REGULARITY PROPERTIES OF VISCOSITY SOLUTIONS FOR FULLY NONLINEAR EQUATIONS ON THE MODEL OF THE ANISOTROPIC p-LAPLACIAN

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REGULARITY PROPERTIES OF VISCOSITY SOLUTIONS FOR FULLY NONLINEAR EQUATIONS ON THE MODEL OF THE ANISOTROPIC $\vec{p}$-LAPLACIAN

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Abstract. This paper is devoted to some Lipschitz estimates between sub-and super-solutions of Fully Nonlinear equations on the model of the anisotropic $\vec{p}$-Laplacian. In particular we derive from the results enclosed that the continuous viscosity solutions for the equation $\sum_1^N \partial_i(\partial_i u|^{p_i-2}\partial_i u) = f$ are Lipschitz continuous when $\sup_i p_i < \inf_i p_i + 1$, where $\vec{p} = \sum_i p_i e_i$.

1. Introduction

This paper is devoted to some Lipschitz estimates between sub- and super-solutions for Fully Nonlinear Degenerate equations on the model of the anisotropic $\vec{p}$-Laplacian. Recall that the equation of the anisotropic $\vec{p}$-Laplacian is

$$\sum_{i=1}^{i=N} \partial_i(\partial_i u|^{p_i-2}\partial_i u) = f$$

where all the p_i are > 1 and f is given, with a regularity to be precised.

This equation has been extensively studied by many authors, with different purposes. If the existence of weak solutions can easily be obtained by classical variational techniques, the regularity is far to be easy to study, and surprisingly, even when the p_i are > 2 and all equal to each others, the Lipschitz regularity was not proved until a recent time. Let us recall some of the results obtained in that case :

Using classical methods in the calculus of variations, equation

$$\tilde{\Delta}_p u := \sum_i \partial_i(|\partial_i u|^{p-2}\partial_i u) = f \tag{1.1}$$

has solutions in $W^{1,p}_{loc}$, when for example $f \in L^{p'}_{loc}$. When $p < 2$, Lipschitz regularity is a consequence of the technics employed in [18].

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When $p > 2$ things are more delicate. If f is sufficiently regular the Lipschitz continuity is a direct consequence of the results in the paper of Bousquet, Brasco and Julin in [10], about the widely degenerate equation

$$\sum_i \partial_i((|\partial_i u| - \delta_i)_+^{p-1} \frac{\partial_i u}{|\partial_i u|}) = f, \tag{1.2}$$

where the δ_i are some given non negative numbers. In that paper, they proved, completing in that way a previous result in [9], the local Lipschitz regularity of the weak solutions of (1.2) under the following assumptions :

-Either $N = 2$, $p \geq 2$ and $f \in W^{1,p'}_{loc}$ - or $N \geq 3$, $p \geq 4$, and $f \in W^{1,\infty}_{loc}$. Of course these regularity assumptions on f, and the gap $2 < p < 4$ when $N \geq 3$, are motivated by the difficulties linked to the presence of the δ_i. In [15] I proved a local Lipschitz estimate between sub- and super-solutions for equation (1.1), ie when $\delta_i = 0$ in (1.2) for all i, under the hypothesis that the right hand side is continuous. One of the consequences of this result is the local Lipschitz continuity of the viscosity solutions when the right hand side f is continuous and bounded, and the same result for weak solutions when $f \in L^\infty_{loc}$.

In [7] we extended these Lipschitz estimates for sub- and super-solutions to some Fully Nonlinear Equations on the model of the pseudo p-Laplacian. An example of such equation is

$$M^{\pm}_\alpha(\nabla u, D^2 u) = f$$

where M^+_α is the pseudo Pucci's operator

$$M^+_\alpha(p, X) = \Lambda tr((\Theta_\alpha(p) X \Theta_\alpha(p))^+) - \lambda tr((\Theta_\alpha(p) X \Theta_\alpha(p))^-),$$

$\Theta_\alpha(p)$ denotes the diagonal matrix with entries $|p_i|^{\frac{\alpha}{2}}$, and $0 < \lambda < \Lambda$, and $\alpha \geq 0$ are some given numbers, while $M^-_\alpha(p, X) = -M^+_\alpha(p, -X)$. X^+ and X^- denote the positive and negative parts of the symmetric matrix X.

When $\lambda = \Lambda$ one recovers the pseudo $(\alpha+2)$-Laplacian, while when $\alpha = 0$, $M^{\pm}_0$ are nothing else that the well known extremal Pucci's operators.

We now turn to the case where the p_i are different and all > 1, and to the variational case, mainly to the case of equations of the form

$$\sum_1^N \partial_i(a_i(x)|\partial_i u|^{p_i-2}\partial_i u) = f \tag{1.3}$$

where the a_i are supposed of the same constant sign, and in general Hölder continuous.

A first step when studying regularity is to get the local boundedness of the solutions, and surprinsingy, if the supremum of the p_i is too large, this can fail : let us cite to that purpose [20] and the paper of Marcellini [26] which exhibits a counterexample to the local boundedness when $a_i = 1$ for all i, $p_i = 2$ for $i \le n-1$ and $p_n > 2\frac{n-1}{n-3}$. This critical value is confirmed by the results obtained later : let us cite in a non exhaustive way [12], [27, 28], [8]. From all these papers it emanates in a first time that a sufficient condition for a local minimizer to be locally bounded is that the supremum of the p_i be strictly less than the critical exponent $\bar{p}^\star$ defined by

$$(\bar{p})^{-1} = \frac{1}{n}\sum_1^n \frac{1}{p_i}, \bar{p}^\star = \frac{n\bar{p}}{n-\bar{p}}.$$

Note that in the case where $p_i = 2$ for $i \le n-1$, the condition $p_n < \bar{p}^\star$ is exactly $p_n < 2\frac{n-1}{n-3}$. In a second time, this local boundedness is extended by Fusco Sbordone in [19] to the case where $\sup p_i = \bar{p}^\star$.

A second step for the regularity is the local higher integrability of the local minimizers : In [16, 17] , Esposito -Leonetti-Mingione consider a large class of functionals, including (1.3) . More recently some authors are interested in the case of the systems, [5] [13, 14], and also in the further regularity $C^{1,\alpha}$ under conditions on the exponent $q > 2$ for the functionals $\int |\nabla u|^2 + |\partial_n u|^q$, ([1], [11]), see also [2] for other more regular functionals.

I want to point out that in the present paper we consider lower semi-continuous (LSC) super-solutions and upper semi-continuous (USC) sub-solutions, then in the case of solutions they are continuous.

We now state the precise assumptions on the Fully Nonlinear operators that will be considered in this paper and we state our main result. Fix $\alpha_i \ge 0$, $1 \le i \le N$, for any $q \in \mathbb{R}^N$, let $\Theta_{\vec{\alpha}}(q)$ be the diagonal matrix with entries $|q_i|^{\frac{\alpha_i}{2}}$ on the diagonal, and let X be a symmetric matrix.

Let S be the space of symmetric matrices on $\mathbb{R}^N$. In the sequel $|x| = \sum_1^N |x_i|^2$, for $x \in \mathbb{R}^N$ and for $X \in S$, $|X| = \sum_{i=1}^{i=N} |\lambda_i(X)|$, the $\lambda_i(X)$ being the eigenvalues of X. Let F be defined on $\mathbb{R}^N \times \mathbb{R}^N \times S$, continuous in all its arguments, which satisfies $F(x, 0, M) = F(x, p, 0) = 0$ and such that :

There exist $0<\lambda<\Lambda$, such that for any $M\in S$ and $N\in S$, $N\geq 0$, for any $x\in \overline{B(0,1)}$

$$\lambda tr(\Theta_{\vec{\alpha}}(q)N\Theta_{\vec{\alpha}}(q))\leq F(x,q,M+N)-F(x,q,M)\leq \Lambda tr(\Theta_{\vec{\alpha}}(q)N\Theta_{\vec{\alpha}}(q)). \tag{1.4}$$

There exist $\gamma_F\in]0,1]$ and $c_{\gamma_F}>0$ such that for any $(q,X)\in \mathbb{R}^N\times S$, for all $(x,y)\in B(0,1)^2$

$$|F(x,q,X)-F(y,q,X)|\leq c_{\gamma_F}|x-y|^{\gamma_F}(\sum_1^N |q_i|^{\alpha_i})|X|. \tag{1.5}$$

There exists c_F such that for all $q,q'\in(\mathbb{R}^N)^2$, $x\in B(0,1)$, and $X\in S$

$$|F(x,q,X)-F(x,q',X)|\leq c_F(\sum_1^N ||q_i|^{\alpha_i}-|q'_i|^{\alpha_i}|)|X|. \tag{1.6}$$

We will also consider a first order term h which satisfy : h is continuous on $\mathbb{R}^N\times\mathbb{R}^N$ and satisfies for some constant c_h :

$$|h(x,q)|\leq c_h\sum |q_i|^{\alpha_i+1}. \tag{1.7}$$

We present some examples of operators that satisfy (1.4), (1.5), and (1.6) :

-Suppose that $L(x)$ is a Lipschitz matrix such that $\sqrt{\lambda}\mathrm{I}\leq L\leq\sqrt{\Lambda}\mathrm{I}$. Then

$$F(x,q,X):=tr(L(x)\Theta_{\vec{\alpha}}(q)X\Theta_{\vec{\alpha}}(q)L(x)),$$

satisfies the hypothesis above.

-For $0<\lambda<\Lambda$

$$\begin{aligned}\mathcal{M}^+_{\vec{\alpha}}(q,X) &= \Lambda tr((\Theta_{\vec{\alpha}}(q)X\Theta_{\vec{\alpha}}(q))^+)-\lambda tr((\Theta_{\vec{\alpha}}(q)X\Theta_{\vec{\alpha}}(q))^-)\\ &= \sup_{\lambda I\leq A\leq\Lambda I} tr(A\Theta_{\vec{\alpha}}(q)X\Theta_{\vec{\alpha}}(q)).\end{aligned}$$

and

$$\mathcal{M}^-_{\vec{\alpha}}(q,X)=-\mathcal{M}^+_{\vec{\alpha}}(q,-X).$$

These operators, denoted as the $\vec{\alpha}$ Pucci's operators, satisfy all the assumptions above. The case where $\alpha_i=0$ for all i reduces to the standard extremizing uniformly elliptic operators. Observe also that for the pseudo anisotropic $\vec{p}$-Laplacian :

$$F(q,X)=\sum_i(p_i-1)|q_i|^{p_i-2}X_{ii} \tag{1.8}$$

satisfies the previous assumptions with $\lambda \leq \inf_i(p_i - 1), \Lambda \geq \sup_i(p_i - 1)$ and $\alpha_i = p_i - 2$ for all i.

-Suppose that a is some Lipschitz function such that $a(x) \geq a_o > 0$. Then

$$F(x, p, X) := a(x)\mathcal{M}^{\pm}_{\vec{\alpha}}(p, X)$$

satisfies all the assumptions before.

We now present the main result of this paper : Suppose that $\bar{\alpha} = \sup \alpha_i$, and $\underline{\alpha} = \inf \alpha_i$.

Theorem 1.1. *Suppose that F is continuous, that $F(x, p, 0) = F(x, 0, X) = 0$, Fsatisfies (1.4), (1.5), (1.6), and that h satisfies (1.7). Suppose that $\bar{\alpha} < \underline{\alpha} + 1$, and that $1 \geq \gamma_F > \bar{\alpha} - \underline{\alpha}$. Suppose that u is USC, bounded and satisfies in $B(0, 1)$*

$$F(x, \nabla u, D^2 u) + h(x, \nabla u) \geq f,$$

that v is LSC, bounded and satisfies in $B(0, 1)$

$$F(x, \nabla v, D^2 v) + h(x, \nabla v) \leq g,$$

and that f and g are continuous and bounded. Then for all $r < 1$, there exists c depending on $(r, N, |u|_\infty, |v|_\infty, |f|_\infty, |g|_\infty)$ and on the data linked to the operator, (say $(\alpha_i, \lambda, \Lambda, \gamma_F, c_{\gamma_F}, c_F, c_h)$), such that for all $(x, y) \in B(0, r)^2$

$$u(x) \leq v(y) + \sup_{B(0,1)} (u - v) + c|x - y|.$$

We intend by weak solution some solution which belongs to $W^{1,p}_{loc}(B(0, 1)$ and satisfies $-\sum_i \partial_i(\partial_i u|^{p_i-2}\partial_i u) = f$ in the distribution sense : Equivalently u satisfies : for any $\varphi \in \mathcal{D}(B(0, 1))$

$$\sum_i \int |\partial_i u|^{p_i-2}\partial_i u \partial_i \varphi = \int f\varphi.$$

As mentioned in (1.8), the Pseudo aniotropic $\vec{p}$ Laplacian satisfies the assumptions in Theorem 1.1, and then

Corollary 1.2. *Suppose that u is a weak, continuous solution in $B(0, 1)$ of*

$$-\sum_i \partial_i(|\partial_i u|^{p_i-2}\partial_i u) = f,$$

that all the p_i are ≥ 2, that f is continuous. Suppose that $\sup_i p_i < \inf_i p_i + 1$. Then for all $r < 1$, u is Lipschitz continuous inside $B(0, r)$, with some Lipschitz constant depending on $(r, p_i, N, |f|_\infty, |u|_\infty)$.

The remainder of this paper is organized as follows : In Section 2 we give some preliminary results, in Section 3 we prove Theorem 1.1 and its corollary.

2. Preliminaries

We suppose in this section and the next one that ω is defined on $\mathbb{R}^+$, continuous on zero, $\mathcal{C}^2$ on $]0,1[$ and such that $\omega(0)=0,\ \omega(s)>0,$ for $s>0,\ \omega'(s)>\frac{1}{2}, \omega''(s)<0$ for $s<1$. We define for some constant $M>1$

$$g(x)=M\omega(|x|).$$

Then

$$Dg(x)=M\omega'(|x|)\frac{x}{|x|}$$

and

$$D^2g(x)=M\left((\omega''(|x|)-\frac{\omega'(|x|)}{|x|})\frac{x\otimes x}{|x|^2}+\frac{\omega'(|x|)}{|x|}\mathrm{I}\right).$$

Taking $\bar{\epsilon}\le\frac{1}{1+4|D^2g|}$ and defining

$$H=D^2g+2\bar{\epsilon}(D^2g)^2, \tag{2.1}$$

one easily sees that there exist $\frac{3}{2}\ge\beta_H\ge\frac{1}{2}$ and $\gamma_H\le\frac{3}{2}$ such that

$$H=M\left((\beta_H\omega''-\gamma_H\frac{\omega'(|x|)}{|x|})\frac{x\otimes x}{|x|^2}+\gamma_H\frac{\omega'(|x|)}{|x|}\mathrm{I}\right).$$

For $\alpha_i\ge 0$ we define the diagonal matrix $(\Theta_{\vec{\alpha}})_{ij}(x)=\left(\frac{M\omega'(|x|)|x_i|}{|x|}\right)^{\frac{\alpha_i}{2}}\delta_i^j$.
Then

$$\begin{aligned}(\Theta_{\vec{\alpha}}H\Theta_{\vec{\alpha}})_{ij} &= M^{1+\frac{\alpha_i+\alpha_j}{2}}(\beta_H\omega''(|x|)-\gamma_H\frac{\omega'(|x|)}{|x|})\left(\frac{\omega'(|x|)}{|x|}\right)^{\frac{\alpha_i+\alpha_j}{2}}\frac{|x_i|^{\frac{\alpha_i}{2}}x_i|x_j|^{\frac{\alpha_j}{2}}x_j}{|x|^2}\\ &+ \gamma_H M^{1+\alpha_i}\left(\frac{\omega'(|x|)}{|x|}\right)^{\alpha_i+1}|x_i|^{\alpha_i}\delta_i^j.\end{aligned}$$

For x a vector in $\mathbb{R}^N$ and for $\epsilon>0$ given, we define

$$I(x,\epsilon)=\{i\in[1,N],|x_i|\ge|x|^{1+\epsilon}\}.$$

Note that since there exists i such that $|x_i|\ge\frac{|x|}{\sqrt{N}}$, as soon as $|x|\le\delta=\exp(-\frac{\log N}{2\epsilon})$, $I(x,\epsilon)\neq\emptyset$.

We then have the following

Proposition 2.1. *Let ω, H, $\Theta_{\vec{\alpha}}$ as above. Let $\bar{\alpha} = \sup \alpha_i, \underline{\alpha} = \inf_i \alpha_i$. For all $x \neq 0$, $|x| < 1$, for any $\epsilon > 0$ such that $I(x,\epsilon) \neq \emptyset$, and such that*

$$\beta_H \omega''(|x|)(1-N|x|^{2\epsilon}) + \gamma_H N|x|^{2\epsilon} \frac{\omega'(|x|)}{|x|} \leq \beta_H \frac{\omega''(|x|)}{2} \leq \frac{\omega''(|x|)}{4} < 0, \quad (2.2)$$

then $\Theta_{\vec{\alpha}} H(x) \Theta_{\vec{\alpha}}$ possesses at least one eigenvalue smaller than

$$2^{-3} M^{1+\underline{\alpha}} (\omega'(|x|))^{\underline{\alpha}} \omega''(|x|)|x|^{\epsilon\overline{\alpha}}. \quad (2.3)$$

Proof. Let us define

$$w = \sum_{i \in I(x,\epsilon)} \left(\frac{\omega'(|x|)|x_i|}{|x|} \right)^{\frac{-\alpha_i}{2}} M^{\frac{-\alpha_i}{2}} x_i e_i.$$

Then using $\omega'(|x|) \geq \frac{1}{2}$,

$$\begin{aligned} |w|^2 &\leq M^{-\underline{\alpha}} 2^{\bar{\alpha}-\underline{\alpha}} \left(\omega'(|x|)\right)^{-\underline{\alpha}} \sum_{i \in I(x,\epsilon)} |x_i|^2 |x|^{-\epsilon\alpha_i} \\ &\leq 2^{\bar{\alpha}-\underline{\alpha}} M^{-\underline{\alpha}} |x|^{-\epsilon\overline{\alpha}} \left(\omega'(|x|)\right)^{-\underline{\alpha}} \sum_{i \in I(x,\epsilon)} |x_i|^2. \end{aligned}$$

Apply ${}^t w$ on the left and w on the right of $\Theta_{\vec{\alpha}} H \Theta_{\vec{\alpha}}$. One gets

$$\begin{aligned} {}^t w(\Theta_{\vec{\alpha}} H \Theta_{\vec{\alpha}}) w &= M \left(\beta_H \omega''(|x|) - \gamma_H \frac{\omega'(|x|)}{|x|} \right) \frac{(\sum_{i \in I(x,\epsilon)} |x_i|^2)^2}{|x|^2} \\ &+ M \gamma_H \frac{\omega'(|x|)}{|x|} (\sum_{i \in I(x,\epsilon)} |x_i|^2) \\ &= M (\sum_{i \in I(x,\epsilon)} |x_i|^2) \left(\beta_H \omega''(|x|)(1 - \frac{\sum_{i \notin I(x,\epsilon)} |x_i|^2}{|x|^2}) \right. \\ &+ \left. \gamma_H \frac{\omega'(|x|)}{|x|} \frac{\sum_{i \notin I(x,\epsilon)} |x_i|^2}{|x|^2} \right) \\ &= M (\sum_{i \in I(x,\epsilon)} |x_i|^2) \left(\beta_H \omega''(|x|) \right. \\ &+ \left. \frac{\sum_{i \notin I(x,\epsilon)} |x_i|^2}{|x|^2} (\gamma_H \frac{\omega'(|x|)}{|x|} - \beta_H \omega''(|x|)) \right) \end{aligned}$$

$$\begin{aligned}
&\leq M(\sum_{i\in I(x,\epsilon)}|x_i|^2)\left(\beta_H\omega''(|x|)\right.\\
&+ \left.N|x|^{2\epsilon}(\gamma_H\frac{\omega'(|x|)}{|x|}-\beta_H\omega''(|x|))\right)\\
&\leq M(\sum_{i\in I(x,\epsilon)}|x_i|^2)\frac{\beta_H}{2}\omega''(|x|),
\end{aligned}$$

since $\omega''<0$, as soon as (2.2) is satisfied. Finally since $\omega''<0$, for $|x|<1$:

$$\begin{aligned}
\frac{{}^t w(\Theta_{\vec{\alpha}}H\Theta_{\vec{\alpha}})w}{|w|^2} &\leq \frac{\beta_H}{2}2^{\underline{\alpha}-\bar{\alpha}}M^{1+\underline{\alpha}}\omega''(|x|)\left(\omega'(|x|)\right)^{\underline{\alpha}}|x|^{\overline{\alpha}\epsilon}\\
&\leq 2^{-3}M^{1+\underline{\alpha}}\omega''(|x|)\left(\omega'(|x|)\right)^{\underline{\alpha}}|x|^{\overline{\alpha}\epsilon}.
\end{aligned}$$

□

We end this section by recalling the definition of viscosity sub- and super-solutions :

Definition 2.2. *u, USC is a sub-solution of $F(x,\nabla u,D^2u)+h(x,\nabla u)=f$ in an open set Ω if for all $\bar{x}\in\Omega$ and for all $\varphi\in\mathcal{C}^2$, such that $(u-\varphi)(x)\leq(u-\varphi)(\bar{x})$ in an open neighborhood of $\bar{x}$ in Ω*

$$F(\bar{x},\nabla\varphi(\bar{x}),D^2\varphi(\bar{x}))+h(\bar{x},\nabla\varphi(\bar{x}))\geq f(\bar{x}),$$

while v, LSC is a super-solution of $F(x,\nabla u,D^2u)+h(x,\nabla u)=f$ in an open set Ω if for all $\bar{x}\in\Omega$ and for all $\varphi\in\mathcal{C}^2$ such that $(u-\varphi)(x)\geq(u-\varphi)(\bar{x})$ in an open neighborhood of $\bar{x}$ in Ω

$$F(\bar{x},\nabla\varphi(\bar{x}),D^2\varphi(\bar{x}))+h(\bar{x},\nabla\varphi(\bar{x}))\leq f(\bar{x}).$$

It is classical in the theory of Second Order Fully Nonlinear Elliptic Equations that one can work with semi-jets, and closed semi-jets in place of $\mathcal{C}^2$ functions. For the convenience of the reader we recall their definition :

Definition 2.3. *Let u be an upper semi-continuous function in a neighbourhood of $\bar{x}$. Then we define the super-jet $(q,X)\in\mathbb{R}^N\times S$ and we note $(q,X)\in J^{2,+}u(\bar{x})$ if there exists $r>0$ such that for all $x\in B_r(\bar{x})$,*

$$u(x)\leq u(\bar{x})+\langle q,x-\bar{x}\rangle+\frac{1}{2}\,{}^t(x-\bar{x})X(x-\bar{x})+o(|x-\bar{x}|^2).$$

Let u be a lower semi-continuous function in a neighbourhood of $\bar{x}$. Then we define the sub-jet $(q,X)\in\mathbb{R}^N\times S$ and we note $(q,X)\in J^{2,-}u(\bar{x})$ if there *exists $r > 0$ such that for all $x \in B_r(\bar{x})$,*

$$u(x) \geq u(\bar{x}) + \langle q, x - \bar{x}\rangle + \frac{1}{2}\ {}^t(x - \bar{x})X(x - \bar{x}) + o(|x - \bar{x}|^2).$$

We also define the "closed semi-jets" :

$$\begin{aligned}\bar{J}^{2,\pm}u(\bar{x}) \quad &= \quad \{(q, X), \exists\ (x_n, q_n, X_n),\ (q_n, X_n) \in J^{2,\pm}u(x_n) \\ &\qquad \text{and } (x_n, q_n, X_n) \to (\bar{x}, q, X)\}.\end{aligned}$$

We refer to the survey of Ishii [23], and to [24] for more complete results about semi-jets: The link between semi-jets and test functions for sub- and super-solutions is the following :

u, USC is a sub-solution if and only if for any $\bar{x}$ and for any $(q, X) \in \bar{J}^{2,+}u(\bar{x})$, then

$$F(\bar{x}, q, X) + h(\bar{x}, q) \geq f(\bar{x})$$

and the same with analogous changes is valid for super-solutions.

Let us now recall Lemma 9 in [23] and one of its consequences for the proofs in the present paper

Lemma 2.4. *Suppose that A is a symmetric matrix on $\mathbb{R}^{2N}$ and that $U \in USC(\mathbb{R}^N)$, $V \in USC(\mathbb{R}^N)$ satisfy $U(0) = V(0)$ and for all $(x, y) \in (\mathbb{R}^N)^2$*

$$U(x) + V(y) \leq \frac{1}{2}({}^t x, {}^t y)A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then for all $\bar{\epsilon} > 0$ there exist $X_{\bar{\epsilon}}^U \in S$, $X_{\bar{\epsilon}}^V \in S$ such that

$$(0, X_{\bar{\epsilon}}^U) \in \bar{J}^{2,+}U(0),\ (0, X_{\bar{\epsilon}}^V) \in \bar{J}^{2,+}V(0)$$

and

$$-(\frac{1}{\bar{\epsilon}} + |A|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_{\bar{\epsilon}}^U & 0 \\ 0 & X_{\bar{\epsilon}}^V \end{pmatrix} \leq (A + \bar{\epsilon}A^2).$$

Lemma 2.5. *Suppose that u and v are respectively USC and LSC functions such that, for some constant $L > 1$ and for some $\mathcal{C}^2$ function Φ*

$$\psi(x, y) := u(x) - v(y) - L|x - x_o|^2 - L|y - x_o|^2 - \Phi(x, y)$$

has a local maximum in $(\bar{x}, \bar{y})$.

Then for any $\bar{\epsilon}$, there exist $X_{\bar{\epsilon}}, Y_{\bar{\epsilon}}$ such that

$$\begin{aligned}(D_1\Phi(\bar{x}, \bar{y}) + 2L(\bar{x} - x_o), X_{\bar{\epsilon}}) &\in \bar{J}^{2,+}u(\bar{x}), \\ (-D_2\Phi(\bar{x}, \bar{y}) - 2L(\bar{y} - x_o), -Y_{\bar{\epsilon}}) &\in \bar{J}^{2,-}v(\bar{y})\end{aligned}$$

with

$$-(\frac{1}{\bar\epsilon}+|A|+1)\begin{pmatrix} \mathrm{I} & 0\\ 0 & \mathrm{I}\end{pmatrix} \leq \begin{pmatrix} X_{\bar\epsilon}-2L\mathrm{I} & 0\\ 0 & Y_{\bar\epsilon}-2L\mathrm{I}\end{pmatrix} \leq (A+\bar\epsilon A^2)+\begin{pmatrix} \mathrm{I} & 0\\ 0 & \mathrm{I}\end{pmatrix}$$

and $A = D^2\Phi(\bar x, \bar y)$.

A proof of Lemma 2.5 is detailed in [7], for example.

3. Proof of Theorem 1.1

In this section we prove the main result of this paper. Before entering the details of the proof let us mention that several Hölder's and Lipschitz regularity results have been obtained for related nonlinear degenerate elliptic but homogeneous in the gradient, let us cite in a non exhaustive manner [23], [6].

Note that Theorem1.1 can be obtained only once we have proven the following Hölder's estimate. So we will prove first

Theorem 3.1. *Suppose that* F *is continuous,* $F(x,p,0) = F(x,0,X) = 0$, *satisfies (1.4), (1.6), (1.5), and that* h *satisfies (1.7). Suppose that* $\bar\alpha < \underline{\alpha}+1$, *that* $\gamma_F > \bar\alpha - \underline{\alpha}$ *and that* u *is USC, bounded and satisfies in* $B(0,1)$ *in the sense of Definition (2.2)*

$$F(x,\nabla u, D^2u) + h(x,\nabla u) \geq f,$$

that v *is LSC, bounded and satisfies in* $B(0,1)$ *in the sense of Definition (2.2)*

$$F(x,\nabla v, D^2v) + h(x,\nabla v) \leq g,$$

and that f *and* g *are continuous and bounded in* $B(0,1)$*.Then for all* $\gamma < 1$ *and for all* $r < 1$ *there exists* c *depending on* $(r,\gamma,N,|u|_\infty,|v|_\infty,|f|_\infty,|g|_\infty)$ *and on the data linked to* F *and* h*, such that for all* $(x,y)\in B(0,r)^2$

$$u(x) \leq v(y) + \sup_{B(0,1)}(u-v) + c|x-y|^\gamma.$$

Remark 3.2. *One can delete in that theorem as well as in Theorem 1.1 the assumptions " bounded" for* u *and* v*. Indeed, with the upper-semi continuity (respectively lower semi continuity) assumption,* u *(respectively* v*) is locally bounded from above, (respectively bounded from below) and one must replace* $|u|_\infty$*,* $|v|_\infty$ *in the previous dependances by* $\sup_{B(0,1)} u$ *and* $-\inf_{B(0,1)} v$*.*

Let us devote a few lines to explain how we will obtain the results.

Suppose that $\omega(s) = s^\gamma$ with $\gamma \in]0,1[$ in the Hölder's case and $\omega(s)$ behaves near zero as s in the Lipschitz case. In a classical way when one deals with viscosity solutions, ([23], [25], [22], [3]), we define

$$\phi(x,y) = u(x) - v(y) - \sup(u-v) - M\omega(|x-y|) - L|x-x_o|^2 - L|y-x_o|^2$$

where $x_o \in B_r$, M and L will be chosen later independently on x_o . As in Section 2 we denote $g(x) = M\omega(|x|)$. We need to prove that $\phi(x,y) \leq 0$ in $B(0,1)$, which will imply the result. Indeed taking $y = x_o$ and making x_o vary, one gets that for all $x \in B_1$ and $y \in B_r$

$$u(x) \quad \leq \quad v(y) + \sup(u-v) + M\omega(|x-y|) + L|x-y|^2$$

which gives the result.

We argue by contradiction and suppose that there exists (x,y) in $B(0,1)$ such that $\phi(x,y) > 0$. The supremum of ϕ is achieved on $(\bar{x},\bar{y}) \in \overline{B(0,1)}^2$. We begin to impose some conditions on L and M in order to be able to use lemma 2.5, in particular we need $(\bar{x},\bar{y})$ to be interior points. So we introduce some $\delta \in]0,1[$ which will be chosen later small depending on $(N, |u|_\infty, |v|_\infty, r, \lambda, \Lambda, |f|_\infty, |g|_\infty)$, define $M = 1 + \frac{2|u|_\infty + 2|v|_\infty}{\omega(\delta)}$, and $L = 1 + \frac{8|u|_\infty + 8|v|_\infty}{(1-r)^2}$. The hypothesis on L ensures that $(\bar{x},\bar{y}) \in B^2_{\frac{1+r}{2}}$. Futhermore by the assumption on M, $|\bar{x}-\bar{y}| \leq \delta$. We shall prove that taking δ small enough depending only on the data, using the fact that u and v are respectively sub-and super-solutions, we get a contradiction with $\phi(\bar{x},\bar{y}) = \sup \phi(x,y) > 0$.

Using Lemma 2.5 , for all $\bar{\epsilon} > 0$, there exist $X_{\bar{\epsilon}}, Y_{\bar{\epsilon}} \in S$ such that, defining $q^x = M\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} + L(\bar{x}-x_o)$, $q^y = M\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} - L(\bar{y}-x_o)$, $q = M\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}$ one has

$$(q^x, X_{\bar{\epsilon}}) \in J^{2,+}u(\bar{x}),\ (q^y, -Y_{\bar{\epsilon}}) \in J^{2,-}v(\bar{y})$$

with (recalling that H is given by (2.1))

$$\begin{aligned} -\left(|D^2 g(\bar{x}-\bar{y})| + \frac{1}{\bar{\epsilon}}\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad &\leq \quad \begin{pmatrix} X_{\bar{\epsilon}} - (2L+1)\mathrm{I} & 0 \\ 0 & Y_{\bar{\epsilon}} - (2L+1)\mathrm{Id} \end{pmatrix} \\ &\leq \quad \begin{pmatrix} H & -H \\ -H & H \end{pmatrix}. \end{aligned} \tag{3.1}$$

We now take $\bar\epsilon = \frac{1}{1+4|D^2 g|}$ and from now we drop the index $\bar\epsilon$ for simplicity for $X_{\bar\epsilon}$, $Y_{\bar\epsilon}$.

We will prove the following claims, both in the Hölder's case and in the lipschitz case

Claims. *There exist $\hat\tau > 0$ and $c > 0$, such that, if δ is small enough and $|\bar x - \bar y| < \delta$ the matrix $\Theta_{\vec\alpha}(X+Y)\Theta_{\vec\alpha}$ has one eigenvalue μ_1 such that*

$$\mu_1(\Theta_{\vec\alpha}(X+Y)\Theta_{\vec\alpha}) \leq -cM^{1+\underline{\alpha}}|\bar x - \bar y|^{-\hat\tau} \tag{3.2}$$

There exist $\tau_i < \hat\tau$ and c_i for $i = 1, \dots, 4$ such that the four following assertions hold :

$$\text{for all } j \geq 2,\ \mu_j(\Theta_{\vec\alpha}(X+Y)\Theta_{\vec\alpha}) \leq c_1 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_1}, \tag{3.3}$$

$$|F(\bar x, q^x, X) - F(\bar x, q, X)|, |F(\bar y, q^y, -Y) - F(\bar y, q, -Y)| \leq c_2 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_2}, \tag{3.4}$$

$$|F(\bar x, q, X) - F(\bar y, q, X)| + |F(\bar x, q, -Y) - F(\bar y, q, -Y)| \leq c_3 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_3}, \tag{3.5}$$

$$|h(\bar x, q^x)| + |h(\bar y, q^y)| \leq c_4 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_4}. \tag{3.6}$$

All these claims permit to obtain a contradiction both for the two cases Lipschitz and Hölder. Indeed remark that by (1.4)

$$\begin{aligned} F(\bar x, q, X) &\leq F(\bar x, q, -Y) + \Lambda \sum_{j\geq 2} \mu_j^+(\Theta_\alpha(q)(X+Y)\Theta_\alpha(q)) \\ &+ \lambda\mu_1(\Theta_\alpha(q)(X+Y)\Theta_\alpha(q)) \end{aligned}$$

hence one has

$$\begin{aligned} f(\bar x) &\leq F(\bar x, q^x, X) + h(\bar x, q^x) \\ &\leq F(\bar x, q, X) + h(\bar x, q^x) + c_2 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_2} \\ &\leq F(\bar x, q, -Y) + h(\bar y, q^y) - c\lambda M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\hat\tau} + N\Lambda c_1 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_1} \\ &+ c_2 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_2} + c_4 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_4} \\ &\leq F(\bar y, q, -Y) + h(\bar y, q^y) - c\lambda M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\hat\tau} + N\Lambda c_1 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_1} \\ &+ c_2 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_2} + c_3 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_3} + c_4 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_4} \\ &\leq F(\bar y, q^y, -Y) + h(\bar y, q^y) - c\lambda M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\hat\tau} + N\Lambda c_1 M^{1+\underline{\alpha}}|\bar x - \bar y|^{-\tau_1} \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2c_2 M^{1+\underline{\alpha}}|\bar{x}-\bar{y}|^{-\tau_2} + c_3 M^{1+\underline{\alpha}}|\bar{x}-\bar{y}|^{-\tau_3} + c_4 M^{1+\underline{\alpha}}|\bar{x}-\bar{y}|^{-\tau_4} \\
&\leq \quad g(\bar{y}) - \frac{c\lambda}{2} M^{1+\underline{\alpha}}|\bar{x}-\bar{y}|^{-\hat{\tau}}, \qquad (3.7)
\end{aligned}
$$

as soon as δ is small enough in order that

$$
c_1 \Lambda N \delta^{-\tau_1} + 2c_2 \delta^{-\tau_2} + c_3 \delta^{-\tau_3} + c_4 \delta^{-\tau_4} < \frac{c}{2} \lambda \delta^{-\hat{\tau}}.
$$

Finally supposing also that δ satisfies $\frac{c\lambda}{2}\delta^{-\hat{\tau}} > |f|_\infty + |g|_\infty$ one gets a contradiction.

So to prove the results in Theorem 3.1 and in Theorem 1.1 it is sufficient to prove (2.2), (3.2), (3.3), (3.4), (3.5), and (3.6) when $\omega(s) = s^\gamma$ and $\gamma \in [0, 1[$. Once this done we obtain for any $\gamma < 1$, the Hölder's estimate. We then define conveniently ω, behaving like s near zero, and prove the above claims in that case.

As a first step to get (3.2), (3.3), both in the Hölder and in the Lipschitz case, let us derive two important consequences of Proposition 2.1 and of (3.1) :

i) All the eigenvalues of $\Theta_{\vec{\alpha}}(X+Y)\Theta_{\vec{\alpha}}$ are less than $cLM^{\bar{\alpha}}\omega'(|\bar{x}-\bar{y}|)^{\bar{\alpha}}$.

ii) There exists at least one eigenvalue of $\Theta_{\vec{\alpha}}(X+Y)\Theta_{\vec{\alpha}}$, less than $\frac{1}{2}M^{1+\underline{\alpha}}\omega''(|\bar{x}-\bar{y}|)\,(\omega'(|\bar{x}-\bar{y}|)))^{\underline{\alpha}}\,|\bar{x}-\bar{y}|^{\epsilon\overline{\alpha}}$.

Indeed to prove i) let us multiply equation (3.1) by $\begin{pmatrix} \Theta_{\vec{\alpha}} & 0 \\ 0 & \Theta_{\vec{\alpha}} \end{pmatrix}$ on the right and on the left. Next apply the resulting inequality to $({}^t x, {}^t x)$ on the left and to $\begin{pmatrix} x \\ x \end{pmatrix}$ the right, x being any vector : One gets the result.

To prove ii) let e be a unit eigenvector for some eigenvalue of $\Theta_{\vec{\alpha}} H \Theta_{\vec{\alpha}}$ which is less than $\frac{1}{2^3}M^{1+\underline{\alpha}}\omega''(\bar{x}-\bar{y})\,(\omega'(|\bar{x}-\bar{y}|))^{\underline{\alpha}}\,|\bar{x}-\bar{y}|^{\epsilon\overline{\alpha}}$. Then by applying to $\begin{pmatrix} \Theta_{\vec{\alpha}} X \Theta_{\vec{\alpha}} & 0 \\ 0 & \Theta_{\vec{\alpha}} Y \Theta_{\vec{\alpha}} \end{pmatrix}$ the vector $\begin{pmatrix} e \\ e \end{pmatrix}$ on the right and to its transpose on the left one gets that $\Theta_{\vec{\alpha}}(X+Y)\Theta_{\vec{\alpha}}$ has at least one eigenvalue less than $4\,{}^t e(\Theta_{\vec{\alpha}} H \Theta_{\vec{\alpha}})e$.

3.1. **Proof of (2.2), (3.2), (3.3), (3.4) and (3.6) in the Hölder's case.** Here $\omega(s) = s^\gamma$ with $\gamma \in]0, 1[$ and then $q^x = M|\bar{x}-\bar{y}|^{\gamma-2}(\bar{x}-\bar{y}) + L(\bar{x}-x_o)$, $q^y = M|\bar{x}-\bar{y}|^{\gamma-2}(\bar{x}-\bar{y}) - L(\bar{y}-x_o)$. For further purposes we also introduce $q = M|\bar{x}-\bar{y}|^{\gamma-2}(\bar{x}-\bar{y})$. Note also that using (3.1) there exists some universal constant c such that

$$|X| + |Y| \leq cM|\bar{x} - \bar{y}|^{\gamma-2} + L.$$

We now take ϵ positive,

$$\epsilon < \inf\left(\frac{\gamma_F - (\bar{\alpha} - \underline{\alpha})}{\bar{\alpha}}, \frac{(1-\gamma)(\underline{\alpha} - (1-\bar{\alpha})^-)}{\bar{\alpha}}\right) \tag{3.8}$$

which is possible since $\gamma_F > \bar{\alpha} - \underline{\alpha}$ and $(1-\gamma)(1-\bar{\alpha}+\underline{\alpha}) > 0$.

Concerning δ, we will suppose first that is enough small in order that $\bar{\alpha}L < M^{1+\underline{\alpha}-\bar{\alpha}}$ and $2L^{\overline{\alpha}} < M^{\underline{\alpha}}$. Note that this implies in particular that

$$|X| + |Y| \leq 2cM|\bar{x} - \bar{y}|^{\gamma-2}. \tag{3.9}$$

Furthermore in order to check (2.2) we will suppose that

$$\delta < \exp\left(\frac{1}{2\epsilon}\log\left(\frac{1-\gamma}{6N(2-\gamma)}\right)\right).$$

Indeed supposing δ so, one has for $|x| < \delta$

$$\begin{aligned} N(-\beta_H\omega''(|x|) + \gamma_H\frac{\omega'(|x|)}{|x|})|x|^{2\epsilon} &\leq \frac{3}{2}N|x|^{\gamma-2+2\epsilon}\gamma(2-\gamma) \\ &\leq \frac{\gamma(1-\gamma)}{4}|x|^{\gamma-2} \\ &\leq \frac{\beta_H|\omega''(|x|)|}{2} \end{aligned}$$

Note that $\delta < e^{\frac{-\log N}{2\epsilon}}$, which implies that as soon as $|x| < \delta$, $I(x,\epsilon) \neq \emptyset$.

To prove (3.2) note that $\hat{\tau} = (2-\gamma) + (1-\gamma)\underline{\alpha} - \bar{\alpha}\epsilon$ is positive by (3.8) and convenient, by using Proposition 2.1.

Secondly (3.3) holds with $\tau_1 = (1-\gamma)\overline{\alpha} < (2-\gamma) + (1-\gamma)\underline{\alpha} - \epsilon\bar{\alpha}$ by (3.8). Indeed one has by the choice of L and for some constant c_1

$$\mu_i(\Theta_{\vec{\alpha}}(X+Y)\Theta_{\vec{\alpha}}) \leq c_1 LM^{\bar{\alpha}}|\bar{x} - \bar{y}|^{(\gamma-1)\bar{\alpha}} \leq c_1 M^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{(\gamma-1)\bar{\alpha}}.$$

To prove (3.4), we need to evaluate $\sum_i ||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}||X|$.

For that aim we use :

-If $\alpha_i \leq 1$ $||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}| \leq |q_i^x - q_i|^{\alpha_i} \leq L^{\alpha_i}|\bar{x} - x_o|^{\alpha_i} \leq 2^{\bar{\alpha}}L^{\bar{\alpha}}$,

hence using (3.9)

$$||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}||X| \leq 2c(2L)^{\bar{\alpha}}M|\bar{x} - \bar{y}|^{\gamma-2} \leq 2cM^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{\gamma-2},$$

by the choice of δ and its consequence on L and M.

-While if $\alpha_i > 1$ $||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}| \leq \alpha_i |q_i^x - q_i|(|q^x| + |q|)^{\alpha_i - 1} \leq c\bar{\alpha} L M^{\bar{\alpha}-1} |\bar{x} - \bar{y}|^{(\gamma-1)(\bar{\alpha}-1)}$, and then $||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}||X| \leq \bar{\alpha} 2cLM^{\bar{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-1)(\bar{\alpha}-1)+\gamma-2} \leq 2cM^{1+\underline{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-1)(\bar{\alpha}-1)+\gamma-2}$. Gathering these two estimates, (3.4) holds with $\tau_2 = (2-\gamma) + (1-\gamma)(\sup(1, \bar{\alpha}) - 1) < 2 - \gamma + (1-\gamma)\underline{\alpha} - \epsilon\bar{\alpha}$ by the choice of ϵ in (3.8).

To prove (3.5) let us observe that

$$\begin{aligned} |F(\bar{x}, q, X) - F(\bar{y}, q, X)| &\leq c_F |\bar{x} - \bar{y}|^{\gamma_F} |q|^{\bar{\alpha}} |X| \\ &\leq c_F \gamma^{\bar{\alpha}} 2c |\bar{x} - \bar{y}|^{\gamma_F} M^{1+\bar{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-1)\bar{\alpha}+\gamma-2}. \end{aligned}$$

Note that by the definition of M there exist some constants c depending only on the data such that

$$\begin{aligned} M^{1+\bar{\alpha}} |\bar{x} - \bar{y}|^{\gamma_F} |\bar{x} - \bar{y}|^{(\gamma-1)\bar{\alpha}+\gamma-2} &\leq cM^{1+\underline{\alpha}} \delta^{-\gamma(\bar{\alpha}-\underline{\alpha})} |\bar{x} - \bar{y}|^{(\gamma-2)+(\gamma-1)\bar{\alpha}+\gamma_F} \\ &\leq cM^{1+\underline{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-2)+\gamma(-\bar{\alpha}+\underline{\alpha})+(\gamma-1)\bar{\alpha}+\gamma_F} \\ &\leq cM^{1+\underline{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-2)+\gamma\underline{\alpha}-\bar{\alpha}+\gamma_F} \end{aligned}$$

and then (3.5) holds with $\tau_3 = (2-\gamma) + \bar{\alpha} - \gamma\underline{\alpha} - \gamma_F < (2-\gamma) + (1-\gamma)\underline{\alpha} - \epsilon\bar{\alpha}$.

We finally check (3.6). One has

$$|h(\bar{x}, q^x)| + |h(\bar{y}, q^y)| \leq c_h N(|q^x|^{\bar{\alpha}+1} + |q^y|^{\bar{\alpha}+1}) \leq 2c_h N M^{1+\bar{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-1)(1+\bar{\alpha})}$$

and by the definition of M one has for some constant which can vary from one line to another

$$\begin{aligned} M^{1+\bar{\alpha}} |\bar{x} - \bar{y}|^{(\gamma-1)(1+\bar{\alpha})} &\leq cM^{1+\underline{\alpha}} \delta^{-\gamma(\bar{\alpha}-\underline{\alpha})} |\bar{x} - \bar{y}|^{(\gamma-1)(1+\bar{\alpha})} \\ &\leq cM^{1+\underline{\alpha}} |\bar{x} - \bar{y}|^{\gamma-1+\gamma\underline{\alpha}-\bar{\alpha}} \end{aligned}$$

and then one has (3.6) by defining $\tau_4 = 1 - \gamma + \bar{\alpha} - \gamma\underline{\alpha} < 2 - \gamma + (1-\gamma)\underline{\alpha} - \bar{\alpha}\epsilon$ since $\epsilon\bar{\alpha} < \gamma_F - (\bar{\alpha} - \underline{\alpha}) \leq 1 - (\bar{\alpha} - \underline{\alpha})$.

3.2. **Proof of (2.2), (3.2), (3.3,) (3.4) and (3.5), (3.6) in the "Lipschitz" case.** We define $\omega(s) = s - \frac{1}{2(1+\tau)} s^{1+\tau}$, where $0 < \tau < 1$ will be precised later, and $s < 1$. ω is extended constantly after 1. Note that for $s < 1$, $\frac{1}{2} \leq \omega'(s) < 1$, and then $\omega(s) \geq \frac{s}{2}$, and ω is $\mathcal{C}^2$ on $]0, 1[$.

We define $\underline{\alpha}^\star = \inf\{\alpha_i, \alpha_i > 0\}$. Let $\epsilon < \inf\left(\frac{\inf(1, \underline{\alpha}^\star)}{2(1+\bar{\alpha})}, \frac{\gamma_F - (\bar{\alpha} - \underline{\alpha})}{1+\bar{\alpha}}\right)$, and τ such that $\tau < \epsilon$, then $\tau + \epsilon\bar{\alpha} < \inf(\underline{\alpha} - \bar{\alpha} + \gamma_F, \frac{\inf(1, \underline{\alpha}^\star)}{2})$.

We suppose

$$M = 1 + \frac{8(|u|_\infty + |v|_\infty)}{\delta},\ L = 1 + \frac{8(|u|_\infty + |v|_\infty)}{(1-r)^2}$$

and that δ is small enough in order that $ML^{\bar\alpha} + L^{\frac{1}{2}}M^{\bar\alpha} \le M^{1+\underline{\alpha}}$. We introduce also $\gamma < 1$ such that $\frac{\gamma}{2}\inf(1, \underline{\alpha}^\star) > \tau + \epsilon\bar\alpha$ which is possible since $\tau + \epsilon\bar\alpha < \epsilon + \epsilon\bar\alpha < \frac{\inf(1,\underline{\alpha}^\star)}{2}$.

We also suppose that $\delta < \exp\frac{\log\frac{\tau}{3N(\tau+2)}}{2\epsilon-\tau}$. Then for $|x| < \delta$,

$$\begin{aligned} N(-\beta_H\omega''(|x|) + \gamma_H\frac{\omega'(|x|)}{|x|})|x|^{2\epsilon} &\le \frac{3N}{2}(\frac{\tau}{2}+1)|x|^{-1+2\epsilon} \\ &\le \frac{\tau}{4}|x|^{-1+\tau} \\ &\le \frac{\beta_H}{2}|\omega''(|x|)| \end{aligned}$$

and then (2.2) is satisfied.

Note that $\delta^{2\epsilon} < \frac{1}{N}$, and then since there exists $i \in [1, N]$ such that $|x_i| > \frac{|x|}{\sqrt{N}}$, for x such that $|x| \le \delta_N$, $I(x,\epsilon) \neq \emptyset$. We introduce as in the last subsection

$$\phi(x,y) = u(x) - v(y) - \sup(u-v) - M\omega(|x-y|) - L|x-x_o|^2 - L|y-x_o|^2$$

and suppose by contradiction that the supremum of ϕ is positive . Then it is achieved on $(\bar x, \bar y)$ which belongs to $B^2_{\frac{1+r}{2}}$ and is such that $|\bar x - \bar y| < \delta$.

Recall that since the estimate $u(x) - v(y) \le \sup(u-v) + c|x-y|^\gamma$ has been proved in the last section, one has $L|\bar x - x_o|^2 \le c_{\frac{1+r}{2},\gamma}|\bar x - \bar y|^\gamma$ and then

$$L|\bar x - x_o| \le c^{\frac{1}{2}}_{\frac{1+r}{2},\gamma} L^{\frac{1}{2}}|\bar x - \bar y|^{\frac{\gamma}{2}}. \tag{3.10}$$

The analogous is true for $\bar y - x_o$. In particular as soon as δ is small enough, $L|\bar x - x_o| \le \frac{M}{4}$.

This will be needed in the estimate (3.4).

Note that here one has $q_i^x = M\frac{\omega'(|\bar x - \bar y|)}{|\bar x - \bar y|}(\bar x_i - \bar y_i) + L(\bar x_i - x_{oi})$, $q_i = M\frac{\omega'(|\bar x-\bar y|)}{|\bar x-\bar y|}(\bar x_i - \bar y_i)$ $q_i^y = M\frac{\omega'(|\bar x-\bar y|)}{|\bar x-\bar y|}(\bar x_i - \bar y_i) - L(\bar y_i - x_{oi})$ and then for any i, $\frac{M}{2} \le |q_i| \le M$ and $\frac{M}{4} \le |q_i^x|, |q_i^y| \le \frac{5M}{4}$.

Applying Proposition 2.3 one gets that (3.2) holds with $\hat\tau = 1 - \tau - \epsilon\overline{\alpha}$. We now prove claim (3.3) : One has for all $i \in [1, N]$

$$\mu_i(\Theta_{\vec\alpha}(X+Y)\Theta_{\vec\alpha}) \le (2L+1)|\Theta_{\vec\alpha}|^2 \le c_1 LM^{\bar\alpha} \le c_1 M^{1+\underline{\alpha}}$$

by the choice of δ and its consequences for L and M and then (3.3) holds with $\tau_1 = 0 < 1 - \tau - \epsilon\bar{\alpha}$.

For the following estimates, we need to observe that inequality (3.1) implies here that there exists c such that

$$|X| + |Y| \le cM|\bar{x} - \bar{y}|^{-1} + 2L \le 3cM|\bar{x} - \bar{y}|^{-1} \tag{3.11}$$

by the assumption on L.

To prove (3.4), let us recall that (3.10) holds.

Suppose that $0 < \alpha_i \le 1$ which implies if such an index exists, that $\underline{\alpha} \le 1$. For such a i one has using (3.11) for some constant c_2 which can vary from one lie to other

$$\begin{aligned} ||q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}||X| &\le |q_i^x - q_i|^{\alpha_i}|X| \le c_2|\bar{x} - \bar{y}|^{\frac{\gamma\alpha_i}{2}} L^{\frac{\alpha_i}{2}} M|\bar{x} - \bar{y}|^{-1} \\ &\le c_2 M^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{\frac{\gamma\alpha_i}{2}-1} \end{aligned}$$

by the choices of L and M, while if $\alpha_i \ge 1$ the mean value's theorem implies, always with the choice of L and M, still using (3.11)

$$\begin{aligned} |q_i^x|^{\alpha_i} - |q_i|^{\alpha_i}||X| &\le c_2|q_i - q_i^x|M^{\alpha_i - 1}M|\bar{x} - \bar{y}|^{-1} \le c_2 L^{\frac{1}{2}}|\bar{x} - \bar{y}|^{\frac{\gamma}{2}-1}M^{\bar{\alpha}} \\ &\le c_2 M^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{\frac{\gamma}{2}-1}. \end{aligned}$$

Combining the two inequalites, (3.4) holds with $\tau_2 = 1 - \frac{\gamma}{2}\inf(1, \underline{\alpha}^\star) < 1 - \tau - \epsilon\bar{\alpha}$.

We now prove (3.5). One has for some constant c_3 which can vary from one line to another

$$\begin{aligned} |F(\bar{x}, q, X) - F(\bar{y}, q, X)| &\le c_F|\bar{x} - \bar{y}|^{\gamma_F} M^{1+\bar{\alpha}}|\bar{x} - \bar{y}|^{-1} \\ &\le c_3 M^{1+\underline{\alpha}}\delta^{-\bar{\alpha}+\underline{\alpha}}|\bar{x} - \bar{y}|^{\gamma_F - 1} \\ &\le c_3 M^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{\gamma_F - 1 + \underline{\alpha} - \bar{\alpha}} \end{aligned}$$

by the choice of δ, hence (3.5) holds with $\tau_2 = 1 - \gamma_F - \underline{\alpha} + \bar{\alpha} < 1 - \tau - \epsilon\bar{\alpha}$ by the choice of τ.

We finally check (3.6): One has for some constant c_4 which can vary from l one line to another

$$\begin{aligned} |h(\bar{x}, q^x)| &\le c_4 M^{1+\bar{\alpha}} \\ &\le c_4 M^{1+\underline{\alpha}}\delta^{-(\bar{\alpha}-\underline{\alpha})} \\ &\le c_4 M^{1+\underline{\alpha}}|\bar{x} - \bar{y}|^{-(\bar{\alpha}-\underline{\alpha})} \end{aligned}$$

a same estimate holds for $|h(\bar{y}, q^y)|$, and then (3.6) holds with $\tau_4 = \bar{\alpha} - \underline{\alpha} < 1 - \tau - \epsilon\bar{\alpha}$ since $\tau + \epsilon\bar{\alpha} < \gamma_F - \bar{\alpha} + \underline{\alpha} < 1 - \bar{\alpha} + \underline{\alpha}$.

Since all the claims are proved one concludes by (3.7).

Proof of Corollary 1.2
We just give a hint of the proof. Adapting arguments as in [4], [15], Proposition 2.7, we obtain that weak continuous solutions are viscosity solutions. Then one applies Theorem 1.1 with $u = v$.

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