the wave vector), a characteristic vertical length h which is the water's height, and the amplitude A of the propagating perturbation. Moreover, we denote U, W, P the characteristic horizontal velocity, vertical velocity and pressure respectively. We may then define:

$$c_0 = \sqrt{gh},\ \alpha = \frac{A}{h},\ \beta = \frac{h^2}{l^2},\ U = \alpha c_0,\ W = \frac{Ul}{h} = \frac{c_0\alpha}{\sqrt{\beta}},\ P = \rho g A,\ \mathrm{Re} = \frac{\rho c_0 h}{\nu}, \tag{3}$$

where c_0 is the phase velocity. As a consequence, one may make the fields dimensionless and unscaled:

$$\tilde{u} = Uu,\ \tilde{w} = Ww,\ \tilde{p} = \tilde{p}_{atm} - \rho g \tilde{z} + Pp,\ \tilde{\eta} = A\eta, \tag{4}$$

and the variables:

$$\tilde{x} = lx,\ \tilde{z} = h(z-1),\ \tilde{t} = t\, l/c_0. \tag{5}$$

With these definitions, the new system with the new fields and variables writes in the new domain $\Omega_t = \{(x,z), x \in \mathbb{R},\ z \in [0, 1+\alpha\eta(x,t)]\}$:

$$\begin{cases} u_t + \alpha u u_x + \frac{\alpha}{\beta} w u_z - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0 & \text{in } \Omega_t, \\ w_t + \alpha u w_x + \frac{\alpha}{\beta} w w_z - \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0 & \text{in } \Omega_t, \\ \beta u_x + w_z = 0 & \text{in } \Omega_t, \\ (\eta - p)\mathbf{n} + \frac{1}{\mathrm{Re}} \begin{pmatrix} 2u_x\sqrt{\beta} & u_z + w_x \\ u_z + w_x & 2/\sqrt{\beta} w_z \end{pmatrix} .\mathbf{n} = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta_t + \alpha u \eta_x - \frac{1}{\beta} w = 0 & \text{on } z = 1 + \alpha\eta, \\ \mathbf{u} = 0 & \text{on } z = 0. \end{cases} \tag{6}$$

Like Kakutani and Matsuuchi [11], we could have eliminated $\eta - p$ in the two equations of stress continuity at the free boundary. After simplification by 1 / Re, this would have led us to the "simplified" system:

$$\begin{cases} \eta - p + \frac{1}{\mathrm{Re}}(-\alpha\eta_x(u_z + w_x) - 2u_x\sqrt{\beta}) = 0, \\ (1 - (\alpha\eta_x)^2)(u_z + w_x) = 4\alpha\sqrt{\beta}\eta_x u_x. \end{cases}$$

Notice that the number of dynamic conditions is linked to the laplacian's presence. If, in a subdomain, the flow is inviscid (Euler or $\mathrm{Re} \to \infty$), then one must not keep the two above equations. Yet, once we have simplified the 1/ Re term above we might forget that the second equation must be swept away as if there remained a 1/Re term before every term. So this "simplification" can be misleading.

Unlike us, the authors of [11] use the same characteristic length in the two space directions and so, for them, $h/l = 1$. Our vertical velocity is not the same as in [11] because we take different characteristic lengths in the x and z directions. It suffices to set $\beta = 1$ in our equations to get those of [11]. Our choice of scales in W raises some $\sqrt{\beta}$ terms that [11] avoid. Although the authors make their system dimensionless, they did not really unscale the fields nor the variables. Our fields are unscaled and so are of the order of unity.

Our characteristic pressure is $P = \rho g A$ while [11] use $\rho g h$. This explains that [11] have an α more before the pressure p in their equations.

2.2 The dispersion relation

Seen our scaling, we are looking for small fields. So we linearize the system (6) and get in the new domain:

$$\begin{cases} u_t - \dfrac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \dfrac{1}{\mathrm{Re}\,\sqrt{\beta}} u_{zz} + p_x = 0 & \text{in } \mathbb{R}\times[0,1], \\ w_t - \dfrac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \dfrac{1}{\mathrm{Re}\,\sqrt{\beta}} w_{zz} + p_z = 0 & \text{in } \mathbb{R}\times[0,1], \\ \beta u_x + w_z = 0 & \text{in } \mathbb{R}\times[0,1], \\ \eta - p - \dfrac{2\sqrt{\beta}u_x}{\mathrm{Re}} = 0 & \text{on } z=1, \\ u_z + w_x = 0 & \text{on } z=1, \\ \eta_t - \frac{1}{\beta} w = 0 & \text{on } z=1, \\ \mathbf{u} = 0 & \text{on } z=0. \end{cases} \tag{7}$$

First, we eliminate the pressure from $(7)_1$ and $(7)_2$:

$$u_{zt} - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xxz} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zzz} - w_{xt} + \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xxx} + \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{xzz} = 0.$$

Then we eliminate u from the previous equation thanks to $(7)_3$ by deriving with respect to x and some simplifications:

$$(\partial_z^2 + \beta\partial_x^2)(\partial_z^2 + \beta\partial_x^2 - \mathrm{Re}\sqrt{\beta}\partial_t) w = 0. \tag{8}$$

If w is of the form $\mathcal{A}(z)\exp ik(x-ct)$ with a non-negative k and a (complex) phase velocity c, we can define a parameter with non-negative real part similar to the one used by [11]:

$$\mu^2 = \beta k^2 - \mathrm{Re}\sqrt{\beta} ikc. \tag{9}$$

Thanks to this notation, the solutions of (8) are such that

$$\mathcal{A}(z) = C_1 \cosh\sqrt{\beta}k(z-1) + C_2 \sinh\sqrt{\beta}k(z-1) + C_3\cosh\mu(z-1) + C_4\sinh\mu(z-1). \tag{10}$$

Up to now we have eliminated u and p only in the volumic equations. We still have to use the boundary conditions of (7) to find the conditions on the remaining field w.

The first equation of $(7)_7$ is $u(0) = 0$. After a differentiation with respect to x and $(7)_3$, we get $w_z(0) = 0$.

The second equation of $(7)_7$ is $w(0) = 0$ and needs no treatment.

The equation $(7)_5$ can be differentiated with respect to x and, thanks to $(7)_3$ leads to $w_{zz} - \beta w_{xx} = 0$ at height $z = 1$.

The equation $(7)_6$ enables to compute/eliminate η.

The equation $(7)_4$ must be differentiated with respect to t for η to be replaced. Then we get

$$\frac{w}{\beta} - p_t + \frac{2}{\mathrm{Re}\sqrt{\beta}} w_{zt} = 0.$$

We may differentiate the previous equation with respect to x so as to have a p_x term which can be replaced thanks to $(7)_1$ to have new u terms. It suffices then to differentiate this equation and use the incompressibility $(7)_3$ to get the last condition. The full conditions on w are:

$$\begin{array}{l} w_z(0)=0, \\ w(0)=0, \\ w_{zz}(1)-\beta w_{xx}(1)=0, \\ w_{xx}(1)-w_{ztt}(1)+\frac{3\sqrt{\beta}}{\mathrm{Re}}w_{xxzt}(1)+\frac{1}{\mathrm{Re}\sqrt{\beta}}w_{zzzt}(1)=0. \end{array} \tag{11}$$

The solutions (10) will satisfy a homogeneous linear system in the constants C_1, C_2, C_3, C_4. Its matrix is:

$$\begin{pmatrix} \sqrt{\beta}k\sinh(\sqrt{\beta}k) & -\sqrt{\beta}k\cosh(\sqrt{\beta}k) & \mu\sinh\mu & -\mu\cosh\mu \\ \cosh(\sqrt{\beta}k) & -\sinh(\sqrt{\beta}k) & \cosh\mu & -\sinh\mu \\ 2k^2\beta & 0 & \mu^2+\beta k^2 & 0 \\ -k^2 & \sqrt{\beta}k^3c^2+\frac{2i\beta k^4 c}{\mathrm{Re}} & -k^2 & \frac{2\mu\sqrt{\beta}ik^3c}{\mathrm{Re}} \end{pmatrix}. \tag{12}$$

It suffices to compute its determinant to get the dispersion relation:

$$\begin{array}{l} 4\beta k^2\mu(\beta k^2+\mu^2)+4\mu k^3\beta^{3/2}(\mu\sinh(k\sqrt{\beta})\sinh\mu-k\sqrt{\beta}\cosh(k\sqrt{\beta})\cosh\mu) \\ \quad -(\beta k^2+\mu^2)^2(\mu\cosh(k\sqrt{\beta})\cosh\mu-k\sqrt{\beta}\sinh(k\sqrt{\beta})\sinh\mu) \\ \quad -k\sqrt{\beta}\mathrm{Re}^2(\mu\sinh(k\sqrt{\beta})\sinh\mu-k\sqrt{\beta}\cosh(k\sqrt{\beta})\sinh\mu)=0. \end{array} \tag{13}$$

This relation is identical to the one of [11] except that our non-dimensionnalizing makes a difference between x and z. So instead of k (for [11]), we have $k\sqrt{\beta}$.

2.3 Asymptotic of the phase velocity (very large Re)

In this subsection, we prove the following Proposition:

Proposition 1. *Under the assumptions*

$$k\sqrt{\beta}Re\,c \rightarrow +\infty \tag{14}$$

$$k = O(1) \tag{15}$$

$$\beta \rightarrow 0 \tag{16}$$

$$Re \rightarrow +\infty \tag{17}$$

$$c = O(1) \ \ (\textit{and } c \textit{ bounded away from } 0) \tag{18}$$

if there exists a complex phase velocity c solution of (13), then it is such that:

$$c=\sqrt{\frac{\tanh(k\sqrt{\beta})}{k\sqrt{\beta}}}-\frac{e^{i\pi/4}Re^{-1/2}(k\sqrt{\beta})^{1/4}}{2\tanh^{3/4}(k\sqrt{\beta})}+o(\beta^{-1/4}Re^{-1/2}). \tag{19}$$

Moreover, the decay rate in our finite-depth geometry, which stems from viscosity, is:

$$Im(kc) = \frac{-1}{2\sqrt{2}} \frac{k^{5/4}\beta^{1/8}}{\sqrt{Re}\tanh^{3/4}(k\sqrt{\beta})} + o(\beta^{-1/4}Re^{-1/2}). \tag{20}$$

We denote $o(f)$ (resp. $O(f)$) a function which ratio with f tends to zero (resp. is bounded).

Our decay rate is not the same as Boussinesq's or Lamb's one. The reason is that our geometry is not infinite. This decay rate, in the regime Re$= R\varepsilon^{-5/2}$ and $\beta = b\,\varepsilon$ with constant R, b gets:

$$\mathrm{Im}(kc) = \frac{-\sqrt{k}}{2\sqrt{2}\sqrt{\mathrm{Re}\sqrt{\beta}}} + o(\beta^{-1/4}\mathrm{Re}^{-1/2}) = \frac{-\sqrt{k}\varepsilon}{2\sqrt{2}\sqrt{R\sqrt{b}}} + o(\varepsilon) \tag{21}$$

Our Proposition is stated in [11] but not rigorously proved. Moreover, one must notice that the viscosity modifies also the real part of the phase velocity at the same order.

Proof. The definition of μ ($\Re(\mu) \geq 0$) and assumptions (14, 15, 16) enable to state that $\mu^2 \to \infty$ and the $k^2\beta$ term tends to zero. So we have:

$$\mu = e^{-i\pi/4}\sqrt{k\sqrt{\beta}\mathrm{Re}\,c} + O(\frac{\beta^{3/4}}{\mathrm{Re}}), \tag{22}$$

where the leading term tends to ∞ and its real part tends to $+\infty$, while the error term tends to zero. As a consequence, $\tanh\mu = 1 + O(e^{-\mu})$ and $1/\cosh\mu = O(e^{-\mu})$. Dividing (13) by $\cosh\mu$ and using a generic notation $P(\beta,\mu)$ for an unspecified polynomial in β,μ, we have:

$$\begin{aligned} O(P(\beta,\mu)e^{-\mu}) + 4\mu k^4\beta^2 \left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} - \cosh(k\sqrt{\beta})\right) - \\ (k^2\beta+\mu^2)^2\left(\mu\cosh(k\sqrt{\beta}) - k\sqrt{\beta}\sinh(k\sqrt{\beta})\right) - \\ k^2\beta\mathrm{Re}^2\left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} - \cosh(k\sqrt{\beta})\right) = 0. \end{aligned} \tag{23}$$

The leading term of the second monomial is $4k^4\beta^2\mu^2\sinh(k\sqrt{\beta})/(k\sqrt{\beta})$ while the leading term of the fourth (last) is $-k^2\beta\mathrm{Re}^2\mu\sinh(k\sqrt{\beta})/(k\sqrt{\beta})$. Seen the assumptions, their ratio is $4k^2\beta\mu\mathrm{Re}^{-2} = O(\beta^{5/4}\mathrm{Re}^{-3/2})$. Under the assumptions (16, 17), this ratio tends to zero. So, in a first step, we can neglect the second monomial with respect to the fourth. If we look for a non-vanishing solution, we need to have a compensation of the only two remaining leading terms. One may then rewrite (23) as:

$$-(\mu^4 + hot)(\mu\cosh(k\sqrt{\beta}) + hot) - k^2\beta\mathrm{Re}^2\left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} + hot\right) + hot = 0.$$

This reads after easy computations:

$$c^2 = \frac{\tanh(k\sqrt{\beta})}{k\sqrt{\beta}} + hot. \tag{24}$$

Such a relation is well-known. It confirms the assumption (18). To pursue the expansion we come back to (23) and expand its various monomials starting with the second:

$$-4ik^5\beta^{5/2}\mathrm{Re}\,c\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}}+O(\beta^3)+O(\beta^{9/4}\mathrm{Re}^{1/2}).$$

Indeed, even the leading term of this second monomial will be negligible in comparison with $O(\beta^{7/4}\mathrm{Re}^{3/2})$ that we will have further. The third monomial of (23) is more complex and we must keep:

$$+e^{-i\pi/4}\left(k\sqrt{\beta}\mathrm{Re}\,c\right)^{5/2}\cosh(k\sqrt{\beta})-k^3\beta^{3/2}\mathrm{Re}^2c^2\sinh(k\sqrt{\beta})+O(\beta^{7/4}\mathrm{Re}^{3/2}).$$

The fourth monomial of (23) is expanded:

$$-k^2\beta\mathrm{Re}^2\left(e^{-i\pi/4}\sqrt{k\sqrt{\beta}\mathrm{Re}\,c}\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}}-\cosh(k\sqrt{\beta})\right)+O(\beta^{7/4}\mathrm{Re}^{3/2}).$$

Using these expansions, the equation (23) can be rewritten:

$$\begin{aligned}&e^{-i\pi/4}\left(k\sqrt{\beta}\mathrm{Re}\right)^{5/2}\sqrt{c}\cosh(k\sqrt{\beta})\left[c^2-\frac{\tanh(k\sqrt{\beta})}{k\sqrt{\beta}}+\frac{e^{i\pi/4}}{\sqrt{k\sqrt{\beta}\mathrm{Re}}\sqrt{c}}\right]+\\&O(P(\beta,\mu)e^{-\mu})-k^3\beta^{3/2}\mathrm{Re}^2c^2\sinh(k\sqrt{\beta})+O(\beta^3)+O(\beta^{7/4}\mathrm{Re}^{3/2})=0.\end{aligned}$$

We would like to state that the term between square brackets vanishes. For that purpose, we must check that the various other monomials are negligible in comparison with the third written between the square brackets which expands in: $O((\sqrt{\beta}\mathrm{Re})^{5/2}[(\sqrt{\beta}\mathrm{Re})^{-1/2}])=O(\beta\mathrm{Re}^2)$ if we assume (24). Once it is checked, we can claim we proved:

$$c^2=\frac{\tanh(k\sqrt{\beta})}{k\sqrt{\beta}}-\frac{e^{i\pi/4}}{\sqrt{k\sqrt{\beta}\mathrm{Re}\,c}}+o(\beta^{-1/4}\mathrm{Re}^{-1/2}),\tag{25}$$

and the proof is complete by computing the square root of (25) and replacing the first order of c in (25) which leads to (19). □

We must stress that the complex phase velocity (19) contains two terms. The first is the classical gravitational term ($\sqrt{\tanh(k\sqrt{\beta})/(k\sqrt{\beta})}$) which may be expanded when β tends to zero: $1-k^2\beta/6+O(\beta^2)$. The second is purely viscous and can be expanded: $-\sqrt{2}(1+i)/(4\sqrt{k})(\mathrm{Re}\sqrt{\beta})^{-1/2}+o(\mathrm{Re}\sqrt{\beta})^{-1/2})$. So the dependences of c both on the gravitational and on the viscous effects are of the same order of magnitude when β and $(\mathrm{Re}\sqrt{\beta})^{-1/2}$ are of the same order. Then the dependence of Re on β is such that:

$$\mathrm{Re}\simeq\beta^{-5/2}.\tag{26}$$

2.4 Second asymptotics of the phase velocity (moderate Re)

The definition of μ^2 is $\mu^2 = k^2\beta - ik\sqrt{\beta}\mathrm{Re}\, c$ and we assume a long-wave asymptotics ($\beta \to 0$). So one term or the other dominates in μ^2. The extremes are either $\mu^2 \to \infty$ (see above) or $\mu^2 \to 0$.
In the present subsection, we investigate the latter case and exhibit a more precise expansion than the one justified in [11]. Indeed, we prove the following Proposition:

Proposition 2. *Under the assumptions*

$$\begin{aligned} k \quad & is \quad \textit{bounded from zero and infinity}, && (27)\\ \mu \quad & \to \quad 0 \textit{ and so } \sqrt{\beta}Re\, c \to 0, && (28)\\ \beta \quad & \to \quad 0 \textit{ (long waves)}, && (29)\\ Re \quad & \to \quad +\infty, && (30) \end{aligned}$$

if there exists complex phase velocities c solutions of (13), then one of them is such that:

$$c = -\frac{ik\sqrt{\beta}Re}{3} - \frac{19ik^3\beta^{3/2}Re^3}{90} + o(\beta^{3/2}Re^3), \tag{31}$$

and necessarily (28) implies:

$$\sqrt{\beta}Re \to 0, \tag{32}$$

and so the phase velocity tends to zero.

Notice that if we assume $\sqrt{\beta}\mathrm{Re} \to 0$, the conclusion is the same and the proof much simpler.

Proof. Since $\mu \to 0$, we can expand all the functions in (13). In this expansion, we pay special attention to the fact that $\mathrm{Re} \to +\infty$ and it may not be considered as a constant parameter of an expansion in β (hidden in $O(\beta^2)$ as [11] did). After tedious expansions, there remains from (13):

$$\begin{aligned} &O(\beta\mathrm{Re}^2c^2\mu^5) + O(\beta\mathrm{Re}^2\mu^7) + O(\beta^{3/2}\mathrm{Re}\, c\mu^5) + O(\beta^2\mu^5) + \\ &O(\beta^3\mathrm{Re}^2c^2\mu) + O(\beta^{7/2}\mathrm{Re}\, c\mu) + O(\beta^4\mathrm{Re}^2\mu) + \\ &+\mu\mathrm{Re}^2k^2\beta c\left[(c + \frac{ik\sqrt{\beta}\mathrm{Re}}{3}) - \frac{ik\sqrt{\beta}\mathrm{Re}\, c^2}{2} + \frac{4k^2\beta\mathrm{Re}^2c}{5} + 2k^2\beta c + \right. \\ &\qquad\qquad \left. + \frac{8ik^3\beta^{3/2}\mathrm{Re}}{5} + \frac{ik^3\beta^{3/2}\mathrm{Re}\,\mu^2}{3\times 5!}\right] = 0. \end{aligned} \tag{33}$$

Thanks to the assumptions (27-28) we know that $\sqrt{\beta}\mathrm{Re}c \to 0$. In a first step we assume the terms denoted with $O(...)$ are really negligible in comparison with the largest written with square brackets $O(\mu\mathrm{Re}^2k^2\beta c[\sqrt{\beta}\mathrm{Re}]) = O(\mu\beta^{3/2}\mathrm{Re}^3c)$. We will check it afterwards. As a consequence, we can look for solutions such that the term inside the square brackets vanishes. After comparison of all the terms, there remains only two terms that may compensate:

$$c = -\frac{ik\sqrt{\beta}\mathrm{Re}}{3} + o(\sqrt{\beta}\mathrm{Re}). \tag{34}$$

Since we assume (28), we may write:

$$\sqrt{\beta}\mathrm{Re}\, c \to 0 \Rightarrow \beta\mathrm{Re}^2 \to 0, \tag{35}$$

which implies (32) and so $c \to 0$. Moreover, $\mu^2 = -ik\sqrt{\beta}\mathrm{Re}\, c(1 + ik\sqrt{\beta}/(\mathrm{Re}c))$, and because of (34) we can write:

$$\mu^2 \sim -k^2\beta\, \mathrm{Re}^2/3. \tag{36}$$

With such properties, we can check *a posteriori* that the assumptions are filled. Indeed, all the "negligible" terms are of the type $O(\beta^{9/2}\mathrm{Re}^9)$, $O(\beta^{9/2}\mathrm{Re}^7)$, $O(\beta^{9/2}\mathrm{Re}^5)$, $O(\beta^{9/2}\mathrm{Re}^3)$, and $O(\beta^{9/2}\mathrm{Re}^2)$. Even the largest (and first in our list) is negligible in comparison with $O(\beta^{5/2}\mathrm{Re}^5)$ which is the order of magnitude of the main term inside the square brackets. So the assumption is consistant with the other results.

Before pursuing the expansion of c, we must check that the already neglected terms can still be neglected in comparison with the next order of the term inside the square brackets. Indeed, the terms inside the square brackets, once isolated, are of the order $O(\beta^{5/2}\mathrm{Re}^5)$, $O(\beta^{7/2}\mathrm{Re}^7)$ (twice), $O(\beta^{7/2}\mathrm{Re}^5)$ (twice), or $O(\beta^{9/2}\mathrm{Re}^7)$ in the order where the terms are written. Since the "neglected" terms are at most of the order $O(\beta^{9/2}\mathrm{Re}^9)$, we can use the informations of the term between square brackets only until the order $O(\beta^{7/2}\mathrm{Re}^7)$ included:

$$\begin{aligned} c &+ \frac{ik\sqrt{\beta}\,\mathrm{Re}}{3} - \frac{ik\sqrt{\beta}\mathrm{Re}\, c^2}{2} + \frac{4k^2\beta\mathrm{Re}^2\, c}{5} = o(\beta^{3/2}\,\mathrm{Re}^3) \\ \Leftrightarrow c &= -\frac{ik\sqrt{\beta}\,\mathrm{Re}}{3} - \frac{19ik^3\beta^{3/2}\mathrm{Re}^3}{90} + o(\beta^{3/2}\,\mathrm{Re}^3). \end{aligned} \tag{37}$$

□

3 Formal derivation

We are going to consider the influence of viscosity on the solution of the Navier-Stokes equations in the domain Ω_t. On the basis of the linear theory of the previous section, we assume

$$\mathrm{Re} \simeq \beta^{-5/2} \tag{38}$$

as justified in (26) which is the case where viscous and gravitationnal effects balance. We further assume

$$\alpha \sim a\varepsilon, \qquad \beta \sim b\varepsilon, \tag{39}$$

where ε is a common measure of smallness. So $\alpha/\beta \simeq 1$ in the sense that it does not vanish nor tends to infinity. Our main purpose here is to derive an asymptotic system of reduced size from the global Navier-Stokes equations in the whole *moving* domain. In the non-viscous case, we would derive the classical Boussinesq system.

In order to prove our main result, we proceed in the same way as [11] and distinguish two subdomains: the upper part ($z > \varepsilon$) where viscosity can be neglected, and the lower part ($0 < z < \varepsilon$) which is a boundary layer at the bottom and where viscosity must be taken into account. All the other geometrical characteristics have already been depicted. Our first main Proposition is stated hereafter.

Proposition 3. *Let $\eta(x,t)$ be the free boundary's height. Let $u^{b,0}(x,\gamma)$ for $\gamma \in [0,+\infty[$ (resp. $u^{u,0}(x,z)$ for $z \in [0, 1+\varepsilon\eta(x,t)[$) be the initial horizontal velocity in the boundary layer (resp. in the upper part of the domain). The solution of the Navier-Stokes equation with this given initial condition, satisfies:*

$$\left\{\begin{array}{lr} u_t + \eta_x + \alpha u u_x - \beta \eta_{xxx}\dfrac{(z^2-1)}{2} & = O(\varepsilon^2), \\ \eta_t + u_x(x,z,t) - \dfrac{\beta}{2}\eta_{xxt}(z^2 - \frac{1}{3}) + \alpha(u\eta)_x - \dfrac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x * \dfrac{1}{\sqrt{t}} + & \\ \quad + \dfrac{2\varepsilon}{\sqrt{\pi}} \displaystyle\int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0)\right) \int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}}{4t}}\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' & = O(\varepsilon^2), \end{array}\right. \tag{40}$$

where the convolution is in time and the parameters α, β, Re have been defined above. If the initial velocity is a Euler flow, then $u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0) = 0$ (there is no viscous flow in the boundary layer) and the system writes:

$$\left\{\begin{array}{lr} u_t + \eta_x + \alpha u u_x - \beta \eta_{xxx}\dfrac{(z^2-1)}{2} & = O(\varepsilon^2), \\ \eta_t + u_x(x,z,t) - \dfrac{\beta}{2}\eta_{xxt}(z^2 - \frac{1}{3}) + \alpha(u\eta)_x - \dfrac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x * \dfrac{1}{\sqrt{t}} & = O(\varepsilon^2), \end{array}\right. \tag{41}$$

where the convolution is still in time.

Of course, the domain in the boundary layer $\gamma \in [0,+\infty[$ is not physical. Indeed, it should be considered as large with respect to ε but small with respect to 1. One could set it to $z = \sqrt{\varepsilon}$ (or $\gamma = 1/\sqrt{\varepsilon}$) or to any value (between ε and 1) large with respect to ε but small with respect to 1 on which our final result should not rely on. This would give the same result, as is classical in boundary layer analysis.

***Remark* 4.** *The double integral term in (40) is new and surprising. In the boundary layer, the initial flow is monotonic and so $u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0) \leq 0$. As a consequence this term vanishes only if the initial condition in the boundary layer is a Euler one. In other words, it happens only if the flow is such that its evolution equations are viscous, but its initial condition non-viscous ... This seems unphysical.*
One could also wonder whether vanishing initial conditions suit. A physical question is then to know whether the initial flow in the boundary layer establishes fast or not. We

claim that the characteristic time for the viscous effects to appear is roughly $T_{NSE} = \rho h_0^2/\nu$ or $T_{NSE} = \rho l^2/\nu$. Its ratio with the characteristic time of the inviscid gravity flow (l/c_0) is either Re $\sqrt{\beta} = \varepsilon^{-2}$ or Re $/\sqrt{\beta} = \varepsilon^{-3}$. In any case, it is large and the boundary layer does not establish fast enough. Khabakhpashev [13] already discussed it but started from rest ! Our double integral term also happens to be negative, increasing from a given value to zero for large time (for which the model is no more valid). Its derivative with respect to time is rather simple and could be used in future numerical simulations.

In the first subsection 3.1 we treat the upper part where convenient equations of (6) are kept. Then in subsection 3.2 we solve in the boundary layer the equations extracted from (6) after a rescaling. Those solutions need to match through a continuity condition at the boundary ($z = \varepsilon$) discussed in Subsection 3.3. At this stage, the system obtained still depends on z. So Subsection 3.4 is devoted to making explicit the dependence on z.

3.1 Resolution in the upper part

The upper part is characterized by $\varepsilon < z < 1 + \alpha\eta(x,t)$ and $x, t \in \mathbb{R}$. We start from the system for the fields in the upper part, written u, w, p instead of u^u, w^u, p^u for sake of simplification. The height of the perturbation η is only defined in the upper part and so will always be denoted the same in the boundary layer. The system of PDE in the upper part is extracted from (6):

$$\left\{\begin{array}{ll} u_t + \alpha u u_x + \frac{\alpha}{\beta} w u_z - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0 & \text{for } \varepsilon < z < 1 + \alpha\eta, \\ w_t + \alpha u w_x + \frac{\alpha}{\beta} w w_z - \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0 & \text{for } \varepsilon < z < 1 + \alpha\eta, \\ \beta u_x + w_z = 0 & \text{for } \varepsilon < z < 1 + \alpha\eta, \\ -\alpha\eta_x(\eta - p) + \frac{1}{\mathrm{Re}}(-2\alpha\sqrt{\beta} u_x \eta_x + (u_z + w_x)) = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta - p + \frac{1}{\mathrm{Re}}(-\alpha\eta_x(u_z + w_x) - 2\sqrt{\beta} u_x) = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta_t + \alpha u \eta_x - \frac{1}{\beta} w = 0 & \text{on } z = 1 + \alpha\eta. \end{array}\right. \tag{42}$$

Since we assume Re $\simeq \varepsilon^{-5/2}$, the terms $\sqrt{\beta}/\mathrm{Re}$ are of the order of ε^3 and the terms Re $\sqrt{\beta}$ of the order of ε^{-2}. This simplifies $(42)_1$ and $(42)_2$ and makes disappear the laplacian. As a consequence, we must not keep the two dynamic conditions $(42)_4$ and $(42)_5$ since they are associated to a laplacian. We decide to drop $(42)_4$.

Alternatively, one can stress that $(42)_5$ gives $\eta - p = O(\varepsilon^3)$ and so the lhs of $(42)_4$ is $O(\varepsilon^4) + O(\varepsilon^{5/2})$. Since we expand until the order two, one may claim the equation reduces to $0 = 0$. But one could also simplify by 1/Re ($\simeq \varepsilon^{5/2}$) and be driven to a new equation. This equation would provide one more condition to the two equations for two fields. It is not surprising to see that the final solution would then be $u = 0$. The error is that we must drop one boundary condition unless we have one additionnal condition. The above argument to get rid of $(42)_4$ is sufficient.

On this topic, the literature uses the same equations as us but the argument for dropping is rarely explicited. In [11], Kakutani and Matsuuchi claim "the condition $[(42)_4]$ is automatically satisfied" (p. 242 al. 3) which is either wrong (the equation disappears) or incomplete (what if they simplify by $\varepsilon^{5/2}$?).
In [8], Dutykh and Dias solve the same problem as us and write two equations (their (3) and (4)) among which they keep only one for the derivation without explaining this drop.

The equation $(42)_3$ gives w up to a constant that can be found in $(42)_6$:

$$w(x,z,t) = -\beta \int_{1+\alpha\eta}^{z} u_x(x,z',t)\,\mathrm{d}z' + \beta(\eta_t + \alpha u(1+\alpha\eta)\eta_x), \tag{43}$$

and we stress that this equation is exact. For the expansions later, we need to expand this equation up to the third order:

$$w(x,z,t) = \beta(\eta_t + \int_0^1 u_x) - \beta \int_0^z u_x + \alpha\beta(u(1)\eta)_x + O(\varepsilon^3). \tag{44}$$

The second order of the previous equations suffices to determine p from $(42)_2$ up to a constant:

$$p(x,z,t) = p(x,1+\alpha\eta,t) - \beta(\eta_{tt} + \int_0^1 u_{xt})(z-1) + \beta \int_1^z \int_0^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2).$$

Thanks to $(42)_5$ the constant may be found ($p(1+\alpha\eta) = \eta + O(\varepsilon^3)$) and so:

$$\begin{aligned} p(x,z,t) &= \eta - \beta(\eta_{tt} + \textstyle\int_0^1 u_{xt})(z-1) + \beta \int_1^z \int_0^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2) \\ &= \eta - \beta\eta_{tt}(z-1) + \beta \textstyle\int_1^z \int_1^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2). \end{aligned} \tag{45}$$

Then the remaining field u satisfies $(42)_1$ at the first order:

$$u_t + \eta_x + \alpha u u_x + \alpha u_z(\eta_t + \int_z^1 u_x) - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2), \tag{46}$$

where we have replaced the u_{xxt} by $-\eta_{xxx}$ as usual.
We still have to solve the equations in the lower part.

3.2 Resolution in the boundary layer

We need first to recall some classical properties of Laplace transforms.

3.2.1 Some useful properties

Before solving the equations in the lower part, we list here some classical properties of the Laplace transform. We start from the definition

$$\mathcal{L}(f)(p) = \hat{f}(p) = \int_{t\in\mathbb{R}^+} f(t)e^{-pt}\,\mathrm{d}t. \tag{47}$$

It is well-known that the Laplace transform of the derivative is given by

$$\mathcal{L}(f')(p) = -f(0) + p\mathcal{L}(f)(p), \tag{48}$$

and the product of two functions transforms into the convolution of the Laplace transforms:

$$\mathcal{L}(fg) = \mathcal{L}(f) * \mathcal{L}(g). \tag{49}$$

Here and below, we use the following definition of the convolution, linked to the Laplace transform:

$$f_1 * f_2(t) = \int_0^t f_1(u)f_2(t-u)\mathrm{d}u. \tag{50}$$

These formulas will be useful in the next subsection.

3.2.2 The fields in the boundary layer

The lower part of the domain ($0 < z < \varepsilon$) is a boundary layer where the viscous effects dominate. We start from the system for the bottom fields, written u, w, p instead of u^b, w^b, p^b for sake of simplification and extracted from (6):

$$\begin{cases} u_t + \alpha u u_x + \dfrac{\alpha}{\beta} w u_z - \dfrac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \dfrac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0 & \text{for } 0 < z < \varepsilon \\ w_t + \alpha u w_x + \dfrac{\alpha}{\beta} w w_z - \dfrac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \dfrac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0 & \text{for } 0 < z < \varepsilon \\ \beta u_x + w_z = 0 & \text{for } 0 < z < \varepsilon \\ u(z=0) = 0 \ \text{ and } w(z=0) = 0. \end{cases} \tag{51}$$

As is justified in subsection 2.3 and equation (26), the viscous and gravitational effects balance when $\mathrm{Re} \simeq \beta^{-5/2}$. So we remind the reader of our assumption $\mathrm{Re} = R\,\varepsilon^{-5/2}$, $\alpha = a\varepsilon$ and $\beta = b\varepsilon$ for constant R, a, b. We are naturally led to change the scale in z as in any boundary layer. Let us introduce a new vertical variable $\gamma = z/\varepsilon$. The new fields should be denoted $\tilde{u}(x, \gamma, t) = u(x, \varepsilon\gamma, t)$ for instance. Nevertheless, we will not change them. The new system writes:

$$\begin{cases} u_t + \alpha u u_x + \frac{a}{b\varepsilon} w u_\gamma - \frac{\sqrt{b}}{R}\varepsilon^3 u_{xx} - \frac{u_{\gamma\gamma}}{R\sqrt{b}} + p_x = 0, \\ w_t + \alpha u w_x + \frac{a}{b\varepsilon} w w_\gamma - \frac{\sqrt{b}}{R}\varepsilon^3 w_{xx} - \frac{w_{\gamma\gamma}}{R\sqrt{b}} + \frac{p_\gamma}{\varepsilon} = 0, \\ \varepsilon\beta u_x + w_\gamma = 0, \\ u(\gamma = 0) = 0 \ \text{ and } w(\gamma = 0) = 0. \end{cases} \tag{52}$$

Like in the upper part we can find the vertical velocity from $(52)_3$ and $(52)_4$:

$$w(x, \gamma, t) = -\varepsilon\beta \int_0^\gamma u_x(x, \gamma', t)\,\mathrm{d}\gamma'. \tag{53}$$

Carrying backward the previous equation in $(52)_2$, one has $p_\gamma = O(\varepsilon^3)$. So as to determine p, we need to use the continuity relation for the pressure ($p(x, \gamma = 1, t) = p^u(x, z = \varepsilon, t)$) unless we cannot go on. As we know the pressure in the upper part p^u from (45), we can write:

$$p(x, \gamma, t) = p(x, \gamma = 1, t) + O(\varepsilon^3) = p^u(x, \varepsilon, t) + O(\varepsilon^3) = \eta(x, t) + O(\varepsilon). \tag{54}$$

Using this equation and (53) in $(52)_1$, we have at first order:

$$u_t + \eta_x - \frac{u_{\gamma\gamma}}{R\sqrt{b}} = O(\varepsilon). \tag{55}$$

This equation must be completed with initial condition

$$u(x, \gamma, t = 0) = u^{b,0}(x, \gamma), \tag{56}$$

and boundary condition:

$$\left\{ \begin{array}{rcl} u(x, \gamma = 0, t) & = & 0 \\ u(x, \gamma \to +\infty, t) & = & u^u(x, z = 0, t) \text{ (continuity condition).} \end{array} \right. \tag{57}$$

Notice that [11] does not take an initial condition, and uses a time-Fourier transform. Since we solve a Cauchy problem, we have an initial condition and so we must replace the Fourier transform by the Laplace one. In all his articles, P.L. Liu (*e.g.* [19]) quotes [22] (pp. 153–159) in which a sine-tranform (in γ) is used, but the initial condition is set to zero. In a separate calculation, not reproduced here, we used the same sine-transform in γ and paid attention to the initial condition. We were led to the very same result.
We solve the system (55-57) in the following Lemma.

Lemma 5. *If the initial conditions $u^{b,0}(x, \gamma)$ and $u^{u,0}(x, z = 0)$ are uniformly continuous in γ and satisfy*

$$\begin{array}{l} \displaystyle\int_0^\infty \mid u^{b,0}(x, \gamma) - u^{u,0}(x, z = 0) \mid \mathrm{d}\gamma < \infty, \\ \displaystyle\int_0^\infty \mid u_x^{b,0}(x, \gamma) - u_x^{u,0}(x, z = 0) \mid \mathrm{d}\gamma < \infty, \end{array} \tag{58}$$

then the solution of (55-57) is

$$\begin{array}{rl} u(x, \gamma, t) = & u^u(x, z = 0, t) + \frac{\sqrt{R\sqrt{b}}}{2} \int_0^{+\infty} f_0(x, \gamma') \frac{e^{-\frac{R\sqrt{b}(\gamma'-\gamma)^2}{4t}}}{\sqrt{\pi t}} \mathrm{d}\gamma' - \\ & u^u(x, 0, .) * \mathcal{L}^{-1}(e^{-\sigma\gamma}) - \frac{\sqrt{R\sqrt{b}}}{2} \int_0^{+\infty} f_0(x, \gamma') \frac{e^{\frac{-R\sqrt{b}(\gamma'+\gamma)^2}{4t}}}{\sqrt{\pi t}} \mathrm{d}\gamma' + O(\varepsilon), \end{array} \tag{59}$$

where $f_0(x, \gamma) = u^{b,0}(x, \gamma) - u^{u,0}(x, z = 0)$, u^u is the horizontal velocity in the upper part, given by (46) and σ is the only root with a positive real part of $R\sqrt{b}\,p$:

$$\sigma = \sigma(p) = \sqrt{R\sqrt{b}p}. \tag{60}$$

where p is the dual variable of time t and the convolution is in time.

***Remark* 6.** *The solution of (55) may be known only up to any function of x. The boundary condition (57) enables to determine this function.*

***Remark* 7.** *The compatibility of the conditions (56) and (57) forces to have, when γ tends to $+\infty$:*

$$u^{b,0}(x,\gamma) \to u^{u,0}(x, z=0),$$

and,when $\gamma \to 0$:

$$u^{b,0}(x,\gamma=0) = 0.$$

Meanwhile we also prove the following Proposition

Proposition 8. *Under the same assumptions as in Lemma 5, the bottom shear stress is*

$$\tau^b = \left(\frac{\partial u^b}{\partial \gamma}\right)_{\gamma=0} = \frac{\sqrt{R\sqrt{b}}u^u(x,z=0,0)}{\sqrt{\pi}}\mathrm{Pf}\frac{1}{\sqrt{t}} + \frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\int_0^t \frac{u_t^u(x,z=0,t-s)}{\sqrt{s}}\mathrm{d}s, \quad (61)$$

where Pf denotes the principal value according to the theory of distributions.

First let us prove Proposition 8.

Proof. The initial condition f_0 may not make any difference (it can be seen through an explicit computation), so the correspondig term is taken off. Then a simple derivative and the following formula (See [7] p. 320)

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{p}}\right) = \frac{a}{2\sqrt{\pi}t^{3/2}}e^{-\frac{a^2}{4t}},$$

applied to (59) for any $a > 0$ leads to

$$\begin{aligned}
\tau^b = & -\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\int_0^t u^u(x,z=0,t-s)\frac{e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\sqrt{R\sqrt{b}}\gamma}{2\sqrt{\pi}s^{3/2}}\mathrm{d}s\right) + O(\varepsilon), \\
= & -\sqrt{R\sqrt{b}}\int_0^t \frac{u^u(x,z=0,t-s)}{2\sqrt{\pi}s^{3/2}}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s - \\
& \sqrt{R\sqrt{b}}\int_0^t \frac{u^u(x,z=0,t-s)}{\sqrt{\pi}s^{1/2}}\left(-\frac{R\sqrt{b}\gamma^2}{4s^2}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\right)\mathrm{d}s + O(\varepsilon).
\end{aligned}$$

The second term may be integrated by parts to get

$$\begin{aligned}
-\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\Bigg(&\frac{u^u(x,z=0,0)}{\sqrt{t}}e^{-\frac{R\sqrt{b}\gamma^2}{4t}} - \\
&\int_0^t\left(-\frac{u_t^u(x,z=0,t-s)}{\sqrt{s}} - \frac{u^u(x,z=0,t-s)}{2s^{3/2}}\right)e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s\Bigg),
\end{aligned}$$

which simplifies partially with the first term. At the end, there remains only

$$\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\frac{u^u(x,z=0,0)}{\sqrt{t}}e^{-\frac{R\sqrt{b}\gamma^2}{4t}} + \frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\int_0^t \frac{u_t^u(x,z=0,t-s)}{\sqrt{s}}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s.$$

This justifies the formula. ∎

The scheme of the proof of Lemma 5 is to solve (55) up to two unknown functions, then to determine these functions so as to satisfy the initial and boundary conditions. This provides a necessary formula. We check in Appendix A the solution satisfies the boundary and initial conditions. Let us prove Lemma 5.

Proof. Let us denote

$$f(x,\gamma,t) = u(x,\gamma,t) - u^u(x,z=0,t). \tag{62}$$

Since $f_t = u_t + \eta_x + O(\varepsilon)$ (thanks to (46)) and $f_\gamma = u_\gamma$, the equation (55) writes:

$$f_t - f_{\gamma\gamma}/(R\sqrt{b}) = O(\varepsilon). \tag{63}$$

The initial condition is

$$f(x,\gamma,t=0) = u^{b,0}(x,\gamma) - u^{u,0}(x,z=0) =: f_0(x,\gamma), \tag{64}$$

and the boundary conditions read

$$\begin{array}{l} f(x,\gamma=0,t) = -u^u(x,0,t), \\ \lim_{\gamma\to+\infty}\lim_{\varepsilon\to 0} f_\varepsilon(x,\gamma,t) = 0 \text{ (continuity condition).} \end{array} \tag{65}$$

So we are driven to a heat equation in a half space with vanishing condition at infinity, and non-homogeneous initial and bottom conditions. Through a Laplace in time transform, (63) becomes

$$-f_0(x,\gamma) + p\hat{f}(p) - \frac{\hat{f}_{\gamma\gamma}}{R\sqrt{b}} = O(\varepsilon). \tag{66}$$

In order to solve this non-homogeneous ODE, we start with the homogeneous one and recall that we define σ as the only root with a positive real part of $R\sqrt{b}p$ in (60). Its solutions are

$$\hat{f}(x,\gamma,\xi) = C_1(x,p)e^{+\sigma\gamma} + C_2(x,p)e^{-\sigma\gamma} + O(\varepsilon).$$

By applying the method of parameters variation, we look for $C_1(x,\gamma,p), C_2(x,\gamma,p)$ such that:

$$-C_{1,\gamma}\sigma e^{\sigma\gamma} + C_{2,\gamma}\sigma e^{-\sigma\gamma} = R\sqrt{b}f_0(x,\gamma) + O(\varepsilon),$$

and solving (66) amounts to solving the system of two equations with two unknown functions C_1 and C_2:

$$\left\{\begin{array}{rcl} C_{1,\gamma}e^{\sigma\gamma} + C_{2,\gamma}e^{-\sigma\gamma} &=& 0 \\ -C_{1,\gamma}e^{\sigma\gamma} + C_{2,\gamma}e^{-\sigma\gamma} &=& \dfrac{R\sqrt{b}}{\sigma}f_0, \end{array}\right.$$

which solution is (thanks to assumption (58)):

$$\left\{\begin{array}{rcl} C_1(x,\gamma,p) &=& -\dfrac{R\sqrt{b}}{2\sigma}\displaystyle\int_{+\infty}^{\gamma} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' + \tilde{C}_1(x,p) \\ C_2(x,\gamma,p) &=& +\dfrac{R\sqrt{b}}{2\sigma}\displaystyle\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' + \tilde{C}_2(x,p). \end{array}\right.$$

The full solution is so

$$\hat{f}(x,\gamma,p) = -\frac{R\sqrt{b}}{2\sigma}\int_{+\infty}^{\gamma} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma} + \tilde{C}_1(x,p)e^{+\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma} + \tilde{C}_2(x,p)e^{-\sigma\gamma'} + O(\varepsilon).$$

We look for $\tilde{C}_1$ first. Since f_0 is bounded, simple bounds prove that the first, third and fourth terms are bounded. So

$$\tilde{C}_1(x,p) = 0.$$

The unknown function $\tilde{C}_2(x,p)$ is then given by the boundary condition $(65)_1$ at the bottom:

$$\tilde{C}_2(x,p) = -u^u(x,z=0,p) - \frac{R\sqrt{b}}{2\sigma}\int_0^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma'.$$

In a necessary way,

$$\hat{f}(x,\gamma,p) = +\frac{R\sqrt{b}}{2\sigma}\int_{\gamma}^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma} - \left(\hat{u}^u(x,z=0,p) + \frac{R\sqrt{b}}{2\sigma}\int_0^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma'\right)e^{-\sigma\gamma} + O(\varepsilon). \tag{67}$$

From the definition of f and the existence of an inverse Laplace transform, one knows that:

$$f(x,\gamma,t) = \frac{R\sqrt{b}}{2}\int_{\gamma}^{+\infty} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{-\sigma(\gamma'-\gamma)}}{\sigma}\right)\mathrm{d}\gamma' + \frac{R\sqrt{b}}{2}\int_0^{\gamma} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{\sigma(\gamma'-\gamma)}}{\sigma}\right)\mathrm{d}\gamma' + f(x,z=0,.)*\mathcal{L}^{-1}\left(e^{-\sigma\gamma}\right) - \frac{R\sqrt{b}}{2}\int_0^{+\infty} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{-\sigma(\gamma'+\gamma)}}{\sigma}\right)\mathrm{d}\gamma' + O(\varepsilon).$$

Owing to formula (see [7]):

$$\mathcal{L}^{-1}\left(\frac{e^{-\tilde{a}\sqrt{p}}}{\sqrt{p}}\right) = \frac{1}{\sqrt{\pi t}}e^{-\frac{\tilde{a}^2}{4t}},$$

if $\tilde{a} > 0$, one may justify the explicit form of u given in (59). Until the end of this article, we denote the function of time t:

$$A = A(t) = \sqrt{\frac{R\sqrt{b}}{4t}}. \tag{68}$$

We still have to check that the initial condition (64) and remaining of the boundary conditions $(65)_2$ are satisfied by u given by (59). This is completed in Appendix A.
So we completed the proof of the whole Lemma 5. $\square$

From (53) and (59), we can then compute the vertical velocity

$$\begin{aligned} w^b(x,\gamma,t) = & -\varepsilon\beta \int_0^\gamma u_x^b(x,\gamma',t)\mathrm{d}\gamma' \\ = & -\varepsilon\beta u_x^u(x,0,t)\gamma + \varepsilon\beta u_x^u(x,0,.) * \mathcal{L}^{-1}\left(\frac{e^{-\sigma\gamma}-1}{-\sigma}\right) - \\ & \varepsilon\beta\frac{A(t)}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma''-\gamma')^2}\,\mathrm{d}\gamma''\mathrm{d}\gamma' + \\ & \varepsilon\beta\frac{A(t)}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma''+\gamma')^2}\,\mathrm{d}\gamma''\mathrm{d}\gamma'. \end{aligned} \tag{69}$$

We still have to satisfy the continuity conditions of all the fields u, w, p.

3.3 The continuity conditions

In the present subsection, we need to write explicitly the superscripts u and b for the upper part and bottom regions respectively. We write the computed fields at the same height ε that is the common frontier of both subdomains.

We already used the continuity of pressure that led us to (54).

Regarding the horizontal velocity, we must notice that the limit when $\gamma \to +\infty$ of $\lim_{\varepsilon\to 0}(u^b(x,\gamma,t) - u^u(x,\varepsilon\gamma,t)) = f(x,\gamma,t)$ has already been computed as vanishing (see Appendix A). So the boundary condition $(65)_2$ is already satisfied and the horizontal velocity is continuous.

Notice that, should we have switched the limits in ε and γ, the limit would be meaningless. Furthemore, it seems more realistic to consider the boundary layer ε to be very small and *then* put its height to one order of magnitude more than ε. This is the meaning of the limits order.

Concerning the vertical velocity, we can use the velocity in the upper part w^u from (44) expanded in ε:

$$\begin{aligned} w^u(x,\varepsilon\gamma,t) = & \beta(\eta_t + \textstyle\int_0^1 u_x^u) - \beta\int_0^{\varepsilon\gamma} u_x^u + \alpha\beta(u^u(1)\eta)_x + O(\varepsilon^3) \\ = & \beta(\eta_t + \textstyle\int_0^1 u_x^u) - \beta\varepsilon\gamma u_x^u(z=0) + \alpha\beta(u^u(1)\eta)_x + O(\varepsilon^3). \end{aligned}$$

The velocity in the bottom w^b is given in (69). The difference $w^u - w^b$ can be expanded in ε:

$$
\begin{aligned}
w^u(x,\varepsilon\gamma,t)-w^b(x,\gamma,t) &= \beta(\eta_t+\int_0^1 u^u_x)-\beta\varepsilon\gamma u^u_x(z=0)+\alpha\beta(u^u(1)\eta)_x+O(\varepsilon^3)\\
&+\varepsilon\beta u^u_x(x,z=0,t)\gamma-\varepsilon\beta u^u_x(x,0,.)*\mathcal{L}^{-1}\left(\frac{e^{-\sigma\gamma}-1}{-\sigma}\right)+\\
&+\varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'-\\
&\varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'+O(\varepsilon^3)\\
&= \beta(\eta_t+\int_0^1 u^u_x)+\alpha\beta(u^u(1)\eta)_x-\varepsilon\beta u^u_x(x,0,.)*\mathcal{L}^{-1}\left(\frac{1}{\sigma}\right)+\\
&+\varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'- \qquad (70)\\
&\varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'+O(\varepsilon^3),
\end{aligned}
$$

up to functions that tend exponentially to zero when $\gamma\to+\infty$.

We still must simplify the two last double integrals. This is made in the following Lemma

Lemma 9. *If $A=A(t)=\sqrt{\frac{R\sqrt{b}}{4t}}$, γ is positive, $f_0(x,\gamma)$ is uniformly continuous in γ and satisfies (58), then*

$$\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'-\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma'$$

tends to

$$\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\int_{\gamma'''=-\gamma''}^{\gamma''}e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma'', \qquad (71)$$

when $\gamma\to+\infty$.

In the proof we apply Fubini's theorem and changes of variables to the two integrals.

Proof. Let us apply the Fubini theorem to the two integrals which rewrite then:

$$\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\int_{\gamma'=0}^{\gamma}e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma'\mathrm{d}\gamma''-\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\int_{\gamma'=0}^{\gamma}e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma'\mathrm{d}\gamma''.$$

Simple changes of variables for each of these integrals give

$$
\begin{aligned}
&\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\int_{\gamma'''=-\gamma''}^{\gamma-\gamma''}e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma''-\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\int_{\gamma'''=\gamma''}^{\gamma+\gamma''}e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma''\\
&=\int_{\gamma''=0}^{+\infty}f_{0,x}(x,\gamma'')\left(\int_{\gamma'''=-\gamma''}^{+\gamma''}e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''-\int_{\gamma'''=\gamma-\gamma''}^{\gamma+\gamma''}e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\right)\mathrm{d}\gamma''.
\end{aligned}
$$

The first of these integrals gives rise to the announced term in (71). So only the second integral (see Figure 2 left) needs to be proved to vanish when γ tends to $+\infty$. One may apply again the Fubini's theorem to the second integral which rewrites as a sum of two integrals (see Figure 2 right):

$$\begin{aligned}
&-\int_{\gamma'''=-\infty}^{\gamma}\int_{\gamma''=\gamma-\gamma'''}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma'' - \\
&\qquad\qquad \int_{\gamma'''=\gamma}^{+\infty}\int_{\gamma''=\gamma'''-\gamma}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma'' \\
&= -\int_{\gamma''''=-\infty}^{0}\int_{\gamma''=-\gamma''''}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma+\gamma'''')^2}\mathrm{d}\gamma''''\mathrm{d}\gamma'' - \\
&\qquad\qquad \int_{\gamma''''=0}^{+\infty}\int_{\gamma''=\gamma''''}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma+\gamma'''')^2}\mathrm{d}\gamma''''\mathrm{d}\gamma'' \\
&= -\int_{\gamma''''=0}^{+\infty} e^{-A^2(\gamma-\gamma'''')^2}\int_{\gamma''=\gamma''''}^{+\infty} f_{0,x}(x,\gamma'')\mathrm{d}\gamma''\mathrm{d}\gamma'''' - \\
&\qquad\qquad \int_{\gamma''''=0}^{+\infty} e^{-A^2(\gamma+\gamma'''')^2}\int_{\gamma''=\gamma''''}^{+\infty} f_{0,x}(x,\gamma'')\mathrm{d}\gamma''\mathrm{d}\gamma'''' \\
&= -\int_{\gamma''''=0}^{+\infty} e^{-A^2(\gamma-\gamma'''')^2}R(\gamma'''')\mathrm{d}\gamma'''' - \int_{\gamma''''=0}^{+\infty} e^{-A^2(\gamma+\gamma'''')^2}R(\gamma'''')\mathrm{d}\gamma'''', \qquad (72)
\end{aligned}$$

where

$$R(\gamma'''') = \int_{\gamma''=\gamma''''}^{+\infty} f_{0,x}(x,\gamma'')\mathrm{d}\gamma'' \to 0 \text{ when } \gamma'''' \to +\infty.$$

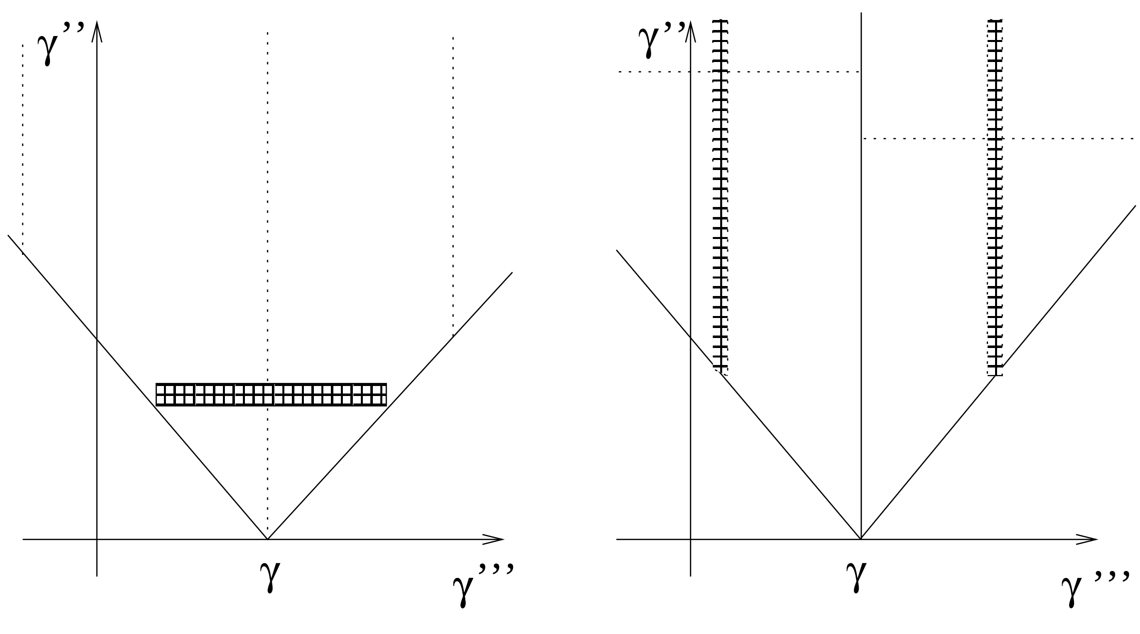

Figure 2: picture of the integrations (right and left)

In the equation (72), we discuss successively the first and second integral in order to prove they vanish. Since we do know that R tends to 0 when γ'''' tends to $+\infty$,

$$\forall \epsilon > 0 \; \exists \Gamma \; / \; \gamma'''' > \Gamma \Rightarrow \mid R(\gamma'''') \mid < \epsilon.$$

Consequently, we can bound the modulus of the first integral of (72) with

$$\begin{aligned} &\int_{\gamma''''=0}^{\Gamma} e^{-A^2(\gamma-\gamma'''')^2} \mid R(\gamma'''') \mid \mathrm{d}\gamma'''' + \epsilon \int_{\gamma''''=\Gamma}^{+\infty} e^{-A^2(\gamma-\gamma'''')^2} \mathrm{d}\gamma'''' \\ &\leq \sup_{\gamma>0} \mid R(\gamma) \mid \int_{\gamma''''=0}^{\Gamma} e^{-A^2(\gamma-\gamma'''')^2} \mathrm{d}\gamma'''' + \epsilon \left[\int_{\Gamma-\gamma}^{0} e^{-A^2\gamma^{(5)2}} \mathrm{d}\gamma^{(5)} + \int_{0}^{+\infty} e^{-A^2\gamma^{(5)2}} \mathrm{d}\gamma^{(5)} \right]. \end{aligned}$$

In the previous formula, the first integral rewrites $\int_{\gamma^{(5)}=\gamma-\Gamma}^{\gamma} e^{-A^2\gamma^{(5)2}} \mathrm{d}\gamma^{(5)}$ which clearly vanishes when γ tends to $+\infty$. The second term with a square bracket can be bounded by ϵ up to a multiplicative constant (for given t and Γ). So the first integral in (72) is as small as wanted when γ is large enough.

Similarly, the second integral of (72) can be bounded by two terms:

$$\sup_{\gamma>0} \mid R(\gamma) \mid \int_{\gamma''''=0}^{\Gamma} e^{-A^2(\gamma+\gamma'''')^2} \mathrm{d}\gamma'''' + \epsilon \int_{\gamma''''=\Gamma}^{+\infty} e^{-A^2(\gamma+\gamma'''')^2} \mathrm{d}\gamma''''.$$

The first of the two terms clearly tends to zero when γ tends to $+\infty$ (for given Γ, t) and the second can be bounded by a constant times ϵ. So the second integral also is as small as wanted when γ is large enough and we completed the proof of Lemma 9.

$\square$

After simplification of β, the continuity of the vertical velocity (70) reads after making $\gamma \to +\infty$ thanks to Lemma 9:

$$\begin{aligned} &\eta_t + \textstyle\int_0^1 u^u_x + \alpha(u^u(1)\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u^u_x(x,0,t) * \frac{1}{\sqrt{t}} + \\ &\quad + \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} f_{0,x}(x,\gamma'') \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' = O(\varepsilon^2), \end{aligned} \tag{73}$$

where the convolution is in time t and the formula $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = 1/\sqrt{\pi t}$ [7] is used.

3.4 The dependence on z of the fields

At this stage, we have reduced the equations but not as much as in the Euler case which leads to a Boussinesq system in 1+1 dimension. We still have derived only a 2+1 dimension problem although we have eliminated w and p. The major difference with the Boussinesq derivation comes from the assumption of irrotationnality of Euler flows. This assumption provides $u_z = O(\varepsilon)$. Such a condition would annihilate the dependence on z and greatly simplify the above computation.

Yet irrotationality and its corollary of a potential flow is incompatible with the number of conditions we set at the bottom, which are needed by the dissipativity of the Navier-Stokes equations.
We are going to determine the dependence on z of u to have a more tractable system.
Starting from now, we drop the u superscripts for the fields in the upper part but keep the superscripts for the boundary layer.
In summary, in case $\mathrm{Re} \simeq \varepsilon^{-5/2}$, we are in the case of the first asymptotic depicted in the subsection 2.3. Then the reduced equations are collected from (46) and (73):

$$u_t + \eta_x + \alpha u u_x + \alpha u_z(\eta_t + \int_z^1 u_x) - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2), \forall z \tag{74}$$

$$\begin{aligned}\eta_t + \int_0^1 u_x(z)\,\mathrm{d}z + \alpha(u(z=1)\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x(x, z=0, t) * \frac{1}{\sqrt{t}} + \\ + \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x, z=0)\right) \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' = O(\varepsilon^2).\end{aligned} \tag{75}$$

The equation (74) can be rewritten thanks to the order 0 of (75):

$$u_t + \eta_x + \alpha u u_x - \alpha u_z \int_0^z u_x - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2), \forall z. \tag{76}$$

Notice that the η_{xxx} term comes from an integral of the shape $\int_1^z \int_1^{z'} u_{xxt}$. As an intermediate result one may see very easily that $\eta_{xx} = \eta_{tt} + O(\varepsilon)$.
We intend to prove the following Lemma:

Lemma 10. *A localized solution of (75), (76) is such that*

$$\begin{aligned}\int_0^1 u &= u(x,z,t) - \beta\eta_{xt}\frac{z^2 - 1/3}{2} + O(\varepsilon^2), & (77)\\ u(x,0,t) &= u(x,z,t) - \beta\eta_{xt}\frac{z^2}{2} + O(\varepsilon^2), & (78)\\ u(x,1,t) &= u(x,z,t) + \beta\eta_{xt}\frac{1-z^2}{2} + O(\varepsilon^2). & (79)\end{aligned}$$

Proof. In a preliminary step, we prove

$$u_z(x,z,t) = \beta\,\eta_{xt}(x,t)\,z + O(\varepsilon^2). \tag{80}$$

To that end, we differentiate (76) with respect to z, so as to have:

$$u_{zt} + \alpha u\, u_{xz} - \alpha u_{zz} \int_0^z u_x - \beta\eta_{xtt} - \beta\eta_{xxx}(z-1) = O(\varepsilon^2),$$

and we can integrate this equation in time using that $\eta_{xx} = \eta_{tt} + O(\varepsilon)$:

$$u_z + \alpha \int_{t_0}^t (u\, u_{xz}) - \alpha \int_{t_0}^t (u_{zz} \int_0^z u_x) - \beta\eta_{xt} - \beta\eta_{xt}(z-1) = C_3(x,z) + O(\varepsilon^2), \tag{81}$$

where C_3 is a function of x, z but it does not depend on t. Since the solution is localized for any x, z, there exists a time t_0 at which $u_z = 0 = \eta_{xt}$, we have

$$C_3(x, z) = O(\varepsilon),$$

in a first attempt to determine C_3. But then the equation (81) implies $u_z = O(\varepsilon)$ and so the quadratic terms are all of second order in (81) since they contain at least one u_z. Hence

$$u_z(x, z, t) - \beta\eta_{xt}\, z = C_4(x, z) + O(\varepsilon^2).$$

Again since for all (x, z) there exists a time at which $u = 0$ and $\eta = 0$, then $C_4(x, z) = O(\varepsilon^2)$ and this completes the proof of (80). We can then go further by integrating between z' and z:

$$u(x, z, t) = u(x, z', t) + \beta\eta_{xt}\frac{z^2 - z'^2}{2} + O(\varepsilon^2),$$

and then, integrating in z' between $z' = 0$ and $z' = 1$, we can state (77). Setting $z' = 0$ we obtain (78) and setting $z' = 1$ gets (79).

$\square$

So the system (75, 76) can be rewritten thanks to (77-79), the formula $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = 1/\sqrt{\pi t}$ [7], and the fact that, as in the Euler case $\eta_{xx} = \eta_{tt} + O(\varepsilon)$:

$$\begin{aligned}
u_t + \eta_x + \alpha u u_x - \beta\eta_{xxx}\frac{(z^2-1)}{2} &= O(\varepsilon^2), \quad (82)\\
\eta_t + u_x(x, z, t) - \frac{\beta}{2}\eta_{xxt}(z^2 - \frac{1}{3}) + \alpha(u\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}u_x * \frac{1}{\sqrt{t}} + &\\
+\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(x, \gamma'') - u_x^{u,0}(x, z=0)\right)\int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' &= O(\varepsilon^2), \quad (83)
\end{aligned}$$

where all the fields u are evaluated at (x, z, t) and the convolution is in time. This is the system stated in Proposition 3 and the proof is complete.

4 Generalization and checkings

In a first subsection, we state the 2-D Boussinesq system and check we may derive the classical Boussinesq systems in the inviscid case. Then, in Subsection 4.2 we derive rigorously the viscous KdV equation and discuss its compatibility with the equation derived by Kakutani and Matsuuchi in [11], by Liu and Orfila in [19], and by Dutykh in [9].

4.1 The full 2-D Boussinesq systems family

One may start from the 3-D Navier-Stokes equations and derive in a way very similar to above a generalization of (82,83):

$$\left\{\begin{aligned} u_t+\eta_x+\alpha u u_x+\alpha v u_y-\beta(\eta_{xxx}+\eta_{xyy})\frac{(z^2-1)}{2} &= O(\varepsilon^2),\\ v_t+\eta_y+\alpha u v_x+\alpha v v_y-\beta(\eta_{yxx}+\eta_{yyy})\frac{(z^2-1)}{2} &= O(\varepsilon^2),\\ \eta_t+u_x+v_y-\frac{\beta}{2}(\eta_{xxt}+\eta_{yyt})(z^2-\tfrac{1}{3})+&\\ +\alpha(u\eta)_x+\alpha(v\eta)_y-\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\eta_t*\left(\frac{1}{\sqrt{t}}\right)+&\\ +\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(x,\gamma'')-u_x^{u,0}(x,z=0)\right)\int_{\gamma'=0}^{A(t)\gamma''}e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' &= O(\varepsilon^2), \end{aligned}\right. \tag{84}$$

In case of a Euler initial condition, the last integral term vanishes. It is well-known thanks to [1] that there is a family of Boussinesq systems, indexed by three free parameters. All these systems are equivalent in the sense that up ot order 1, they can be derived one from the other by using their own $O(\varepsilon^0)$ part and by replacing partially η_t,η_x and η_y by u_x,u_t. We are going to prove the same for our system. Namely, the order 0 of (84) enables to interpolate with a_{int},b_{int},c_{int}:

$$\left\{\begin{array}{l} \eta_x=a_{int}\eta_x-(1-a_{int})u_t+O(\varepsilon),\\ \eta_y=b_{int}\eta_y-(1-b_{int})v_t+O(\varepsilon),\\ \eta_t=c_{int}\eta_t-(1-c_{int})(u_x+v_y)+O(\varepsilon). \end{array}\right.$$

These formulas are reported in the full 2D system (84), where we drop the convolution term and the integral on the initial condition:

$$\left\{\begin{aligned} &u_t+\eta_x+\alpha u u_x+\alpha v u_y-a_{int}\beta\Delta\eta_x\frac{(z^2-1)}{2}+(1-a_{int})\beta\Delta u_t\frac{(z^2-1)}{2} &&= O(\varepsilon^2),\\ &v_t+\eta_y+\alpha u v_x+\alpha v v_y-b_{int}\beta\Delta\eta_y\frac{(z^2-1)}{2}+(1-b_{int})\beta\Delta v_t\frac{(z^2-1)}{2} &&= O(\varepsilon^2),\\ &\eta_t+u_x+v_y-c_{int}\frac{\beta}{2}\Delta\eta_t(z^2-\tfrac{1}{3})+(1-c_{int})\frac{\beta}{2}\Delta(u_x+v_y)(z^2-\tfrac{1}{3})+ &&\\ &+\alpha(u\eta)_x+\alpha(v\eta)_y &&= O(\varepsilon^2), \end{aligned}\right. \tag{85}$$

where we denote Δ the x,y laplacian.

This is the general Boussinesq system as can be seen in [1] (p. 285 equation (1.6)). Indeed if we denote a_{BCS},b_{BCS},c_{BCS} and d_{BCS} the interpolation parameters of this article, we can identify the 1D version of our interpolated (85) with

$$\begin{array}{ll} a_{BCS}=\frac{\beta}{2}(1-c_{int})(z^2-\frac{1}{3}) & b_{BCS}=\frac{\beta}{2}c_{int}(z^2-\frac{1}{3}),\\ c_{BCS}=-\beta a_{int}\frac{z^2-1}{2} & d_{BCS}=-\beta(1-a_{int})\frac{z^2-1}{2}. \end{array}$$

The meaning of our height z is the same as the θ of [1] and the relation between a_{BCS},b_{BCS},c_{BCS} and d_{BCS} (see (1.8) of this article) is satisfied.

4.2 About the KdV-like equation

Various authors have derived either a viscous Boussinesq system or a viscous KdV equation.

One may wonder what is the viscous KdV equation derived from our viscous Boussinesq system and compare it with what may be found in the literature. First, we state and prove the following Proposition.

Proposition 11. *If the initial flow is localized, the KdV change of variables applied to the system (82, 83) leads to*

$$2\tilde{\eta}_\tau + 3a\tilde{\eta}\tilde{\eta}_\xi + \frac{b}{3}\tilde{\eta}_{\xi\xi\xi} - \frac{1}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\tilde{\eta}_\xi(\xi+\xi',\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = O(\varepsilon), \tag{86}$$

for not too small times τ, where we set $\alpha = a\varepsilon$, Re$= R\,\varepsilon^{-5/2}$ and $\beta = b\varepsilon$.

In formula (86), since it has been proved in [15] that KdV is a good approximation of Euler for times up to $1/\varepsilon^2$, and that the velocity is localized, we could replace the integral term by

$$-\frac{1}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{+\infty} \frac{\tilde{\eta}_\xi(\xi+\xi',\tau)}{\sqrt{\xi'}}\mathrm{d}\xi'.$$

This is the term found in [11]. Indeed, if we had not raised the question of the initial condition, we could have used a Fourier transform as [11]. Then, the remaining convolution term would be a clear convolution in ξ over all space.

Proof. We start from the most general form of (40) and use the KdV change of variables

$$(\xi = x - t,\ \tau = \varepsilon t) \Leftrightarrow (x = \xi + \tau/\varepsilon,\ t = \tau/\varepsilon), \tag{87}$$

and change of fields

$$\Phi(x,z,t) = \tilde{\Phi}(x-t,z,\varepsilon t) \Rightarrow \Phi_t = -\tilde{\Phi}_\xi + \varepsilon\tilde{\Phi}_\tau(x-t,z,\varepsilon t), \tag{88}$$

where the generic field Φ is tilded when it depends on the (ξ,τ) variables.

There are only two difficult terms in the system (82, 83) (equivalent to (40)). The first is the convolution which we denote T_1:

$$\begin{aligned} T_1(x,z,t) &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{t} \frac{u_x(x,z,t-t')}{\sqrt{t'}}\mathrm{d}t' \\ &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{t} \frac{\tilde{u}_\xi(x-t+t',z,\varepsilon t-\varepsilon t')}{\sqrt{t'}}\mathrm{d}t', \end{aligned}$$

because of (87). But then it suffices to recognize the function of $(x-t,\varepsilon t) = (\xi,\tau)$ in the last equation to have the term after the KdV change of variables:

$$\begin{aligned} \tilde{T}_1(\xi,z,\tau) &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{\tau/\varepsilon} \frac{\tilde{u}_\xi(\xi+t',z,\tau-\varepsilon t')}{\sqrt{t'}}\mathrm{d}t', \\ &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\tilde{u}_\xi(\xi+\xi',z,\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' + O(\varepsilon^2). \end{aligned} \tag{89}$$

Since the t' variable is in place of a ξ, we changed the notation to ξ'. This term is odd because it has an integration variable (ξ') that has a physical meaning and yet stems from a time (t'). We will discuss it below.
The second difficult term is the one that keeps the initial conditions.

$$\begin{aligned}
T_2(x,z,t) &= +\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0)\right)\int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}}{4t}}\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' \\
\tilde{T}_2(\xi,z,\tau) &= +\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(\xi+\frac{\tau}{\varepsilon},\gamma'') - u_x^{u,0}(\xi+\frac{\tau}{\varepsilon},z=0)\right)\int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}\,\varepsilon}{4\tau}}\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma''.
\end{aligned}$$

If the initial boundary layer is localized (or even if it vanishes), for τ not too small, $u_x^{b,0}(\xi+\frac{\tau}{\varepsilon},\gamma'') - u_x^{u,0}(\xi+\frac{\tau}{\varepsilon},z=0)$ will be small and $\tilde{T}_2$ will be negligible in comparison with ε and so can be dropped.
We can then claim that the Boussinesq system after the KdV change of variables and fields is

$$\left\{\begin{array}{l}
-\tilde{u}_\xi + \varepsilon\tilde{u}_\tau + \tilde{\eta}_\xi + \alpha\tilde{u}\tilde{u}_\xi - \beta\tilde{\eta}_{\xi\xi\xi}\left(\frac{z^2-1}{2}\right) = O(\varepsilon^2), \\
-\tilde{\eta}_\xi + \varepsilon\tilde{\eta}_\tau + \tilde{u}_\xi + \frac{\beta}{2}\tilde{\eta}_{\xi\xi\xi}\left(z^2-\frac{1}{3}\right) + \alpha(\tilde{u}\tilde{\eta})_\xi - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\tilde{u}_\xi(\xi+\xi',z,\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = O(\varepsilon^2).
\end{array}\right. \tag{90}$$

We may notice that at the first order, and as in the derivation of the KdV equation,

$$\tilde{u}_\xi = \tilde{\eta}_\xi + O(\varepsilon) \Rightarrow \tilde{u} = \tilde{\eta} + O(\varepsilon),$$

thanks to a simple and classical integration. But then the sum of the two equations of (90) gives:

$$\varepsilon\tilde{u}_\tau + \varepsilon\tilde{\eta}_\tau + \alpha\tilde{u}\tilde{u}_\xi + \alpha(\tilde{u}\tilde{\eta})_\xi + \frac{\beta}{3}\tilde{\eta}_{\xi\xi\xi} - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\tilde{u}_\xi(\xi+\xi',z,\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = O(\varepsilon^2).$$

Using now the fact that $\tilde{u} = \tilde{\eta} + O(\varepsilon)$, dividing by ε, one states exactly the equation (86). The convolution that used to be on time is now on ξ' and the proof is complete. $\square$

What can be found in the literature ?
As stated in the introduction, various authors already derived either a viscous Boussinesq system or a viscous KdV equation. Yet, none of them have the very same equation as us. We must clarify why there are such differences.
The first article is [23] which proposed

$$-\alpha_3\int_{\xi'=-\infty}^{+\infty}\frac{\tilde{u}_\xi(\xi',\tau)\,\mathrm{sgn}(\xi'-\xi)}{\sqrt{\mid \xi'-\xi \mid}}\mathrm{d}\xi'.$$

but Ott and Sudan made an error in their viscous KdV corrected by [11].
Later, Kakutani and Matsuuchi [11] derive rather rigorously the KdV equation from Navier-Stokes and set the same regime as us. Yet, they do not raise the problem of

the initial condition. As a consequence, they use a Fourier (in time) transform to solve the heat-like equation. They may not have the same equation as us since they model a different reality. Yet, they have the same principal part of the evolution operator (half a derivative). They propose:

$$-\frac{1}{4\sqrt{\pi R}}\int_{\xi'=-\infty}^{+\infty}\frac{\tilde{\eta}_{\xi}(\xi',\tau)(1-\mathrm{sgn}(\xi-\xi'))}{\sqrt{\mid \xi-\xi'\mid}}\mathrm{d}\xi'.$$

Liu and Orfila in [19] (and subsequent articles) derive a Boussinesq system for a regime different from ours (Re$=R\,\varepsilon^{-7/2}$). They solve their heat equation with a sine-transform in the vertical coordinate by quoting [21] where is assumed vanishing initial conditions. Given their regime, their Boussinesq system is right. But when they derive a KdV equation (see [19] p. 89), they do not make explicit their change of variables in the T_1 term. With the change of variable $\xi_{LO}=x-t, \tau_{LO}=(\alpha_{LO}/\mu_{LO})t$, they exhibit (see their (3.19) or (3.21)):

$$-\frac{1}{2\sqrt{\pi}}\int_0^t\frac{\eta_{\xi_{LO}}}{\sqrt{t-T}}\mathrm{d}T,$$

where there remains the former variable t inside the integral and in the bounds. Moreover, the dependence of $\eta_{\xi_{LO}}$ on the variables $(t,\tau_{LO},...\ ?)$ is not written. This explains that they do not see that the time convolution transforms into a *space* one.

Dutykh derives a Boussinesq system by a Leray-Helmholtz decomposition from a Linearized Navier-Stokes [9]. In order to derive the associated KdV (see Sec. 3.2), he assumes $u=\eta+\varepsilon P+\beta Q+...$ and finds P and Q. In this process, he uses only the assumption that waves go right $(\eta_t+\eta_x=O(\varepsilon))$. So he *does not* use the change of time $(\tau=\varepsilon t)$ and writes a formula with unscaled time t (his (14)):

$$-\sqrt{\frac{\nu}{\pi}\frac{g}{h}}\int_0^t\frac{\eta_x}{\sqrt{t-\tau}}\mathrm{d}\tau.$$

5 Conclusion

In this article, we derive the viscous Boussinesq system for surface waves from Navier-Stokes equations with non-vanishing initial conditions (see Proposition 3). One of our by-product is the bottom shear stress as a function of the velocity (cf. Proposition 8) and the decay rate for shallow water (see Proposition 1). We also state the system in 3-D case in (84), and derive the viscous KdV equation from our viscous Boussinesq system (cf. Proposition 11). The differences of our viscous KdV with other equations already derived in the literature are highlighted and explained.

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A Boundary and initial conditions in Lemma 5

As is said in the proof of Lemma 5, we must check that u, given by the necessary equation (59), satisfies the initial condition (64) and the remaining of the boundary conditions $(65)_2$. Concerning the initial condition (64). We try to find the limit when t tends to 0^+ and so $A = A(t)$ tends to $+\infty$. Since one assumes below $\gamma > 0$, the term $-u^u(x,0,.) * \mathcal{L}^{-1}(e^{-\sigma\gamma})$ tends to zero. Then, one can come back to the formula of f and make one change of variables in every integral:

$$f(x,\gamma,t) = \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{+\infty} f_0(x,\Gamma'+\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' - \frac{A}{\sqrt{\pi}}\int_{\gamma}^{+\infty} f_0(x,\Gamma'-\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + O(\varepsilon),$$

up to an exponentially tending to zero function when t tends to 0. This can be rewritten

$$\begin{aligned} f(x,\gamma,t) = & \frac{A}{\sqrt{\pi}}\int_{\gamma}^{+\infty} \left(f_0(x,\Gamma'+\gamma) - f_0(x,\Gamma'-\gamma)\right)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ & +\frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma} f_0(x,\Gamma'+\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + O(\varepsilon) \end{aligned}$$

The first integral may be bounded by

$$\begin{aligned} & \frac{2A}{\sqrt{\pi}}\sup_{\gamma>0} \mid f_0(x,\gamma)\mid \int_{\gamma}^{+\infty} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ \leq\ & \frac{2}{\pi}\sup_{\gamma>0} \mid f_0(x,\gamma)\mid \int_{A\gamma}^{+\infty} e^{-\Gamma''^2}\mathrm{d}\Gamma'', \end{aligned}$$

which clearly tends to zero when t tends to zero thanks to $A(t)$.
For the second integral denoted I_2, one may compute a similar integral where the integration variable of f_0 is frozen:

$$\begin{aligned} I_2' = & \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma} f_0(x,\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ = & f_0(x,\gamma)\frac{1}{\sqrt{\pi}}\int_{-A\gamma}^{A\gamma} e^{-\Gamma''^2}\mathrm{d}\Gamma'', \end{aligned}$$

which clearly tends to $f_0(x,\gamma)$ if $\gamma > 0$ when $t \to 0^+$. So one may make the difference of the second integral with the previous integral (which tends to $f_0(x,\gamma)$) and find:

$$I_2 - I_2' = \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma}\left(f_0(x,\Gamma'+\gamma) - f_0(x,\gamma)\right)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + o_{t\to 0^+}(1).$$

Here we must use the assumption of uniform continuity of the initial data:

$$\forall\epsilon > 0\ \exists\delta > 0\ /\ \mid\gamma'-\gamma\mid < \delta \Rightarrow \mid f_0(x,\gamma') - f_0(x,\gamma)\mid < \epsilon.$$

Then, for any $\epsilon > 0$, there exists a δ such that $I_2 - I_2'$ can be splitted into two parts and bounded by

$$\begin{aligned} & \frac{2A}{\sqrt{\pi}}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid \int_{|\Gamma'|>\delta \bigcap |\Gamma'|<\gamma} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + \frac{A}{\sqrt{\pi}}\epsilon\int_{\Gamma'=-\delta}^{\delta} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ \leq\ & \frac{2A}{\sqrt{\pi}}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid 2\gamma e^{-A^2\delta^2} + \epsilon. \end{aligned}$$

So we have proved that the f given by (67) or u given by (59 satisfies the initial condition. Concerning the boundary condition $(65)_2$). Now we look for the limit when γ tends to $+\infty$. The formula (67) can be written:

$$\hat{f}(x,\gamma,p) = +\frac{R\sqrt{b}}{2\sigma}\int_{\gamma}^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma},$$

up to exponentially tending to zero functions of γ. In this formula, the first integral is bounded by

$$\begin{aligned} & \frac{R\sqrt{b}}{2\sigma}\sup_{\gamma'\geq\gamma} \mid f_0(x,\gamma') \mid \int_{\gamma}^{+\infty} e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{\sigma\gamma} \\ \leq \; & \frac{R\sqrt{b}}{2\sigma^2}\sup_{\gamma'\geq\gamma} \mid f_0(x,\gamma') \mid, \end{aligned}$$

which clearly tends to zero when γ tends to $+\infty$ because $f_0(x,\gamma)$ tends to zero when γ tends to $+\infty$.

For the second integral, one needs to cut it at a value Γ given by the definition of $f_0 \to 0$ when γ tends to $+\infty$ ($\forall\epsilon > 0 \; \exists\Gamma > 0 \; / \mid \gamma \mid > \Gamma \Rightarrow \mid f_0 \mid < \epsilon$). We can bound it with:

$$\frac{R\sqrt{b}}{2\sigma}\int_0^{\Gamma} \mid f_0(x,\gamma') \mid e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\epsilon\int_{\Gamma}^{\gamma} e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma}.$$

Since the first term tends to zero when γ tends to $+\infty$ (Γ fixed) and the second term is less than $R\sqrt{b}\epsilon/(2\sigma^2)$, the whole tends to zero with ϵ.
So we completed the proof that $(65)_2$ is satisfied.

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