A note on decay rates of solutions to a system of cubic nonlinear Schrödinger equations in one space dimension

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Abstract: We consider the initial value problem for a system of cubic nonlinear Schrödinger equations with different masses in one space dimension. Under a suitable structural condition on the nonlinearity, we will show that the small amplitude solution exists globally and decays of the rate $O(t^{-1/2}(\log t)^{-1/2})$ in L^∞ as t tends to infinity, if the system satisfies certain mass relations.

Key Words: NLS system; Nonlinear dissipation; Time-decay; Critical nonlinearity.
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1 Introduction

We consider the initial value problem for a system of cubic nonlinear Schrödinger equations in one space dimension:

$$i\partial_t u_j + \frac{1}{2m_j}\partial_x^2 u_j = F_j(u), \qquad (t,x) \in (0,\infty)\times\mathbb{R} \tag{1.1}$$

with the initial condition

$$u_j(0,x) = u_j^\circ(x), \qquad x \in \mathbb{R} \tag{1.2}$$

for $j = 1,\ldots,N$, where $i = \sqrt{-1}$, $u = (u_j)_{1\le j\le N}$ is a $\mathbb{C}^N$-valued unknown function of $(t,x) \in [0,\infty)\times\mathbb{R}$ and the masses $m_1,\ldots,m_N$ are positive constants. Simply we assume that the nonlinear term $F = (F_j)_{1\le j\le N} : \mathbb{C}^N \to \mathbb{C}^N$ is a cubic homogeneous polynomial in $(u,\overline{u})$ with some complex coefficients, i.e.,

$$F_j(u) = \sum_{k,l,m=1}^{N} \sum_{\sigma\in\{+,-\}^3} C_{jklm}^{\sigma} u_k^{(\sigma_1)} u_l^{(\sigma_2)} u_m^{(\sigma_3)}$$

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with complex constants C^{σ}_{jklm}, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $u_j^{(+)} = u_j$, $u_j^{(-)} = \overline{u_j}$ respectively. Also we assume that the system satisfies *gauge invariance*, i.e.,

$$F_j\left(e^{im_1\theta}z_1, \dots, e^{im_N\theta}z_N\right) = e^{im_j\theta}F_j\left(z_1, \dots, z_N\right) \tag{1.3}$$

for each $j = 1, \dots, N$ and any $\theta \in \mathbb{R}$, $z = (z_j)_{1\le j\le N} \in \mathbb{C}^N$. In the present paper, we are interested in large-time behavior of the small amplitude solution for (1.1)–(1.2).

Let us recall some previous results briefly. There is a large body of literature discussing global existence and large-time behavior of solutions for the single nonlinear Schrödinger equations in n-space dimensions of the form

$$i\partial_t u + \frac{1}{2}\Delta u = G(u), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \tag{1.4}$$

where Δ is the Laplace operator in $x \in \mathbb{R}^n$ and $G(u)$ is a nonlinear term. We refer the readers to [2] concerning the recent development on studies of (1.4). Let us denote by $p_S(n)$ the Strauss exponent, which is defined by $p_S(n) = \frac{n+2+\sqrt{n^2+12n+4}}{2n}$. Strauss showed in [13] that if the nonlinear term $G(u)$ satisfies $|G'(u)| \le C|u|^{p-1}$ with $p > p_S(n)$, then there exists a unique global solution for (1.4) with a suitable small initial data. Note that for one-dimensional case, we have $p_S(1) = \frac{3+\sqrt{17}}{2} \approx 3.56$. Now we concentrate our attention to the power-type nonlinearity, i.e. the case that $G(u) = \lambda|u|^{p-1}u$ with $\lambda \in \mathbb{C}\setminus\{0\}$, because it is a typical one satisfying the gauge invariance condition (1.3). In this case, when $p > 1 + 2/n$, it is well-known that any solution $u(t)$ of (1.4) behaves like a free solution as $t \to \infty$, if the data belongs to suitable weighted Sobolev spaces (see e.g. [14]). On the other hand, Barab [1] showed that there is no asymptotically free solution for (1.4) if $1 \le p \le 1 + 2/n$. So we can see that cubic nonlinearities are critical for (1.4) when $n = 1$.

Now we consider (1.4) with $G(u) = \lambda|u|^2u$, $\lambda \in \mathbb{C}$ in one space dimension. In this case, we can find an asymptotic profile of the solution to (1.4) in [11] for sufficiently small final data, if $\lambda \in \mathbb{R}$. We note that the asymptotic profile given there is just a phase-shifted free profile, so the amplitude of the solution still behaves like a free solution. Similar results can be found in [3]. More precisely, Hayashi and Naumkin proved in [3] that the solution decays like $O(t^{-1/2})$ in L^∞ and behaves like a phase-shifted free solution when $\lambda \in \mathbb{R}$, if the initial data is sufficiently small and in suitable weighted Sobolev spaces. On the other hand, according to the result by Shimomura [12], the solution decays like $O(t^{-1/2}(\log t)^{-1/2})$ in L^∞, if $\operatorname{Im}\lambda < 0$ and the initial data is small enough. Remember that the L^∞-decay rate of the free evolution is $t^{-n/2}$ for n-dimensional cases. Therefore this gain of additional logarithmic time-decay can be read as a kind of the long-range effect. His result was extended by [4], which considers the nonlinear terms including derivative types also.

Next we turn our attention to the case of systems, i.e. (1.1) with $N \ge 2$ where ∂_x^2 is replaced by Δ. In this case, the problem becomes more complicated because global existence and large-time behavior of the solution are affected by the ratio of masses as well as the structure of nonlinearities. Li found some structural conditions on the quadratic nonlinearities and the masses in [9] under which the solution to (1.1) exists globally and decays like a free solution in two space dimension, if the data is small and belongs to a suitable weighted Sobolev space (note that quadratic nonlinearities are critical in two space dimension). And by a minor modification of the method there, we can obtain similar results for one-dimensional cases also. Recently, Katayama, Li and Sunagawa considered the quadratic two-dimensional NLS system

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \lambda_1 |u_1| u_1 + \mu_1 \overline{u_2} u_3, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = \lambda_2 |u_2| u_2 + \mu_2 \overline{u_1} u_3, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^2 \\ i\partial_t u_3 + \dfrac{1}{2m_3}\Delta u_3 = \lambda_3 |u_3| u_3 + \mu_3 u_1 u_2 \end{cases} \tag{1.5}$$

in [5] with complex coefficients on the nonlinearities under the mass relation $m_1 + m_2 = m_3$. They showed if $\operatorname{Im}\lambda_j < 0$ for $j = 1,2,3$ and $\kappa_1\mu_1 + \kappa_2\mu_2 = \kappa_3\overline{\mu_3}$ with some $\kappa_1, \kappa_2, \kappa_3 > 0$, then the solution of (1.5) decays like $O(t^{-1}(\log t)^{-1})$ in L^∞ for sufficiently small data which belongs to certain weighted Sobolev spaces. We refer the readers to [10] and the references cited therein for the recent progress on two-dimensional NLS systems with critical nonlinearities.

This paper can be regarded as a one-dimensional version of the paper [5] or a piece of extension of the paper [12]. The aim of the present work is to introduce a structural condition of the cubic nonlinearities and the masses under which (1.1)–(1.2) admits a unique global solution and it decays like $O(t^{-1/2}(\log t)^{-1/2})$ in L^∞ as $t \to \infty$, if the initial data is small enough and belongs to suitable weighted Sobolev spaces.

2 Main Results

In order to state our main results, we introduce some notations here. We denote the usual Lebesgue space by $L^p(\mathbb{R})$ equipped with the norm $\|\phi\|_{L^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p \, dx\right)^{1/p}$ if $p \in [1,\infty)$ and $\|\phi\|_{L^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space is defined by

$$H_p^{s,q}(\mathbb{R}) = \left\{ \phi = (\phi_1, \ldots, \phi_N) \in L^p(\mathbb{R}) : \|\phi\|_{H_p^{s,q}} = \sum_{j=1}^N \|\phi_j\|_{H_p^{s,q}} < \infty \right\}$$

with the norm $\|\phi_j\|_{H_p^{s,q}} = \|\langle x\rangle^q \langle i\partial_x\rangle^s \phi_j\|_{L^p}$ for $s, q \in \mathbb{R}$ and $p \in [1,\infty]$, where $\langle\cdot\rangle = \sqrt{1+\cdot^2}$. For simplicity, we write $H^{s,q} = H_2^{s,q}$, $H_p^s = H_p^{s,0}$ and the usual Sobolev space as $H^s = H^{s,0}$. We define the Fourier transform $\hat{\phi}(\xi)$ of a function $\phi(x)$ by

$$(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x) \, dx.$$

Then the inverse Fourier transform is given by

$$(\mathcal{F}^{-1}\phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(\xi) \, d\xi.$$

We denote by $y \cdot z$ the standard scalar product in $\mathbb{C}^N$ for $y, z \in \mathbb{C}^N$ and write $|z| = \sqrt{z \cdot z}$ as usual. We can now formulate the main results.

Theorem 2.1. *Let* $u^\circ \in H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$ *and* $\|u^\circ\|_{H^{1,0}} + \|u^\circ\|_{H^{0,1}} = \varepsilon$. *Assume the condition* (1.3) *holds and suppose that there exists an* $N \times N$ *positive Hermitian matrix* A *such that*

$$\operatorname{Im}\left(F(z) \cdot Az\right) \leq 0 \tag{2.1}$$

for all $z \in \mathbb{C}^N$. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *the initial value problem* (1.1)–(1.2) *admits a unique global solution*

$$u(t) \in C^0\left([0,\infty); H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})\right)$$

satisfying the time-decay estimate

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-1/2}$$

for all $t \geq 0$.

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 are fulfilled. Moreover, suppose that there exist an* $N \times N$ *positive Hermitian matrix* A *and constants* $C_*, C^* > 0$ *such that*

$$-C_*|z|^4 \leq \operatorname{Im}\left(F(z) \cdot Az\right) \leq -C^*|z|^4 \tag{2.2}$$

for all $z \in \mathbb{C}^N$. *Then the global solution of* (1.1)–(1.2), *which is guaranteed by Theorem 2.1, satisfies the time-decay estimate*

$$\|u(t)\|_{L^\infty} \leq \frac{C(1+t)^{-1/2}}{\sqrt{\log(2+t)}}$$

for all $t \geq 0$.

Here we give some examples satisfying the assumption of Theorem 2.1 or Theorem 2.2 with suitable mass relations.

Example 2.1. We consider the following two-component system

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\partial_x^2 u_1 = F_1(u) = \lambda_1 |u|^2 u_1 + \mu_1 \overline{u_1}^2 u_2, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\partial_x^2 u_2 = F_2(u) = \lambda_2 |u|^2 u_2 + \mu_2 u_1^3 \end{cases} \tag{2.3}$$

in $(t,x) \in (0,\infty) \times \mathbb{R}$ under the mass relation $m_2 = 3m_1$, where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Then we can see that the system (2.3) satisfies the gauge invariance condition (1.3). Also we assume that the constants satisfy the following conditions:

$$\operatorname{Im}\lambda_j < 0 \ \text{ for } \ j = 1,2, \qquad \kappa_1\mu_1 = \kappa_2\overline{\mu_2} \ \text{ with some } \ \kappa_1, \kappa_2 > 0.$$

Then we have

$$-C_*|z|^4 \leq \operatorname{Im}\left(F(z)\cdot Az\right) = \sum_{j=1}^{2} \kappa_j \left(\operatorname{Im}\lambda_j\right)|z|^2|z_j|^2 + \operatorname{Im}\left(\kappa_1\mu_1\overline{z_1}^3 z_2 + \kappa_2\mu_2 z_1^3\overline{z_2}\right) \leq -C^*|z|^4$$

for all $z \in \mathbb{C}^2$ with $A = \operatorname{diag}(\kappa_1, \kappa_2)$. Therefore we can conclude from Theorem 2.1 and Theorem 2.2 that there exists a unique global solution $u(t) \in C^0\left([0,\infty); H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})\right)$ to the initial value problem (2.3)–(1.2) and the solution decays like $O\left(t^{-1/2}(\log t)^{-1/2}\right)$ in L^∞ as $t \to \infty$, if $u^\circ \in H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$ and it is sufficiently small.

Remark 2.1. Since the Klein-Gordon equation is a relativisitic version of the Schrödinger equation, it is interesting to compare our results with a system of nonlinear Klein-Gordon equations. Here we consider the following two-component cubic nonlinear Klein-Gordon system including dissipative nonlinearities

$$\begin{cases} (\partial_t^2 - \partial_x^2 + m_1^2)u_1 = \lambda_1|\partial_t u|^2\partial_t u_1 - (\partial_t u_1)^2\partial_t u_2 \\ (\partial_t^2 - \partial_x^2 + m_2^2)u_2 = \lambda_2|\partial_t u|^2\partial_t u_2 + (\partial_t u_1)^3 \end{cases} \tag{2.4}$$

in $(t,x) \in (0,\infty)\times\mathbb{R}$ with the same mass relation $m_2 = 3m_1$ (often called *mass resonance* relation) as above, where $u = (u_1, u_2)$ is real-valued and $\lambda_1, \lambda_2 < 0$. Then as pointed out in [7], we can modify the proof of [8] to see that (2.4) admits a unique global solution $u(t)$ and it decays like $O\left(t^{-1/2}(\log t)^{-1/2}\right)$ in L^∞ as $t \to \infty$, if the Cauchy data are sufficiently smooth, small and compactly-supported. We remark that the condition $\lambda_1, \lambda_2 < 0$ reflects a dissipative character in this case, as the condition $\operatorname{Im}\lambda_1, \operatorname{Im}\lambda_2 < 0$ implies a dissipative property in (2.3).

We end this section by giving an example which satisfies the condition (2.1) but violates (2.2).

Example 2.2. We consider the following four-component system

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\partial_x^2 u_1 = F_1(u) = \mu_1\overline{u_2 u_3}u_4, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\partial_x^2 u_2 = F_2(u) = \mu_2\overline{u_3}u_4\overline{u_1}, \\ i\partial_t u_3 + \dfrac{1}{2m_3}\partial_x^2 u_3 = F_3(u) = \mu_3 u_4\overline{u_1 u_2}, \\ i\partial_t u_4 + \dfrac{1}{2m_4}\partial_x^2 u_4 = F_4(u) = \mu_4 u_1 u_2 u_3 \end{cases} \tag{2.5}$$

in $(t,x) \in (0,\infty)\times\mathbb{R}$ under the mass relation $m_4 = m_1 + m_2 + m_3$, where $\mu_1, \ldots, \mu_4 \in \mathbb{C}$. Then we can see that the condition (1.3) holds and $\operatorname{Im}\left(F(z)\cdot Az\right) = 0$ for all $z \in \mathbb{C}^4$ with $A = \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$, if there exist some positive constants $\kappa_1, \ldots, \kappa_4$ such that $\kappa_1\mu_1 + \kappa_2\mu_2 + \kappa_3\mu_3 = \kappa_4\overline{\mu_4}$. Therefore in this case, we can conclude from Theorem 2.1 that the global solution of (2.5) decays like a free solution if the data is small enough.

The rest of this paper is organized as follows. In Section 3, we compile some basic facts concerning the free Schrödinger evolution group. Section 4 is devoted to obtain a suitable a priori estimate from which Theorem 2.1 follows immediately. After that, we prove Theorem 2.2 in Section 5 and discuss the optimality of the decay-rate of the solution. In what follows, all non-negative constants will be denoted by C which may vary from line to line unless otherwise specified.

3 Preliminaries

In this section, we introduce some notations and useful estimates which will be used in Section 4 and Section 5 for the proof of the main results. In what follows, we denote $\mathcal{AB} = (\mathcal{A}_j\mathcal{B}_j)_{1\le j\le N}$ for N-dimensional column vectors $\mathcal{A} = (\mathcal{A}_j)_{1\le j\le N}$ and $\mathcal{B} = (\mathcal{B}_j)_{1\le j\le N}$. First we introduce the free Schrödinger evolution group $\mathcal{U}(t) = (\mathcal{U}_j(t))_{1\le j\le N}$ defined by

$$\mathcal{U}_j(t) = e^{\frac{it}{2m_j}\partial_x^2} = \mathcal{F}^{-1}e^{-\frac{it}{2m_j}\xi^2}\mathcal{F}.$$

It is well-known that $\mathcal{U}(t)$ is decomposed into $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{G}\mathcal{M}(t)$, where the multiplication factor $\mathcal{M}(t) = (\mathcal{M}_j(t))_{1\le j\le N}$ is defined by $\mathcal{M}_j(t)\phi(x) = \exp(\frac{im_j}{2t}x^2)\phi(x)$, the Fourier-like transform $\mathcal{G} = (\mathcal{G}_j)_{1\le j\le N}$ (see e.g. [5]) and the dilation operator $\mathcal{D}(t)$ are given by

$$(\mathcal{G}_j\phi)(\xi) = \sqrt{\frac{m_j}{i}}(\mathcal{F}\phi)(m_j\xi), \qquad \mathcal{D}(t)\phi(x) = \frac{1}{\sqrt{t}}\phi\left(xt^{-1}\right).$$

Also we define $\mathcal{W}(t) = \mathcal{G}\mathcal{M}(t)\mathcal{G}^{-1}$ so that $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\mathcal{G}$. Then we have

$$\|(\mathcal{W}(t)-1)\,\phi\|_{L^\infty} \le Ct^{-1/4}\|\phi\|_{H^1}, \qquad \left\|\left(\mathcal{W}^{-1}(t)-1\right)\phi\right\|_{L^\infty} \le Ct^{-1/4}\|\phi\|_{H^1} \tag{3.1}$$

for $\phi = (\phi_1,\dots,\phi_N) \in H^1(\mathbb{R})$. Indeed it is easy to check that the estimates (3.1) hold. Since

$$|\mathcal{M}_j(t)-1| = \left|e^{\frac{im_j}{2t}x^2}-1\right| = 2\left|\sin\frac{m_jx^2}{4t}\right| \le C\frac{|x|^{2\beta}}{t^\beta}$$

where $\beta \in [0,1]$, we find

$$\begin{aligned}\|(\mathcal{W}(t)-1)\,\phi\|_{L^\infty} &= \left\|\mathcal{G}(\mathcal{M}(t)-1)\mathcal{G}^{-1}\phi\right\|_{L^\infty} \le C\left\|(\mathcal{M}(t)-1)\mathcal{G}^{-1}\phi\right\|_{L^1}\\ &\le Ct^{-\beta}\left\|\langle x\rangle^{-\eta}\langle x\rangle^{\eta}|x|^{2\beta}\mathcal{F}^{-1}\phi\right\|_{L^1}\\ &\le Ct^{-\beta}\left\|\langle x\rangle^{-\eta}\right\|_{L^2}\left\|\langle x\rangle^{2\beta+\eta}\mathcal{F}^{-1}\phi\right\|_{L^2}\\ &\le Ct^{-\beta}\|\phi\|_{H^{2\beta+\eta}}\end{aligned}$$

holds for any $\eta > 1/2$. So by choosing $\beta = 1/4$, we get the first estimate of (3.1). In view of the relation $\mathcal{W}^{-1}(t)-1 = -(\mathcal{W}(t)-1)\mathcal{W}^{-1}(t)$ and $\|\mathcal{W}^{-1}(t)\phi\|_{H^1} \le C\|\phi\|_{H^1}$, the second estimate of (3.1) follows immediately.

4 A Priori Estimates

The argument of this section is similar to those of the previous works, for example [5], [9] and [12]. Let $u(t)$ be the solution to (1.1)–(1.2) in $[0,T]$ and we define

$$\|u\|_{X_T} = \sup_{t\in[0,T]} \Big(\langle t\rangle^{-\gamma} \left(\|u(t)\|_{H^1} + \|\mathcal{U}(-t)u(t)\|_{H^{0,1}} \right) + \langle t\rangle^{1/2} \|u(t)\|_{L^\infty} \Big)$$

where $0<\gamma\ll 1$ small.

Lemma 4.1. *Under the assumption of Theorem 2.1, there exist $\varepsilon_1>0$ and $C_0>0$ such that $\|u\|_{X_T}\le\sqrt{\varepsilon}$ implies $\|u\|_{X_T}\le C_0\varepsilon$ for any $\varepsilon\in(0,\varepsilon_1]$. Here the constant C_0 does not depend on T.*

Once this lemma is proved, we can obtain the global existence part of Theorem 2.1 in the following way: By taking $\varepsilon_0\in(0,\varepsilon_1]$ so that $2C_0\sqrt{\varepsilon_0}\le 1$, we deduce that $\|u\|_{X_T}\le\sqrt{\varepsilon}$ implies $\|u\|_{X_T}\le\sqrt{\varepsilon}/2$ for any $\varepsilon\in(0,\varepsilon_0]$. Then by the continuity argument, we have $\|u\|_{X_T}\le C_0\varepsilon$ as long as the solution exists. So the local solution to (1.1)–(1.2) can be extended to the global one.

From now on, we will prove Lemma 4.1. Since $i\partial_t\mathcal{U}_j(-t)=\mathcal{U}_j(-t)(i\partial_t+\frac{1}{2m_j}\partial_x^2)$, taking $\mathcal{U}(-t)$ to the both sides of (1.1), we get the following integral equation

$$u(t)=\mathcal{U}(t)u^\circ - i\int_0^t \mathcal{U}(t-\tau)F(u(\tau))\,d\tau.$$

Taking the H^1 norm, we obtain

$$\begin{aligned}
\|u(t)\|_{H^1} &\le \|u^\circ\|_{H^1} + \int_0^t \|F(u(\tau))\|_{H^1}\,d\tau \\
&\le C\varepsilon + C\int_0^t \|u(\tau)\|_{L^\infty}^2 \|u(\tau)\|_{H^1}\,d\tau \\
&\le C\varepsilon + C\int_0^t \varepsilon^{3/2}\langle\tau\rangle^{-1+\gamma}\,d\tau \le C\varepsilon\langle t\rangle^{\gamma}, \qquad (4.1)
\end{aligned}$$

where we used the assumption $\|u\|_{X_T}\le\sqrt{\varepsilon}$. Now we are going to estimate $\|\mathcal{U}(-t)u(t)\|_{H^{0,1}}$. Since $\mathcal{M}(-t)F(u)=F(\mathcal{M}(-t)u)$ by (1.3), using the relation

$$\mathcal{U}_j(t)x\mathcal{U}_j(-t)=\mathcal{M}_j(t)itm_j^{-1}\partial_x\mathcal{M}_j(-t),$$

we have

$$\begin{aligned}
\|x\mathcal{U}(-t)F(u)\|_{L^2} &= \|\mathcal{U}(t)x\mathcal{U}(-t)F(u)\|_{L^2} \le Ct\|\partial_x\mathcal{M}(-t)F(u)\|_{L^2} \\
&= Ct\|\partial_x F(\mathcal{M}(-t)u)\|_{L^2} \le Ct\|\mathcal{M}(-t)u\|_{L^\infty}^2\|\partial_x\mathcal{M}(-t)u\|_{L^2} \\
&\le C\|u\|_{L^\infty}^2\|x\mathcal{U}(-t)u\|_{L^2}. \qquad (4.2)
\end{aligned}$$

Therefore by the similar way as above, we obtain

$$\begin{aligned}
\|\mathcal{U}(-t)u(t)\|_{H^{0,1}} &\le \|u^\circ\|_{H^{0,1}} + \int_0^t \|\mathcal{U}(-\tau)F(u(\tau))\|_{H^{0,1}}\, d\tau \\
&\le C\varepsilon + C\int_0^t \|u(\tau)\|_{L^\infty}^2 \|\mathcal{U}(-\tau)u(\tau)\|_{H^{0,1}}\, d\tau \\
&\le C\varepsilon \langle t\rangle^\gamma, \qquad (4.3)
\end{aligned}$$

where we used (4.2) for the second inequality. Next, we consider the term $\|u(t)\|_{L^\infty}$. If $t \le 1$, the standard Sobolev embedding and (4.1) suffice to obtain

$$\langle t\rangle^{1/2+\gamma} \|u(t)\|_{L^\infty} \le 2^{1/4+\gamma/2} \|u(t)\|_{H^1} \le C\varepsilon \langle t\rangle^\gamma. \qquad (4.4)$$

So from now on we consider the case that $t \ge 1$. We define the new function $\alpha = (\alpha_j)_{1\le j\le N}$ by

$$\alpha(t,\xi) = \mathcal{G}\left(\mathcal{U}(-t)u(t)\right)(\xi).$$

Then from the decomposition of the free Schrödinger evolution group and (1.3), we have

$$\begin{aligned}
i\partial_t\alpha(t,\xi) &= \mathcal{G}\mathcal{U}(-t)F(u(t))(\xi) \\
&= \mathcal{W}^{-1}(t)\mathcal{D}^{-1}(t)\mathcal{M}(-t)F(\mathcal{M}(t)\mathcal{D}(t)\mathcal{W}(t)\alpha(t,\xi)) \\
&= t^{-1}\mathcal{W}^{-1}(t)F(\mathcal{W}(t)\alpha(t,\xi)) \\
&= t^{-1}F(\alpha(t,\xi)) + R(t,\xi), \qquad (4.5)
\end{aligned}$$

where

$$R(t,\xi) = t^{-1}\mathcal{W}^{-1}(t)F(\mathcal{W}(t)\alpha(t,\xi)) - t^{-1}F(\alpha(t,\xi)).$$

As we shall see below, R can be regarded as a remainder because it decays strictly faster than $O(t^{-1})$ in L^∞, while the first term of the right-hand side of (4.5) plays a role as a main term. Since we have by the estimate (3.1), the Sobolev embedding and (4.3),

$$\begin{aligned}
t^{-1}\left\|\left(\mathcal{W}^{-1}(t)-1\right)F(\mathcal{W}(t)\alpha(t))\right\|_{L^\infty} &\le Ct^{-5/4}\|F(\mathcal{W}(t)\alpha(t))\|_{H^1} \le Ct^{-5/4}\|\mathcal{W}(t)\alpha(t)\|_{H^1}^3 \\
&\le Ct^{-5/4}\|\alpha(t)\|_{H^1}^3 \le Ct^{-5/4}\|\mathcal{G}\mathcal{U}(-t)u(t)\|_{H^1}^3 \\
&\le Ct^{-5/4}\|\mathcal{U}(-t)u(t)\|_{H^{0,1}}^3 \\
&\le C\varepsilon^3 t^{-5/4+3\gamma}
\end{aligned}$$

and similarly

$$\begin{aligned}
&t^{-1}\|F(\mathcal{W}(t)\alpha(t)) - F(\alpha(t))\|_{L^\infty} \\
&\quad\le Ct^{-1}\|(\mathcal{W}(t)-1)\alpha(t)\|_{L^\infty}\left(\|(\mathcal{W}(t)-1)\alpha(t)\|_{L^\infty}^2 + \|\mathcal{W}(t)\alpha(t)\|_{L^\infty}\|\alpha(t)\|_{L^\infty}\right) \\
&\quad\le Ct^{-5/4}\|\alpha(t)\|_{H^1}\left(t^{-1/2}\|\alpha(t)\|_{H^1}^2 + \|\alpha(t)\|_{H^1}^2\right) \\
&\quad\le C\varepsilon^3 t^{-5/4+3\gamma},
\end{aligned}$$

we deduce that the remainder satisfies the estimate

$$\begin{aligned}\|R(t)\|_{L^\infty} &\le t^{-1}\left(\left\|\left(\mathcal{W}^{-1}(t)-1\right)F(\mathcal{W}(t)\alpha(t))\right\|_{L^\infty}+\|F(\mathcal{W}(t)\alpha(t))-F(\alpha(t))\|_{L^\infty}\right)\\ &\le C\varepsilon^3 t^{-5/4+3\gamma}\end{aligned}\tag{4.6}$$

for $t \ge 1$. Here we note that

$$|y\cdot Az| \le (y\cdot Ay)^{1/2}(z\cdot Az)^{1/2}, \qquad \lambda_*|z|^2 \le z\cdot Az \le \lambda^*|z|^2 \tag{4.7}$$

hold for $y, z \in \mathbb{C}^N$, where the matrix A is in Theorem 2.1 and λ^* (resp. λ_*) is the largest (resp. smallest) eigenvalue of A. Therefore it follows from (4.5), (2.1), (4.7) and (4.6) that

$$\begin{aligned}\partial_t\left(\alpha(t,\xi)\cdot A\alpha(t,\xi)\right) &= 2\,\mathrm{Im}\left(i\partial_t\alpha(t,\xi)\cdot A\alpha(t,\xi)\right)\\ &= 2t^{-1}\,\mathrm{Im}\left(F(\alpha(t,\xi))\cdot A\alpha(t,\xi)\right)+2\,\mathrm{Im}\left(R(t,\xi)\cdot A\alpha(t,\xi)\right)\\ &\le Ct^{-5/4}\left|t^{5/4}R(t,\xi)\cdot A\alpha(t,\xi)\right|\\ &\le Ct^{-5/4}\left(\alpha(t,\xi)\cdot A\alpha(t,\xi)+t^{5/2}R(t,\xi)\cdot AR(t,\xi)\right)\\ &\le Ct^{-5/4}\alpha(t,\xi)\cdot A\alpha(t,\xi)+C\varepsilon^6 t^{-5/4+6\gamma}.\end{aligned}\tag{4.8}$$

Noting that

$$\|\alpha(1)\|_{L^\infty} = \|\mathcal{G}\mathcal{U}(-1)u(1)\|_{L^\infty} \le C\,\|\mathcal{G}\mathcal{U}(-1)u(1)\|_{H^1} \le C\,\|\mathcal{U}(-1)u(1)\|_{H^{0,1}} \le C\varepsilon,$$

integrating (4.8) with respect to time lead to

$$\alpha(t,\xi)\cdot A\alpha(t,\xi) \le C\varepsilon^2 + \int_1^t \tau^{-5/4}\alpha(t,\xi)\cdot A\alpha(t,\xi)\,d\tau.$$

Thus the Gronwall lemma and (4.7) yield

$$\|\alpha(t)\|_{L^\infty} \le C\varepsilon \tag{4.9}$$

for $t \ge 1$. Hence with the estimates (4.9), (4.3) and the inequality

$$\begin{aligned}\|\mathcal{G}(\mathcal{M}(t)-1)\mathcal{U}(-t)u(t)\|_{L^\infty} &= \|(\mathcal{W}(t)-1)\mathcal{G}\mathcal{U}(-t)u(t)\|_{L^\infty}\\ &\le Ct^{-1/4}\,\|\mathcal{G}\mathcal{U}(-t)u(t)\|_{H^1},\end{aligned}\tag{4.10}$$

we finally obtain

$$\begin{aligned}\|u(t)\|_{L^\infty} &= \|\mathcal{M}(t)\mathcal{D}(t)\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)u(t)\|_{L^\infty}\\ &\le Ct^{-1/2}\,\|\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)u(t)\|_{L^\infty}\\ &\le Ct^{-1/2}\left(\|\mathcal{G}\mathcal{U}(-t)u(t)\|_{L^\infty}+\|\mathcal{G}(\mathcal{M}(t)-1)\mathcal{U}(-t)u(t)\|_{L^\infty}\right)\\ &\le Ct^{-1/2}\left(\|\alpha(t)\|_{L^\infty}+t^{-1/4}\,\|\mathcal{U}(-t)u(t)\|_{H^{0,1}}\right)\\ &\le C\varepsilon t^{-1/2}\end{aligned}\tag{4.11}$$

for $t \ge 1$. By (4.1), (4.3), (4.4) and (4.11), we arrive at Lemma 4.1 and the L^∞-decay estimate in Theorem 2.1 follows immediately. □

5 Proof of Theorem 2.2

Now we are in a position to prove Theorem 2.2. Note that the similar arguments of this section are also used in the previous works [5], [6] and [8]. If $t \le 2$ then by (4.1) we have

$$\langle t \rangle^{1/2+\gamma} \sqrt{\log(2+t)} \left\| u(t) \right\|_{L^\infty} \le C \left\| u(t) \right\|_{H^1} \le C\varepsilon \langle t \rangle^{\gamma},$$

so we only consider the case that $t \ge 2$. First we note that

$$\partial_t \left((\log t)^2 \left(\alpha(t,\xi) \cdot A\alpha(t,\xi) \right) \right) = (\log t)^2 \partial_t (\alpha(t,\xi) \cdot A\alpha(t,\xi)) + \frac{2\log t}{t} \alpha(t,\xi) \cdot A\alpha(t,\xi).$$

Similarly to (4.8), we have

$$\begin{aligned} \partial_t (\alpha(t,\xi) \cdot A\alpha(t,\xi)) &= 2t^{-1} \operatorname{Im} \left(F(\alpha(t,\xi)) \cdot A\alpha(t,\xi) \right) + 2 \operatorname{Im} \left(R(t,\xi) \cdot A\alpha(t,\xi) \right) \\ &\le -2C^* t^{-1} |\alpha(t,\xi)|^4 + C|R(t,\xi)||\alpha(t,\xi)| \\ &\le -2C^* t^{-1} |\alpha(t,\xi)|^4 + C\varepsilon^4 t^{-5/4+3\gamma}, \end{aligned}$$

where we used (4.6), (4.7), (4.9) and the assumption (2.2) with the constant C^* appearing in Theorem 2.2. Also we have

$$\begin{aligned} \alpha(t,\xi) \cdot A\alpha(t,\xi) \le \lambda^* |\alpha(t,\xi)|^2 &= \frac{\lambda^*}{\sqrt{2C^* \log t}} \sqrt{2C^* \log t} |\alpha(t,\xi)|^2 \\ &\le \frac{(\lambda^*)^2}{4C^* \log t} + C^* \log t |\alpha(t,\xi)|^4 \end{aligned}$$

by the Young inequality. Piecing them together, we obtain

$$\partial_t \left((\log t)^2 \left(\alpha(t,\xi) \cdot A\alpha(t,\xi) \right) \right) \le Ct^{-1} + C\varepsilon^4 (\log t)^2 t^{-5/4+3\gamma}.$$

Integrating with respect to time, we get

$$(\log t)^2 \left(\alpha(t,\xi) \cdot A\alpha(t,\xi) \right) \le C\varepsilon^2 + C \int_2^t \left(\tau^{-1} + \frac{(\log \tau)^2}{\tau^{5/4-3\gamma}} \right) d\tau \le C \log t$$

for $t \ge 2$. Thus (4.7) yields

$$\left\| \alpha(t) \right\|_{L^\infty} \le C(\log t)^{-1/2}.$$

Therefore by the same arguments as in (4.11), we arrive at

$$\left\| u(t) \right\|_{L^\infty} \le Ct^{-1/2} \left(\left\| \alpha(t) \right\|_{L^\infty} + t^{-1/4} \left\| \mathcal{U}(-t) u(t) \right\|_{H^{0,1}} \right) \le Ct^{-1/2} (\log t)^{-1/2},$$

which proves Theorem 2.2. □

Finally, we discuss the optimality of the decay rate $O(t^{-1/2}(\log t)^{-1/2})$. We put $u^\circ(x) = \delta v^\circ(x) (\not\equiv 0) \in H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$ with $\delta > 0$ (note that $\varepsilon = \left\| u^\circ \right\|_{H^{1,0}} + \left\| u^\circ \right\|_{H^{0,1}} \le C\delta$). Here we will show that the solution does not decay strictly faster than $t^{-1/2}(\log t)^{-1/2}$ as $t \to \infty$, if δ is sufficiently small. Suppose that

$$\lim_{t\to\infty} (t \log t)^{1/2} \|u(t)\|_{L^\infty} = 0 \tag{5.1}$$

holds. By (4.10) and (4.3), we have

$$\begin{aligned}
\|\alpha(t)\|_{L^\infty} &= \|\mathcal{G}\mathcal{U}(-t)u(t)\|_{L^\infty} \\
&\le \|\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)u(t)\|_{L^\infty} + \|\mathcal{G}(\mathcal{M}(t)-1)\mathcal{U}(-t)u(t)\|_{L^\infty} \\
&\le Ct^{1/2} \|\mathcal{M}(t)\mathcal{D}(t)\mathcal{G}\mathcal{M}(t)\mathcal{U}(-t)u(t)\|_{L^\infty} + Ct^{-1/4} \|\mathcal{U}(-t)u(t)\|_{H^{0,1}} \\
&\le Ct^{1/2} \|u(t)\|_{L^\infty} + C\delta t^{-1/4+\gamma}.
\end{aligned}$$

Thus from (5.1), we get

$$(\log t)^{1/2}|\alpha(t,\xi)| \le C(t\log t)^{1/2} \|u(t)\|_{L^\infty} + C\delta t^{-1/4+\gamma}(\log t)^{1/2} \to 0 \tag{5.2}$$

as $t \to \infty$ uniformly with respect to $\xi \in \mathbb{R}$. Hence if δ is sufficiently small, we have

$$\sqrt{\frac{2C_*}{\lambda_*}}(\log t)^{1/2}|\alpha(t,\xi)| \le 1 \tag{5.3}$$

for all $t \ge 2$ and $\xi \in \mathbb{R}$, where C_* and λ_* are the constants appearing in (2.2) and (4.7) respectively. Therefore as in (4.8), it follows from (2.2), (4.7), (4.6), (4.9) and (5.3) that

$$\begin{aligned}
&\partial_t \left((\log t)\alpha(t)\cdot A\alpha(t)\right) = (\log t)\partial_t(\alpha(t)\cdot A\alpha(t)) + t^{-1}\alpha(t)\cdot A\alpha(t) \\
&\quad = 2(\log t)\left(t^{-1}\,\mathrm{Im}\,(F(\alpha(t))\cdot A\alpha(t)) + \mathrm{Im}\,(R(t)\cdot A\alpha(t))\right) + t^{-1}\alpha(t)\cdot A\alpha(t) \\
&\quad \ge -2C_* t^{-1}(\log t)|\alpha(t)|^4 + \lambda_* t^{-1}|\alpha(t)|^2 - 2(\log t)|R(t)\cdot A\alpha(t)| \\
&\quad \ge \lambda_* t^{-1}|\alpha(t)|^2\left(1 - \frac{2C_*\log t}{\lambda_*}|\alpha(t)|^2\right) - C\delta^4 t^{-5/4+3\gamma}\log t \\
&\quad \ge -C\delta^4 t^{-5/4+3\gamma}\log t,
\end{aligned}$$

which yields

$$(\log t)\alpha(t)\cdot A\alpha(t) \ge (\log 2)\alpha(2)\cdot A\alpha(2) - C\delta^4\int_2^t \frac{\log\tau}{\tau^{5/4-3\gamma}}\,d\tau \ge C\delta^2 - C'\delta^4 > 0$$

for sufficiently small δ with some positive constants C and C'. This contradicts (5.2).

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