# MOMENTS OF 2D PARABOLIC ANDERSON MODEL 

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#### Abstract

In this note, we use the Feynman-Kac formula to derive a moment representation for the 2D parabolic Anderson model in small time, which is related to the intersection local time of planar Brownian motions.


Keywords: Feynman-Kac formula, renormalization, intersection local time.

## 1. Introduction

The aim of this note is to study the existence of moments of the solution to the parabolic Anderson model (PAM) in two spatial dimensions, formally given by

$$
\begin{equation*}
\partial_{t} u=\frac{1}{2} \Delta u+u \cdot \xi, \quad(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{2}, \tag{1.1}
\end{equation*}
$$

where $\xi$ is the two dimensional spatial white noise, that is, a generalized Gaussian process with covariance $\mathbb{E}[\xi(x) \xi(y)]=\delta(x-y)$.

The equation is well-posed in dimension 1 , but the product between $u$ and $\xi$ becomes ill-defined as soon as $d \geq 2$. For $d=2$, the solution $u$ is defined in $[7,8,10]$ as the limit of a sequence of the regularized and renormalized equations. More precisely, fix a symmetric mollifier $\rho: \mathbb{R}^{2} \rightarrow \mathbb{R}^{+}$with $\rho(x)=\rho(-x)$ and $\int \rho=1$. Let

$$
\rho_{\varepsilon}(x)=\varepsilon^{-2} \rho(x / \varepsilon), \quad \xi_{\varepsilon}=\xi \star \rho_{\varepsilon}
$$

and consider the equation

$$
\begin{equation*}
\partial_{t} u_{\varepsilon}=\frac{1}{2} \Delta u_{\varepsilon}+\left(\xi_{\varepsilon}-C_{\varepsilon}\right) u_{\varepsilon}, \tag{1.2}
\end{equation*}
$$

for some large constant $C_{\varepsilon}$. Then, for

$$
\begin{equation*}
C_{\varepsilon}=\frac{1}{\pi} \log \varepsilon^{-1} \tag{1.3}
\end{equation*}
$$

the sequence of solutions $\left\{u_{\varepsilon}\right\}$ converges in some weighted Hölder space in probability to a limit $u$ that is independent of the mollification, see e.g. [10, Theorem 4.1], and we call this limit $u$ the solution to 2D PAM. In $d=3$, the mollifier $\rho_{\varepsilon}(x)=\varepsilon^{-3} \rho(x / \varepsilon)$, and the renormalization constant takes the form $C_{\varepsilon}=c_{1} \varepsilon^{-1}+c_{2} \log \varepsilon^{-1}+O(1)[9]$.

So far, most of the results mentioned above focused on the existence of the solution and the convergence of the regularized PDE after renormalization. The statistical properties of $u$ remains a challenge; see [1, 2, 4] for some relevant discussions. The goal of this note is to show that the $n$-th moment of the solution
$u$ to 2D PAM exists for small time, and we present a Feynman-Kac formula for $\mathbb{E}\left[u^{n}\right]$. The following is our main result.

Theorem 1.1. There exists a universal constant $\delta>0$ such that for every $n \in \mathbb{N}$, the $n$-th moment of $u$ exists for $t \in\left(0, \frac{\delta}{n^{2}}\right)$ with $\mathbb{E}\left[u(t, x)^{n}\right]$ given by (1.10).
1.1. Heuristic argument. We first give a heuristic derivation of $\mathbb{E}[u(t, x)]^{n}$ by writing down a representation for $\mathbb{E}\left[u_{\varepsilon}(t, x)\right]^{n}$ and passing to the limit formally.

Suppose $u_{\varepsilon}(0, x)=u_{0}(x)$ for some continuous function $u_{0}$ with $\left\|u_{0}\right\|_{\infty} \leq 1$, we write the solution to (1.2) by the Feynman-Kac formula

$$
\begin{equation*}
u_{\varepsilon}(t, x)=\mathbb{E}_{\mathbf{B}}\left[u_{0}\left(x+B_{t}\right) \exp \left(\int_{0}^{t} \xi_{\varepsilon}\left(x+B_{s}\right) d s-C_{\varepsilon} t\right)\right] . \tag{1.4}
\end{equation*}
$$

Here, $B=\left(B_{t}\right)_{t \geq 0}$ is a standard planar Brownian motion starting from the origin and independent of the white noise $\xi$, and $C_{\varepsilon}$ is the constant defined in (1.3). We use $\mathbb{E}_{\mathbf{B}}$ to denote the expectation with respect to $B$. We now proceed to calculating the $n$-th moment of $u_{\varepsilon}(t, x)$. First of all, the covariance function of $\xi_{\varepsilon}$ satisfies

$$
\mathbb{E}\left[\xi_{\varepsilon}(x) \xi_{\varepsilon}(y)\right]=R_{\varepsilon}(x-y):=\varepsilon^{-2} R\left(\frac{x-y}{\varepsilon}\right),
$$

where $R=\rho \star \rho$, and $\rho$ is the mollifier used to regularize the noise $\xi$. Next, one raises the expression (1.4) to the $n$-th power, and take a further expectation with respect to $\xi_{\varepsilon}$. Since $B$ is independent of $\xi_{\varepsilon}$, one can interchange this expectation with the one with respect to the Brownian motions, and get

$$
\begin{equation*}
\mathbb{E}\left[u_{\varepsilon}(t, x)^{n}\right]=\mathbb{E}_{\mathbf{B}}\left[\exp \left(I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t\right) \prod_{k=1}^{n} u_{0}\left(x+B_{t}^{k}\right)\right] \tag{1.5}
\end{equation*}
$$

Here, $B^{k}, k=1, \ldots, n$ are independent Brownian motions, and $\mathbb{E}_{\mathrm{B}}$ denotes the expectation with respect to these $B^{k}$ s. Also, $I_{n}^{\varepsilon}(t)$ is given by

$$
\begin{equation*}
I_{n}^{\varepsilon}(t)=\sum_{k=1}^{n} \int_{0}^{t} \int_{0}^{s} R_{\varepsilon}\left(B_{s}^{k}-B_{u}^{k}\right) d u d s+\sum_{1 \leq i<j \leq n} \int_{0}^{t} \int_{0}^{t} R_{\varepsilon}\left(B_{s}^{i}-B_{u}^{j}\right) d s d u \tag{1.6}
\end{equation*}
$$

where $R_{\varepsilon}(x)=\varepsilon^{-2} R(x / \varepsilon)$ converges to the Dirac function as $\varepsilon \rightarrow 0$. Note that we do not have the factor $\frac{1}{2}$ in front of the first term since the integration is on the simplex rather than the square $[0, t]^{2}$. It is well known (see for example $[3$, Chapter 2]) that each term in the second term above (when $i \neq j$ ) converges to the mutual intersection local time of Brownian motion, formally written as $\int_{[0, t]^{2}} \delta\left(B_{s}^{i}-B_{u}^{j}\right) d s d u$. The first term above (when one has the same Brownian motion in the argument of $R_{\varepsilon}$ ) unfortunately does not converge as $\varepsilon \rightarrow 0$, but it does when one subtracts its mean (see [12, 14, 15]). Thus, we define

$$
\begin{equation*}
\nu_{\varepsilon}(t)=\int_{0}^{t} \int_{0}^{s} \mathbb{E}_{\mathbf{B}}\left[R_{\varepsilon}\left(B_{s}-B_{u}\right)\right] d u d s \tag{1.7}
\end{equation*}
$$

and for every $t \geq 0$, we have

$$
\begin{equation*}
I_{n}^{\varepsilon}(t)-n \nu_{\varepsilon}(t) \rightarrow \mathcal{X}_{n}(t) \tag{1.8}
\end{equation*}
$$

in probability, where $\mathcal{X}_{n}(t)$ is a linear combination of self- and mutual-intersection local times of planar Brownian motions, formally written as

$$
\begin{align*}
\mathcal{X}_{n}(t)= & \sum_{k=1}^{n} \int_{0}^{t} \int_{0}^{s}\left(\delta\left(B_{s}^{k}-B_{u}^{k}\right)-\mathbb{E}_{\mathbf{B}}\left[\delta\left(B_{s}^{k}-B_{u}^{k}\right)\right]\right) d u d s  \tag{1.9}\\
& +\sum_{1 \leq i<j \leq n} \int_{0}^{t} \int_{0}^{t} \delta\left(B_{s}^{i}-B_{u}^{j}\right) d s d u .
\end{align*}
$$

It is well known from [12] that $\mathcal{X}_{n}(t)$ has exponential moments for small enough $t$ (depending on $n$ ). In order for the expression (1.5) to converge, one needs the divergent constant $C_{\varepsilon} t$ coincides with $\nu_{\varepsilon}(t)$. A simple calculations shows that this is indeed the case up to an $O(1)$ correction.

Lemma 1.2. There exists constants $\mu_{1}$ and $\mu_{2}$ such that

$$
\nu_{\varepsilon}(t)-C_{\varepsilon} t \rightarrow t\left(\mu_{1}+\mu_{2} \log t\right)
$$

as $\varepsilon \rightarrow 0$.
By (1.8) and Lemma 1.2, we have

$$
\begin{aligned}
I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t & =I_{n}^{\varepsilon}(t)-n \nu_{\varepsilon}(t)+n\left(\nu_{\varepsilon}(t)-C_{\varepsilon} t\right) \\
& \rightarrow \mathcal{X}_{n}(t)+n t\left(\mu_{1}+\mu_{2} \log t\right)
\end{aligned}
$$

in probability. If the families $\left\{u_{\varepsilon}(t, x)^{n}\right\}$ and $\left\{e^{I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t}\right\}$ are both uniformly integrable, then we can pass both sides of (1.5) to the limit, and obtain

$$
\begin{equation*}
\mathbb{E}\left[u(t, x)^{n}\right]=\mathbb{E}_{\mathbf{B}}\left[\exp \left(\mathcal{X}_{n}(t)+n t\left(\mu_{1}+\mu_{2} \log t\right)\right) \prod_{k=1}^{n} u_{0}\left(x+B_{t}^{k}\right)\right] \tag{1.10}
\end{equation*}
$$

The rest of the note is to show the uniform integrability of $\left\{u_{\varepsilon}(t, x)^{n}\right\}$ and $\left\{e^{I_{n}^{( }(t)-n C_{\varepsilon} t}\right\}$ for small time $t$, so (1.10) does hold.

### 1.2. Discussions.

Remark 1.3. The same argument leads to a similar result in $d=1$, where we choose $C_{\varepsilon}=0$ and do not have the small time constraint. The renormalized self-intersection local time can be written as
$\int_{0}^{t} \int_{0}^{s}\left(\delta\left(B_{s}-B_{u}\right)-\mathbb{E}_{\mathbf{B}}\left[\delta\left(B_{s}-B_{u}\right)\right]\right) d u d s=\frac{1}{2} \int_{\mathbb{R}} L_{t}(x)^{2} d x-\frac{1}{2} \int_{\mathbb{R}} \mathbb{E}_{\mathbf{B}}\left[L_{t}(x)^{2}\right] d x$, with $L_{t}(x)$ denoting the local time of 1D Brownian motion up to $t$.
Remark 1.4. For $n=1$, the moment formula reads

$$
\mathbb{E}[u(t, x)]=\mathbb{E}_{\mathbf{B}}\left[u_{0}\left(x+B_{t}\right) e^{\gamma\left([0, t]^{2}\right)+t\left(\mu_{1}+\mu_{2} \log t\right)}\right],
$$

with $\gamma\left([0, t]_{<}^{2}\right)=\int_{0}^{t} \int_{0}^{s}\left(\delta\left(B_{s}-B_{u}\right)-\mathbb{E}_{\mathbf{B}}\left[\delta\left(B_{s}-B_{u}\right)\right]\right) d u d s$ representing the selfintersection local time of $B$. It was proved in [13] that there exists $t_{0}>0$ such that

$$
\mathbb{E}_{\mathbf{B}}\left[e^{\gamma\left([0, t]^{2}\right)}\right] \left\lvert\, \begin{array}{cc}
<\infty & t<t_{0}, \\
=\infty & t>t_{0} .
\end{array}\right.
$$

Thus, it is natural to expect that the moments of $u$ do not exist for large $t$, although we do not have a rigorous proof of it.

Remark 1.5. In [1], the authors defined the 2D Anderson Hamiltonian $\mathscr{H}=$ $-\Delta+\xi$ on the torus $\mathbb{T}^{2}=\mathbb{R}^{2} / \mathbb{Z}^{2}$ using para-controlled calculus. An interesting application is the exponential tail bounds for the ground state eigenvalue $\Lambda_{1}$. It was proved in [1, Proposition 5.4] that there exists $C_{1}, C_{2}>0$ such that

$$
e^{C_{1} x} \leq \mathbb{P}\left[\Lambda_{1} \leq x\right] \leq e^{C_{2} x}
$$

as $x \rightarrow-\infty$. Using the orthonormal eigenvectors of $\mathscr{H}$, denoted by $\left\{e_{n}\right\}$, we write the solution to PAM as

$$
u(t, x)=\sum_{n=1}^{\infty} e^{-\Lambda_{n} t}\left\langle u_{0}, e_{n}\right\rangle e_{n}(x),
$$

therefore,

$$
\int_{\mathbb{T}^{2}} \mathbb{E}\left[|u(t, x)|^{2}\right] d x \leq \mathbb{E}\left[e^{-2 \Lambda_{1} t}\right] \int_{\mathbb{T}^{2}}\left|u_{0}(t, x)\right|^{2} d x
$$

By the exponential tail bounds on $\Lambda_{1}$, it is clear the r.h.s. of the above display is only finite for small $t$, which is consistent with our result.

Remark 1.6. In the forthcoming article [5], the authors consider the 2D PAM with a small noise

$$
\begin{equation*}
\partial_{t} u=\Delta u+\beta u \cdot \xi, \quad(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{2} . \tag{1.11}
\end{equation*}
$$

They obtain an explicit chaos expansion of certain polymer measure associated with (1.11) for $\beta \ll 1$. In particular, this implies that the second moment of $u$ exists for $t \in[0,1], x \in \mathbb{R}^{2}$ and $\beta$ sufficiently small. The restriction of $\beta \ll 1$ is equivalent with our small time restriction. Indeed, define

$$
u_{\beta}(t, x):=u\left(t / \beta^{2}, x / \beta\right),
$$

one sees that $u_{\beta}$ satisfies (1.1), hence for $u_{\beta}(t, x)$ to be square integrable, we need $t / \beta^{2} \leq 1$, i.e., $t \leq \beta^{2} \ll 1$.

Remark 1.7. A simple calculation shows that the moments of the approximations to 3D PAM explode as $\varepsilon \rightarrow 0$, and indicates that the solution to 3D PAM may not have a moment. To see this, we consider the constant initial condition $u_{0} \equiv 1$, so

$$
\mathbb{E}\left[u_{\varepsilon}(t, x)\right]=e^{-C_{\varepsilon} t} \mathbb{E}_{\mathbf{B}}\left[\exp \left(\int_{0}^{t} \int_{0}^{s} R_{\varepsilon}\left(B_{s}-B_{u}\right) d u d s\right)\right]
$$

where $R_{\varepsilon}(x)=\varepsilon^{-3} R(x / \varepsilon)$.
Since $R(x)$ is continuous and $R(0)>0$, without loss of generality we assume there exists $\delta>0$ such that $R(x)>\delta>0$ for $|x| \leq 2$. Thus, by considering the event that $\left|B_{s}\right|<\varepsilon$ for all $s \in[0, t]$, we have

$$
\mathbb{E}_{\mathbf{B}}\left[\exp \left(\int_{0}^{t} \int_{0}^{s} R_{\varepsilon}\left(B_{s}-B_{u}\right) d u d s\right)\right] \geq \exp \left(\frac{\delta t^{2}}{2 \varepsilon^{3}}\right) \mathbb{P}\left[\sup _{s \in[0, t]}\left|B_{s}\right|<\varepsilon\right] .
$$

The probability $\mathbb{P}\left[\sup _{s \in[0, t]}\left|B_{s}\right|<\varepsilon\right]$ is bounded from below by $e^{-c^{\prime} t \varepsilon^{-2}}$ for some $c^{\prime}>0$ depending on the dimension. When $d=3$, the renormalization constant
$C_{\varepsilon}=c_{1} \varepsilon^{-1}+c_{2}|\log \varepsilon|+O(1)$. It implies that for any $t>0, x \in \mathbb{R}^{3}$, we have $\lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[u_{\varepsilon}(t, x)\right]=\infty$. The same discussion applies to $d=2$, where

$$
\mathbb{E}\left[u_{\varepsilon}(t, x)\right] \geq \exp \left(\frac{\delta t^{2}}{2 \varepsilon^{2}}-\frac{c^{\prime} t}{\varepsilon^{2}}-C_{\varepsilon} t\right)
$$

If $t>2 c^{\prime} / \delta$, we also have $\lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[u_{\varepsilon}(t, x)\right]=\infty$. Since we do not have a proof of $\mathbb{E}[u(t, x)]=\lim _{\varepsilon \rightarrow 0} \mathbb{E}\left[u_{\varepsilon}(t, x)\right]$ in $d=3$ or $d=2$ for large $t$, we only conjecture that $\mathbb{E}[u(t, x)]=\infty$ in those cases.
Remark 1.8. When $d=2$, the small time constraint for the existence of moments in our context also appears in [11, Theorem 4.1], where the usual product $u \cdot \xi$ is replaced by the Wick product $u \diamond \xi$.

Remark 1.9. In [6], a similar result is derived for the random Schrödinger equation $i \partial_{t} \phi+\frac{1}{2} \Delta \phi-\phi \cdot \xi=0$.

## 2. Proof of Lemma 1.2 and Theorem 1.1

We denote $[0, t]_{<}^{n}=\left\{0 \leq s_{1}<\ldots<s_{n} \leq t\right\}$, and write $a \lesssim b$ if $a \leq C b$ with some constant $C$ independent of $\varepsilon$.
Proof of Lemma 1.2. By scaling property of Brownian motion, we have

$$
R_{\varepsilon}\left(B_{s}-B_{u}\right)=\varepsilon^{-2} R\left(\frac{B_{s}-B_{u}}{\varepsilon}\right) \stackrel{\text { law }}{=} \varepsilon^{-2} R\left(B_{s / \varepsilon^{2}}-B_{u / \varepsilon^{2}}\right) .
$$

A change of variable $\left(u / \varepsilon^{2}, s / \varepsilon^{2}\right) \mapsto(u, s)$ then yields

$$
\nu_{\varepsilon}(t)=\varepsilon^{2} \int_{0}^{t / \varepsilon^{2}} \int_{0}^{s} \mathbb{E}_{\mathbf{B}}\left[R\left(B_{s}-B_{u}\right)\right] d u d s
$$

Now, $B_{s}-B_{u}$ has the normal density $x \mapsto(2 \pi(s-u))^{-1} e^{-\frac{|x|^{2}}{2(s-u)}}$. We then do another change of variable $s-u \mapsto v$, integrate $s$ out, and rescale $v \rightarrow v \varepsilon^{2}$. This leads us to

$$
\begin{aligned}
\nu_{\varepsilon}(t) & =\frac{t}{2 \pi} \int_{\mathbb{R}^{2}} R(x)\left(\int_{0}^{t} v^{-1} e^{-\frac{\varepsilon^{2}|x|^{2}}{2 v}} d v\right) d x-\frac{1}{2 \pi} \int_{0}^{t}\left(\int_{\mathbb{R}^{2}} R(x) e^{-\frac{\varepsilon^{2}|x|^{2}}{2 v}} d x\right) d v \\
& :=(\mathrm{i})-\text { (ii). }
\end{aligned}
$$

Since $R$ integrates to 1 , it is clear that

$$
\text { (ii) } \rightarrow \frac{t}{2 \pi}
$$

as $\varepsilon \rightarrow 0$. As for (i), a substitution of variable $\frac{\varepsilon^{2}|x|^{2}}{2 v} \mapsto \lambda$ and then an integration by parts yields

$$
\begin{aligned}
(\mathrm{i}) & =\frac{t}{2 \pi} \int_{\mathbb{R}^{2}} R(x)\left(\int_{\frac{\varepsilon^{2} \mid x x^{2}}{2 t}}^{\infty} \lambda^{-1} e^{-\lambda} d \lambda\right) d x \\
& =\frac{t}{2 \pi} \int_{\mathbb{R}^{2}} R(x)\left(\int_{\frac{\varepsilon^{2} \mid x x^{2}}{2 t}}^{\infty} e^{-\lambda} \log \lambda d \lambda-e^{-\frac{\varepsilon^{2}|x|^{2}}{2 t}} \log \left(\frac{\varepsilon^{2}|x|^{2}}{2 t}\right)\right) d x .
\end{aligned}
$$

It is clear from the above expression that as $\varepsilon \rightarrow 0$, the only divergent part of (i) is from the term $\log \left(\varepsilon^{2}\right)$, and a direct calculation shows

$$
\nu_{\varepsilon}(t)-\frac{t}{\pi} \cdot|\log \varepsilon| \rightarrow \mu_{1} t+\mu_{2} t \log t
$$

for some constant $\mu_{1}, \mu_{2}$.
Proof of Theorem 1.1. Fix $(t, x)$ and $n$, and recall that

$$
\begin{equation*}
\mathbb{E}\left[u_{\varepsilon}(t, x)^{n}\right]=\mathbb{E}_{\mathbf{B}}\left[\exp \left(I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t\right) \prod_{k=1}^{n} u_{0}\left(x+B_{t}^{k}\right)\right], \tag{2.1}
\end{equation*}
$$

where $\mathbb{E}_{\mathbf{B}}$ is the expectation with respect to independent planar Brownian motions $B^{k}$ 's, and $I_{n}^{\varepsilon}$ is given by the expression (1.6). Note that $u_{\varepsilon}(t, x)^{n} \rightarrow u(t, x)^{n}$ in probability, and that by (1.8) and Lemma 1.2, we have

$$
I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t \rightarrow \mathcal{X}_{n}(t)+n t\left(\mu_{1}+\mu_{2} \log t\right)
$$

in probability. Thus, in view of (2.1), it suffices to show the uniform integrability of $u_{\varepsilon}(t, x)^{n}$ and $\exp \left(I_{n}^{\varepsilon}(t)-n C_{\varepsilon} t\right) \prod_{k=1}^{n} u_{0}\left(x+B_{t}^{k}\right)$. This allows us to pass both sides of (2.1) to the limit and conclude Theorem 1.1.

To prove the uniform integrability, we bound the second moment of these two objects:

$$
\mathbb{E}\left[\left|u_{\varepsilon}(t, x)\right|^{2 n}\right] \lesssim \mathbb{E}_{\mathbf{B}}\left[e^{I_{2 n}^{\varepsilon}(t)-2 n C_{\varepsilon} t}\right] \lesssim \mathbb{E}_{\mathbf{B}}\left[e^{I_{2 n}^{\varepsilon}(t)-2 n \nu_{\varepsilon}(t)}\right]
$$

and

$$
\mathbb{E}_{\mathbf{B}}\left[\left|e^{I_{n}^{\varepsilon}(t)} e^{-n C_{\varepsilon} t} \prod_{k=1}^{n} u_{0}\left(x+B_{t}^{k}\right)\right|^{2}\right] \lesssim \mathbb{E}_{\mathbf{B}}\left[e^{2 I_{n}^{\varepsilon}(t)-2 n C_{\varepsilon} t}\right] \lesssim \mathbb{E}_{\mathbf{B}}\left[e^{2\left(I_{n}^{\varepsilon}(t)-n \nu_{\varepsilon}(t)\right)}\right]
$$

where we have used $\left\|u_{0}\right\|_{\infty} \leq 1$. Thus, it suffices to show that for every $n$ and $\theta$, there exists $t_{0}$ small enough such that $\mathbb{E}_{\mathbf{B}}\left[e^{\theta\left(I_{n}^{\varepsilon}(t)-n \nu_{\varepsilon}(t)\right)}\right]$ is uniformly bounded in $\varepsilon$ for all $t<t_{0}$. To see this, using Hölder's inequality, we get
$\mathbb{E}_{\mathbf{B}}\left[e^{\theta\left(I_{n}^{\varepsilon}(t)-n \nu_{\varepsilon}(t)\right)}\right] \leq \prod_{k=1}^{n}\left[\mathbb{E}_{\mathbf{B}} e^{\theta N\left[\beta_{\varepsilon}^{k}\left([0, t]^{2}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}^{k}\left([0, t]^{2}\right)\right]}\right]^{\frac{1}{N}} \prod_{1 \leq i<j \leq n}\left(\mathbb{E}_{\mathbf{B}} e^{\theta N \alpha_{\varepsilon}^{i, j}\left([0, t]^{2}\right)}\right)^{\frac{1}{N}}$,
where $N=\frac{n(n+1)}{2}$, and we have used the notations

$$
\beta_{\varepsilon}^{k}\left([0, t]_{<}^{2}\right)=\int_{0}^{t} \int_{0}^{s} R_{\varepsilon}\left(B_{s}^{k}-B_{u}^{k}\right) d u d s, \quad \alpha_{\varepsilon}^{i, j}\left([0, t]^{2}\right)=\int_{0}^{t} \int_{0}^{t} R_{\varepsilon}\left(B_{s}^{i}-B_{u}^{j}\right) d s d u .
$$

By change of variables and the scaling property of the Brownian motion, we have

$$
\beta_{\varepsilon}^{k}\left([0, t]_{<}^{2}\right) \stackrel{\text { law }}{=} t \beta_{\varepsilon / \sqrt{t}}^{k}\left([0,1]_{<}^{2}\right), \quad \alpha_{\varepsilon}^{i, j}\left([0, t]^{2}\right) \stackrel{\text { law }}{=} t \alpha_{\varepsilon / \sqrt{t}}^{i, j}\left([0,1]^{2}\right) .
$$

Then, Lemma A. 1 implies that there exists $\lambda, C>0$ such that

$$
t<\frac{\lambda}{\theta N} \Rightarrow \sup _{\varepsilon \in(0,1)} \mathbb{E}_{\mathbf{B}}\left[e^{\theta\left(I_{n}^{E}(t)-n \nu_{\varepsilon}(t)\right)}\right] \leq C
$$

This completes the proof.

## Appendix A. Exponential moments of intersection local time of planar Brownian motions

Recall that $R_{\varepsilon}(x)=\varepsilon^{-2} R\left(\frac{x}{\varepsilon}\right)$, we define

$$
\alpha_{\varepsilon}(A)=\int_{A} R_{\varepsilon}\left(B_{s}^{1}-B_{u}^{2}\right) d s d u, \quad \beta_{\varepsilon}(A)=\int_{A} R_{\varepsilon}\left(B_{s}-B_{u}\right) d s d u
$$

for any set $A \subset \mathbb{R}_{+}^{2}$, and

$$
X_{\varepsilon}=\beta_{\varepsilon}\left([0,1]_{<}^{2}\right)-\mathbb{E}_{\mathbf{B}}\left[\beta_{\varepsilon}\left([0,1]_{<}^{2}\right)\right], \quad Y_{\varepsilon}=\alpha_{\varepsilon}\left([0,1]^{2}\right)
$$

Lemma A.1. There exists universal constants $\lambda, C>0$ such that

$$
\sup _{\varepsilon \in(0,1)}\left(\mathbb{E}_{\mathbf{B}}\left[e^{\lambda X_{\varepsilon}}\right]+\mathbb{E}_{\mathbf{B}}\left[e^{\lambda Y_{\varepsilon}}\right]\right) \leq C
$$

The above result is standard. The case $\varepsilon=0$, i.e., the exponential integrability of intersection local time, was addressed in the classical work [13]. We could not find a direct reference for $\varepsilon>0$, though the proof follows essentially in the same line as the case of $\varepsilon=0$. For the convenience of the reader, we present the details here.
Proof. We consider $Y_{\varepsilon}$ first. Since $R=\rho \star \rho$, we can write

$$
Y_{\varepsilon}=\int_{[0,1]^{2}} \int_{\mathbb{R}^{2}} \rho_{\varepsilon}\left(B_{s}^{1}-x\right) \rho_{\varepsilon}\left(B_{u}^{2}-x\right) d x d s d u
$$

with $\rho_{\varepsilon}(x)=\varepsilon^{-2} \rho(x / \varepsilon)$. For any $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{B}}\left[Y_{\varepsilon}^{n}\right] & =\int_{\mathbb{R}^{2 n}}\left(\int_{[0,1]^{2 n}} \mathbb{E}_{\mathbf{B}}\left[\prod_{k=1}^{n} \rho_{\varepsilon}\left(B_{s_{k}}^{1}-x_{k}\right) \rho_{\varepsilon}\left(B_{u_{k}}^{2}-x_{k}\right)\right] d \mathbf{s} d \mathbf{u}\right) d \mathbf{x} \\
& =\int_{\mathbb{R}^{2 n}}\left(\int_{[0,1]^{n}} \mathbb{E}_{\mathbf{B}}\left[\prod_{k=1}^{n} \rho_{\varepsilon}\left(B_{s_{k}}-x_{k}\right)\right] d \mathbf{s}\right)^{2} d \mathbf{x} .
\end{aligned}
$$

By [3, (2.2.11)], we have
$\mathbb{E}_{\mathbf{B}}\left[Y_{\varepsilon}^{n}\right]=\int_{\mathbb{R}^{2 n}}\left(\int_{\mathbb{R}^{2 n}} \prod_{k=1}^{n} \rho_{\varepsilon}\left(z_{k}-x_{k}\right) \sum_{\sigma} \int_{[0,1]^{n}} \prod_{k=1}^{n} p_{s_{k}-s_{k-1}}\left(z_{\sigma(k)}-z_{\sigma(k-1)}\right) d \mathbf{s} d \mathbf{z}\right)^{2} d \mathbf{x}$,
where $p_{t}(x)$ is the density of $N(0, t),[0, t]_{<}^{n}=\left\{0 \leq s_{1}<\ldots<s_{n} \leq t\right\}$, and $\sum_{\sigma}$ denotes the summation over all permutations over $\{1, \ldots, n\}$. If we denote
$h\left(z_{1}, \ldots, z_{n}\right)=\sum_{\sigma} \int_{[0,1]^{n} \in} \prod_{k=1}^{n} p_{s_{k}-s_{k-1}}\left(z_{\sigma(k)}-z_{\sigma(k-1)}\right) d \mathbf{s}, \quad Q_{\varepsilon}\left(z_{1}, \ldots, z_{n}\right)=\prod_{k=1}^{n} \rho_{\varepsilon}\left(z_{k}\right)$, then $\mathbb{E}_{\mathbf{B}}\left[Y_{\varepsilon}^{n}\right]$ equals to
(A.1)

$$
\begin{aligned}
\int_{\mathbb{R}^{2 n}}\left|Q_{\varepsilon} \star h\left(x_{1}, \ldots, x_{n}\right)\right|^{2} d \mathbf{x} & \leq\left(\int_{\mathbb{R}^{2 n}} Q_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right) d \mathbf{x}\right)^{2} \int_{\mathbb{R}^{2 n}}\left|h\left(x_{1}, \ldots, x_{n}\right)\right|^{2} d \mathbf{x} \\
& =\int_{\mathbb{R}^{2 n}}\left|h\left(x_{1}, \ldots, x_{n}\right)\right|^{2} d \mathbf{x}=\mathbb{E}_{\mathbf{B}}\left[\alpha\left([0,1]^{2}\right)^{n}\right]
\end{aligned}
$$

where $\alpha\left([0,1]^{2}\right)$ is the mutual-intersection local time formally written as

$$
\alpha\left([0,1]^{2}\right)=\int_{0}^{1} \int_{0}^{1} \delta\left(B_{s}^{1}-B_{u}^{2}\right) d s d u
$$

and we used the Le Gall's moment formula in the second line of (A.1). By [13], we have

$$
\mathbb{E}_{\mathbf{B}}\left[\exp \left(\mu \alpha\left([0,1]^{2}\right)\right)\right]<C
$$

for some $\mu>0$, hence we only need to choose $\lambda=\mu$ to get

$$
\mathbb{E}_{\mathbf{B}}\left[e^{\lambda Y_{\varepsilon}}\right]=\sum_{n=0}^{\infty} \frac{\lambda^{n} \mathbb{E}_{\mathbf{B}}\left[Y_{\varepsilon}^{n}\right]}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^{n}}{n!} \mathbb{E}_{\mathbf{B}}\left[\left|\alpha\left([0,1]^{2}\right)\right|^{n}\right]=\mathbb{E}_{\mathbf{B}}\left[e^{\mu \alpha\left([0,1]^{2}\right)}\right]<\infty
$$

Next, we consider $X_{\varepsilon}$. We define the triangle approximation of $\{(u, s): 0 \leq$ $u<s \leq 1\}$ :

$$
A_{l}^{k}=\left[\frac{2 l}{2^{k+1}}, \frac{2 l+1}{2^{k+1}}\right) \times\left[\frac{2 l+1}{2^{k+1}}, \frac{2 l+2}{2^{k+1}}\right), \quad l=0,1, \ldots, 2^{k-1}, k=0,1, \ldots
$$

We will use the following three properties:
(i) Fix any $k,\left\{\beta_{\varepsilon}\left(A_{l}^{k}\right)\right\}_{l=0, \ldots, 2^{k}-1}$ are i.i.d. random variables.
(ii) $\beta_{\varepsilon}\left(A_{l}^{k}\right) \stackrel{\text { law }}{=} 2^{-(k+1)} \beta_{\varepsilon 2^{(k+1) / 2}}([0,1] \times[1,2]) \stackrel{\text { law }}{=} 2^{-(k+1)} \alpha_{\varepsilon 2^{(k+1) / 2}}\left([0,1]^{2}\right)$
(iii) $\sup _{\varepsilon>0} \mathbb{E}_{\mathbf{B}}\left[e^{\lambda \alpha_{\varepsilon}\left([0,1]^{2}\right)}\right] \leq C$ for some $\lambda, C>0$.

By (iii) and a Taylor expansion, there exists $C>0$ such that for sufficiently small $\lambda$

$$
\begin{equation*}
\sup _{\varepsilon>0} \mathbb{E}_{\mathbf{B}}\left[e^{\lambda\left(\alpha_{\varepsilon}\left([0,1]^{2}\right)-\mathbb{E}_{\mathbf{B}}\left[\alpha_{\varepsilon}\left([0,1]^{2}\right)\right]\right)}\right] \leq e^{C \lambda^{2}} \tag{A.2}
\end{equation*}
$$

We fix the constants $\lambda, C$ from now on, and write

$$
X_{\varepsilon}=\sum_{k=0}^{\infty} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}}\left[\beta_{\varepsilon}\left(A_{l}^{k}\right)\right]\right)
$$

Fix $a \in(0,1)$ and define a sequence of constants

$$
b_{1}=2 \lambda, \quad b_{N}=2 \lambda \prod_{j=2}^{N}\left(1-2^{-a(j-1)}\right), N=2,3, \ldots
$$

we have

$$
\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[b_{N} \sum_{k=0}^{N} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right] \\
\leq & \left(\mathbb{E}_{\mathbf{B}} \exp \left[b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right]\right)^{1-2^{-a(N-1)}} \\
& \times\left(\mathbb{E}_{\mathbf{B}} \exp \left[2^{a(N-1)} b_{N} \sum_{l=0}^{2^{N}-1}\left(\beta_{\varepsilon}\left(A_{l}^{N}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{N}\right)\right)\right]\right)^{2^{-a(N-1)}} \\
\leq & \times \mathbb{E}_{\mathbf{B}} \exp \left[b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right] \\
& \times\left(\mathbb{E}_{\mathbf{B}} \exp \left[2^{a(N-1)} b_{N}\left(\beta_{\varepsilon}\left(A_{0}^{N}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{0}^{N}\right)\right)\right]\right)^{2^{N-a(N-1)}}
\end{aligned}
$$

Since $\beta_{\varepsilon}\left(A_{0}^{N}\right) \stackrel{\text { law }}{=} 2^{-(N+1)} \alpha_{\varepsilon 2^{(N+1) / 2}}\left([0,1]^{2}\right)$, we have

$$
\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[2^{a(N-1)} b_{N}\left(\beta_{\varepsilon}\left(A_{0}^{N}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{0}^{N}\right)\right)\right] \\
= & \mathbb{E}_{\mathbf{B}} \exp \left[2^{a(N-1)} b_{N} 2^{-(N+1)}\left(\alpha_{\varepsilon 2^{(N+1) / 2}}\left([0,1]^{2}\right)-\mathbb{E}_{\mathbf{B}} \alpha_{\varepsilon 2^{(N+1) / 2}}\left([0,1]^{2}\right)\right)\right] .
\end{aligned}
$$

Using the fact that $2^{a(N-1)} b_{N} 2^{-(N+1)}<\lambda$ and (A.2), we derive for all $\varepsilon>0$ that

$$
\mathbb{E}_{\mathbf{B}} \exp \left[2^{a(N-1)} b_{N}\left(\beta_{\varepsilon}\left(A_{0}^{N}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{0}^{N}\right)\right)\right] \leq e^{C b_{N}^{2} 2^{-2 N+2 a(N-1)}}
$$

so there exists $C^{\prime}>0$ such that

$$
\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[b_{N} \sum_{k=0}^{N} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right] \\
& \leq \mathbb{E}_{\mathbf{B}} \exp \left[b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right] e^{C^{\prime} 2^{(a-1) N}} .
\end{aligned}
$$

Iterating the above inequality, we get

$$
\mathbb{E}_{\mathbf{B}} \exp \left[b_{N} \sum_{k=0}^{N} \sum_{l=0}^{2^{k}-1}\left(\beta_{\varepsilon}\left(A_{l}^{k}\right)-\mathbb{E}_{\mathbf{B}} \beta_{\varepsilon}\left(A_{l}^{k}\right)\right)\right] \leq \exp \left(C^{\prime}\left(1-2^{a-1}\right)^{-1}\right)
$$

Since $b_{N} \rightarrow b_{\infty}$ for some $b_{\infty}>0$, we have

$$
\mathbb{E}_{\mathbf{B}}\left[\exp \left(b_{\infty} X_{\varepsilon}\right)\right] \leq \exp \left(C^{\prime}\left(1-2^{a-1}\right)^{-1}\right)
$$

which completes the proof.
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