## MOMENTS OF 2D PARABOLIC ANDERSON MODEL

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ABSTRACT. In this note, we use the Feynman-Kac formula to derive a moment representation for the 2D parabolic Anderson model in small time, which is related to the intersection local time of planar Brownian motions.

KEYWORDS: Feynman-Kac formula, renormalization, intersection local time.

### 1. INTRODUCTION

The aim of this note is to study the existence of moments of the solution to the parabolic Anderson model (PAM) in two spatial dimensions, formally given by

(1.1) 
$$\partial_t u = \frac{1}{2} \Delta u + u \cdot \xi, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

where  $\xi$  is the two dimensional spatial white noise, that is, a generalized Gaussian process with covariance  $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$ .

The equation is well-posed in dimension 1, but the product between u and  $\xi$  becomes ill-defined as soon as  $d \geq 2$ . For d = 2, the solution u is defined in [7, 8, 10] as the limit of a sequence of the regularized and renormalized equations. More precisely, fix a symmetric mollifier  $\rho : \mathbb{R}^2 \to \mathbb{R}^+$  with  $\rho(x) = \rho(-x)$  and  $\int \rho = 1$ . Let

$$\rho_{\varepsilon}(x) = \varepsilon^{-2}\rho(x/\varepsilon), \qquad \xi_{\varepsilon} = \xi \star \rho_{\varepsilon},$$

and consider the equation

(1.2) 
$$\partial_t u_{\varepsilon} = \frac{1}{2} \Delta u_{\varepsilon} + (\xi_{\varepsilon} - C_{\varepsilon}) u_{\varepsilon}$$

for some large constant  $C_{\varepsilon}$ . Then, for

(1.3) 
$$C_{\varepsilon} = \frac{1}{\pi} \log \varepsilon^{-1},$$

the sequence of solutions  $\{u_{\varepsilon}\}$  converges in some weighted Hölder space in probability to a limit u that is independent of the mollification, see e.g. [10, Theorem 4.1], and we call this limit u the solution to 2D PAM. In d = 3, the mollifier  $\rho_{\varepsilon}(x) = \varepsilon^{-3}\rho(x/\varepsilon)$ , and the renormalization constant takes the form  $C_{\varepsilon} = c_1\varepsilon^{-1} + c_2\log\varepsilon^{-1} + O(1)$  [9].

So far, most of the results mentioned above focused on the existence of the solution and the convergence of the regularized PDE after renormalization. The statistical properties of u remains a challenge; see [1, 2, 4] for some relevant discussions. The goal of this note is to show that the *n*-th moment of the solution

u to 2D PAM exists for small time, and we present a Feynman-Kac formula for  $\mathbb{E}[u^n]$ . The following is our main result.

**Theorem 1.1.** There exists a universal constant  $\delta > 0$  such that for every  $n \in \mathbb{N}$ , the n-th moment of u exists for  $t \in (0, \frac{\delta}{n^2})$  with  $\mathbb{E}[u(t, x)^n]$  given by (1.10).

1.1. Heuristic argument. We first give a heuristic derivation of  $\mathbb{E}[u(t,x)]^n$  by writing down a representation for  $\mathbb{E}[u_{\varepsilon}(t,x)]^n$  and passing to the limit formally.

Suppose  $u_{\varepsilon}(0, x) = u_0(x)$  for some continuous function  $u_0$  with  $||u_0||_{\infty} \leq 1$ , we write the solution to (1.2) by the Feynman-Kac formula

(1.4) 
$$u_{\varepsilon}(t,x) = \mathbb{E}_{\mathbf{B}} \left[ u_0(x+B_t) \exp\left(\int_0^t \xi_{\varepsilon}(x+B_s)ds - C_{\varepsilon}t\right) \right]$$

Here,  $B = (B_t)_{t\geq 0}$  is a standard planar Brownian motion starting from the origin and independent of the white noise  $\xi$ , and  $C_{\varepsilon}$  is the constant defined in (1.3). We use  $\mathbb{E}_{\mathbf{B}}$  to denote the expectation with respect to B. We now proceed to calculating the *n*-th moment of  $u_{\varepsilon}(t, x)$ . First of all, the covariance function of  $\xi_{\varepsilon}$  satisfies

$$\mathbb{E}[\xi_{\varepsilon}(x)\xi_{\varepsilon}(y)] = R_{\varepsilon}(x-y) := \varepsilon^{-2}R\left(\frac{x-y}{\varepsilon}\right).$$

where  $R = \rho \star \rho$ , and  $\rho$  is the mollifier used to regularize the noise  $\xi$ . Next, one raises the expression (1.4) to the *n*-th power, and take a further expectation with respect to  $\xi_{\varepsilon}$ . Since *B* is independent of  $\xi_{\varepsilon}$ , one can interchange this expectation with the one with respect to the Brownian motions, and get

(1.5) 
$$\mathbb{E}[u_{\varepsilon}(t,x)^{n}] = \mathbb{E}_{\mathbf{B}}\left[\exp\left(I_{n}^{\varepsilon}(t) - nC_{\varepsilon}t\right)\prod_{k=1}^{n}u_{0}(x+B_{t}^{k})\right].$$

Here,  $B^k, k = 1, ..., n$  are independent Brownian motions, and  $\mathbb{E}_{\mathbf{B}}$  denotes the expectation with respect to these  $B^k$ 's. Also,  $I_n^{\varepsilon}(t)$  is given by

(1.6) 
$$I_n^{\varepsilon}(t) = \sum_{k=1}^n \int_0^t \int_0^s R_{\varepsilon} (B_s^k - B_u^k) du ds + \sum_{1 \le i < j \le n} \int_0^t \int_0^t R_{\varepsilon} (B_s^i - B_u^j) ds du,$$

where  $R_{\varepsilon}(x) = \varepsilon^{-2}R(x/\varepsilon)$  converges to the Dirac function as  $\varepsilon \to 0$ . Note that we do not have the factor  $\frac{1}{2}$  in front of the first term since the integration is on the simplex rather than the square  $[0, t]^2$ . It is well known (see for example [3, Chapter 2]) that each term in the second term above (when  $i \neq j$ ) converges to the mutual intersection local time of Brownian motion, formally written as  $\int_{[0,t]^2} \delta(B_s^i - B_u^j) ds du$ . The first term above (when one has the same Brownian motion in the argument of  $R_{\varepsilon}$ ) unfortunately does not converge as  $\varepsilon \to 0$ , but it does when one subtracts its mean (see [12, 14, 15]). Thus, we define

(1.7) 
$$\nu_{\varepsilon}(t) = \int_0^t \int_0^s \mathbb{E}_{\mathbf{B}}[R_{\varepsilon}(B_s - B_u)] du ds,$$

and for every  $t \ge 0$ , we have

(1.8) 
$$I_n^{\varepsilon}(t) - n\nu_{\varepsilon}(t) \to \mathcal{X}_n(t)$$

in probability, where  $\mathcal{X}_n(t)$  is a linear combination of self- and mutual-intersection local times of planar Brownian motions, formally written as

(1.9) 
$$\begin{aligned} \mathcal{X}_{n}(t) &= \sum_{k=1}^{n} \int_{0}^{t} \int_{0}^{s} \left( \delta(B_{s}^{k} - B_{u}^{k}) - \mathbb{E}_{\mathbf{B}}[\delta(B_{s}^{k} - B_{u}^{k})] \right) du ds \\ &+ \sum_{1 \leq i < j \leq n} \int_{0}^{t} \int_{0}^{t} \delta(B_{s}^{i} - B_{u}^{j}) ds du. \end{aligned}$$

It is well known from [12] that  $\mathcal{X}_n(t)$  has exponential moments for small enough t (depending on n). In order for the expression (1.5) to converge, one needs the divergent constant  $C_{\varepsilon}t$  coincides with  $\nu_{\varepsilon}(t)$ . A simple calculations shows that this is indeed the case up to an O(1) correction.

**Lemma 1.2.** There exists constants  $\mu_1$  and  $\mu_2$  such that

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$$u_{\varepsilon}(t) - C_{\varepsilon}t \to t(\mu_1 + \mu_2 \log t)$$

 $as \; \varepsilon \to 0.$ 

By (1.8) and Lemma 1.2, we have

$$I_n^{\varepsilon}(t) - nC_{\varepsilon}t = I_n^{\varepsilon}(t) - n\nu_{\varepsilon}(t) + n(\nu_{\varepsilon}(t) - C_{\varepsilon}t)$$
  
$$\rightarrow \mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t)$$

in probability. If the families  $\{u_{\varepsilon}(t,x)^n\}$  and  $\{e^{I_n^{\varepsilon}(t)-nC_{\varepsilon}t}\}$  are both uniformly integrable, then we can pass both sides of (1.5) to the limit, and obtain

(1.10) 
$$\mathbb{E}[u(t,x)^{n}] = \mathbb{E}_{\mathbf{B}}\bigg[\exp\big(\mathcal{X}_{n}(t) + nt(\mu_{1} + \mu_{2}\log t)\big)\prod_{k=1}^{n}u_{0}(x + B_{t}^{k})\bigg].$$

The rest of the note is to show the uniform integrability of  $\{u_{\varepsilon}(t,x)^n\}$  and  $\{e^{I_n^{\varepsilon}(t)-nC_{\varepsilon}t}\}$  for small time t, so (1.10) does hold.

#### 1.2. Discussions.

Remark 1.3. The same argument leads to a similar result in d = 1, where we choose  $C_{\varepsilon} = 0$  and do not have the small time constraint. The renormalized self-intersection local time can be written as

$$\int_{0}^{t} \int_{0}^{s} (\delta(B_{s} - B_{u}) - \mathbb{E}_{\mathbf{B}}[\delta(B_{s} - B_{u})]) du ds = \frac{1}{2} \int_{\mathbb{R}} L_{t}(x)^{2} dx - \frac{1}{2} \int_{\mathbb{R}} \mathbb{E}_{\mathbf{B}}[L_{t}(x)^{2}] dx$$

with  $L_t(x)$  denoting the local time of 1D Brownian motion up to t.

*Remark* 1.4. For n = 1, the moment formula reads

$$\mathbb{E}[u(t,x)] = \mathbb{E}_{\mathbf{B}}[u_0(x+B_t)e^{\gamma([0,t]_{<}^2)+t(\mu_1+\mu_2\log t)}].$$

with  $\gamma([0,t]_{\leq}^2) = \int_0^t \int_0^s (\delta(B_s - B_u) - \mathbb{E}_{\mathbf{B}}[\delta(B_s - B_u)]) duds$  representing the selfintersection local time of B. It was proved in [13] that there exists  $t_0 > 0$  such that

$$\mathbb{E}_{\mathbf{B}}\left[e^{\gamma([0,t]_{\leq}^{2})}\right] \left| \begin{array}{cc} < \infty & t < t_{0}, \\ = \infty & t > t_{0}. \end{array} \right.$$

Thus, it is natural to expect that the moments of u do not exist for large t, although we do not have a rigorous proof of it.

Remark 1.5. In [1], the authors defined the 2D Anderson Hamiltonian  $\mathscr{H} = -\Delta + \xi$  on the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  using para-controlled calculus. An interesting application is the exponential tail bounds for the ground state eigenvalue  $\Lambda_1$ . It was proved in [1, Proposition 5.4] that there exists  $C_1, C_2 > 0$  such that

$$e^{C_1 x} \le \mathbb{P}[\Lambda_1 \le x] \le e^{C_2 x}$$

as  $x \to -\infty$ . Using the orthonormal eigenvectors of  $\mathscr{H}$ , denoted by  $\{e_n\}$ , we write the solution to PAM as

$$u(t,x) = \sum_{n=1}^{\infty} e^{-\Lambda_n t} \langle u_0, e_n \rangle e_n(x),$$

therefore,

$$\int_{\mathbb{T}^2} \mathbb{E}[|u(t,x)|^2] dx \le \mathbb{E}[e^{-2\Lambda_1 t}] \int_{\mathbb{T}^2} |u_0(t,x)|^2 dx.$$

By the exponential tail bounds on  $\Lambda_1$ , it is clear the r.h.s. of the above display is only finite for small t, which is consistent with our result.

*Remark* 1.6. In the forthcoming article [5], the authors consider the 2D PAM with a small noise

(1.11) 
$$\partial_t u = \Delta u + \beta u \cdot \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

They obtain an explicit chaos expansion of certain polymer measure associated with (1.11) for  $\beta \ll 1$ . In particular, this implies that the second moment of uexists for  $t \in [0, 1], x \in \mathbb{R}^2$  and  $\beta$  sufficiently small. The restriction of  $\beta \ll 1$  is equivalent with our small time restriction. Indeed, define

$$u_{\beta}(t,x) := u(t/\beta^2, x/\beta),$$

one sees that  $u_{\beta}$  satisfies (1.1), hence for  $u_{\beta}(t, x)$  to be square integrable, we need  $t/\beta^2 \leq 1$ , i.e.,  $t \leq \beta^2 \ll 1$ .

Remark 1.7. A simple calculation shows that the moments of the approximations to 3D PAM explode as  $\varepsilon \to 0$ , and indicates that the solution to 3D PAM may not have a moment. To see this, we consider the constant initial condition  $u_0 \equiv 1$ , so

$$\mathbb{E}[u_{\varepsilon}(t,x)] = e^{-C_{\varepsilon}t} \mathbb{E}_{\mathbf{B}}\Big[\exp\Big(\int_{0}^{t}\int_{0}^{s} R_{\varepsilon}(B_{s}-B_{u})duds\Big)\Big],$$

where  $R_{\varepsilon}(x) = \varepsilon^{-3} R(x/\varepsilon)$ .

Since R(x) is continuous and R(0) > 0, without loss of generality we assume there exists  $\delta > 0$  such that  $R(x) > \delta > 0$  for  $|x| \le 2$ . Thus, by considering the event that  $|B_s| < \varepsilon$  for all  $s \in [0, t]$ , we have

$$\mathbb{E}_{\mathbf{B}}\Big[\exp\left(\int_{0}^{t}\int_{0}^{s}R_{\varepsilon}(B_{s}-B_{u})duds\right)\Big]\geq\exp\left(\frac{\delta t^{2}}{2\varepsilon^{3}}\right)\mathbb{P}\Big[\sup_{s\in[0,t]}|B_{s}|<\varepsilon\Big].$$

The probability  $\mathbb{P}[\sup_{s \in [0,t]} |B_s| < \varepsilon]$  is bounded from below by  $e^{-c't\varepsilon^{-2}}$  for some c' > 0 depending on the dimension. When d = 3, the renormalization constant

 $C_{\varepsilon} = c_1 \varepsilon^{-1} + c_2 |\log \varepsilon| + O(1)$ . It implies that for any  $t > 0, x \in \mathbb{R}^3$ , we have  $\lim_{\varepsilon \to 0} \mathbb{E}[u_{\varepsilon}(t, x)] = \infty$ . The same discussion applies to d = 2, where

$$\mathbb{E}[u_{\varepsilon}(t,x)] \ge \exp\left(\frac{\delta t^2}{2\varepsilon^2} - \frac{c't}{\varepsilon^2} - C_{\varepsilon}t\right).$$

If  $t > 2c'/\delta$ , we also have  $\lim_{\varepsilon \to 0} \mathbb{E}[u_{\varepsilon}(t, x)] = \infty$ . Since we do not have a proof of  $\mathbb{E}[u(t, x)] = \lim_{\varepsilon \to 0} \mathbb{E}[u_{\varepsilon}(t, x)]$  in d = 3 or d = 2 for large t, we only conjecture that  $\mathbb{E}[u(t, x)] = \infty$  in those cases.

Remark 1.8. When d = 2, the small time constraint for the existence of moments in our context also appears in [11, Theorem 4.1], where the usual product  $u \cdot \xi$ is replaced by the Wick product  $u \diamond \xi$ .

*Remark* 1.9. In [6], a similar result is derived for the random Schrödinger equation  $i\partial_t \phi + \frac{1}{2}\Delta\phi - \phi \cdot \xi = 0.$ 

## 2. PROOF OF LEMMA 1.2 AND THEOREM 1.1

We denote  $[0, t]_{\leq}^{n} = \{0 \leq s_{1} < \ldots < s_{n} \leq t\}$ , and write  $a \leq b$  if  $a \leq Cb$  with some constant C independent of  $\varepsilon$ .

*Proof of Lemma 1.2.* By scaling property of Brownian motion, we have

$$R_{\varepsilon}(B_s - B_u) = \varepsilon^{-2} R\left(\frac{B_s - B_u}{\varepsilon}\right) \stackrel{\text{law}}{=} \varepsilon^{-2} R\left(B_{s/\varepsilon^2} - B_{u/\varepsilon^2}\right).$$

A change of variable  $(u/\varepsilon^2, s/\varepsilon^2) \mapsto (u, s)$  then yields

$$\nu_{\varepsilon}(t) = \varepsilon^2 \int_0^{t/\varepsilon^2} \int_0^s \mathbb{E}_{\mathbf{B}}[R(B_s - B_u)] du ds.$$

Now,  $B_s - B_u$  has the normal density  $x \mapsto (2\pi(s-u))^{-1}e^{-\frac{|x|^2}{2(s-u)}}$ . We then do another change of variable  $s - u \mapsto v$ , integrate s out, and rescale  $v \to v\varepsilon^2$ . This leads us to

$$\nu_{\varepsilon}(t) = \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_0^t v^{-1} e^{-\frac{\varepsilon^2 |x|^2}{2v}} dv \right) dx - \frac{1}{2\pi} \int_0^t \left( \int_{\mathbb{R}^2} R(x) e^{-\frac{\varepsilon^2 |x|^2}{2v}} dx \right) dv$$
  
:= (i) - (ii).

Since R integrates to 1, it is clear that

(ii) 
$$\rightarrow \frac{t}{2\pi}$$

as  $\varepsilon \to 0$ . As for (i), a substitution of variable  $\frac{\varepsilon^2 |x|^2}{2v} \mapsto \lambda$  and then an integration by parts yields

$$\begin{aligned} (\mathbf{i}) &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^{\infty} \lambda^{-1} e^{-\lambda} d\lambda \right) dx \\ &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^{\infty} e^{-\lambda} \log \lambda d\lambda - e^{-\frac{\varepsilon^2 |x|^2}{2t}} \log \left(\frac{\varepsilon^2 |x|^2}{2t}\right) \right) dx. \end{aligned}$$

It is clear from the above expression that as  $\varepsilon \to 0$ , the only divergent part of (i) is from the term  $\log(\varepsilon^2)$ , and a direct calculation shows

$$\nu_{\varepsilon}(t) - \frac{t}{\pi} \cdot |\log \varepsilon| \to \mu_1 t + \mu_2 t \log t$$

for some constant  $\mu_1, \mu_2$ .  $\Box$ 

Proof of Theorem 1.1. Fix (t, x) and n, and recall that

(2.1) 
$$\mathbb{E}[u_{\varepsilon}(t,x)^{n}] = \mathbb{E}_{\mathbf{B}}\left[\exp(I_{n}^{\varepsilon}(t) - nC_{\varepsilon}t)\prod_{k=1}^{n}u_{0}(x+B_{t}^{k})\right],$$

where  $\mathbb{E}_{\mathbf{B}}$  is the expectation with respect to independent planar Brownian motions  $B^k$ 's, and  $I_n^{\varepsilon}$  is given by the expression (1.6). Note that  $u_{\varepsilon}(t, x)^n \to u(t, x)^n$ in probability, and that by (1.8) and Lemma 1.2, we have

$$I_n^{\varepsilon}(t) - nC_{\varepsilon}t \to \mathcal{X}_n(t) + nt(\mu_1 + \mu_2\log t)$$

in probability. Thus, in view of (2.1), it suffices to show the uniform integrability of  $u_{\varepsilon}(t,x)^n$  and  $\exp(I_n^{\varepsilon}(t) - nC_{\varepsilon}t) \prod_{k=1}^n u_0(x+B_t^k)$ . This allows us to pass both sides of (2.1) to the limit and conclude Theorem 1.1.

To prove the uniform integrability, we bound the second moment of these two objects:

$$\mathbb{E}\big[|u_{\varepsilon}(t,x)|^{2n}\big] \lesssim \mathbb{E}_{\mathbf{B}}\big[e^{I_{2n}^{\varepsilon}(t)-2nC_{\varepsilon}t}\big] \lesssim \mathbb{E}_{\mathbf{B}}\big[e^{I_{2n}^{\varepsilon}(t)-2n\nu_{\varepsilon}(t)}\big],$$

and

$$\mathbb{E}_{\mathbf{B}}\Big[\left|e^{I_n^{\varepsilon}(t)}e^{-nC_{\varepsilon}t}\prod_{k=1}^n u_0(x+B_t^k)\right|^2\Big] \lesssim \mathbb{E}_{\mathbf{B}}\Big[e^{2I_n^{\varepsilon}(t)-2nC_{\varepsilon}t}\Big] \lesssim \mathbb{E}_{\mathbf{B}}\Big[e^{2(I_n^{\varepsilon}(t)-n\nu_{\varepsilon}(t))}\Big],$$

where we have used  $||u_0||_{\infty} \leq 1$ . Thus, it suffices to show that for every n and  $\theta$ , there exists  $t_0$  small enough such that  $\mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^{\varepsilon}(t)-n\nu_{\varepsilon}(t))}]$  is uniformly bounded in  $\varepsilon$  for all  $t < t_0$ . To see this, using Hölder's inequality, we get

$$\mathbb{E}_{\mathbf{B}}\left[e^{\theta(I_{n}^{\varepsilon}(t)-n\nu_{\varepsilon}(t))}\right] \leq \prod_{k=1}^{n} \left[\mathbb{E}_{\mathbf{B}}e^{\theta N[\beta_{\varepsilon}^{k}([0,t]_{\varsigma}^{2})-\mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}^{k}([0,t]_{\varsigma}^{2})]}\right]^{\frac{1}{N}} \prod_{1\leq i< j\leq n} \left(\mathbb{E}_{\mathbf{B}}e^{\theta N\alpha_{\varepsilon}^{i,j}([0,t]^{2})}\right)^{\frac{1}{N}},$$

where  $N = \frac{n(n+1)}{2}$ , and we have used the notations

$$\beta_{\varepsilon}^{k}([0,t]_{\leq}^{2}) = \int_{0}^{t} \int_{0}^{s} R_{\varepsilon}(B_{s}^{k} - B_{u}^{k}) du ds, \quad \alpha_{\varepsilon}^{i,j}([0,t]^{2}) = \int_{0}^{t} \int_{0}^{t} R_{\varepsilon}(B_{s}^{i} - B_{u}^{j}) ds du.$$

By change of variables and the scaling property of the Brownian motion, we have

$$\beta_{\varepsilon}^{k}([0,t]_{<}^{2}) \stackrel{\text{law}}{=} t\beta_{\varepsilon/\sqrt{t}}^{k}([0,1]_{<}^{2}), \qquad \alpha_{\varepsilon}^{i,j}([0,t]^{2}) \stackrel{\text{law}}{=} t\alpha_{\varepsilon/\sqrt{t}}^{i,j}([0,1]^{2}).$$

Then, Lemma A.1 implies that there exists  $\lambda, C > 0$  such that

$$t < \frac{\lambda}{\theta N} \Rightarrow \sup_{\varepsilon \in (0,1)} \mathbb{E}_{\mathbf{B}} \left[ e^{\theta (I_n^{\varepsilon}(t) - n\nu_{\varepsilon}(t))} \right] \le C.$$

This completes the proof.  $\Box$ 

# Appendix A. Exponential moments of intersection local time of planar Brownian motions

Recall that  $R_{\varepsilon}(x) = \varepsilon^{-2} R(\frac{x}{\varepsilon})$ , we define

$$\alpha_{\varepsilon}(A) = \int_{A} R_{\varepsilon}(B_{s}^{1} - B_{u}^{2}) ds du, \quad \beta_{\varepsilon}(A) = \int_{A} R_{\varepsilon}(B_{s} - B_{u}) ds du$$

for any set  $A \subset \mathbb{R}^2_+$ , and

$$X_{\varepsilon} = \beta_{\varepsilon}([0,1]_{<}^2) - \mathbb{E}_{\mathbf{B}}[\beta_{\varepsilon}([0,1]_{<}^2)], \quad Y_{\varepsilon} = \alpha_{\varepsilon}([0,1]^2)$$

**Lemma A.1.** There exists universal constants  $\lambda, C > 0$  such that

$$\sup_{\varepsilon \in (0,1)} \left( \mathbb{E}_{\mathbf{B}}[e^{\lambda X_{\varepsilon}}] + \mathbb{E}_{\mathbf{B}}[e^{\lambda Y_{\varepsilon}}] \right) \le C.$$

The above result is standard. The case  $\varepsilon = 0$ , i.e., the exponential integrability of intersection local time, was addressed in the classical work [13]. We could not find a direct reference for  $\varepsilon > 0$ , though the proof follows essentially in the same line as the case of  $\varepsilon = 0$ . For the convenience of the reader, we present the details here.

*Proof.* We consider  $Y_{\varepsilon}$  first. Since  $R = \rho \star \rho$ , we can write

$$Y_{\varepsilon} = \int_{[0,1]^2} \int_{\mathbb{R}^2} \rho_{\varepsilon} (B_s^1 - x) \rho_{\varepsilon} (B_u^2 - x) dx ds du,$$

with  $\rho_{\varepsilon}(x) = \varepsilon^{-2} \rho(x/\varepsilon)$ . For any  $n \in \mathbb{N}$ ,

$$\mathbb{E}_{\mathbf{B}}[Y_{\varepsilon}^{n}] = \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^{2n}} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^{n} \rho_{\varepsilon} (B_{s_{k}}^{1} - x_{k}) \rho_{\varepsilon} (B_{u_{k}}^{2} - x_{k}) \right] d\mathbf{s} d\mathbf{u} \right) d\mathbf{x}$$
$$= \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^{n}} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^{n} \rho_{\varepsilon} (B_{s_{k}} - x_{k}) \right] d\mathbf{s} \right)^{2} d\mathbf{x}.$$

By [3, (2.2.11)], we have

$$\mathbb{E}_{\mathbf{B}}[Y_{\varepsilon}^{n}] = \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \rho_{\varepsilon}(z_{k} - x_{k}) \sum_{\sigma} \int_{[0,1]_{<}^{n}} \prod_{k=1}^{n} p_{s_{k}-s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) d\mathbf{s} d\mathbf{z} \right)^{2} d\mathbf{x},$$

where  $p_t(x)$  is the density of N(0,t),  $[0,t]^n_{\leq} = \{0 \leq s_1 < \ldots < s_n \leq t\}$ , and  $\sum_{\sigma}$  denotes the summation over all permutations over  $\{1,\ldots,n\}$ . If we denote

$$h(z_1, \dots, z_n) = \sum_{\sigma} \int_{[0,1]_{<}^n} \prod_{k=1}^n p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) d\mathbf{s}, \ Q_{\varepsilon}(z_1, \dots, z_n) = \prod_{k=1}^n \rho_{\varepsilon}(z_k),$$

then  $\mathbb{E}_{\mathbf{B}}[Y_{\varepsilon}^n]$  equals to (A.1)

$$\int_{\mathbb{R}^{2n}} |Q_{\varepsilon} \star h(x_1, \dots, x_n)|^2 d\mathbf{x} \le \left( \int_{\mathbb{R}^{2n}} Q_{\varepsilon}(x_1, \dots, x_n) d\mathbf{x} \right)^2 \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x}$$
$$= \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x} = \mathbb{E}_{\mathbf{B}}[\alpha([0, 1]^2)^n],$$

where  $\alpha([0,1]^2)$  is the mutual-intersection local time formally written as

$$\alpha([0,1]^2) = \int_0^1 \int_0^1 \delta(B_s^1 - B_u^2) ds du,$$

and we used the Le Gall's moment formula in the second line of (A.1). By [13], we have

$$\mathbb{E}_{\mathbf{B}}[\exp(\mu\alpha([0,1]^2))] < C$$

for some  $\mu > 0$ , hence we only need to choose  $\lambda = \mu$  to get

$$\mathbb{E}_{\mathbf{B}}[e^{\lambda Y_{\varepsilon}}] = \sum_{n=0}^{\infty} \frac{\lambda^{n} \mathbb{E}_{\mathbf{B}}[Y_{\varepsilon}^{n}]}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^{n}}{n!} \mathbb{E}_{\mathbf{B}}[|\alpha([0,1]^{2})|^{n}] = \mathbb{E}_{\mathbf{B}}[e^{\mu\alpha([0,1]^{2})}] < \infty.$$

Next, we consider  $X_{\varepsilon}$ . We define the triangle approximation of  $\{(u, s) : 0 \le u < s \le 1\}$ :

$$A_{l}^{k} = \left[\frac{2l}{2^{k+1}}, \frac{2l+1}{2^{k+1}}\right) \times \left[\frac{2l+1}{2^{k+1}}, \frac{2l+2}{2^{k+1}}\right), \quad l = 0, 1, \dots, 2^{k-1}, k = 0, 1, \dots$$

We will use the following three properties:

- (i) Fix any k,  $\{\beta_{\varepsilon}(A_l^k)\}_{l=0,\dots,2^k-1}$  are i.i.d. random variables.
- (ii)  $\beta_{\varepsilon}(A_l^k) \stackrel{\text{law}}{=} 2^{-(k+1)} \beta_{\varepsilon 2^{(k+1)/2}}([0,1] \times [1,2]) \stackrel{\text{law}}{=} 2^{-(k+1)} \alpha_{\varepsilon 2^{(k+1)/2}}([0,1]^2)$
- (iii)  $\sup_{\varepsilon>0} \mathbb{E}_{\mathbf{B}}[e^{\lambda \alpha_{\varepsilon}([0,1]^2)}] \leq C$  for some  $\lambda, C > 0$ .

By (iii) and a Taylor expansion, there exists C>0 such that for sufficiently small  $\lambda$ 

(A.2) 
$$\sup_{\varepsilon>0} \mathbb{E}_{\mathbf{B}}[e^{\lambda(\alpha_{\varepsilon}([0,1]^2) - \mathbb{E}_{\mathbf{B}}[\alpha_{\varepsilon}([0,1]^2)])}] \le e^{C\lambda^2}.$$

We fix the constants  $\lambda, C$  from now on, and write

$$X_{\varepsilon} = \sum_{k=0}^{\infty} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}[\beta_{\varepsilon}(A_{l}^{k})]).$$

Fix  $a \in (0, 1)$  and define a sequence of constants

$$b_1 = 2\lambda, \ b_N = 2\lambda \prod_{j=2}^N (1 - 2^{-a(j-1)}), N = 2, 3, \dots,$$

we have

$$\mathbb{E}_{\mathbf{B}} \exp\left[b_{N} \sum_{k=0}^{N} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{k}))\right]$$

$$\leq \left(\mathbb{E}_{\mathbf{B}} \exp\left[b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{k}))\right]\right)^{1-2^{-a(N-1)}}$$

$$\times \left(\mathbb{E}_{\mathbf{B}} \exp\left[2^{a(N-1)}b_{N} \sum_{l=0}^{2^{N-1}} (\beta_{\varepsilon}(A_{l}^{N}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{N}))\right]\right)^{2^{-a(N-1)}}$$

$$\leq \mathbb{E}_{\mathbf{B}} \exp\left[b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{k}))\right]$$

$$\times \left(\mathbb{E}_{\mathbf{B}} \exp\left[2^{a(N-1)}b_{N}(\beta_{\varepsilon}(A_{0}^{N}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{0}^{N}))\right]\right)^{2^{N-a(N-1)}}.$$

Since  $\beta_{\varepsilon}(A_0^N) \stackrel{\text{law}}{=} 2^{-(N+1)} \alpha_{\varepsilon 2^{(N+1)/2}}([0,1]^2)$ , we have

$$\mathbb{E}_{\mathbf{B}} \exp\left[2^{a(N-1)}b_N(\beta_{\varepsilon}(A_0^N) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_0^N))\right]$$
$$= \mathbb{E}_{\mathbf{B}} \exp\left[2^{a(N-1)}b_N 2^{-(N+1)}(\alpha_{\varepsilon 2^{(N+1)/2}}([0,1]^2) - \mathbb{E}_{\mathbf{B}}\alpha_{\varepsilon 2^{(N+1)/2}}([0,1]^2))\right].$$

Using the fact that  $2^{a(N-1)}b_N 2^{-(N+1)} < \lambda$  and (A.2), we derive for all  $\varepsilon > 0$  that

$$\mathbb{E}_{\mathbf{B}} \exp\left[2^{a(N-1)}b_N(\beta_{\varepsilon}(A_0^N) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_0^N))\right] \le e^{Cb_N^2 2^{-2N+2a(N-1)}},$$

so there exists C' > 0 such that

$$\mathbb{E}_{\mathbf{B}} \exp \left[ b_{N} \sum_{k=0}^{N} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{k})) \right]$$
$$\leq \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^{k}-1} (\beta_{\varepsilon}(A_{l}^{k}) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_{l}^{k})) \right] e^{C'2^{(a-1)N}}$$

Iterating the above inequality, we get

$$\mathbb{E}_{\mathbf{B}} \exp\left[b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_{\varepsilon}(A_l^k) - \mathbb{E}_{\mathbf{B}}\beta_{\varepsilon}(A_l^k))\right] \le \exp(C'(1-2^{a-1})^{-1})$$

Since  $b_N \to b_\infty$  for some  $b_\infty > 0$ , we have

$$\mathbb{E}_{\mathbf{B}}[\exp(b_{\infty}X_{\varepsilon})] \le \exp(C'(1-2^{a-1})^{-1}),$$

which completes the proof.  $\Box$ 

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