Trapped modes in thin and infinite ladder like domains. Part 1 : existence results

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Abstract

The present paper deals with the wave propagation in a particular two dimensional structure, obtained from a localized perturbation of a reference periodic medium. This reference medium is a ladder like domain, namely a thin periodic structure (the thickness being characterized by a small parameter $\varepsilon > 0$) whose limit (as ε tends to 0) is a periodic graph. The localized perturbation consists in changing the geometry of the reference medium by modifying the thickness of one rung of the ladder. Considering the scalar Helmholtz equation with Neumann boundary conditions in this domain, we wonder whether such a geometrical perturbation is able to produce localized eigenmodes. To address this question, we use a standard approach of asymptotic analysis that consists of three main steps. We first find the formal limit of the eigenvalue problem as the ε tends to 0. In the present case, it corresponds to an eigenvalue problem for a second order differential operator defined along the periodic graph. Then, we proceed to an explicit calculation of the spectrum of the limit operator. Finally, we prove that the spectrum of the initial operator is close to the spectrum of the limit operator. In particular, we prove the existence of localized modes provided that the geometrical perturbation consists in diminishing the width of one rung of the periodic thin structure. Moreover, in that case, it is possible to create as many eigenvalues as one wants, provided that ε is small enough. Numerical experiments illustrate the theoretical results.

Keywords: spectral theory, periodic media, quantum graphs, trapped modes

1 Introduction

Photonic crystals, also known as electromagnetic bandgap metamaterials, are 2D or 3D periodic media designed to control the light propagation. Indeed, the multiple scattering resulting from the periodicity of the material can give rise to destructive interferences at some range of frequencies. It follows that there might exist intervals of frequencies (called gaps) wherein the monochromatic waves cannot propagate. At the same time, a local perturbation of the crystal can produce defect mid-gap modes, that is to say solutions to the homogeneous time-harmonic wave equation, at a fixed frequency located inside one gap, that remains strongly localized in the vicinity of the perturbation. This localization phenomenon is of particular interest for a variety of promising applications in optics, for instance the design of highly efficient waveguides [26, 27].

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From a mathematical point of view, the presence of gaps is theoretically explained by the band-gap structure of the spectrum of the periodic partial differential operator associated with the wave propagation in such materials (see for instance [12, 32]). In turn, the localization effect is directly linked to the possible presence of discrete spectrum appearing when perturbing the perfectly periodic operator. A thorough mathematical description of photonic crystals can be found in [38]. Without being exhaustive, let us remind the reader about a few important results on the topic. In the one dimensional case, it is well-known [7] that a periodic material has infinitely many gaps unless it is constant. By contrast, in 2D and 3D, a periodic medium might or might not have gaps. Nevertheless, several configurations where at least one gap do exist can be found in [17, 18, 25, 42, 44, 3, 29, 30] and references therein. In any case, except in dimension one, the number of gaps is expected to be finite. This statement, known as the Bethe Sommerfeld conjecture is fully demonstrated in [49, 50] for the periodic Schrödinger operator but is still partially open for Maxwell equations (see [58]). For the localization effect, [15, 16, 1, 33, 37, 45, 8, 9] several papers exhibit situations where a compact (resp. lineic) perturbation of a periodic medium give rise to localized (resp. guided) modes. It seems that the first results concern strong material perturbations : for local perturbation [15, 16] and for lineic perturbation [33, 37]. There exist fewer results about weak material perturbations : [8, 9] deal with 2D lineic perturbations. Finally, geometrical perturbations are considered in [40, 45], where the geometrical domain under investigation is exactly the same as ours but with homogeneous Dirichlet boundary conditions on the boundary of the ladder. As in our case, changing the size of one or several rungs of the ladder can create eigenvalues inside a gap (see also remark 7).

The aim of this paper is to complement the references mentioned above by proving the existence of localized midgap modes created by a geometrical perturbation of a particular periodic medium. We shall use a standard approach of analysis (used in [17, 44]) that consists in comparing the periodic medium with a reference one, for which theoretical results are available. To be more specific, we are interested in the Laplace operator with Neumann boundary condition in a ladder-like periodic waveguide. As the thickness of the rungs (proportional to a small parameter ε) tends to zeros, the domain shrinks to an (infinite) periodic graph. More precisely, the spectrum of the operator posed on the 2D domain tends to the spectrum of a self-adjoint operator posed on the limit graph ([54, 39, 55, 51, 48]). This limit operator consists of the second order derivative operator on each edge of the graph together with transmission conditions (called Kirchhoff conditions) at its vertices ([14, 10, 39]). As opposed to the initial operator, the spectrum of the limit operator can be explicitly determined using a finite difference scheme ([2, 13]). From a mode of the limit operator, we construct a so called *quasi-mode* and we are able to prove that, for ε sufficiently small, the diminution of the thickness of one rung of the ladder gives rise to a localized mode. Moreover, diminushing ε, it is possible to create as many eigenvalues as one wants. We point out that the analysis of quantum graphs has been a very active research area for the last three decades and we refer the reader to the surveys [34, 35, 36] as well as the books [4, 52] for an overview and an exhaustive bibliography of this field.

2 Presentation of the problem

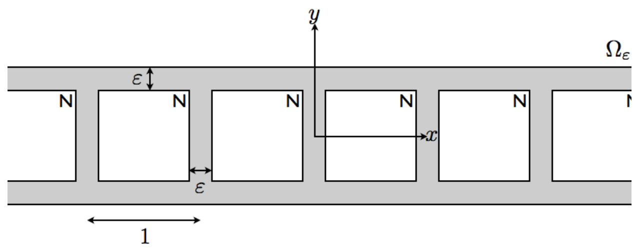
In the present work we study the propagation of waves in a ladder-like periodic medium (see figure 1a). The homogeneous domain Ω_ε (we will call it ladder) consists of the infinite band of height L minus an infinite set of equispaced rectangular obstacles. The domain is 1-periodic in one space direction, corresponding to the variable x. The distance between two consecutive obstacles is equal to the distance between the obstacles and the boundary of the band and is denoted by ε.

Remark 1. *Some extensions We can change the distance between 2 consecutive obstacles from ε to $\nu\varepsilon$. The study will depend on this new parameter but its conclusions remain the same.*

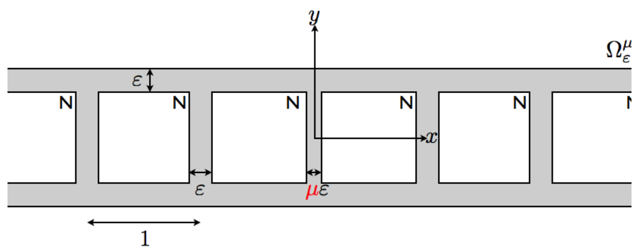
The aim of this work is to find localized modes, that is solutions of the homogeneous scalar wave equation with Neumann boundary condition

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0, \qquad \partial_n v = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{1}$$

which are confined in the x-direction.



(a) The unperturbed periodic ladder



(b) The perturbed ladder

Figure 1: The unperturbed and the perturbed periodic ladders

Without giving a strict mathematical formulation (this will be done in the following section) a localized mode can be understood as a solution of the wave equation (1), which is harmonic in time

$$v(x, y, t) = u(x, y)\, e^{i\omega t}, \qquad v \in L_2(\Omega_\varepsilon). \tag{2}$$

where the function u (which does not depend on time) belongs to L_2. The factor $e^{i\omega t}$ shows the harmonic dependence on time. Injecting (2) into (1) leads to the following problem for the function u:

$$\begin{cases} -\Delta u = \omega^2 u & \text{in } \Omega_\varepsilon \ , \\ \partial_n u = 0 & \text{in } \partial\Omega_\varepsilon. \end{cases} \tag{3}$$

Problem (3) is an eigenvalue problem posed in the unbounded domain Ω_ε. In order to create eigenvalues, we introduce a local perturbation in this perfectly periodic domain (the delicate question of existence of eigenvalues (or flat bands) for the unperturbed problem is not addressed in this paper, see for the absence of flat bands in waveguide problems (or the absolute continuity of the spectrum) for instance [56, 22, 57] and for the existence [19]). The perturbed domain is obtained by changing the distance between two consecutive obstacles from ε to $\mu\varepsilon$, $\mu > 0$ (see Figure 1b in the case where $\mu < 1$). It corresponds to modify the width of one vertical rung of the ladder from $|x| < \varepsilon/2$ to $|x| < \mu\varepsilon/2$.

As we will see such a perturbation does not change the continuous spectrum of the underlying operator but it can introduce a non-empty discrete spectrum. Our aim is to find eigenvalues by playing with the values of μ and ε, ε being treated as a small parameter.

Remark 2. *Imitating the approach developped in this article, it is also possible to study sufficient conditions which ensures the existence of guided modes in a ladder-like open periodic waveguides. More precisely, the domain is $\mathbb{R}^2$ minus an infinite set of equispaced perfect conductor rectangular obstacles with Neumann boundary conditions. And this domain is perturbed by a lineic defect, by changing the distance between two consecutive columns of obstacles. There exists a guided mode with a β wave number if and only if there exists a localized mode in a perturbed periodic ladder where β-boundary conditions are imposed. The results of the present paper can be extended and the sufficient condition which ensures the existence of guided modes remains basically the same.*

3 Mathematical formulation of the problem

This section describes a mathematical framework for the analysis of the spectral problem formulated above. We introduce the operator A^μ_ε, acting in the space $L_2(\Omega^\mu_\varepsilon)$, associated with the eigenvalue problem (3) in the perturbed domain:

$$A^\mu_\varepsilon u = -\Delta u, \qquad D(A^\mu_\varepsilon) = \left\{u \in H^1_\Delta(\Omega^\mu_\varepsilon), \quad \partial_n u|_{\partial\Omega^\mu_\varepsilon} = 0\right\}.$$

Here $H^1_\Delta(\Omega^\mu_\varepsilon) = \left\{u \in H^1(\Omega^\mu_\varepsilon), \quad \Delta u \in L_2(\Omega^\mu_\varepsilon)\right\}$. The operator A^μ_ε is self-adjoint and positive. Our goal is to characterize its spectrum and, more precisely, to find sufficient conditions which ensures the existence of eigenvalues.

3.1 The essential spectrum of A^μ_ε

To determine the essential spectrum of the operator A^μ_ε, we consider the case $\mu = 1$, where the domain Ω_ε is perfectly periodic (see Figure 1a). We will denote the corresponding operator A_ε. The Floquet-Bloch theory shows that the spectrum of periodic elliptic operators is reduced to its essential spectrum and has a band-gap structure [12, 53, 32]:

$$\sigma(A_\varepsilon) = \sigma_{ess}(A_\varepsilon) = \mathbb{R} \setminus \bigcup_{1\leqslant n\leqslant N}]a_n, b_n[, \tag{4}$$

where, in the previous formula, the union disappears if $N = 0$. For $N > 0$, the intervals $]a_n, b_n[$ are called spectral gaps. Their number N is conjectured to be finite (Bethe-Sommerfeld, 1933, [49, 50, 58]). The band-gap structure of the spectrum is a consequence of the following result given by the Floquet-Bloch theory:

$$\sigma(A_\varepsilon) = \bigcup_{\theta\in[-\pi,\pi[} \sigma\left(A_\varepsilon(\theta)\right). \tag{5}$$

Here $A_\varepsilon(\theta)$ is the Laplace operator defined on the periodicity cell $\mathcal{C}_\varepsilon = \Omega_\varepsilon \cap \left\{x \in (-1/2, 1/2)\right\}$ (see Figure 2) with θ-quasiperiodic boundary conditions on the lateral boundaries $\Gamma^\pm_\varepsilon = \partial\mathcal{C}_\varepsilon \cap \{x = \pm 1/2\}$ and homogeneous Neumann boundary conditions on the remaining part $\Gamma_\varepsilon = \partial\mathcal{C}_\varepsilon \setminus (\Gamma^-_\varepsilon \cup \Gamma^+_\varepsilon)$ of the boundary: for $\theta \in [-\pi, \pi]$,

$$\begin{aligned}
&A_\varepsilon(\theta) \,:\, L_2(\mathcal{C}_\varepsilon) \longrightarrow L_2(\mathcal{C}_\varepsilon), \qquad A_\varepsilon(\theta)u = -\Delta u,\\
&D\big(A_\varepsilon(\theta)\big) = \left\{u \in H^1_\Delta(\mathcal{C}_\varepsilon), \quad \partial_n u|_{\Gamma_\varepsilon} = 0, \quad u|_{\Gamma^+_\varepsilon} = e^{-i\theta}\, u|_{\Gamma^-_\varepsilon}, \quad \partial_x u|_{\Gamma^+_\varepsilon} = e^{-i\theta}\, \partial_x u|_{\Gamma^-_\varepsilon}\right\}.
\end{aligned}$$

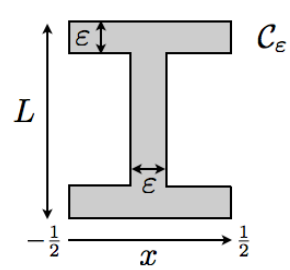


Figure 2: Periodicity cell

For each $\theta \in [-\pi, \pi[$ the operator $A_\varepsilon(\theta)$ is self-adjoint, positive and its resolvent is compact. Its spectrum is then a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda^{(1)}_\varepsilon(\theta) \leqslant \lambda^{(2)}_\varepsilon(\theta) \leqslant \cdots \leqslant \lambda^{(n)}_\varepsilon(\theta) \leqslant \ldots, \qquad \lim_{n\to\infty} \lambda^{(n)}_\varepsilon(\theta) = +\infty. \tag{6}$$

In (6) the eigenvalues are repeated with their multiplicity. The representative curves of the functions $\theta \mapsto \lambda_n(\varepsilon, \theta)$ are called dispersion curves and are known to be continuous and non-constant (cf. Theorem

XIII.86, volume IV in [53]). The fact that the dispersion curves are non-constant implies that the operator A_ε has no eigenvalues of infinite multiplicity. Finally, (5) can be rewritten as

$$\sigma(A_\varepsilon) = \bigcup_{n\in\mathbb{N}} \lambda_\varepsilon^{(n)}\big([-\pi,\pi]\big),$$

which gives (4). The conjecture of Bethe-Sommerfeld means that for n large enough the intervals $\lambda_n\big(\varepsilon,[-\pi,\pi]\big)$ overlap or only touch.

Since $D(A_\varepsilon(-\theta)) = \overline{D(A_\varepsilon(\theta))}$ and the operators A_ε have real coefficients, the function $\lambda_n(\theta)$ are even. Thus, it is sufficient to consider $\theta\in[0,\pi]$ in (5). This will be used systematically in the rest of the paper.

As expected (this is related to Weyl's Theorem, see in [53, Chapter 13, Volume 4], [6, Chapter 9] and [15, Theorem 1]), the essential spectrum is stable under a perturbation of the thickness of one rung of the ladder. In the present case, the domains of definition of the resolvents of A_ε^μ and A_ε^μ are not the same. As a result, we cannot directly apply the standard results (in [6, Chapter 9]).

Proposition 1. $\sigma_{ess}(A_\varepsilon^\mu) = \sigma_{ess}(A_\varepsilon)$.

This stability result is given in [47, §4 Chapter 3, Theorem 4.1 Chapter 5] and [41, Theorem 5]. For the sake of completeness, we provide a constructive proof based on the following assertion.

Lemma 1. *Let $\chi\in C^\infty(\Omega_\varepsilon)$ be a function such that*

(a) $\partial_n\chi|_{\partial\Omega_\varepsilon} = 0$,

(b) $\exists\, M>0$ such that $|x|>M \;\Rightarrow\; \chi(x,y)=1$.

If $\{u_j\}_{j\in\mathbb{N}}$ is a singular sequence for the operator A_ε corresponding to the value λ, then there exists a subsequence of $\{\chi u_j\}_{j\in\mathbb{N}}$ which is also a singular sequence for the operator A_ε corresponding to the value λ.

Proof. By definition of a singular sequence, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has the following properties:

1. $u_j\in D(A_\varepsilon), \quad j\in\mathbb{N}$;
2. $\inf\limits_{j\in\mathbb{N}} \|u_j\|_{L_2(\Omega_\varepsilon)} > 0$;
3. $u_j \xrightarrow{w} 0$ in $L_2(\Omega_\varepsilon)$;
4. $A_\varepsilon\, u_j - \lambda\, u_j \longrightarrow 0$ in $L_2(\Omega_\varepsilon)$.

Let us show that there exists a subsequence of $\{\chi u_j\}_{j\in\mathbb{N}}$ which has the same properties. The property 1 is verified by the whole sequence χu_j thanks to property (a) . To prove property 2, it suffices to show that there exists a subsequence, still denoted u_j, such that

$$\|u_j\|_{L_2(K_\varepsilon)} \longrightarrow 0, \quad j\longrightarrow\infty, \quad \text{with } K_\varepsilon := \big\{(x,y)\in\overline{\Omega_\varepsilon}\ /\ |x|\le M\big\}. \tag{7}$$

Indeed, (7) and property (b) imply $\inf\limits_{j\in\mathbb{N}}\|\chi u_j\|_{L_2(\Omega_\varepsilon)} \geqslant \inf\limits_{j\in\mathbb{N}}\|u_j\|_{L_2(\Omega_\varepsilon\cap\{|x|>M\})} > 0$. To prove (7), we write

$$\|\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} = (A_\varepsilon\, u_j - \lambda\, u_j, u_j)_{L_2(\Omega_\varepsilon)} + \lambda\, \|u_j\|^2_{L_2(\Omega_\varepsilon)}.$$

Then properties 3 and 4 imply that u_j is bounded in $H^1(\Omega_\varepsilon)$. By compactness, one can thus extract a subsequence that converges weakly $H^1(\Omega_\varepsilon)$ and strongly in $L^2(K_\varepsilon)$ to a limit which is necessarily 0 thanks to property 3, which proves (7). For the sequel, we work with the above subsequence.

The property 3 being obvious the only thing to show is the property 4 for the sequence $\{\chi u_j\}_{j\in\mathbb{N}}$. We have:

$$\|A_\varepsilon(\chi u_j) - \lambda(\chi u_j)\|^2_{L_2(\Omega_\varepsilon)} \leqslant \|\chi(A_\varepsilon u_j - \lambda u_j)\|^2_{L_2(\Omega_\varepsilon)} + 2\,\|\nabla\chi\cdot\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} + \|u_j\,\Delta\chi\|^2_{L_2(\Omega_\varepsilon)}.$$

The first and the last terms in the right-hand side tend to zero thanks to property 4 and (7)). Let us estimate the second term. Using first property (b), then properties (a) and (1) together with an integration by parts, we obtain

$$\|\nabla\chi\cdot\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} = \int\limits_{K_\varepsilon} \nabla u_j\cdot\nabla\overline{u}_j\,|\nabla\chi|^2\,d\mathbf{x} = -\int\limits_{K_\varepsilon} u_j\,\Big(|\nabla\chi|^2\,\Delta\overline{u_j} + \nabla\big(|\nabla\chi|^2\big)\cdot\nabla\overline{u_j}\Big)\,d\mathbf{x}$$

which tends to 0 due to (7) since ∇u_j is bounded in $L^2(\Omega_\varepsilon)$ as well as Δu_j (by properties 3 and 4). $\square$

Proof of Proposition 1. It is sufficient to take a function χ in the previous lemma which does not depend on y, vanishes in a neighbourhood of the perturbed edge and such that $\nabla\chi$ vanishes in a neighbourhood of all vertical edges. Then, it follows from Lemma 1 that any singular sequence associated to λ of the operator A_ε provides the construction of a singular sequence of the operator A_ε^μ for the same λ and vice versa. □

The essential spectrum of the operator $\sigma_{ess}(A_\varepsilon^\mu)$ having a band-gap structure, we will be interested in finding eigenvalues inside gaps (once the existence of gaps is established).

3.2 Towards the existence of eigenvalues: the method of study.

Our analysis consists of three main steps.

- First, we find a formal limit of the eigenvalue problem (3) when $\varepsilon \to 0$ (Section 4.1). To do so, we use the fact that, as ε goes to zero, the domain Ω_ε^μ shrinks to a graph $\mathcal{G}$. As a consequence, the formal limit problem will involve a self-adjoint operator $\mathcal{A}^\mu$ associated with a second order differential operator along the graph. Its definition is strongly related to the fact that homogeneous Neumann boundary conditions are considered in the original problem. More precisely, at the limit $\varepsilon \to 0$, looking for an eigenvalue of A_ε^μ leads to search an eigenvalue of $\mathcal{A}^\mu$. This operator, that is well known (see the works of [14, 10, 39]), will be described more rigorously in the next section.
- The second step is an explicit calculation of the spectrum of the limit operator. The essential spectrum is determined using the Floquet-Bloch theory (by solving a set of cell problems) (Section 4.2) while the discrete spectrum of the perturbed operator is found using a reduction to a finite difference equation (Section 4.3). In particular, we shall show that the limit operator has infinitely many eigenvalues of finite multplicity as soon as $\mu < 1$ (and no one when $\mu \geq 1$), which form a discrete subset of $\mathbb{R}^+$.
- Finally, when $\mu < 1$, we deduce the existence of an eigenvalue of A_ε^μ close to the eigenvalue of $\mathcal{A}^\mu$ as soon as ε is small enough (Section 5). The proof will be based on the construction of a quasi-mode (a kind approximation of the eigenfunction) and a criterion for the existence of eigenvalues of self-adjoint operators (see for instance Lemma 4 in [43]). It can be seen as a generalization of the well-known min-max principle for eigenvalues located below the lower bound of the essential spectrum.

An essential preliminary step is the decomposition of the operator A_ε^μ as the sum of two operators, namely its symmetric and antisymmetric parts. To do so, we introduce the following decomposition of $L_2(\Omega_\varepsilon^\mu)$:

$$L_2(\Omega_\varepsilon^\mu) = L_{2,s}(\Omega_\varepsilon^\mu) \oplus L_{2,a}(\Omega_\varepsilon^\mu),$$

where $L_{2,s}(\Omega_\varepsilon^\mu)$ and $L_{2,a}(\Omega_\varepsilon^\mu)$ are subspaces consisting of functions respectively symmetric and antisymmetric with respect to the axis $y = 0$:

$$L_{2,s}(\Omega_\varepsilon^\mu) = \{u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = u(x,-y),\ \forall\,(x,y) \in \Omega_\varepsilon^\mu\}\,,$$

$$L_{2,a}(\Omega_\varepsilon^\mu) = \{u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = -u(x,-y),\ \forall\,(x,y) \in \Omega_\varepsilon^\mu\}\,.$$

The operator A_ε^μ is then decomposed into the orthogonal sum

$$A_\varepsilon^\mu = A_{\varepsilon,s}^\mu \oplus A_{\varepsilon,a}^\mu, \quad A_{\varepsilon,s}^\mu = A_\varepsilon^\mu\Big|_{L_{2,s}(\Omega_\varepsilon^\mu)}, \qquad A_{\varepsilon,a}^\mu = A_\varepsilon^\mu\Big|_{L_{2,a}(\Omega_\varepsilon^\mu)}.$$

Accordingly, the limit operator $\mathcal{A}^\mu$ is decomposed as:

$$\mathcal{A}^\mu = \mathcal{A}_s^\mu \oplus \mathcal{A}_a^\mu \tag{8}$$

The key point is that, as we shall see, contrary to the full operator $\mathcal{A}^\mu$ whose spectrum is $\mathbb{R}^+$, both operators $\mathcal{A}_s^\mu$ and $\mathcal{A}_a^\mu$ have spectral gaps (an infinity of them), each of them containing eigenvalues : these are isolated eigenvalues for $\mathcal{A}_s^\mu$ or $\mathcal{A}_a^\mu$, but embedded eigenvalues for $\mathcal{A}^\mu$. One deduces that the operators $A_{\varepsilon,s}^\mu$ and $A_{\varepsilon,a}^\mu$ have at least finitely many spectral gaps, the number of gaps tending to $+\infty$ when ε goes to 0: this is an important fact for applying the quasi-mode approach.

At this stage, it is worthwhile mentionning that the convergence of the spectrum of differential operators in thin domains degenerating into a graph is not a new subject, particularly in the case of elliptic operators. In particular, for the Laplace operator with Neumann boundary conditions and in the case of compact domains, the convergence results (which are reduced to the convergence of eigenvalues) have been known since the works of Rubinstein-Schatzman [54] and Kuchment-Zheng [39]. Thanks to the Floquet Bloch theory, such results are transformed into analogous results for thin periodic domains (in [39, Theorem 5.1]), since in this case, only continuous spectrum is involved. For general unbounded domains, a general (and somewhat abstract) theory has been developed by Post in [51] for the convergence of all spectral components. This theory can be applied to our problem, however, for the sake of simplicity, we have chosen to use here a more direct approach (based on the construction quasi-modes).

4 Spectral problem on the graph

4.1 The operator $\mathcal{A}^\mu$.

As $\varepsilon \to 0$, the domain Ω_ε tends to the periodic graph $\mathcal{G}$ represented on Figure 3. Let us number the vertical edges of the graph $\mathcal{G}$ from left to right so that the set of the vertical edges is $\{e_j = \{j\} \times (-L/2, L/2)\}_{j\in\mathbb{Z}}$. The upper end of the edge e_j is denoted by M_j^+ and the lower one by M_j^-. The set of all the vertices of the graphs is then

$$\mathcal{M} = \{M_j^\pm\}_{j\in\mathbb{Z}}.$$

The horizontal edge joining the vertices $M_j^\pm$ and $M_{j+1}^\pm$ is denoted by $e^\pm_{j+\frac{1}{2}} = (j, j+1) \times \{\pm L/2\}$. The set of all the edges of the graph is

$$\mathcal{E} = \{e_j, e^\pm_{j+\frac{1}{2}}\}_{j\in\mathbb{Z}}$$

and we denote by $\mathcal{E}(M)$ the set of all the edges of the graph containing the vertex M.

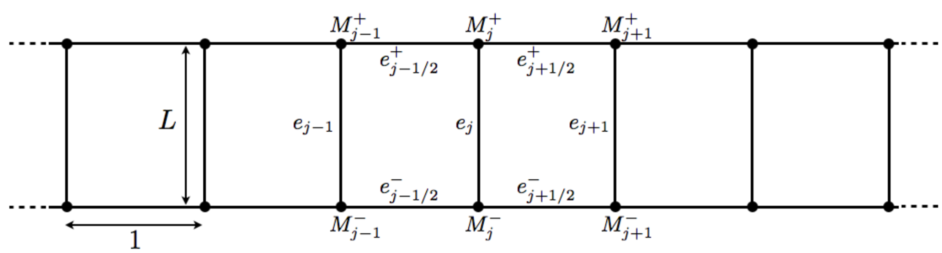


Figure 3: Limit graph $\mathcal{G}$

If u is a function defined on $\mathcal{G}$ we will use the following notation:

$$\mathbf{u}_j^\pm = u(M_j^\pm), \quad u_j(y) = u|_{e_j}, \quad u^\pm_{j+\frac{1}{2}}(x) = u|_{e^\pm_{j+\frac{1}{2}}}.$$

Let $w^\mu : \mathcal{E} \longrightarrow \mathbb{R}^+$ be a weight function which is equal to μ on the "perturbed edge" e_0, i.e. the limit of the perturbed rung $|x| < \mu\varepsilon/2$, and to 1 on the other edges:

$$w^\mu(e_0) = \mu, \quad w^\mu(e) = 1, \quad \forall\, e \in \mathcal{E},\, e \neq e_0. \tag{9}$$

Let us now introduce the following functional spaces

$$L_2^\mu(\mathcal{G}) = \left\{ u \,/\, u \in L_2(e),\, \forall e \in \mathcal{E}; \quad \|u\|^2_{L_2^\mu(\mathcal{G})} = \sum_{e\in\mathcal{E}} w^\mu(e)\, \|u\|^2_{L_2(e)} < \infty \right\}, \tag{10}$$

$$H^1(\mathcal{G}) = \left\{ u \in L_2^\mu(\mathcal{G}) \,/\, u \in C(\mathcal{G}); \quad u \in H^1(e),\, \forall e \in \mathcal{E}; \quad \|u\|^2_{H^1(\mathcal{G})} = \sum_{e\in\mathcal{E}} \|u\|^2_{H^1(e)} < \infty \right\},$$

$$H^2(\mathcal{G}) = \left\{ u \in L_2^\mu(\mathcal{G}) \,/\, u \in C(\mathcal{G}); \quad u \in H^2(e),\, \forall e \in \mathcal{E}; \quad \|u\|^2_{H^2(\mathcal{G})} = \sum_{e\in\mathcal{E}} \|u\|^2_{H^2(e)} < \infty \right\}, \tag{11}$$

where $C(\mathcal{G})$ denotes the space of continuous functions on $\mathcal{G}$.:

We define the limit operator $\mathcal{A}^\mu$ in $L_2^\mu(\mathcal{G})$ as follows. Denoting u_e the restriction of u to e,

$$(\mathcal{A}^\mu u)_e = -u_e'', \qquad \forall e \in \mathcal{E}, \tag{12}$$

$$D(\mathcal{A}^\mu) = \big\{ u \in H^2(\mathcal{G}) \,/\, \sum_{e\in\mathcal{E}(M)} w^\mu(e) u_e'(M) = 0, \quad \forall M \in \mathcal{M} \big\}, \tag{13}$$

where $u_e'(M)$ stands for the derivative of the function u_e at the point M in the *outgoing direction*. The vertex relations in (13) are called Kirchhoff's conditions. Note that they all have an identical expression except at the vertices $M_0^\pm$. The following assertion as well as its proof can be found in [34, Section 3.3].

Proposition 2 (Kuchment)**.** *The operator $\mathcal{A}^\mu$ in the space $L_2^\mu(\mathcal{G})$ is self-adjoint. The corresponding closed sesquilinear form has the following form:*

$$a^\mu[f,g] = (f', g')_{L_2^\mu(\mathcal{G})}, \quad \forall f,g \in D[a^\mu], \qquad D[a^\mu] = H^1(\mathcal{G}).$$

As for the ladder, we introduce the following decomposition of the space $L_2^\mu(\mathcal{G})$ into the spaces of symmetric and antisymmetric functions:

$$L_2^\mu(\mathcal{G}) = L_{2,s}^\mu(\mathcal{G}) \oplus L_{2,a}^\mu(\mathcal{G}),$$

$$L_{2,s}^\mu(\mathcal{G}) = \{ u \in L_2(\mathcal{G}) \,/\, u(x,y) = u(x,-y),\ \forall\, (x,y) \in \mathcal{G} \},$$
$$L_{2,a}^\mu(\mathcal{G}) = \{ u \in L_2(\mathcal{G}) \,/\, u(x,y) = -u(x,-y),\ \forall\, (x,y) \in \mathcal{G} \}.$$

Again, the operator $\mathcal{A}^\mu$ can be decomposed into the orthogonal sum

$$\mathcal{A}^\mu = \mathcal{A}_s^\mu \oplus \mathcal{A}_a^\mu,$$

with

$$\mathcal{A}_s^\mu = \mathcal{A}^\mu \Big|_{L_{2,s}^\mu(\mathcal{G})}, \qquad \mathcal{A}_a^\mu = \mathcal{A}^\mu \Big|_{L_{2,a}^\mu(\mathcal{G})},$$

which implies

$$\sigma(\mathcal{A}^\mu) = \sigma(\mathcal{A}_s^\mu) \cup \sigma(\mathcal{A}_a^\mu).$$

Thus, it is sufficient to study the spectrum of the operators $\mathcal{A}_s^\mu$ and $\mathcal{A}_a^\mu$ separately. The analysis of these two operators being analogous, we will present a detailed study of $\mathcal{A}_s^\mu$ (Section 4.2 and Section 4.3) and state the results for $\mathcal{A}_a^\mu$ (Section 4.4).

4.2 The essential spectrum of the operator $\mathcal{A}_s^\mu$

We shall study the spectrum of the operator $\mathcal{A}_s^\mu$ by a perturbation technique with respect to the case $\mu = 1$ which corresponds to the purely periodic case. The corresponding operator will be denoted by $\mathcal{A}_s$. Indeed, based on compact perturbation arguments (in [6, Theorem 4, Chapter 9]), we can prove the following proposition:

Proposition 3. *The essential spectra of $\mathcal{A}_s^\mu$ and $\mathcal{A}_s$ coincide:*

$$\sigma_{ess}(\mathcal{A}_s^\mu) = \sigma_{ess}(\mathcal{A}_s). \tag{14}$$

This reduces the study of the essential spectrum of $\mathcal{A}_s^\mu$ to the study of the spectrum of the purely periodic operator $\mathcal{A}_s$, which can be done through the Floquet-Bloch theory.

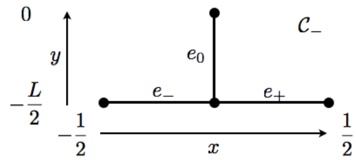


Figure 4: Periodicity cell

4.2.1 Description of the spectrum of $\mathcal{A}_s$ through Floquet-Bloch theory

As previously explained, the spectrum of the operator $\mathcal{A}_s$ can be studied using the Floquet-Bloch theory. One has then to study a set of problems set on the periodicity cell of $\mathcal{G}$. Since we consider the subspace of symmetric functions with respect to the axis $y = 0$, this enables to reduce the problem to the lower half part of the periodicity cell $\mathcal{C}_- = \mathcal{G} \cap [-1/2, 1/2] \times [-L/2, 0]$ (see Figure 4).

We introduce the spaces $L_2(\mathcal{C}_-)$ and $H^2(\mathcal{C}_-)$ analogously to (10), (11):

$$L_2(\mathcal{C}_-) = \big\{u \,/\, u_\ell := u|_{e_\ell} \in L_2(e_\ell),\, \ell \in \{0,+,-\}\big\},$$

$$H^2(\mathcal{C}_-) = \big\{u \in C(\mathcal{C}_-)/u_\ell \in H^2(e_\ell), \ell \in \{0,+,-\}\big\}.$$

We have then

$$\sigma(\mathcal{A}_s) = \bigcup_{\theta \in [0,\pi]} \sigma\left(\mathcal{A}_s(\theta)\right) \tag{15}$$

where $\mathcal{A}_s(\theta)$ is the following unbounded operator in $L_2(\mathcal{C}_-)$

$$\left|\begin{array}{l} \big[\mathcal{A}_s(\theta)u\big]_\ell = -u_\ell'',\ \ell \in \{0,+,-\}, \\ D(\mathcal{A}_s(\theta)) = \big\{u \in H^2(\mathcal{C}_-) \,/\, u \text{ satisfies (16) and } u_0'(0) = 0\big\} \end{array}\right.$$

$$\begin{array}{c} (a) \quad u_+'(0) - u_-'(0) + u_0'\left(-L/2\right) = 0, \\ (b) \quad u_+\left(1/2\right) = e^{-i\theta} u_-\left(-1/2\right), \quad u_+'\left(1/2\right) = e^{-i\theta} u_-'\left(-1/2\right). \end{array} \tag{16}$$

In the definition of $D\big(\mathcal{A}_s(\theta)\big)$, the condition $u_0'(0) = 0$ corresponds to the symmetry with respect to $y = 0$, (16)-(a) is the Kirchhoff's condition with $\mu = 1$ and (16)-(b) are the θ-quasiperiodicity conditions. For each $\theta \in [0, \pi]$, the operator $\mathcal{A}_s(\theta)$ is self-adjoint and positive and its resolvent is compact due to the compactness of the embedding $H^1(\mathcal{C}_-) \subset L_2(\mathcal{C}_-)$. Consequently, its spectrum is a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda_s^{(1)}(\theta) \leqslant \lambda_s^{(2)}(\theta) \leqslant \cdots \leqslant \lambda_s^{(n)}(\theta) \leqslant \ldots, \qquad \lim_{n\to\infty} \lambda_s^{(n)}(\theta) = +\infty. \tag{17}$$

In the present case, the eigenvalues can be computed explicitly.

Proposition 4. *For $\theta \in [0, \pi]$, $\omega^2 \in \sigma(\mathcal{A}_s(\theta))$ if and only if ω is a solution of the equation*

$$2\cos\left(\omega L/2\right)\left(\cos\omega - \cos\theta\right) = \sin\omega\, \sin\left(\omega L/2\right). \tag{18}$$

Proof. If $\omega^2 \neq 0$ is an eigenvalue of the operator $\mathcal{A}_s(\theta)$ then the corresponding eigenfunction $u = \{u_0, u_+, u_-\}$ is of the form

$$u_-(x) = a_-\, e^{i\omega x} + b_-\, e^{-i\omega x}, \qquad x \in [-1/2, 0], \tag{19}$$

$$u_+(x) = a_+\, e^{i\omega x} + b_+\, e^{-i\omega x}, \qquad x \in [0, 1/2], \tag{20}$$

$$u_0(y) = a_0\, e^{i\omega y} + b_0\, e^{-i\omega y}, \qquad y \in [-L/2, 0]. \tag{21}$$

Taking into account that $u \in D(\mathcal{A}_s(\theta))$, we arrive at the following linear system

$$a_- + b_- = a_+ + b_+ = a_0\, e^{-i\omega L/2} + b_0\, e^{i\omega L/2}, \tag{22}$$
$$a_0 = b_0, \tag{23}$$
$$b_- - a_- + a_+ - b_+ + a_0\, e^{-i\omega L/2} - b_0\, e^{i\omega L/2} = 0, \tag{24}$$
$$a_+ e^{i\omega/2} + b_+ e^{-i\omega/2} = e^{-i\theta}\big(a_- e^{-i\omega/2} + b_- e^{i\omega/2}\big), \tag{25}$$
$$a_+ e^{i\omega/2} - b_+ e^{-i\omega/2} = e^{-i\theta}\big(a_- e^{-i\omega/2} - b_- e^{i\omega/2}\big). \tag{26}$$

The relations (22) express the continuity of the eigenfunction at the vertex $(0, -L/2)$. The equation (23) comes from the condition $u_0'(0) = 0$. The relation (24) corresponds to (16)-(a) while (25) and (26) correspond to (16)-(b). Adding and substracting (25) and (26) lead to $a_- = a_+\, e^{i(\theta+\omega)}$ and $b_- = b_+\, e^{i(\theta-\omega)}$, which we can substitute into (22)–(24) to obtain the following system in (a_+, b_+, a_0)

$$M(\theta,\omega,L)\begin{pmatrix} a_+ \\ b_+ \\ a_0 \end{pmatrix} = 0 \quad \text{where} \quad M(\theta,\omega,L) := \begin{pmatrix} 1 - e^{i(\theta+\omega)} & 1 - e^{i(\theta-\omega)} & 0 \\ 1 & 1 & -2\cos(\frac{\omega L}{2}) \\ 1 - e^{i(\theta+\omega)} & -1 + e^{i(\theta-\omega)} & -2i\sin(\frac{\omega L}{2}) \end{pmatrix} \tag{27}$$

It is then easy to conclude since one obtains, after some computations omitted here

$$\det\, M(\theta,\omega,L) = 4e^{i\theta}\,\left(2\cos\left(\tfrac{\omega L}{2}\right)(\cos\omega - \cos\theta) - \sin\omega \sin\left(\tfrac{\omega L}{2}\right)\right).$$

For $\omega = 0$, the relations (19)–(21) are replaced by

$$\begin{aligned} u_-(x) &= a_- + b_-\, x, && x \in [-1/2, 0], \\ u_+(x) &= a_+ + b_+\, x, && x \in [0, 1/2], \\ u_0(y) &= a_0 + b_0, y, && y \in [-L/2, 0]. \end{aligned}$$

Using the fact that $u \in D(\mathcal{A}_s(\theta))$ we have (instead of (22-26)):

$$a_- = a_+ = a_0, \qquad b_0 = 0, \qquad b_- = b_+, \qquad b_+ = b_-\, e^{-i\theta}, \qquad b_+ = a_-\,(e^{-i\theta} - 1).$$

One then easily sees that there exists a non-trivial solution if and only if $\theta = 0$ and that the corresponding eigenfunction is constant. Noticing that, for $\theta = 0$, $\omega = 0$ is solution of (18) allows us to conclude. $\square$

The reader will notice that when $L \in \mathbb{Q}$, the spectrum of $\mathcal{A}_s(\theta)$ has a particular structure: it is the image by the function $x \mapsto x^2$ of a periodic countable subset of $\mathbb{R}$. To see that, it suffices to remark that both functions at the left and right hand sides of (18) are periodic with a common period. As a consequence of (15), the spectrum of $\mathcal{A}_s(\theta)$ is the image by the function $x \mapsto x^2$ of a periodic subset of $\mathbb{R}$.

4.2.2 Characterization of the spectrum of $\mathcal{A}_s$

Using (15), Proposition 4 allows us to describe the structure of the spectrum of the operator $\mathcal{A}_s$. We first prove the existence of a countable infinity of gaps.

Proposition 5. *The following properties hold*

1. $\sigma_{2,s} \cup \sigma_{L,s} \subset \sigma(\mathcal{A}_s)$, *where* $\sigma_{2,s} = \big\{(\pi n)^2,\ n \in \mathbb{N}\big\}$ *and* $\sigma_{L,s} = \big\{(2\pi n/L)^2,\ n \in \mathbb{N}\big\}$.
2. *The operator* $\mathcal{A}_s$ *has infinitely many gaps whose ends tend to infinity.*

Proof. 1. For $\sin\omega = 0$ or $\sin(\omega L/2) = 0$, the equation (18) is satisfied for $\cos(\theta) = \cos(\omega)$ so that ω belongs to $\sigma(\mathcal{A}_s(\theta)) \subset \sigma(\mathcal{A}_s)$.

2. Let $\omega_n = (2n+1)\pi/L$ (such that $\cos(\omega_n L/2) = 0$), let us distinguish two cases:

   (a) $\sin(\omega_n) \neq 0$: the left hand side of equation (18) vanishes for all θ and, as $\sin(\omega_n L/2) \neq 0$, the right hand side does not. Then ω_n^2 does not belong to the spectrum of $\sigma(\mathcal{A}_s)$. Since $(2\pi n/L)^2$ and $(2\pi(n+1)/L)^2$ belong to $\sigma(\mathcal{A}_s)$ (in view of the point 1) there exists a gap which contains ω_n^2, strictly included in $((2\pi n/L)^2, (2\pi(n+1)/L)^2)$.

(b) $\sin(\omega_n) = 0$: (this case can occur only for special values of L, see remark 3), we know by point 1, that $\omega_n^2 \in \sigma(\mathcal{A}_s)$ and we are going to show that it exists $\delta > 0$ such that $((\omega_n - \delta)^2, \omega_n^2)$ and $(\omega_n^2, (\omega_n + \delta)^2)$ are in the resolvant set of $\mathcal{A}_s$. This will show the existence of two disjoint gaps of the form $(\omega_n^2 - l_n^-, \omega_n^2) \subset ((2\pi n/L)^2, \omega_n^2)$ and $(\omega_n^2, \omega_n^2 + l_n^+) \subset (\omega_n^2, (2\pi(n+1)/L)^2)$. Setting $\omega = \omega_n + z$ in relation (18) leads to

$$f_n(z) = 0 \quad \text{where} \quad f_n(z) := 2\sin(zL/2)(\cos\omega_n \cos z - \cos\theta) + \cos\omega_n \sin z \cos(zL/2)\,. \tag{28}$$

We have

$$f_n(z) = z(\cos\omega_n + L(\cos\omega_n - \cos\theta)) + o(z)$$

which cannot vanish for $0 < |z| < \delta$ for δ small enough, since $\cos\omega_n = \pm 1$. This implies that $\cos\omega_n + L(\cos\omega_n - \cos\theta) \neq 0$ for all θ.

The conclusion follows from the fact that the intervals $((2\pi n/L)^2, (2\pi(n+1)/L)^2)$ are disjoint, go to infinity with n, and contain one or two gaps.

□

Remark 3. *The case 2.(b) of the above proof can occur only for special values of L. Indeed, the reader will easily verify that the existence of ω such that $\sin(\omega) = cos(\omega L/2) = 0$ is equivalent to the fact that*

$$L \in \mathbb{Q}_c := \left\{ q \in \mathbb{Q} \;/\; \exists\,(m,k) \in \mathbb{N} \times \mathbb{N}^* \textit{ such that } q = \frac{2m+1}{k} \textit{ (irreducible fraction)} \right\}. \tag{29}$$

In fact, the condition (29) also influences the nature of the spectrum of $\mathcal{A}_s$. Indeed it can be shown that when L does not belong to $\mathbb{Q}_c$, the point spectrum of $\mathcal{A}_s$ is empty (i. e. the spectrum of $\mathcal{A}_s$ is purely continuous). When L belongs to $\mathbb{Q}_c$, it coincides with an infinity of eigenvalues of infinite multiplicity, associated with compactly supported eigenfunctions. It is worth noting that the presence of such eigenvalues is a specific feature of periodic graphs (see [35, Section 5]).

Remark 4. *In the proof, in the case 2.(a), gaps are located in the vicinity of the points λ satisfying $\cos(\lambda L/2) = 0$. These points are nothing else but the eigenvalues of the 1d Laplace operator defined on the vertical half edges $\{(x,y), x = j, -L/2 < y < 0\}$ with Dirichlet boundary condition at $y = -L/2$ and Neumann boundary condition at $y = 0$. The presence of gaps is therefore consistent with [35, Theorem 5] dealing with gaps created by so-called graph decorations. Indeed, the vertical half edges can be seen as decorations of the infinite periodic graph $\mathcal{G}_0$ made of the set of the horizontal edges $e^+_{j+1/2}$.*

Next, we give a more precise description of the gap structure of $\sigma(\mathcal{A}_s)$ through a geometrical interpretation of (18). We first remark that as soon as $\omega \notin \{\pi\mathbb{Z}\} \cup \{2\pi\mathbb{Z}/L\}$, $\lambda = \omega^2$ belongs to $\sigma(\mathcal{A}_s)$ (i. e. ω is solution of (18)) if and only if

$$\exists\,\theta \in [0,\pi] \quad \text{such that} \quad \cos\theta \neq \cos\omega \quad \text{and} \quad \phi_L(\omega) = f(\theta,\omega), \tag{30}$$

where the functions ϕ_L and f are defined by

$$\phi_L(\omega) := \frac{2}{\tan(\omega L/2)}, \quad f(\theta,\omega) := \frac{\sin\omega}{\cos\omega - \cos\theta}. \tag{31}$$

In the following we reason in the (ω, y)-plane with y an additional auxiliary variable. We introduce the domain D

$$D = \left\{ \big(\omega, f(\theta,\omega)\big), (\omega,\theta) \in \mathbb{R} \times [0,\pi] \text{ and } \cos\omega \neq \cos\theta \right\}. \tag{32}$$

Lemma 2. *The domain D is the domain of the (ω, y)-plane, π-periodic with respect to ω, given by*

$$D = \bigcup_{n\in\mathbb{Z}} \{D_0 + (n\pi, 0)\}\,, \qquad D_0 := [0,\pi] \times \mathbb{R} \setminus \left\{ (\omega,y) \;/\; 0 < \omega < \pi, -\tan(\frac{\omega}{2})^{-1} < y < \tan(\frac{\omega}{2}) \right\}. \tag{33}$$

Proof. The π-periodicity of the domain D with respect to ω follows from the identity $f(\theta, \omega + \pi) = f(\pi - \theta, \omega)$. To conclude, it suffices to remark that, for a given $\omega \in (0,\pi)$, if θ varies in the interval $[0,\omega)$ $\theta \mapsto f(\theta,\omega)$ is continuous and strictly decreasing from $-(\tan(\omega/2))^{-1}$ to $-\infty$ while, if θ varies in the interval $(\omega,\pi]$, $\theta \mapsto f(\theta,\omega)$ is continuous and strictly decreasing from $+\infty$ to $\tan(\omega/2)$. □

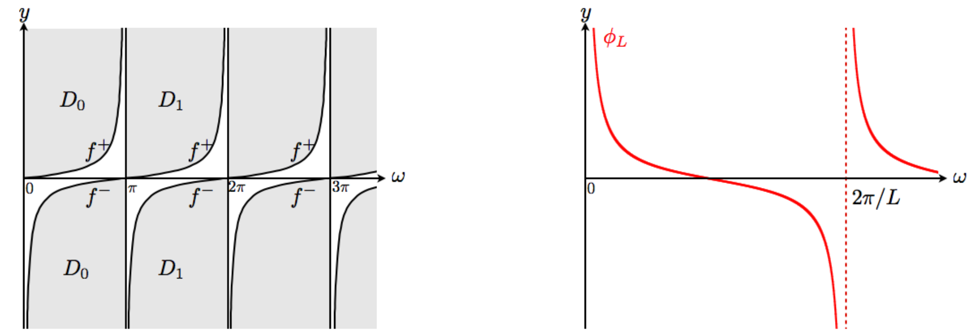
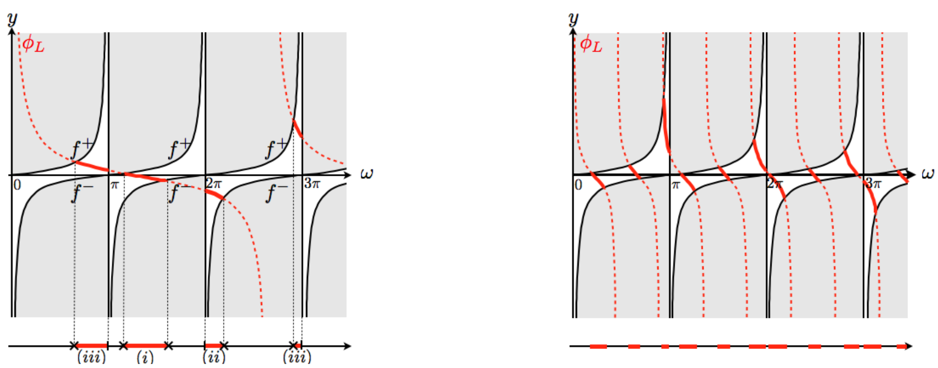


Figure 5: Representation of the D (grey part) and the curve $\mathcal{C}_L$ (for $L = 8$).



(a) $L = 8$

(b) $L = 10\pi/7$

Figure 6: The images of the spectral gaps by $x \mapsto \sqrt{x}$. In the left picture, the three types of gaps are distinguished (according to the legend).

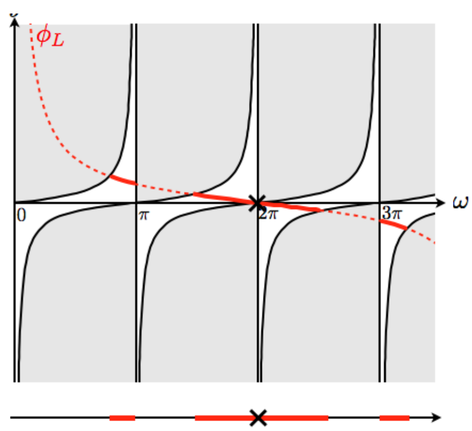


Figure 7: An example of eigenvalue of infinite multiplicity ($\omega = 2\pi$) obtained for $L = 1/2$. This eigenvalue separates a gap of type (ii) on the left from a gap of type (iii) on the right. This occurs for $L \in \mathbb{Q}_c$.

Thanks to Proposition 5 and the characterization (30), we have

$$\sigma(\mathcal{A}_s) = \sigma_{2,s} \cup \sigma_{L,s} \cup \left\{\omega^2 \notin \sigma_{L,s} \,/\, (\omega, \phi_L(\omega)) \in D\right\}. \tag{34}$$

In other words, $\sigma(\mathcal{A}_s)$ is the union of $\sigma_{2,s}$, $\sigma_{L,s}$, and the image by the application $x \mapsto x^2$ of the projection on the line $y = 0$ of the intersection of the domain D with the curve $\mathcal{C}_L = \{(\omega, \phi_L(\omega)), \omega \in \mathbb{R}\}$. Thanks to this geometrical characterization, we shall be able to describe the structure of the gaps of the operator $\mathcal{A}_s$.

Let us introduce the π-periodic functions $f^{\pm} : \mathbb{R} \to \mathbb{R}^{\pm}$, such that, for any $\omega \in [0, \pi)$,

$$f^{+}(\omega) = \tan \frac{\omega}{2} \quad \text{and} \quad f^{-}(\omega) = -\left(\tan \frac{\omega}{2}\right)^{-1}.$$

The easy proof of the following result is left to the reader (see also Figures 5, 6 and 7):

Proposition 6. *An interval (ω_b^2, ω_t^2) is a gap of the operator $\mathcal{A}_s$ if and only if $[\omega_b, \omega_t] \cap \frac{2\pi\mathbb{Z}}{L} = \emptyset$ and one of the following three possibilities holds:*

(i) There exists $n \in \mathbb{Z}$ such that $\pi n < \omega_b < \omega_t < \pi(n+1)$, and, $\phi_L(\omega_b) = f^{+}(\omega_b)$, $\phi_L(\omega_t) = f^{-}(\omega_t)$.

(ii) There exists $n \in \mathbb{Z}$ such that $\pi n = \omega_b < \omega_t < \pi(n+1)$, and $\phi_L(\omega_b) \leq 0$, $\phi_L(\omega_t) = f^{-}(\omega_t)$.

(iii) There exists $n \in \mathbb{Z}$ such that $\pi n < \omega_b < \omega_t = \pi(n+1)$, and $\phi_L(\omega_b) = f^{+}(\omega_b)$, $\phi_L(\omega_t) \geq 0$.

4.3 The discrete spectrum of $\mathcal{A}_s^{\mu}$

We are now interested in determining the discrete spectrum of $\mathcal{A}_s^{\mu}$. Suppose that ω^2 is not in the essential spectrum of $\mathcal{A}_s^{\mu}$, which implies in particular that $\omega \notin \pi\mathbb{Z}$ (see Prop. 5). Let u be a corresponding eigenfunction and let $\mathbf{u}_j = u(M_j^-) = u(M_j^+)$ (we consider symmetric functions). Since the eigenfunction u verifies the equation $-u'' + \omega^2 u = 0$ on each horizontal edge of the graph $\mathcal{G}$, one has

$$u_{j+\frac{1}{2}}(s) = \mathbf{u}_j \frac{\sin(\omega(1-s))}{\sin\omega} + \mathbf{u}_{j+1} \frac{\sin(\omega s)}{\sin\omega}, \qquad s := x - j \in [0,1], \qquad \forall j \in \mathbb{Z}. \tag{35}$$

We first begin by excluding some particular cases:

Lemma 3. *If $\cos\frac{\omega L}{2} = 0$, then ω^2 is not in the discrete spectrum of $\mathcal{A}_s^{\mu}$.*

Proof. If $\cos\frac{\omega L}{2} = 0$ and $\omega \notin \pi\mathbb{Z}$, then ω^2 is an eigenvalue of infinite multiplicity (see Remark 3). Thus, it does not belong to the discrete spectrum of $\mathcal{A}_s^{\mu}$. Similarly, if $\cos\frac{\omega L}{2} = 0$ and $\omega \in \pi\mathbb{Z}$, then $\omega^2 \in \sigma_{ess}(\mathcal{A}_s^{\mu})$ (Prop. 5), which implies that it does not belong to the discrete spectrum of $\mathcal{A}_s^{\mu}$. □

Thus we can assume that $\cos\frac{\omega L}{2} \neq 0$. In this case, on the vertical edges e_j, u_j is given by

$$u_j(y) = \mathbf{u}_j \frac{\cos(\omega y)}{\cos(\omega L/2)}, \qquad y \in [-L/2, L/2], \qquad \forall j \in \mathbb{Z}. \tag{36}$$

According to (35)-(36), the function u is completely determined by the point values $\mathbf{u}_j$. Moreover, in order to ensure that $u \in L^2(\mathcal{G})$, the sequence $\mathbf{u}_j$ must be square integrable:

$$\sum_{j \in \mathbb{Z}} |\mathbf{u}_j|^2 < +\infty. \tag{37}$$

It remains to express that u belongs to $D(\mathcal{A}_s^{\mu})$ (see (13)), i.e. Kirchhoff's conditions are satisfied. Doing so, we obtain the following set of finite difference equations:

$$\mathbf{u}_{j+1} + 2\,g(\omega)\,\mathbf{u}_j + \mathbf{u}_{j-1} = 0, \qquad j \in \mathbb{Z}^*, \tag{38}$$

$$\mathbf{u}_1 + 2\,g_{\mu}(\omega)\,\mathbf{u}_0 + \mathbf{u}_{-1} = 0, \tag{39}$$

with

$$g(\omega) = -\cos\omega + \frac{\sin\omega}{\phi_L(\omega)}, \tag{40}$$

$$g^{\mu}(\omega) = -\cos\omega + \mu\frac{\sin\omega}{\phi_L(\omega)}. \tag{41}$$

where ϕ_L is defined in (31). Thus, we reduced the initial problem for a differential operator on the graph to a problem for a finite difference operator acting on sequences $\{\mathbf{u}_j\}_{j\in\mathbb{Z}}$. Looking for particular solutions of (38) for $j<0$ and $j>0$ under the form $\mathbf{u_j}=r^j$ leads to the characteristic equation

$$r^2+2\,g(\omega)\,r+1=0. \tag{42}$$

At this point, we observe the following property

Lemma 4. *As soon as* $\cos(\omega L/2)\neq 0$*, one has the equivalence*

$$\omega^2\in\sigma_{ess}(\mathcal{A}_s) \quad\Leftrightarrow\quad |g(\omega)|\leq 1.$$

Proof. Indeed, $|g(\omega)|\leq 1$ is equivalent to the existence of $\theta\in[0,\pi]$ such that

$$\cos(\theta)=g(\omega)=-\cos\omega+\frac{1}{2}\sin\omega\tan(\omega L/2).$$

Since $\cos(\omega L/2)\neq 0$, this is equivalent to the characterization (18) of the essential spectrum. □

Since ω^2 does not belong to the essential spectrum of $\mathcal{A}_s^\mu$, $|g(\omega)|>1$ and the discriminant $D(\omega)$ of (42) is strictly positive, which means that (42) has two distinct real solutions. Since the product of these solutions is equal to one, (42) has a unique solution $r(\omega)\in(-1,1)$ given by

$$r(\omega)=-g(\omega)+sign(g(\omega))\sqrt{g^2(\omega)-1}. \tag{43}$$

Joining (37) and (38), we deduce that there exists a constant $A\neq 0$

$$\mathbf{u}_j=A\,r(\omega)^{|j|},\qquad j\in\mathbb{Z}. \tag{44}$$

It remains to enforce the Kirchhoff condition (39), which leads to

$$r(\omega)=-g^\mu(\omega). \tag{45}$$

Taking into account (40), (41) and (43), we arrive at the following relation:

$$sign(g(\omega))\sqrt{g^2(\omega)-1}=(1-\mu)(g(\omega)+\cos\omega).$$

Since $|g(\omega)|>1$, $sign(g(\omega))=sign(g(\omega)+\cos\omega)$, we can rewrite the previous equality as

$$F(\omega)=\mu\quad\text{where } F(\omega):=1-\sqrt{\frac{g^2(\omega)-1}{(g(\omega)+\cos\omega)^2}}. \tag{46}$$

For the rest of the analysis, it is useful to rewrite $F(\omega)$ (using (40)) as

$$F(\omega)=1-\sqrt{1-\phi_L(\omega)\,(\phi_L(\omega)+\phi_2(\omega))},$$

where $\phi_2(\omega)=2/\tan\omega$.

Remark 5. *Let* (ω_b^2,ω_t^2) *be a gap of the operator* $\mathcal{A}_s^\mu$*. Since* $|g(\omega)|>1$ *in* (ω_b,ω_t)*,* F *is well defined and continuous in* (ω_b,ω_t)*. However,* F *might blow up (together with the function* ϕ_2*) as* ω *tends to* ω_t *or* ω_b*, i.e. at the extremities of the gap.*

Theorem 1. *For* $\mu\geq 1$*, the discrete spectrum of the operator* $\mathcal{A}_s^\mu$ *is empty. For* $0<\mu<1$*, let* (ω_b^2,ω_t^2) *be a gap of the operator* $\mathcal{A}_s^\mu$*:*

(a) If (ω_b^2,ω_t^2) *is a gap of type (i), then* $\mathcal{A}_s^\mu$ *has exactly two simple eigenvalues* $\lambda_1=\omega_1^2$ *and* $\lambda_2=\omega_2^2$ *that satisfy* $\omega_b<\omega_1<\omega_2<\omega_t$*.*

(b) If (ω_b^2,ω_t^2) *is a gap of type (ii) or (iii), then* $\mathcal{A}_s^\mu$ *has exactly one simple eigenvalue* $\lambda_1=\omega_1^2$ *such that* $\omega_b<\omega_1<\omega_t$*.*

Proof. Assume that $\mu\geqslant 1$. If ω^2 belongs to the discrete sprectrum of $\mathcal{A}_s^\mu$, then ω^2 is in a gap of $\mathcal{A}_s^\mu$ and Equation (46) is satisfied. But this is impossible because $|g(\omega)|>1$, which means in particular that $\mu=F(\omega)<1$.

Then, we consider the case $0<\mu<1$. We investigate the variations of F for the different types of gaps described in Prop. 6:

- Gap of type (i): as a preliminary step, one can verify that $|g(\omega_b)| = |g(\omega_t)| = 1$ (using for instance the definition (40) of g together with the fact that $\phi_L(\omega_b) = f^+(\omega_b)$ and $\phi_L(\omega_t) = f^-(\omega_t)$, see Prop. 6), which implies that
$$F(\omega_b) = F(\omega_t) = 1. \tag{47}$$
Then, let us investigate the variations of the function $\phi_L\varphi$, with $\varphi = \phi_L + \phi_2$: first, since $\phi_L(\omega_b) = f^+(\omega_b) > 0$ and $\phi_L(\omega_t) = f^-(\omega_t) < 0$ (see Prop. 6), the strictly decaying function ϕ_L, which is continuous in the interval $[\omega_b, \omega_t]$, has exactly one zero in (ω_b, ω_t). We denote it by c. Besides, the fonction φ is continuous and strictly decaying in the interval $[\omega_b, \omega_t]$ (Prop. 6 ensuring the existence of $n \in \mathbb{Z}$ such that $[\omega_b, \omega_t] \subset (n\pi, (n+1)\pi)$, we deduce that ϕ_2 is continuous in $[\omega_b, \omega_t]$). Moreover, it satisfies $\varphi(\omega_b) > 0$ and $\varphi(\omega_t) < 0$. Indeed, a direct computation shows that
$$\forall\, n \in \mathbb{Z}, \quad \forall\, \omega \in (n\pi, (n+1)\pi), \quad -f^+(\omega) < \phi_2(\omega) < -f^-(\omega). \tag{48}$$
As a consequence, $\varphi(\omega_b) = f^+(\omega_b) + \phi_2(\omega_b) > 0$ and $\varphi(\omega_t) = f^-(\omega_t) + \phi_2(\omega_t) < 0$. As a result, φ has exactly one zero in (ω_b, ω_t). We denote it by d.
Noting that (47) implies that $\phi_L(\omega_b)\varphi(\omega_b) = \phi_L(\omega_t)\varphi(\omega_t) = 1$, we deduce that the function $\phi_L\varphi$, which is continuous on $[\omega_b, \omega_t]$, is strictly decaying from 1 to 0 in the interval $[\omega_b, \min(c, d)]$, is strictly increasing from 0 to 1 in the interval $[\max(c, d), \omega_t]$, and is negative in the interval $(\min(c, d), \max(c, d))$. It follows that F, which is therefore also continuous in $[\omega_b, \omega_t]$, is strictly decaying from 1 to 0 in the interval $[\omega_b, \min(c, d)]$, is negative in the interval $(\min(c, d), \max(c, d))$, and is strictly increasing from 0 to 1 in $[\max(c, d), \omega_t]$. As a result, for any $\mu \in (0, 1)$, Equation (46) has exactly two solutions in (ω_b, ω_t), the first one belonging to $(\omega_b, \min(c, d))$ and the second one to $(\max(c, d), \omega_t)$.

- Gap of type (ii): in this case, $\omega_b \in \mathbb{Z}\pi$ and the function F blows up in the neighborhood ω_b unless $\cos(\omega_b L/2) = 0$. More precisely, we can prove that
$$\lim_{\omega \to \omega_b^+} F(\omega) = \begin{cases} 1 - \sqrt{1 + 2L} < 0 & \text{if } \cos(\omega_b L/2) = 0, \\ -\infty & \text{otherwise.} \end{cases}$$
By contrast, since $\omega_t \notin \mathbb{Z}\pi$ and as for the first kind of gap, we can prove that
$$F(\omega_t) = 1. \tag{49}$$
Then, here again, we investigate the variations of the function $\phi_L\varphi$, with $\varphi = \phi_L + \phi_2$. In view of Prop 6, the function ϕ_L is continuous, strictly decaying and negative in the intervall $(\omega_b, \omega_t]$. Then, the function $\varphi = \phi_L + \phi_2$ is continuous in $(\omega_b, \omega_t]$, strictly decaying, and (thanks to (48)) satisfies
$$\lim_{\omega \to \omega_b^+} \varphi(\omega) = +\infty \quad \text{and} \lim_{\omega \to \omega_t^-} \varphi(\omega) = f^-(\omega_t) + \phi_2(\omega_t) < 0.$$
As result, φ has still exactly one zero in (ω_b, ω_t). We denote it by d.
Noting that (49) implies that $\phi_L(\omega_t)\varphi(\omega_t) = 1$, we deduce that the function $\phi_L\varphi$, which is continuous in $(\omega_b, \omega_t]$, is negative in (ω_b, d), and stricly increasing from 0 to 1 in $[d, \omega_t]$. Thus, the function F, which is continuous in (ω_b, ω_t), is negative in (ω_b, d), and strictly increasing from 0 to 1 on $[d, \omega_t]$. Consequently, for any $\mu \in (0, 1)$, Equation (46) has exactly one solution (that belongs to (d, ω_t)). The proof for the gaps of type (iii) follows the same way.

□

4.4 The spectrum of the operator $\mathcal{A}_a^\mu$.

We will now briefly describe the modifications of the previous considerations in the case of the operator $\mathcal{A}_a^\mu$. The operator corresponding to the periodic case $\mu = 1$ is denoted by $\mathcal{A}_a$. First, based on compact perturbation arguments, we can prove the proposition, which is analogous to Proposition 3:

Proposition 7. *The essential spectra of $\mathcal{A}_a^\mu$ and $\mathcal{A}_a$ coincide:*
$$\sigma_{ess}(\mathcal{A}_a^\mu) = \sigma_{ess}(\mathcal{A}_a). \tag{50}$$

Besides, using the Floquet-Bloch Theory, we obtain the analogue of Proposition 4 in the antisymmetric case (we refer the reader to Section 4.2 for the definition of $\mathcal{A}_a(\theta)$):

Proposition 8. *For $\theta \in [0, \pi]$, $\omega^2 \in \sigma(\mathcal{A}_a(\theta))$ if and only if $\omega \neq 0$ and ω is a solution of the equation*

$$2\sin(\omega L/2)\left(\cos\omega - \cos\theta\right) = -\sin\omega\cos(\omega L/2). \tag{51}$$

Thanks to the previous characterization, and similarly to the results of Proposition 5, we can describe the structure of the spectrum of $\mathcal{A}_a$:

Proposition 9. *The following properties hold:*

1. *$\sigma_{2,a} \cup \sigma_{L,a} \subset \sigma(\mathcal{A}_s)$, where $\sigma_{2,a} = \left\{(\pi n)^2,\ n \in \mathbb{N}^*\right\}$ and $\sigma_{L,a} = \left\{\left((2n+1)\pi/L\right)^2,\ n \in \mathbb{N}\right\}$.*
2. *The operator $\mathcal{A}_a$ has infinitely many gaps whose ends tend to infinity.*

Then, excluding here again the particular cases $\omega \in \{\pi\mathbb{Z}\} \cup \{2\pi\mathbb{Z}/L\}$, the computation of the discrete spectrum $\mathcal{A}_a$ leads to the set (38)-(39) of finite-difference equations substituting $g_a(\omega)$ and $g_a^\mu(\lambda)$ for respectively $g(\omega)$ and $g^\mu(\omega)$:

$$\begin{aligned} g_a(\lambda) &= -\cos\lambda + \frac{1}{2}\sin\lambda\tan\left(\lambda L/2 + \pi/2\right), \\ g_a^\mu(\lambda) &= -\cos\lambda + \frac{\mu}{2}\sin\lambda\tan\left(\lambda L/2 + \pi/2\right). \end{aligned}$$

The investigation of the characteristic equation (42) then provides the following characterization for the discrete spectrum of $\mathcal{A}_a$:

$$\omega^2 \in \sigma_d(\mathcal{A}_a^\mu) \quad \Leftrightarrow \quad \mu = 1 - \sqrt{\frac{g_a^2(\omega) - 1}{(g_a(\omega) + \cos\omega)^2}}. \tag{52}$$

Finally, as in the symmetric case (see Theorem 1), a detailed analysis of (52) allows us to prove the following result of the existence of eigenvalues:

Theorem 2. *For $\mu \geqslant 1$ the discrete sprectrum of $\mathcal{A}_a^\mu$ is empty. For $0 < \mu < 1$, there exists either one or two eigenvalues in each gap of $\mathcal{A}_a^\mu$.*

4.5 The spectrum of the operator $\mathcal{A}$.

As we have seen, both of the operators $\mathcal{A}_s$ and $\mathcal{A}_a$ have infinitely many gaps. However, it turns out that the gaps of one operator overlap with the spectral bands of the other one, so that the full operator $\mathcal{A}$ have no gap.

Proposition 10.

$$\sigma(\mathcal{A}) = \mathbb{R}^+.$$

Proof. Let us suppose that there exists ω such that $\omega^2 \notin \sigma(\mathcal{A})$ (of course, the same is true for some open neighborhood of ω). We first note that $\omega \notin \sigma_L \cup \sigma_{L,a}$, since these sets are either in the spectrum of $\mathcal{A}_s^\mu$ or in the spectrum $\mathcal{A}_a^\mu$ (Propositions 5 and 9). As a consequence, $\cos(\omega L/2) \neq 0$ and $\sin(\omega L/2) \neq 0$. Then, since $\omega^2 \notin \sigma(\mathcal{A})$, the characterizations (18)-(51) of the essential spectrum of $\mathcal{A}_s$ and $\mathcal{A}_a$ (divided respectively by $\cos(\omega L/2)$ and $\sin(\omega L/2)$) imply that

$$\left| -\cos\omega + \frac{1}{2}\sin\omega\tan\left(\omega L/2\right) \right| > 1 \quad \text{and} \quad \left| \cos\omega + \frac{\sin\omega}{2\tan\left(\omega L/2\right)} \right| > 1. \tag{53}$$

Introducing $a = \tan(\omega L/2)$, the system (53) can be rewritten as

$$\left\{ \begin{aligned} &\frac{a^2}{4}\sin^2\omega - a\sin\omega\cos\omega + \cos^2\omega > 1, &(54) \\ &\frac{1}{4a^2}\sin^2\omega + \frac{1}{a}\sin\omega\cos\omega + \cos^2\omega > 1. &(55) \end{aligned} \right.$$

Multiplying (55) by a^2 and taking the sum with (54) we obtain

$$\frac{1}{4}(1 + a^2)\sin^2\omega + (1 + a^2)\cos^2\omega > 1 + a^2,$$

which is impossible. $\square$

Let us then remark that the set of eigenvalues of $\mathcal{A}^\mu$, which is the union of the sets of eigenvalues of $\mathcal{A}^\mu_s$ and $\mathcal{A}^\mu_a$, is embedded in the essential spectrum of $\mathcal{A}^\mu$.

5 Existence of eigenvalues for the operator on the ladder

5.1 Main result

We return now to the case of the ladder. As it was mentioned before, instead of studying the full operator A^μ_ε we will study separately the operators $A^\mu_{\varepsilon,s}$, $A^\mu_{\varepsilon,a}$. Let us remind first the result, already proven for instance in [39], which states the convergence of the essential spectrum of the periodic operators $A_{\varepsilon,s}$ (resp. $A_{\varepsilon,a}$) to the essential spectrum of $\mathcal{A}_s$ (resp. $\mathcal{A}_a$). The proof in [39] is based on the convergence of the eigenvalues of the reduced operator $A_\varepsilon(\theta)$. We point out that the construction of the asymptotic expansion of these eigenvalues in the vicinity of the intersection point of the dispersion curves of $A_\varepsilon(\theta)$ is delicate and we refer the reader to [43] for an example of detailed asymptotic in that case.

Theorem 3 (Essential spectrum)**.** *Let $\{(a_m, b_m),\ m \in \mathbb{N}^*\}$ be the gaps of the operator $\mathcal{A}_s$ (respectively $\mathcal{A}_a$) on the limit graph $\mathcal{G}$. Then, for each $m_0 \in \mathbb{N}^*$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}$ (respectively $A_{\varepsilon,a}$) has at least m_0 gaps $\{(a_{\varepsilon,m}, b_{\varepsilon,m}),\ 1 \leqslant m \leqslant m_0\}$ such that*

$$a_{\varepsilon,m} = a_m + O(\varepsilon), \quad b_{\varepsilon,m} = b_m + O(\varepsilon), \qquad \varepsilon \to 0, \qquad 1 \leqslant m \leqslant m_0.$$

In [51], O. Post proves the norm convergence of the resolvent of the laplacian with Neumann boundary conditions for a large class of thin domains shrinking to graphs. It consequently demonstrates the existence of eigenvalues of $A^\mu_{\varepsilon,s}$ (respectively $A^\mu_{\varepsilon,a}$) located in the gap of the essential spectrum. This paper provides a simple and constructive alternative proof of this result. In the following proof, we consider the eigenvalues of the operator $A^\mu_{\varepsilon,s}$, the case of the operator $A^\mu_{\varepsilon,a}$ being treated analogously.

Theorem 4 (Discrete spectrum)**.** *Let (a, b) be a gap of the operator $\mathcal{A}^\mu_s$ (respectively $\mathcal{A}^\mu_a$) on the limit graph $\mathcal{G}$ and $\lambda \in (a, b)$ an eigenvalue of this operator. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A^\mu_{\varepsilon,s}$ (resp. $A^\mu_{\varepsilon,a}$) has an eigenvalue λ_ε inside a gap $(a_\varepsilon, b_\varepsilon)$. Moreover, for $\varepsilon < \varepsilon_0$, there exists $C > 0$ such that*

$$|\lambda_\varepsilon - \lambda| \leq C\sqrt{\varepsilon}. \tag{56}$$

Remark 6. *As every eigenvalue of the operators $\mathcal{A}^\mu_s$ (resp. $\mathcal{A}^\mu_a$) is simple (as established in Theorem 1), for ε small enough, λ_ε will be a simple eigenvalue of $A^\mu_{\varepsilon,s}$ (resp. $A^\mu_{\varepsilon,a}$), see [51]. In addition, it is worth noting that the error estimate* (56) *is suboptimal. In fact, writing a high order asymptotic expansion of λ^ε restores the optimal convergence rate:*

$$|\lambda_\varepsilon - \lambda^{(0)}| \leq C\varepsilon$$

The construction and the justification of this high order asymptotic expansion will be detailed in a forthcoming paper.

Remark 7. *We point out that imposing Dirichlet conditions leads to an entirely different asymptotic analysis. In [24], the asymptotic of the eigenvalues is obtained in the case of a compact 'thickened' graph with different types of boundary conditions, including the Dirichlet and Robin ones (see also [5] for non standard boundary conditions). Besides, the Dirichlet ladder is investigated in [40]-[45]: as in our case, changing the size of one or several rungs of the ladder can create eigenvalues inside the first gap (see [45, Theorem 8.1]). The analysis is deeply linked to the presence of a non empty discrete spectrum for the Laplace problem posed in a T-shape waveguide (cf. [46]).*

Thanks to Theorem 4, it is easy to show the existence of as many eigenvalues as one wants.

Corollary 1. *For any number $m \in \mathbb{N}$, it exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$, $A^\mu_{\varepsilon,s}$ (respectively $A^\mu_{\varepsilon,a}$) has at least m eigenvalues.*

In the next section, we give the constructive proof and in Section 5.3, we illustrate these theoretical results by numerical illustrations.

5.2 Proof of Theorem 4

Our proof of Theorem 4 relies on the construction of a pseudo-mode, that is to say a symmetric function $u_\varepsilon \in H^1(\Omega_\varepsilon^\mu)$ such that for every symmetric function $v \in H^1(\Omega_\varepsilon^\mu)$

$$\Big| \int_{\Omega_\varepsilon^\mu} (\nabla u_\varepsilon \nabla v - \lambda u_\varepsilon v)\, d\mathbf{x} \Big| \leqslant C\,\sqrt{\varepsilon}\,\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^\mu)} \|v\|_{H^1(\Omega_\varepsilon^\mu)}, \tag{57}$$

By adapting the Lemma 4 for [43] (see Appendix A in [11]) the existence of such a function provides an estimate of the distance from λ to the spectrum of $A^\mu_{\varepsilon,s}$, namely

$$dist(\sigma(A^\mu_{\varepsilon,s}), \lambda) \leqslant \widetilde{C}\sqrt{\varepsilon}, \tag{58}$$

with some constant $\widetilde{C}$ that does not depend on ε, but depends on λ.

According to Theorem 3, for ε small enough, there exists a constant C such that $\sigma_{ess}(A^\mu_{\varepsilon,s}) \cap [a + C\varepsilon,\, b - C\varepsilon] = \emptyset$. As a consequence, the intersection between the discrete spectrum $\sigma_d(A^\mu_{\varepsilon,s})$ and the interval $[\lambda - \widetilde{C}\sqrt{\varepsilon},\, \lambda + \widetilde{C}\sqrt{\varepsilon}]$ is non empty, which proves the existence of an eigenvalue in the neighborhood of λ.

5.2.1 Construction of a pseudo-mode

Since we consider the symmetric case, it suffices to construct the pseudo-mode u_ε on the lower half part $\Omega_\varepsilon^{\mu,-}$ of Ω_ε^μ (comb shape domain, see Figure 8):

$$\Omega_\varepsilon^{\mu,-} = \{(x,y) \in \Omega_\varepsilon^\mu \quad \text{s.t.} \quad y < 0\}\,.$$

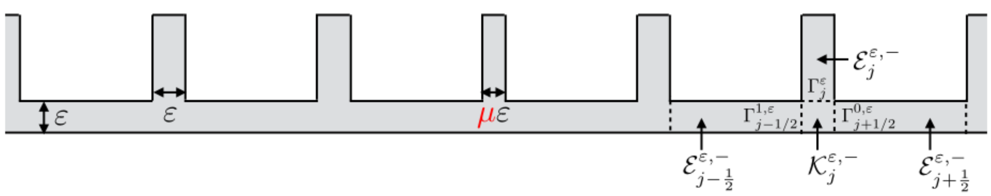


Figure 8: The domain $\Omega_\varepsilon^{\mu,-}$

As represented on Figure 8, we denote by $\mathcal{E}^{\varepsilon,-}_{j+\frac{1}{2}}$, $j \in \mathbb{Z}$, the horizontal edges of the domain $\Omega_\varepsilon^{\mu,-}$

$$\mathcal{E}^{\varepsilon,-}_{j+\frac{1}{2}} = (j + \varepsilon\mu_j/2, (j+1) - \varepsilon\mu_{j+1}/2) \times (-L/2, -L/2 + \varepsilon)\,,$$

by $\mathcal{E}_j^{\varepsilon,-}$, $j \in \mathbb{Z}$, its vertical edges

$$\mathcal{E}_j^{\varepsilon,-} = (j - \varepsilon\mu_j/2, j + \varepsilon\mu_j/2) \times (-L/2 + \varepsilon, 0)\,,$$

and by $\mathcal{K}_j^{\varepsilon,-}$, $j \in \mathbb{Z}$, the junctions

$$\mathcal{K}_j^{\varepsilon,-} = (j - \varepsilon\mu_j/2, j + \varepsilon\mu_j/2) \times (-L/2, -L/2 + \varepsilon)\,.$$

Here, $\mu_j = 1$ if $j \neq 0$ and $\mu_0 = \mu$ (with the notations of Section 4.1 $\mu_j = w^\mu(e_j)$, the function w^μ being defined by (9)).

Denoting by u an eigenfunction of the limite operator $\mathcal{A}_s^\mu$ associated with the eigenvalue λ (see formula (36)), we construct the pseudo-mode u_ε on $\Omega_\varepsilon^{\mu,-}$ by "fattening" u (with an appropriate rescaling) as follows:

$$u_\varepsilon(x,y) = \begin{cases} u_{j+\frac12}(s^\varepsilon_{j+1/2}(x)), & (x,y)\in\mathcal{E}^{\varepsilon,-}_{j+\frac12}, \\ u_j(t^\varepsilon(y)), & (x,y)\in\mathcal{E}^{\varepsilon,-}_{j}, \\ \mathbf{u}_j, & (x,y)\in\mathcal{K}^{\varepsilon,-}_{j}. \end{cases}$$

Here the rescaling functions $s^\varepsilon_{j+1/2}$ and t^ε are linear functions given by the relations

$$s^\varepsilon_{j+1/2}(x) = \frac{x-j-w^\mu(e_j)\varepsilon/2}{1-\left(w^\mu(j)+w^\mu(e_{j+1})\right)\varepsilon/2}, \qquad t^\varepsilon(y) = \frac{y}{1-2\varepsilon/L}. \tag{59}$$

We remark that the function $u_\varepsilon\in H^1(\Omega_\varepsilon^{\mu,-})$ since the function u is continuous at the vertices of the graph.

5.2.2 Proof of Estimate (57)

The pseudo-mode being constructed, it remains to prove (57). We notice that it is sufficient to prove it for any test function $v\in\mathcal{C}^1_s(\overline{\Omega^\mu_\varepsilon})$ ($\mathcal{C}^1_s$ standing for the symmetric subspace of $\mathcal{C}^1$). Indeed, $\mathcal{C}^1_s(\overline{\Omega^\mu_\varepsilon})$ is dense in the subset of $H^1(\Omega^\mu_\varepsilon)$ made of symmetric functions. Let us then estimate the left-hand side of (57) for $v\in\mathcal{C}^1_s(\overline{\Omega^\mu_\varepsilon})$. First, an integration by parts gives

$$\begin{aligned}\int_{\mathcal{E}^{\varepsilon,-}_{j-1/2}} \nabla u_\varepsilon\nabla v-\lambda u_\varepsilon v\,d\mathbf{x} &= \int_{\mathcal{E}^{\varepsilon,-}_{j-1/2}} \underbrace{\left({u_{j-1/2}}''(s^\varepsilon_{j-1/2}(x))-\lambda u_{j-1/2}(s^\varepsilon_{j-1/2}(x))\right)}_{=0} v dx \\ &+\int_{\mathcal{E}^{\varepsilon,-}_{j-1/2}} \left((s^\varepsilon_{j-1/2})'(x)-1\right) {u_{j-1/2}}''(s^\varepsilon_{j-1/2}(x)) v dx + \int_{\Gamma^{1,\varepsilon}_{j-1/2}\cup\Gamma^{0,\varepsilon}_{j-1/2}} \partial_n u_\varepsilon v dx \\ &= \lambda\int_{\mathcal{E}^{\varepsilon,-}_{j-1/2}} \left((s^\varepsilon_{j-1/2})'(x)-1\right) u_{j-1/2}(s^\varepsilon_{j-1/2}(x)) v dx + \int_{\Gamma^{1,\varepsilon}_{j-1/2}\cup\Gamma^{0,\varepsilon}_{j-1/2}} \partial_n u_\varepsilon v dx, \end{aligned} \tag{60}$$

where, for any $j\in\mathbb{Z}$, $\Gamma^{0,\varepsilon}_{j-\frac12}=\partial\mathcal{E}^{\varepsilon,-}_{j-\frac12}\cap\partial\mathcal{K}^{\varepsilon,-}_{j-1}$, $\Gamma^{1,\varepsilon}_{j-\frac12}=\partial\mathcal{E}^{\varepsilon,-}_{j-\frac12}\cap\partial\mathcal{K}^{\varepsilon,-}_{j}$ (see Figure 8). Here, we used the fact that $s^\varepsilon_{j-1/2}$ is a linear function of x. Similarly,

$$\int_{\mathcal{E}^{\varepsilon,-}_j} \nabla u_\varepsilon\nabla v \;-\; \lambda u_\varepsilon v\,d\mathbf{x} \quad=\quad \lambda\int_{\mathcal{E}^{\varepsilon,-}_j} \left((t^\varepsilon_j)'(x)-1\right) u_{j-1/2}(t^\varepsilon_j(x)) v dx \;+\; \int_{\Gamma^\varepsilon_j} \partial_n u_\varepsilon v dx \tag{61}$$

where $\Gamma^\varepsilon_j=\partial\mathcal{E}^\varepsilon_j\cap\mathcal{K}^{\varepsilon,-}_j$. Summing over $j\in\mathbb{Z}$ (noting that $|(s^\varepsilon_{j-1/2})'(x)-1|$ and $|(t^\varepsilon)'(y)-1|$ are of order ε), we obtain

$$\begin{aligned}\Big|\int\limits_{\Omega^{\mu,-}_\varepsilon} (\nabla u_\varepsilon\nabla v-\lambda u_\varepsilon v)\;d\mathbf{x}\Big| &\leqslant \sum_{j\in\mathbb{Z}}\lambda\mathbf{u}_j\Big|\int\limits_{\mathcal{K}^{\varepsilon,-}_j} v\,d\mathbf{x}\Big| + \lambda\|u_\varepsilon\|_{L_2(\Omega^{\mu,-}_\varepsilon)}\|v\|_{L_2(\Omega^{\mu,-}_\varepsilon)}O(\varepsilon) \\ &+\sum_{j\in\mathbb{Z}}\Big(-u'_{j+\frac12}(0)\int\limits_{\Gamma^{0,\varepsilon}_{j+\frac12}} v(x,y)dy + u'_{j-\frac12}(1)\int\limits_{\Gamma^{1,\varepsilon}_{j-\frac12}} v(x,y)dy - u'_j\left(-\tfrac{L}{2}\right)\int\limits_{\Gamma^\varepsilon_j} v(x,y)dx\Big)\left(1+O(\varepsilon)\right), \end{aligned} \tag{62}$$

Since $\mathbf{u}_j$ is a geometrical progression (according to (44)), and using the Cauchy Schwarz inequality (the size of the junction $\mathcal{K}^{\varepsilon,-}_j$ is of order ε^2), we obtain

$$\sum_{j\in\mathbb{Z}}\lambda\mathbf{u}_j\Big|\int\limits_{\mathcal{K}^{\varepsilon,-}_j} v\,d\mathbf{x}\Big| \leqslant C_1\varepsilon\|v\|_{L_2(\Omega^{\mu,-}_\varepsilon)}. \tag{63}$$

where C_1 is a constant that does not depend on ε. Next, denoting by $M^{\varepsilon,-}_j$ the barycenter of $\mathcal{K}^{\varepsilon,-}_j$, we remark that we can replace $v(x,y)$ by $v(x,y)-v\left(M^{\varepsilon,-}_j\right)$ in the integrals over the boundaries in the right-hand side of (62) because u satisfies the Kirchhoff's conditions (13). Moreover,

$$\Big|\int_{\Gamma_j^\varepsilon} \left(v(x,y) - v\left(M_j^{\varepsilon,-}\right)\right)\, dx\Big| \leqslant \int_{\Gamma_j^\varepsilon} \int_{M_j^{\varepsilon,-}}^{(x,y)} |\nabla v|\, dx dt \leqslant C_2 \varepsilon \|v\|_{H^1(\mathcal{K}_j^{\varepsilon,-})}. \tag{64}$$

In the previous formula, $\int_{M_j^{\varepsilon,-}}^{(x,y)}$ stands for the integral on the segment linking $M_j^{\varepsilon,-}$ to the point of coordinates (x,y). Combining (62-63-64) and taking into account (35-36-44), we obtain that

$$\Big|\int_{\Omega_\varepsilon^{\mu,-}} (\nabla u_\varepsilon \nabla v - \lambda u_\varepsilon v)\, d\mathbf{x}\Big| \leqslant C_3 \varepsilon \|v\|_{H^1(\Omega_\varepsilon^{\mu,-})}, \qquad \forall v \in \mathcal{C}_s^1(\Omega_\varepsilon^\mu). \tag{65}$$

To conclude, we notice that by definition of the pseudo-mode u_ε,

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^{\mu,-})} \geqslant C_4 \sqrt{\varepsilon} \|u\|_{H^1(\mathcal{G}^-)}, \qquad C_4 > 0,$$

($\mathcal{G}^-$ standing for the lower half part of the graph $\mathcal{G}$), which, together with (65) and the density argument mentioned above finishes the proof of (57).

5.3 Numerical illustration

To illustrate and validate the results of Theorem 4, we have computed a part of the essential spectrum, some eigenvalues and their associated eigenvectors of the operator $A_{\varepsilon,s}^\mu$ for several values of ε and several values of μ.

Essential spectrum

To compute the essential spectrum of the operator $A_{\varepsilon,s}^\mu$ (resp. $\mathcal{A}_s$), one method consists in computing the eigenvalues $\lambda_\varepsilon^{(n)}(\theta) = (\omega_\varepsilon^{(n)}(\theta))^2$ defined in (6) (resp. $\lambda_s^{(n)}(\theta) = (\omega_s^{(n)}(\theta))^2$) defined in (17)) for a discrete set of θ included in $[-\pi,\pi[$ (or equivalently in $[-\pi,0]$). From a numerical point of view, this is done using the standard P_1 conform finite element method ([23]). We have represented in Figure 9 the dispersive curves $\theta \mapsto \omega_\varepsilon^{(n)}(\theta)$ for $n \in [\![1,5]\!]$, when $L=2$ and $\varepsilon = 0.1$ (left figure), and $\theta \mapsto \omega_s^{(n)}(\theta)$ for $n \in [\![1,5]\!]$, corresponding to the graph with the same L (right figure). The essential spectrum can be easily deduced from the dispersion curves: indeed, as explained by the Floquet-Bloch theory, it is the image of the segment ($[-\pi,\pi]$) by the functions $\lambda_\varepsilon^{(n)}$ (resp. $\lambda_s^{(n)}$). In Figure 10, we have represented a part of the essential spectrum of the operator $A_{\varepsilon,s}^\mu$ for different values of ε: the blue bars correspond to the values ω such that $\lambda = \omega^2$ is in the essential spectrum of $A_{\varepsilon,s}^\mu$. Obviously, for small values of ε, the essential spectrum of the operator $A_{\varepsilon,s}^\mu$ is very close to the essential spectrum of the limit operator $\mathcal{A}_s^\mu$. More precisely, to each gap of the limit operator $\mathcal{A}_s^\mu$, corresponds a gap of the operator $A_{\varepsilon,s}^\mu$, which is close to it for small ε . The convergence with respect to ε is linear as it is predicted by the theory, see Theorem 3. We also remark a phenomenon that has not been detected by our approach: the opening of a gap near the values $\omega = \pi\mathbb{N}^*$, points where the dispersion curves of the limit operator $\mathcal{A}_s^\mu$ touch (see right figure of Figure 9). This phenomenon could be probably proven using the techniques of [43].

Another interesting phenomenon concerns the eigenvalues of infinite multiplicity for the limit operator. As explained in Remark 3, the operator $\mathcal{A}_s^\mu$ might have eigenvalues of infinite multiplicity when L is rational. For instance in the case of $L = 0.5$, the set of eigenvalues of infinite multiplicity is given by

$$\{\lambda = \omega^2,\ \omega = 2(2n+1)\pi,\ n \in \mathbb{N}\}.$$

We can predict that such an eigenvalue becomes a (small) spectral band in the 2D case for ε small enough. Indeed, as shown in [51], the dimension of the spectral projector on any interval is preserved for ε small enough. On the other hand, in most cases, a periodic 2D operator does not have eigenvalues (see for instance [56, 57, 22] for the proof of the absolute continuity of the spectrum of classes of periodic operator defined in waveguides, but see also the counterexample [19]). Thus the most likely possibility

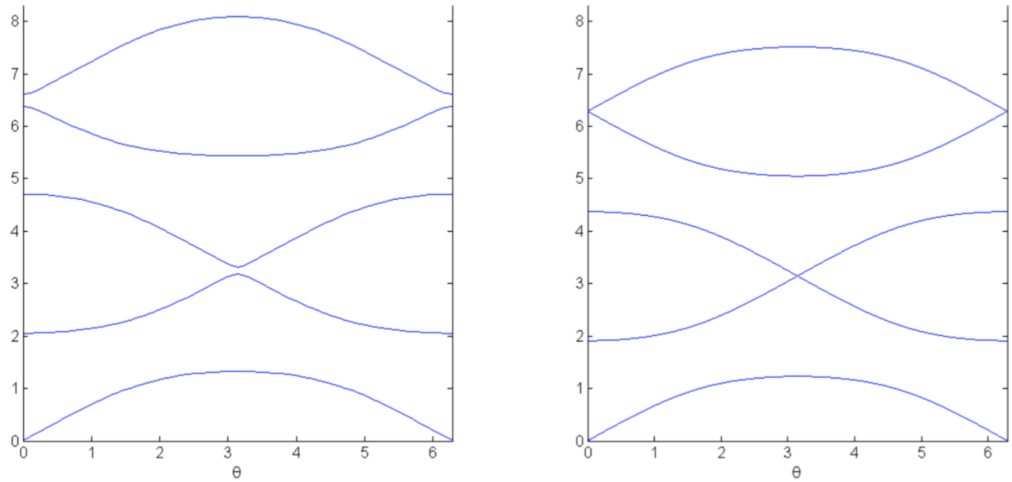


Figure 9: Dispersive curves for the ladder of thickness $\varepsilon = 0.1$ (left figure) and for the graph (right figure) when $L = 2$.

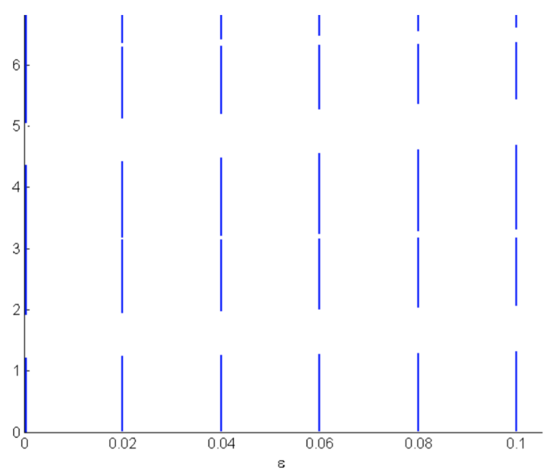


Figure 10: Representation of the essential spectrum of $A^{\mu}_{\varepsilon,s}$ for different values of ε for $L = 2$

is that the operator $A^{\mu}_{\varepsilon,s}$ has a small spectral band (of width $O(\varepsilon)$, see Theorem 3) in a neighborhood of the eigenvalues of infinite multiplicity. This phenomenon can be seen on Figure 11 where the essential spectrum of $A^{\mu}_{\varepsilon,s}$ is represented for different values of ε and for $L = 0.5$. A small spectral band appears in the vicinity of $\omega = 2\pi$, corresponding to the first eigenvalue of infinite multiplicity of the limit operator.

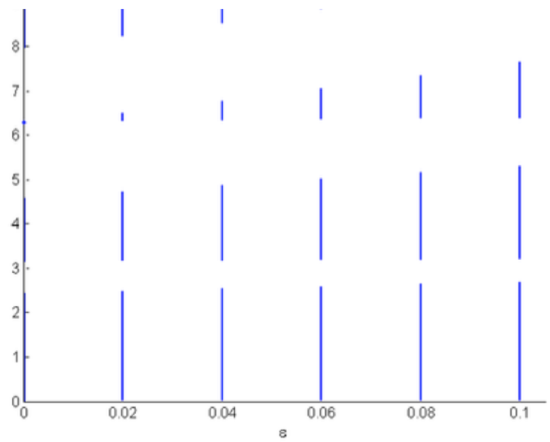


Figure 11: Representation of the essential spectrum of $A^{\mu}_{\varepsilon,s}$ for different values of ε for $L = 0.5$

Discrete spectrum

It is less easy to compute the discrete spectrum because one has to solve an eigenvalue problem set on an unbounded domain. To address this difficulty, we have used a method based on the construction of Dirichlet-to-Neumann operators in periodic waveguides (see [28, 20]): this requires the solution of cell problems (discretized here again using the standard P_1 finite element methods) and the solution of a stationary Ricatti equation. The construction of these Dirichlet-to-Neumann operators enables us to reduce the numerical computation to a small neighborhood of the perturbation independently from the confinement of the mode (which depends on the distance between the eigenvalue and the essential spectrum of the operator). However the reduction of the problem leads to a non linear eigenvalue problem (since the DtN operators depend on the eigenvalue) of a fixed point nature. It is solved using a Newton-type procedure, each iteration needing a finite element computation, see [21] for more details. In Figure 12, we represent the eigenvalues computed for different values of ε: here again, the blue bars correspond to the values ω such that $\lambda = \omega^2$ is in the essential spectrum of $A^{\mu}_{\varepsilon,s}$. The red asterisks stand for the values ω such that $\lambda = \omega^2$ is in the discrete spectrum of $A^{\mu}_{\varepsilon,s}$. In Figure 13, we make a zoom on the eigenvalues and we observe the linear convergence of one of this eigenvalue toward the limit one: as explained in Remark 6, the error estimate (56) is suboptimal. Indeed, a high order asymptotic expansion of λ^{ε} would restore the linear convergence rate. In Figure 14, the eigenfunction corresponding to the first eigenvalue of the operator $A^{\mu}_{\varepsilon,s}$ is represented. In Figure 15, we study the dependence of the eigenvalues with respect to $\mu \in (0, 1)$. As it is natural to expect, the smaller μ is (so, the stronger the perturbation is), the better the eigenvalues are separated from the essential spectrum. When μ is close to 1, the computation becomes more costly: since the distance between the eigenvalue and the essential spectrum becomes very small, the mesh size has to be small enough in order to make the distinction between the two different kinds of spectrum.

A last natural question for which no theoretical answer has been given yet is what happens for larger values of ε, i.e. when the spectrum of the operator $A^{\mu}_{\varepsilon,s}$ is not close to the spectrum of the limit operator. In particular, if a gap exists for small values of ε, does it still exist for large values of ε (until the obstacles disappear)? Similarly, do the eigenvalues still exist when ε increases or do they immerse into the essential spectrum? In the cases that we have tested, the gaps seem to keep present for any value of ε for which the obstacles are present (for $\varepsilon \in (0, \min(1, L/2))$). In Figure 16, we represent the dependence of the first two gaps with respect to ε in the case $L = 2$. The limit case $\varepsilon \to \min(1, L/2)$ has been studied by S.

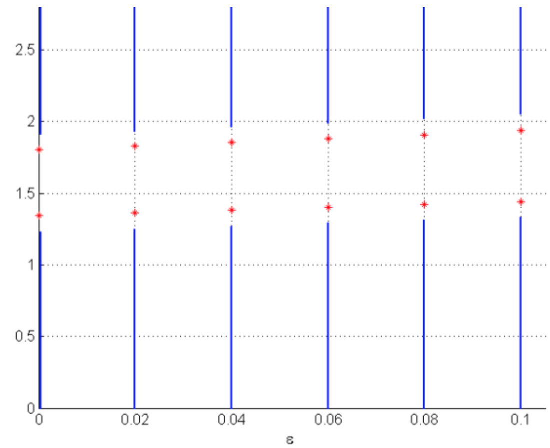


Figure 12: Representation of the eigenvalues appearing in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ for different values of ε for $L = 2$ and $\mu = 0.25$ (red asterisks). The values for $\varepsilon = 0$ correspond to the limit operator A^{μ}_{s}.

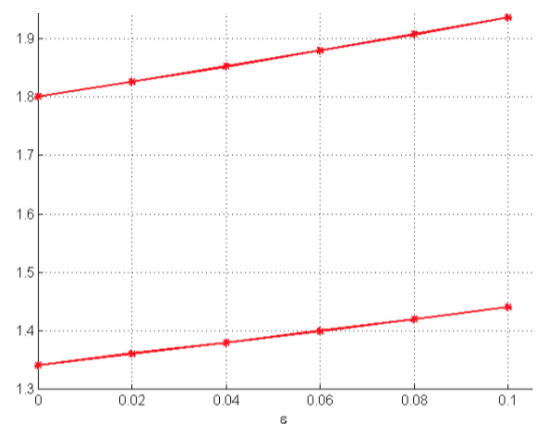


Figure 13: Linear convergence of the eigenvalues represented in figure 12 as $\varepsilon \to 0$.

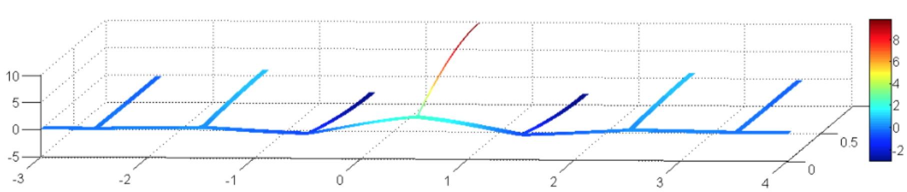


Figure 14: Eigenfunction corresponding to the first eigenvalue of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$, $\varepsilon = 0.06$ and $\mu = 0.25$.

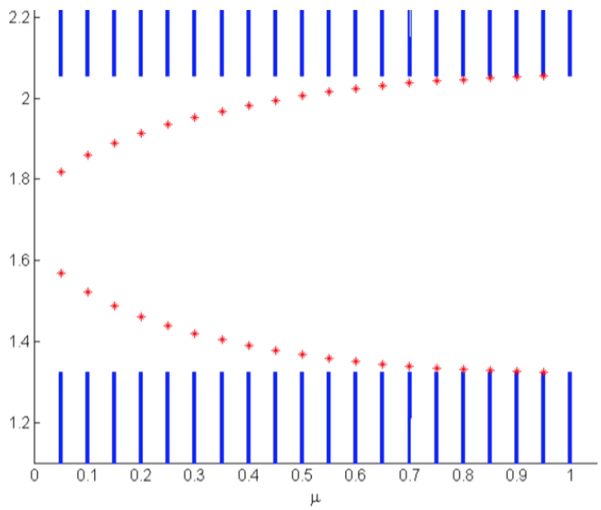


Figure 15: Dependence of the eigenvalues in the first gap with respect to μ for $L = 2$ and $\varepsilon = 0.1$.

Nazarov [43] where opening of a gap is proven when Dirichlet conditions are imposed on the boundary of the periodic waveguide instead of Neumann boundary conditions. The behaviour of the eigenvalues is by contrast more unclear. In Figure 17, we show the eigenvalues in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$ and $\mu = 0.25$. We might think that the eigenvalues disappear for some values of $\varepsilon < 1$: however, as previously mentioned, the numerical computation becomes costly when the eigenvalues approach the essential spectrum. For this reason, it is difficult to make the distinction between the case when the eigenvalues do not exist any more and the case when they exist but are very close to the essential spectrum. Moreover, if they really disappear, do they immerse in the essential spectrum? Do they move in the complex plane? Let us point out that the use of the sophisticated numerical method (based on an automatic choice of the mesh size) presented in [31] might help to answer these questions.

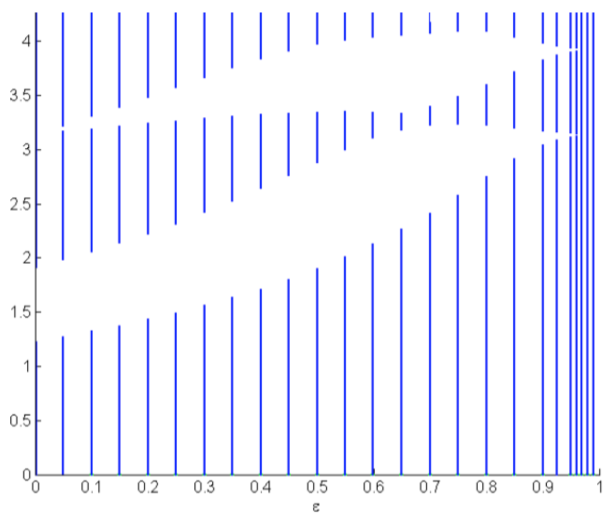


Figure 16: Dependence of the first gaps with respect to $\varepsilon < 1$ for $L = 2$.

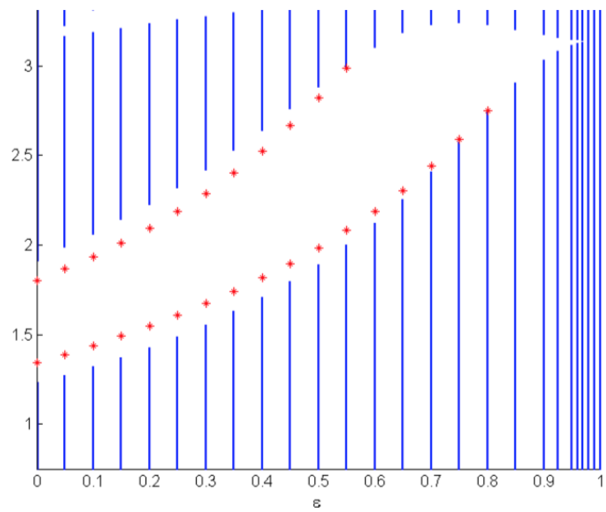


Figure 17: Dependence of the eigenvalues in the first gap with respect to $\varepsilon < 1$ for $L = 2$ and $\mu = 0.25$.

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