Global well-posedness and asymptotic behavior for Navier-Stokes-Coriolis equations in homogeneous Besov spaces

Lucas C. F. Ferreira¹*, Vladimir Angulo-Castillo²

^{1,2} Universidade Estadual de Campinas, IMECC-Departamento de Matemática, Rua Sérgio Buarque de Holanda, CEP 13083-859, Campinas, SP, Brazil.

Abstract

We are concerned with the 3D-Navier-Stokes equations with Coriolis force. Existence and uniqueness of global solutions in homogeneous Besov spaces are obtained for large speed of rotation. In the critical case of the regularity, we consider a suitable initial data class whose definition is based on the Stokes-Coriolis semigroup and Besov spaces. Moreover, we analyze the asymptotic behavior of solutions in that setting as the speed of rotation goes to infinity.

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1 Introduction

In this paper we are concerned with the incompressible Navier-Stokes equations in the rotational framework

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ \nabla \cdot u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ u(x, 0) = u_0(x) \text{ in } \mathbb{R}^3 \end{cases}$$

$$(1.1)$$

where $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ and p=p(x,t) stand for the velocity field and the pressure of the fluid, respectively. The initial data $u_0=(u_{0,1}(x),u_{0,2}(x),u_{0,3}(x))$ satisfies the divergence-free condition $\nabla \cdot u_0=0$. The letter $\Omega \in \mathbb{R}$ represents the Coriolis parameter while its modulus $|\Omega|$ is the speed of rotation around the vertical vector $e_3=(0,0,1)$. For more details about the physical model, we refer the reader to the book [9]. Here, we will use the same notation for spaces of scalar and vector functions, e.g., we write $u_0 \in H^s$ instead of $u_0 \in (H^s)^3$.

Invoking Duhamel's principle, the system (1.1) can be converted to the integral equation (see e.g. [12])

$$u(t) = T_{\Omega}(t)u_0 - \mathfrak{B}(u, u)(t), \tag{1.2}$$

E-mail addresses: lcff@ime.unicamp.br (L.C.F. Ferreira), vladimirangulo01@gmail.com (V. Angulo-Castillo).

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^{*}Corresponding author.

where the bilinear operator B is defined by

$$\mathfrak{B}(u,v)(t) = \int_0^t T_{\Omega}(t-\tau) \mathbb{P}\nabla \cdot (u \otimes v)(\tau) d\tau. \tag{1.3}$$

In (1.3), $\mathbb{P} = (\delta_{i,j} + R_i R_j)_{1 \leq i,j \leq 3}$ is the Leray-Helmholtz projector, $\{R_i\}_{1 \leq i \leq 3}$ are the Riesz transforms, and $T_{\Omega}(\cdot)$ stands for the semigroup corresponding to the linear part of (1.1) (Stokes-Coriolis semigroup). More explicitly, we have that

$$T_{\Omega}(t)f = \left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2t}I\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^2t}\mathcal{R}(\xi)\widehat{f}(\xi)\right]^{\vee}$$

for divergence-free vector fields f, where I is the identity matrix in \mathbb{R}^3 and $\mathcal{R}(\xi)$ is the skew-symmetric matrix symbol

$$\mathcal{R}(\xi) = \frac{1}{|\xi|} \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \text{ for } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Vector-fields u satisfying the formulation (1.2) are called mild solutions for (1.1).

In the last decades, the global well-posedness of models in fluid mechanics has been studied by several authors of the mathematical community, particularly in physical models of rotating fluids as the system (1.1). In what follows, we give a brief review on some of these results. We start with the works of Babin, Mahalov and Nicolaenko [2, 3, 4], who showed the global existence and regularity of solutions for (1.1) with periodic initial velocity provided that the speed of rotation $|\Omega|$ is sufficiently large. In [8, 9], Chemin $et\,al$. obtained a unique global strong Leray-type solution for large $|\Omega|$ and initial data $u_0(x) \in L^2(\mathbb{R}^2)^3 + H^{\frac{1}{2}}(\mathbb{R}^3)^3$ (notice that the first parcel of $u_0(x)$ depends on (x_1,x_2) where $x=(x_1,x_2,x_3)$). For almost periodic initial data and using the l^1 -norm of amplitudes with sum closed frequency set, Yoneda [20] proved the existence of solutions for large times and sufficiently large $|\Omega|$. Considering the mild (semigroup) formulation, the global well-posedness in homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$ with $1/2 \le s < 3/4$ was obtained by Iwabuchi and Takada [14]. They considered sufficiently large $|\Omega|$ (depending on the size of $|u_0|_{\dot{H}^s}$) when 1/2 < s < 3/4. In the critical case s=1/2, they used a class of precompact subsets in $\dot{H}^{1/2}(\mathbb{R}^3)$ in order to get similar results. Local versions (T large but finite) of the results in [14] can be found in [16] for 1/2 < s < 5/4.

Another type of results for (1.1) is the uniform global solvability (or well-posedness) in which the smallness condition on u_0 is independent of $|\Omega|$. Giga et~al.~[11] obtained the uniform global solvability for small data u_0 in $FM_0^{-1}(\mathbb{R}^3)=\operatorname{div}(FM_0(\mathbb{R}^3))^3$, where $FM_0(\mathbb{R}^3)$ denotes the space of the finite Radon measures with no point mass at the origin. The space $FM_0^{-1}(\mathbb{R}^3)$ is an example of critical space for the 3D Navier-Stokes equations (NS) ((NSC) with $\Omega=0$), i.e., its norm is invariant by the scaling $u_0^{\lambda}(x)\to \lambda u_0(\lambda x)$, for all $\lambda>0$. The uniform global well-posedness for small u_0 in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$ was proved by Hieber and Shibata [12] and for small initial data in the critical Fourier-Besov space $FB_{p,\infty}^{2-\frac{3}{p}}(\mathbb{R}^3)$ with $1< p\le \infty$ and in $FB_{1,1}^{-1}(\mathbb{R}^3)\cap FB_{1,1}^0(\mathbb{R}^3)$ was proved by Konieczny and Yoneda [18]. Iwabuchi and Takada [15] obtained the uniform global well-posedness with small initial velocity in the Fourier-Besov $FB_{1,2}^{-1}(\mathbb{R}^3)$ as well as the ill-posedness in $FB_{1,q}^{-1}(\mathbb{R}^3)$ for $2< q\le \infty$. These results were extended to the framework of critical Fourier-Besov-Morrey spaces by Almeida, Ferreira and Lima [1].

Concerning the asymptotic behavior for (1.1), we quote the work of Iwabuchi, Mahalov and Takada [17], where they treated the high-rotating cases and proved the asymptotic stability of large time periodic solutions for large initial perturbations. We also mention [9] where the reader can find convergence results of solutions towards a two-dimensional model as $|\Omega| \to \infty$ (see also references therein).

It is worthy to highlight that global existence of strong, mild or smooth solutions for the Navier-Stokes equations $(\Omega = 0)$, without assume smallness conditions on u_0 , are outstanding open problems. Thus, global solvability results for (1.1) with arbitrary data in suitable spaces show an interesting "smoothing effect" due to the Coriolis parameter Ω .

In this paper, we show the global well-posedness of (1.1) for large $|\Omega|$ and arbitrary initial data u_0 belonging to homogeneous Besov spaces $\dot{B}^s_{2,q}(\mathbb{R}^3)$ where $1 \leq q \leq \infty$ and $1/2 \leq s < 3/4$. In fact, for the cases $s \in (1/2,3/4)$ with $q = \infty$ and s = 1/2 with $q \in [2,\infty]$, we introduce the suitable initial-data classes \mathcal{I} and \mathcal{F}_0 (see (4.1) and (4.12)), respectively, whose definitions depend on the Stokes-Coriolis semigroup and Besov spaces. Also, we analyze the asymptotic behavior of solutions as $|\Omega| \to \infty$. For the case 1/2 < s < 3/4, we use some spacetime estimates of Strichartz type for the Stokes-Coriolis semigroup, and also the condition of $|\Omega|$ being large with respect to the $\dot{B}^s_{2,q}$ -norm (\mathcal{I} -norm for $q = \infty$) of the initial data u_0 (a power-type dependence). For the critical case s = 1/2, $|\Omega|$ depends on initial data belonging to precompact sets $D \subset \mathcal{F}_0$. In view of the strict continuous inclusions $\dot{H}^{1/2} \subset \mathcal{F}_0$ and

$$\dot{B}_{2,1}^s \subset \dot{B}_{2,q_1}^s \subset \dot{H}^s = \dot{B}_{2,2}^s \subset \dot{B}_{2,q_2}^s \subset \dot{B}_{2,\infty}^s,$$

for $1 \le q_1 \le 2 \le q_2 < \infty$, our results provide a new initial data class for the global well-posedness of (1.1) and, in particular, a class larger than that of [14].

Throughout this paper, we denote by C > 0 constants that may differ even on the same line. Also, the notation $C = C(a_1, \ldots, a_k)$ indicates that C depends on the quantities a_1, \ldots, a_k .

The outline of this paper is as follows. Section 2 is devoted to review some basic facts about homogeneous Besov spaces and certain mixed space-time functional settings. Estimates in Besov norms for the semigroup T_{Ω} and the Duhamel integral term in (1.2) are the subject of Section 3. In Section 4, we state and prove our global well-posedness and asymptotic behavior results for (1.1).

2 Function spaces

This section is devoted to some preliminaries about homogeneous Besov spaces and some mixed space-time functional settings.

We start with the definition of the homogeneous Besov spaces. For this, let $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)$ stand for the Schwartz class and the space of tempered distributions, respectively. Let \widehat{f} denote the Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^3)$.

Consider a nonnegative radial function $\phi_0 \in \mathcal{S}(\mathbb{R}^3)$ satisfying

$$0 \leq \widehat{\phi}_0(\xi) \leq 1 \text{ for all } \xi \in \mathbb{R}^3, \text{ supp } \widehat{\phi}_0 \subset \{\xi \in \mathbb{R}^3 : \frac{1}{2} \leq |\xi| \leq 2\} \text{ and } \sum_{j \in \mathbb{Z}} \widehat{\phi}_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\phi_j(x)=2^{3j}\phi_0(2^jx)$. For $f\in\mathcal{S}'(\mathbb{R}^3)$, the Littlewood-Paley operator $\{\Delta_j\}_{j\in\mathbb{Z}}$ is defined by $\Delta_j f=\phi_j*f$. Let $s\in\mathbb{R}$ and $1\leq p,q\leq\infty$ and let \mathcal{P} denote the set of polynomials with 3 variables. The homogeneous Besov space, denoted by $\dot{B}^s_{p,q}(\mathbb{R}^3)$, is defined as the set of all $f\in\mathcal{S}'(\mathbb{R}^3)/\mathcal{P}$ such that the following norm is finite

$$||f||_{\dot{B}_{2,q}^{s}} = ||\{2^{sj}||\Delta_{j}f||_{L^{p}}\}_{j\in\mathbb{Z}}||_{l^{q}(\mathbb{Z})}.$$

The pair $(\dot{B}^s_{p,q},\|\cdot\|_{\dot{B}^s_{p,q}})$ is a Banach space. We will denote abusively distributions in $\mathcal{S}'(\mathbb{R}^3)$ and their equivalence classes in $\mathcal{S}'(\mathbb{R}^3)/\mathcal{P}$ in the same way. The space $\mathcal{S}_0(\mathbb{R}^3)$ of functions in $\mathcal{S}(\mathbb{R}^3)$ whose Fourier transforms are supported away from 0 is dense in $\dot{B}^s_{p,q}(\mathbb{R}^3)$ for $1 \leq p,q < \infty$. For more details, see [5].

Using a duality argument, the norm $\|u\|_{\dot{B}^s_{p,q}}$ can be estimated as follows

$$||u||_{\dot{B}_{p,q}^{s}} \le C \sup_{\phi \in Q_{p',q'}^{-s}} |\langle u, \phi \rangle| \tag{2.1}$$

where $Q_{p',q'}^{-s}(\mathbb{R}^3)$ denotes the set of all functions $\phi \in \mathcal{S}(\mathbb{R}^3) \cap \dot{B}_{p',q'}^{-s}$ such that $\|\phi\|_{\dot{B}_{p',q'}^{-s}} \leq 1$ and $\langle \cdot, \cdot \rangle$ is defined by

$$\langle u, \phi \rangle := \sum_{|j-j'| \le 1} \int_{\mathbb{R}^3} \Delta_j u(x) \Delta_{j'} \phi(x) \ dx$$

for $u \in \dot{B}^{s}_{p,q}(\mathbb{R}^3)$ and $\phi \in Q^{-s}_{p',q'}(\mathbb{R}^3)$.

The next lemma contains a Leibniz type rule in the framework of Besov spaces.

Lemma 2.1 (see [7]). Let s > 0, $1 \le q \le \infty$ and $1 \le p, p_1, p_2, r_1, r_2 \le \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, there exists a universal constant C > 0 such that

$$\|fg\|_{\dot{B}^{s}_{p,q}} \leq C \left(\|f\|_{L^{p_{1}}} \|g\|_{\dot{B}^{s}_{p_{2},q}} + \|g\|_{L^{r_{1}}} \|f\|_{\dot{B}^{s}_{r_{2},q}} \right).$$

Considering in particular p = r and $p_i = r_i$ in Lemma 2.1, we have that

$$||fg||_{\dot{B}^{s}_{r,q}} \leq C \left(||f||_{L^{r_{1}}} ||g||_{\dot{B}^{s}_{r_{2},q}} + ||g||_{L^{r_{1}}} ||f||_{\dot{B}^{s}_{r_{2},q}} \right).$$

If $\frac{1}{r}=\frac{2}{r_2}-\frac{s}{3}$, then $\frac{1}{r_1}=\frac{2}{r_2}-\frac{s}{3}$ and we can use the embedding $\dot{B}^s_{r_2,q}(\mathbb{R}^3)\hookrightarrow L^{r_1}(\mathbb{R}^3)$ to obtain

$$||fg||_{\dot{B}^{s}_{r,q}} \le C||f||_{\dot{B}^{s}_{r_{2},q}} ||g||_{\dot{B}^{s}_{r_{2},q}}.$$
 (2.2)

The reader is referred to [6] for more details on $\dot{B}^s_{p,q}$ -spaces and their properties.

We finish this section by recalling some mixed space-time functional spaces. Let $\theta \geq 1$, we denote by $L^{\theta}(0,\infty;\dot{B}^{s}_{p,q}(\mathbb{R}^{3}))$ the set of all distributions f such that

$$||f||_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} = |||f(t)||_{\dot{B}^{s}_{p,q}}||_{L^{\theta}_{t}(0,\infty)} < \infty.$$

Also, we denote by $\tilde{L}^{\theta}(0,\infty;\dot{B}^{s}_{p,q}(\mathbb{R}^{3}))$ the set of all distributions f such that

$$||f||_{\tilde{L}^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} = ||\{2^{js}||\Delta_{j}f||_{L^{\theta}(0,\infty;L^{p})}\}_{j\in\mathbb{Z}}||_{l^{q}(\mathbb{Z})} < \infty.$$

As consequence of the Minkowski inequality, we have the following embeddings

$$L^{\theta}(0,\infty;\dot{B}^{s}_{p,q}) \hookrightarrow \tilde{L}^{\theta}(0,\infty;\dot{B}^{s}_{p,q}), \text{ if } \theta \leq q,$$

$$\tilde{L}^{\theta}(0,\infty;\dot{B}^{s}_{p,q}) \hookrightarrow L^{\theta}(0,\infty;\dot{B}^{s}_{p,q}), \text{ if } \theta \geq q.$$
(2.3)

3 Estimates

Firstly, we recall some estimates for the heat semigroup $e^{t\Delta}$ in Besov spaces [19] and the dispersive estimates for $T_{\Omega}(t)$ obtained in [13].

Lemma 3.1 (see [19]). Let $-\infty < s_0 \le s_1 < \infty$, $1 \le p, q \le \infty$ and $f \in \dot{B}^{s_0}_{p,q}(\mathbb{R}^3)$. Then, there exists a positive constant $C = C(s_0, s_1)$ such that

$$||e^{t\Delta}f||_{\dot{B}^{s_1}_{p,q}} \le Ct^{-\frac{3}{2}(s_1-s_0)}||f||_{\dot{B}^{s_0}_{p,q}}, for \ all \ t>0.$$

Before stating the dispersive estimates of [13], we need to define the operators

$$\mathcal{G}_{\pm}(\tau)[f] = \left[e^{\pm i\tau \frac{\xi_3}{|\xi|}} \widehat{f} \right]^{\vee}, \text{ for } \tau \in \mathbb{R},$$
(3.1)

and the matrix R of singular integral operators

$$\mathcal{R} = \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}. \tag{3.2}$$

Using (3.1) and (3.2), $T_{\Omega}(t)$ can be expressed as

$$T_{\Omega}(t)f = \frac{1}{2}\mathcal{G}_{+}(\Omega t)[e^{t\Delta}(I+\mathcal{R})f] + \frac{1}{2}\mathcal{G}_{-}(\Omega t)[e^{t\Delta}(I-\mathcal{R})f]$$
(3.3)

for $t \geq 0$ and $\Omega \in \mathbb{R}$.

Notice that the operators $\mathcal{G}_{\pm}(t\Omega)$ correspond to the oscillating parts of $T_{\Omega}(t)$.

Lemma 3.2 (see [13]). Let $s, t \in \mathbb{R}$, $2 \le p \le \infty$ and $f \in \dot{B}^{s+3\left(1-\frac{2}{p}\right)}_{p',q}(\mathbb{R}^3)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then, there exists a constant C = C(p) > 0 such that

$$\|\mathcal{G}_{\pm}(t)[f]\|_{\dot{B}^{s}_{p,q}} \le C \left(\frac{\log\left(e+|t|\right)}{1+|t|}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|f\|_{\dot{B}^{s+3}_{p',q}\left(1-\frac{2}{p}\right)}.$$

In what follows, we establish our estimates in Besov spaces for $T_{\Omega}(t)$ and the Duhamel term $\int_0^t T_{\Omega}(t-\tau)\mathbb{P}\nabla f(\tau)\ d\tau$. We start with three lemmas for $T_{\Omega}(t)$.

Lemma 3.3. Assume that $s, \Omega \in \mathbb{R}$, t > 0, $1 < r \le p' \le 2 \le p < \infty$ and $1 \le q \le \infty$, and let k be a multi-index. Then, there exists a constant C > 0 (independent of Ω and t) such that

$$\|\nabla_x^k T_{\Omega}(t)f\|_{\dot{B}^s_{p,q}} \le C \left(\frac{\log (e + |t\Omega|)}{1 + |t\Omega|}\right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} t^{-\frac{|k|}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{p}\right)} \|f\|_{\dot{B}^s_{r,q}},$$

for all $f \in \dot{B}^{s}_{r,q}(\mathbb{R}^3)$.

Proof. Using the representation (3.3), Lemma 3.2, the embedding $\dot{B}_{r,q}^{s+3\left(\frac{1}{r}-\frac{1}{p}\right)}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p',q}^{s+3\left(1-\frac{2}{p}\right)}(\mathbb{R}^3)$ and Lemma 3.1, we obtain

$$\begin{split} \|\nabla_{x}^{k}T_{\Omega}(t)f\|_{\dot{B}_{p,q}^{s}} &\leq C\|\mathcal{G}_{\pm}(t\Omega)[\nabla_{x}^{k}e^{t\Delta}f]\|_{\dot{B}_{p,q}^{s}} \\ &\leq C\left(\frac{\log\left(e+|t\Omega|\right)}{1+|t\Omega|}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|\nabla_{x}^{k}e^{t\Delta}f\|_{\dot{B}_{p',q}^{s+3}\left(1-\frac{2}{p}\right)} \\ &\leq C\left(\frac{\log\left(e+|t\Omega|\right)}{1+|t\Omega|}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|\nabla_{x}^{k}e^{t\Delta}f\|_{\dot{B}_{r,q}^{s+3}\left(\frac{1}{r}-\frac{1}{p}\right)} \\ &\leq C\left(\frac{\log\left(e+|t\Omega|\right)}{1+|t\Omega|}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} t^{-\frac{|k|}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|f\|_{\dot{B}_{r,q}^{s}}. \end{split}$$

Lemma 3.4. Let $1 \leq q < \infty$. Consider $s, p, \theta \in \mathbb{R}$ satisfying

$$0 \leq s < \frac{3}{p}, \ 2 < p < 6 \ \ \text{and} \ \ \frac{3}{4} - \frac{3}{2p} \leq \frac{1}{\theta} < \min\left\{\frac{1}{2}, 1 - \frac{2}{p}, \frac{1}{q}\right\}.$$

Then, there exists C > 0 (independent of $t \ge 0$ and $\Omega \in \mathbb{R}$) such that

$$||T_{\Omega}(t)f||_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \le C|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}}||f||_{\dot{B}^{s}_{2,q}},\tag{3.4}$$

for all $f \in \dot{B}^s_{2,a}(\mathbb{R}^3)$.

Proof. By duality and estimate (2.1), notice that (3.4) holds true provided that

$$I := \left| \int_{0}^{\infty} \sum_{|j-k| \le 1} \int_{\mathbb{R}^{3}} \Delta_{j} \mathcal{G}_{\pm}(\Omega t) [e^{t\Delta} f](x) \overline{\Delta_{k} \phi(x, t)} \, dx dt \right|$$

$$\leq C |\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} ||f||_{\dot{B}_{2,q}^{s}} ||\phi||_{L^{\theta'}(0,\infty; \dot{B}_{p',q'}^{-s})},$$
(3.5)

for all $\phi \in C_0^{\infty}(\mathbb{R}^3 \times (0,\infty))$ with $0 \notin \operatorname{supp}(\widehat{\phi}(\xi,t))$ for each t > 0, where 1/p + 1/p' = 1, $1/\theta + 1/\theta' = 1$ and 1/q + 1/q' = 1.

For (3.5), we use Parseval formula, Hölder inequality, the inclusion $\dot{B}^0_{p,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and Lemma 3.3 in order to estimate

$$I \leq \sum_{|j-k|\leq 1} \left| \int_{0}^{\infty} \int_{\mathbb{R}^{3}} \Delta_{j} \mathcal{G}_{\pm}(\Omega t) [e^{t\Delta} f](x) \overline{\Delta_{k} \phi(x,t)} \, dx dt \right|$$

$$= \sum_{|j-k|\leq 1} \left| \int_{0}^{\infty} \int_{\mathbb{R}^{3}} \Delta_{j} f(x) \overline{\Delta_{k} \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)](x)} \, dx dt \right|$$

$$= \sum_{|j-k|\leq 1} \left| \int_{\mathbb{R}^{3}} \Delta_{j} f(x) \int_{0}^{\infty} \overline{\Delta_{k} \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)](x)} \, dt dx \right|$$

$$\leq C \sum_{|j-k|\leq 1} \|\Delta_{j} f\|_{L^{2}} \left\| \int_{0}^{\infty} \Delta_{k} \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)] \, dt \right\|_{L^{2}}$$

$$\leq C 2^{|s|} \sum_{|j-k|\leq 1} 2^{js} \|\Delta_{j} f\|_{L^{2}} 2^{-ks} \left\| \int_{0}^{\infty} \Delta_{k} \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)] \, dt \right\|_{L^{2}}$$

$$\leq C 2^{|s|} \|f\|_{\dot{B}^{s}_{2,q}} \left(\sum_{k\in\mathbb{Z}} 2^{-ksq'} \left\| \int_{0}^{\infty} \Delta_{k} \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)] \, dt \right\|_{L^{2}}^{q'} \right).$$

$$(3.6)$$

Now, we are going to prove that

$$I_k^2 \le C|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|\Delta_k \phi\|_{L^{\theta'}(0,\infty;L^{p'})}^2,$$

where

$$I_k := \left\| \int_0^\infty \Delta_k \mathcal{G}_{\mp}(\Omega t) [e^{t\Delta} \phi(t)] \ dt \right\|_{L^2}.$$

In fact, using the Parseval formula, Hölder inequality, the embedding $\dot{B}^0_{p,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and Lemma 3.3, we have

$$\begin{split} I_k^2 &= \left\langle \int_0^\infty \Delta_k \mathcal{G}_\mp(\Omega t) [e^{t\Delta}\phi(t)] \ dt, \int_0^\infty \Delta_k \mathcal{G}_\mp(\Omega \tau) [e^{\tau\Delta}\phi(\tau)] \ d\tau \right\rangle_{L^2} \\ &= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^3} \Delta_k \mathcal{G}_\mp(\Omega t) [e^{t\Delta}\phi(t)](x) \overline{\Delta_k \mathcal{G}_\mp(\Omega \tau) [e^{\tau\Delta}\phi(\tau)](x)} \ dx d\tau dt \\ &\leq \int_0^\infty \int_0^\infty \|\Delta_k \phi(t)\|_{L^{p'}} \|\Delta_k \mathcal{G}_\pm(\Omega (t-\tau)) [e^{(t+\tau)\Delta}\phi(\tau)]\|_{L^p} \ d\tau dt \\ &\leq C \int_0^\infty \int_0^\infty \|\Delta_k \phi(t)\|_{L^{p'}} \|\Delta_k \mathcal{G}_\pm(\Omega (t-\tau)) [e^{(t+\tau)\Delta}\phi(\tau)]\|_{\dot{B}_{p,2}^0} \ d\tau dt \\ &\leq C \int_0^\infty \int_0^\infty \|\Delta_k \phi(t)\|_{L^{p'}} \left(\frac{\log \left(e + |\Omega| |t-\tau|\right)}{1 + |\Omega| |t-\tau|}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|e^{(t+\tau)\Delta}\Delta_k \phi(\tau)\|_{\dot{B}_{p',2}^3}^{3\left(1-\frac{2}{p}\right)} \ d\tau dt. \end{split}$$

By Lemma 3.1 and the embedding $L^{p'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{p',2}(\mathbb{R}^3)$ for p'<2, it follows that

$$||e^{(t+\tau)\Delta}\Delta_k\phi(\tau)||_{\dot{B}^{3\left(1-\frac{2}{p}\right)}_{p',2}} \le C(t+\tau)^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}||\Delta_k\phi(\tau)||_{\dot{B}^{0}_{p',2}}$$

$$\le C|t-\tau|^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}||\Delta_k\phi(\tau)||_{L^{p'}}.$$

Thus,

$$I_{k}^{2} \leq C \int_{0}^{\infty} \int_{0}^{\infty} \|\Delta_{k} \phi(t)\|_{L^{p'}} \left(\frac{\log\left(e + |\Omega||t - \tau|\right)}{1 + |\Omega||t - \tau|} \right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} |t - \tau|^{-\frac{3}{2}\left(1 - \frac{2}{p}\right)} \|\Delta_{k} \phi(\tau)\|_{L^{p'}} d\tau dt$$

$$\leq C \|\Delta_{k} \phi\|_{L^{\theta'}(0,\infty;L^{p'})} \left\| \int_{0}^{\infty} h(\cdot - \tau) \|\Delta_{k} \phi(\tau)\|_{L^{p'}} d\tau \right\|_{L^{\theta}(0,\infty)},$$
(3.7)

where

$$h(t) = \left(\frac{\log(e + |\Omega||t|)}{1 + |\Omega||t|}\right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} |t|^{-\frac{3}{2}\left(1 - \frac{2}{p}\right)}.$$

We consider the cases $\frac{1}{\theta} > \frac{3}{4} - \frac{3}{2p}$ and $\frac{1}{\theta} = \frac{3}{4} - \frac{3}{2p}$. In the first case, notice that

$$||h||_{L^{\frac{\theta}{2}}} = C|\Omega|^{-\frac{2}{\theta} + \frac{3}{2} - \frac{3}{p}}.$$

Therefore, using Young inequality in (3.7) and the above equality, we obtain

$$I_k^2 \le C|\Omega|^{-\frac{2}{\theta} + \frac{3}{2} - \frac{3}{p}} \|\Delta_k \phi\|_{L^{\theta'}(0,\infty;L^{p'})}^2.$$

Now, multiplying by 2^{-ks} , applying the $\ell^{q'}(\mathbb{Z})$ -norm and using (2.3), we arrive at

$$\left(\sum_{k\in\mathbb{Z}} 2^{-ksq'} I_k^{q'}\right)^{\frac{1}{q'}} \leq C|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \left(\sum_{k\in\mathbb{Z}} 2^{-ksq'} \|\Delta_k \phi\|_{L^{\theta'}(0,\infty;L^{p'})}^{q'}\right)^{\frac{1}{q'}} \\
\leq C|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|\phi\|_{L^{\theta'}(0,\infty;\dot{B}_{n',q'}^{-s})}.$$
(3.8)

It follows from (3.6) and (3.8) that

$$I \le C|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|f\|_{\dot{B}^{s}_{r,q}} \|\phi\|_{L^{\theta'}(0,\infty;\dot{B}^{-s}_{p',q'})},\tag{3.9}$$

with C > 0 independent of ϕ and f.

In the second case $\frac{1}{\theta} = \frac{3}{4} - \frac{3}{2p}$, we use the fact $h(t) \leq |t|^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}$ and Hardy-Littlewood-Sobolev inequality in (3.7) to obtain

$$I_k^2 \le C \|\Delta_k \phi\|_{L^{\theta'}(0,\infty;L^{p'})}^2. \tag{3.10}$$

Thus, using (3.10) and proceeding as in (3.8), we obtain a constant C > 0 (independent of ϕ and f) such that

$$I \le C \|f\|_{\dot{B}^{s}_{2,q}} \|\phi\|_{L^{\theta'}(0,\infty;\dot{B}^{-s}_{p',q'})}. \tag{3.11}$$

 \Diamond

 \Diamond

Estimates (3.9) and (3.11) give the desired result.

Lemma 3.5. Assume that $1 \leq q < 4$ and $f \in \dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3)$. Then,

$$\lim_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)f\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} = 0.$$
(3.12)

Proof. Since $\overline{\mathcal{S}_0(\mathbb{R}^3)}^{\|\cdot\|_{\dot{B}^{1/2}_{2,q}}} = \dot{B}^{\frac{1}{2}}_{2,q}$ for $q \neq \infty$ (see Section 2), there exists $(w_k)_{k \in \mathbb{N}}$ in $\mathcal{S}_0(\mathbb{R}^3)$ such that $w_k \to f$ in $\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3)$ as $k \to \infty$. Next, using Lemma 3.4, we obtain

$$\limsup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)f\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \leq \limsup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)(f-w_{k})\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + \limsup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)w_{k}\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \\
\leq C\|w_{k} - f\|_{\dot{B}_{2,q}^{\frac{1}{2}}} + \limsup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)w_{k}\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}.$$
(3.13)

Choosing $p \in (\frac{8}{3}, 3)$, we have the conditions

$$\frac{3}{4} - \frac{3}{2p} < \frac{1}{4} < \min\left\{1 - \frac{2}{p}, \frac{1}{q}\right\} \text{ and } \frac{1}{2} - \frac{3}{2p} < 0.$$

Then, we can use $\dot{B}^{-\frac{1}{2}+\frac{3}{p}}_{p,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{\frac{1}{2}}_{3,q}(\mathbb{R}^3)$ and Lemma 3.4 to estimate

$$\lim \sup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)w_{k}\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \leq C \lim \sup_{|\Omega| \to \infty} \|T_{\Omega}(\cdot)w_{k}\|_{L^{4}(0,\infty;\dot{B}_{p,q}^{-\frac{1}{2} + \frac{3}{p}})} \\
\leq C|\Omega|^{\frac{1}{2} - \frac{3}{2p}} \|w_{k}\|_{\dot{B}_{2,q}^{-\frac{1}{2} + \frac{3}{p}}} \to 0, \text{ as } |\Omega| \to \infty. \tag{3.14}$$

By (3.13), (3.14) and $\|w_k - f\|_{\dot{B}^{\frac{1}{2}}_{2,q}} \to 0$, it follows (3.12).

The next two lemmas are concerned with the Duhamel term $\int_0^t T_{\Omega}(t-\tau) \mathbb{P} \nabla f(\tau) \ d\tau$.

Lemma 3.6. Let $s \in \mathbb{R}$ and $\Omega \in \mathbb{R} \setminus \{0\}$ and let p, r, q, θ be real numbers satisfying

$$2
$$\max\left\{0, \frac{1}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{p}\right) - \frac{1}{2}\left(1 - \frac{2}{p}\right)\right\} < \frac{1}{\theta} \le \frac{1}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{p}\right).$$$$

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Then, there exists a universal constant C > 0 such that

$$\left\| \int_{0}^{t} T_{\Omega}(t-\tau) \mathbb{P} \nabla f(\tau) \, d\tau \right\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})} \le C |\Omega|^{-\frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p}\right) + \frac{1}{\theta}} \|f\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{B}_{r,q}^{s})}. \tag{3.15}$$

Proof. Using Lemma 3.3 it follows that

$$\left\| \int_{0}^{t} T_{\Omega}(t-\tau) \mathbb{P} \nabla f(\tau) d\tau \right\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \leq C \left\| \int_{0}^{t} \| T_{\Omega}(t-\tau) \mathbb{P} \nabla f(\tau) \|_{\dot{B}^{s}_{p,q}} d\tau \right\|_{L^{\theta}(0,\infty)}$$

$$\leq C \left\| \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p}\right)} \left(\frac{\log\left(e + |\Omega||t-\tau|\right)}{1 + |\Omega||t-\tau|} \right)^{\frac{1}{2} \left(1 - \frac{2}{p}\right)} \| f(\tau) \|_{\dot{B}^{s}_{r,q}} d\tau \right\|_{L^{\theta}(0,\infty)}. \tag{3.16}$$

We are going to prove (3.6) in two cases. First we consider the case $\frac{1}{\theta} = \frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right)$. Here, we note that

$$\left(\frac{\log\left(e + |\Omega||t - \tau|\right)}{1 + |\Omega||t - \tau|}\right)^{\frac{1}{2}\left(1 - \frac{2}{p}\right)} \le 1$$

and employ Hardy-Littlewood-Sobolev inequality to estimate

$$\left\| \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p}\right)} \left(\frac{\log\left(e + |\Omega||t-\tau|\right)}{1 + |\Omega||t-\tau|} \right)^{\frac{1}{2} \left(1 - \frac{2}{p}\right)} \|f(\tau)\|_{\dot{B}_{r,q}^{s}} d\tau \right\|_{L^{\theta}(0,\infty)} \le C \|f\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{B}_{r,q}^{s})}. \tag{3.17}$$

Consider now the case $\frac{1}{\theta} < \frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right)$. Selecting ℓ such that $\frac{1}{\theta} = \frac{1}{\ell} + \frac{2}{\theta} - 1$, a direct computation gives

$$\left\| (t - \tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p}\right)} \left(\frac{\log\left(e + |\Omega||t - \tau|\right)}{1 + |\Omega||t - \tau|} \right)^{\frac{1}{2} \left(1 - \frac{2}{p}\right)} \right\|_{L^{\ell}(0, \infty)} = C|\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p}\right)}.$$
 (3.18)

By Young inequality and (3.18), we have that

$$\left\| \int_{0}^{t} (t-\tau)^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \left(\frac{\log\left(e+|\Omega||t-\tau|\right)}{1+|\Omega||t-\tau|} \right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \|f(\tau)\|_{\dot{B}_{r,q}^{s}} d\tau \right\|_{L^{\theta}(0,\infty)}$$

$$\leq \left\| t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \left(\frac{\log\left(e+|\Omega|t\right)}{1+|\Omega|t} \right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} \right\|_{L^{\theta}(0,\infty)} \|f\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{B}_{r,q}^{s})}$$

$$= C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|f\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{B}_{r,q}^{s})}.$$
(3.19)

The proof is completed by substituting (3.17) and (3.19) into (3.16).

Lemma 3.7. Let $s, \Omega \in \mathbb{R}$ and $2 \leq q \leq \infty$. Then, there exists a universal constant C > 0 such that

$$\left\| \int_0^t T_{\Omega}(t-\tau) \nabla f(\tau) \, d\tau \right\|_{L^{\infty}(0,\infty;\dot{B}^s_{2,q}) \cap L^4(0,\infty;\dot{B}^s_{3,q})} \le C \|f\|_{L^2(0,\infty;\dot{B}^s_{2,q})}. \tag{3.20}$$

 \Diamond

Proof. We denote $X=X_1\cap X_2$ where $X_1=L^\infty(0,\infty;\dot{B}^s_{2,q})$ and $X_2=L^4(0,\infty;\dot{B}^s_{3,q})$. We start with estimates for the X_1 -norm. We have that

$$\begin{split} \left\| \Delta_{j} \int_{0}^{t} T_{\Omega}(t-\tau) \nabla f(\tau) \ d\tau \right\|_{L^{2}} &= \left\| \int_{0}^{t} T_{\Omega}(t-\tau) \nabla \Delta_{j} f(\tau) \ d\tau \right\|_{L^{2}} \\ &\leq C \left\| \int_{0}^{t} e^{-(t-\tau)|\xi|^{2}} |\xi| |\widehat{\phi}_{j}(\xi) \widehat{f}(\tau)| \ d\tau \right\|_{L^{2}} \\ &\leq C \left\| \|e^{-(t-\tau)|\xi|^{2}} \|_{L^{2}_{\tau}(0,t)} |\xi| \|\widehat{\phi}_{j}(\xi) \widehat{f}(\tau)\|_{L^{2}_{\tau}(0,t)} \right\|_{L^{2}} \\ &\leq C \|\Delta_{j} f\|_{L^{2}(0,\infty;L^{2})}. \end{split}$$

Multiplying by 2^{sj} , applying $l^q(\mathbb{Z})$ -norm and using inequality (2.3), we arrive at

$$\left\| \int_{0}^{t} T_{\Omega}(t-\tau) \nabla f(\tau) \, d\tau \right\|_{\dot{B}_{2,q}^{s}} \leq C \left(\sum_{j \in \mathbb{Z}} 2^{sjq} \|\Delta_{j} f\|_{L^{2}(0,\infty;L^{2})}^{q} \right)^{\frac{1}{q}}$$

$$\leq C \|f\|_{L^{2}(0,\infty;\dot{B}_{2,q}^{s})}$$

and then

$$\left\| \int_0^t T_{\Omega}(t-\tau) \nabla f(\tau) \, d\tau \right\|_{X_1} \le C \|f\|_{L^2(0,\infty;\dot{B}^s_{2,q})}. \tag{3.21}$$

 \Diamond

In order to estimate the X_2 -norm, we use Lemma 3.3 and Hardy-Littlewood-Sobolev inequality to obtain

$$\left\| \int_{0}^{t} T_{\Omega}(t-\tau) \nabla f(\tau) \, d\tau \right\|_{X_{2}} \leq \left\| \int_{0}^{t} \| T_{\Omega}(t-\tau) \nabla f(\tau) \|_{\dot{B}_{3,q}^{s}} \, d\tau \right\|_{L^{4}(0,\infty)}$$

$$\leq C \left\| \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{2} - \frac{1}{3}\right)} \| f(\tau) \|_{\dot{B}_{2,q}^{s}} \, d\tau \right\|_{L^{4}(0,\infty)}$$

$$\leq C \| f \|_{L^{2}(0,\infty;\dot{B}_{2,q}^{s})}. \tag{3.22}$$

Putting together (3.21) and (3.22), we arrive at (3.20).

4 Global existence

In this section we state and prove our results about existence and uniqueness of global solutions to (1.1). Basically, we have two cases 1/2 < s < 3/4 and s = 1/2. We start with the former.

Theorem 4.1. (i) For $1 \le q < \infty$, consider s, p and θ satisfying

$$\frac{1}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3},$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < \min\left\{1 - \frac{2}{p}, \frac{1}{q}\right\}.$$

Let $\Omega \in \mathbb{R} \setminus \{0\}$ and $u_0 \in \dot{B}^s_{2,q}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. There is a constant $C = C(s,p,\theta) > 0$ such that if $\|u_0\|_{\dot{B}^s_{2,q}} \leq C|\Omega|^{\frac{s}{2}-\frac{1}{4}}$, then there exists a unique global solution $u \in C([0,\infty); \dot{B}^s_{2,q}(\mathbb{R}^3))$ to (1.1).

(ii) For $q = \infty$, consider s, p and θ satisfying

$$\frac{1}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3},$$
$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}.$$

Let $\Omega_0 > 0$ and $u_0 \in \mathcal{I}$ with $\nabla \cdot u_0 = 0$, where

$$\mathcal{I} := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \colon \|f\|_{\mathcal{I}} := \sup_{|\Omega| \ge \Omega_0} |\Omega|^{\frac{1}{\theta} - \frac{3}{4} + \frac{3}{2p}} \|T_{\Omega}(t)f\|_{L^{\theta}(0,\infty;\dot{B}^s_{p,\infty})} < \infty \right\}. \tag{4.1}$$

There is a constant $C = C(s, p, \theta) > 0$ such that if $||u_0||_{\mathcal{I}} \leq C|\Omega|^{\frac{s}{2} - \frac{1}{4}}$ for $|\Omega| \geq \Omega_0$, then the system (1.1) has a unique global solution $u \in L^{\theta}(0, \infty; \dot{B}^s_{p,\infty}(\mathbb{R}^3))$. Moreover, if in addition $u_0 \in \dot{B}^s_{2,\infty}(\mathbb{R}^3)$ then $u \in C_{\omega}([0, \infty); \dot{B}^s_{2,\infty}(\mathbb{R}^3))$ where C_{ω} stands to time weakly continuous functions.

Remark 4.2. Notice that the space \mathcal{I} depends on the parameters Ω_0 , θ , p and s, but for simplicity we have omitted them in the notation.

Proof of Theorem 4.1.

Part (i): By Lemma 3.3, it follows that

$$||T_{\Omega}(t)u_0||_{L^{\theta}(0,\infty;\dot{B}^s_{p,q})} \le C_0|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} ||u_0||_{\dot{B}^s_{2,q}}.$$
(4.2)

Now, we define the operator Γ and the set Z by

$$\Gamma(u)(t) = T_{\Omega}(t)u_0 - \mathfrak{B}(u, u)(t) \tag{4.3}$$

and

$$Z = \left\{ u \in L^{\theta}(0, \infty; \dot{B}_{p,q}^{s}(\mathbb{R}^{3})) : \|u\|_{L^{\theta}(0, \infty; \dot{B}_{p,q}^{s})} \le 2C_{0} |\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}_{2,q}^{s}}, \ \nabla \cdot u = 0 \right\}.$$

Taking $\frac{1}{r} = \frac{2}{p} - \frac{s}{3}$, we can employ Lemma 3.6 and (2.2) to estimate $\Gamma(\cdot)$ as follows

$$\begin{split} \|\Gamma(u) - \Gamma(v)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} &= \left\| \int_{0}^{t} T_{\Omega}(t-\tau) \mathbb{P} \nabla \cdot \left(u \otimes (u-v)(\tau) + (u-v) \otimes v(\tau) \right) \tau \right\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \\ &\leq C |\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right)} \|u \otimes (u-v) + (u-v) \otimes v\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{B}^{s}_{r,q})} \\ &\leq C |\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right)} \left(\|u\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} + \|v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \right) \|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \\ &\leq C |\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right)} 4C_{0} |\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}^{s}_{2,q}} \|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \\ &= C_{2} |\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right) - \frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}^{s}_{2,q}} \|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \\ &= C_{2} |\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\dot{B}^{s}_{2,q}} \|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}, \end{split} \tag{4.4}$$

for all $u, v \in \mathbb{Z}$, where $C_2 = C_2(s, p, \theta)$. Moreover, using (4.2) and (4.4) with v = 0, we obtain

$$\|\Gamma(u)\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})} \leq \|T_{\Omega}(t)u_{0}\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})} + \|\Gamma(u) - \Gamma(0)\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})}$$

$$\leq C_{0}|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}_{2,q}^{s}} + C_{2}|\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\dot{B}_{2,q}^{s}} \|u\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})}$$

$$\leq C_{0}|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}_{2,q}^{s}} + C_{2}|\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\dot{B}_{2,q}^{s}} 2C_{0}|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\dot{B}_{2,q}^{s}}$$

$$= C_{0}\|u_{0}\|_{\dot{B}_{2,q}^{s}} |\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \left(1 + 2C_{2}|\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\dot{B}_{2,q}^{s}}\right)$$

$$(4.5)$$

for all $u \in Z$. Thus, for Ω and u_0 satisfying

$$C_2|\Omega|^{\frac{1}{4}-\frac{s}{2}}||u_0||_{\dot{B}_{2,q}^s} \le \frac{1}{2},$$

we get

$$\|\Gamma(u)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \leq 2C_{0}|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}-\frac{3}{2p}}\|u_{0}\|_{\dot{B}^{s}_{2,q}} \text{ and } \|\Gamma(u)-\Gamma(v)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} \leq \frac{1}{2}\|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}.$$

Then, Banach fixed point theorem implies that there exists a unique mild solution $u \in Z$ to (1.1), i.e.,

$$u(t) = T_{\Omega}(t)u_0 - \mathfrak{B}(u, u)(t).$$

It remains to prove that $u \in C([0,\infty); \dot{B}^s_{2,q}(\mathbb{R}^3))$. Basically, we need to estimate the $\dot{B}^s_{2,q}$ -norm of the linear and nonlinear parts in (4.3). For the linear one, we use Lemma 3.3 to get

$$||T_{\Omega}(t)u_0||_{\dot{B}_{2,q}^s} \le C_0||u_0||_{\dot{B}_{2,q}^s}. \tag{4.6}$$

For the nonlinear part, taking $\frac{1}{r} = \frac{2}{p} - \frac{s}{3}$, we use Lemma 3.3, (2.2) and Hölder inequality to obtain

$$\left\| \int_{0}^{t} T_{\Omega}(t-\tau) \mathbb{P} \nabla \cdot (u \otimes u)(\tau) \, d\tau \right\|_{\dot{B}_{2,q}^{s}} \leq C \int_{0}^{t} \|T_{\Omega}(t-\tau) \mathbb{P} \nabla \cdot (u \otimes u)(\tau)\|_{\dot{B}_{2,q}^{s}} \, d\tau \\
\leq C \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \|(u \otimes u)(\tau)\|_{\dot{B}_{r,q}^{s}} \, d\tau \\
\leq C \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \|u(\tau)\|_{\dot{B}_{p,q}^{s}}^{2} \, d\tau \\
\leq C \left\| (t-\tau)^{-\frac{1}{2} - \frac{3}{2r} + \frac{3}{4}} \right\|_{L^{\frac{\theta}{\theta - 2}}(0 < \tau < t)} \|\|u(\tau)\|_{\dot{B}_{p,q}^{s}}^{2} \|_{L^{\frac{\theta}{2}}(0,\infty)} \\
\leq C t^{\frac{\theta - 2}{\theta} \left(1 + \frac{\theta}{\theta - 2} \left(-\frac{1}{2} - \frac{3}{2r} + \frac{3}{4}\right)\right)} \|u\|_{L^{\theta}(0,\infty;\dot{B}_{p,q}^{s})}^{2}, \tag{4.7}$$

where we need $\frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}$ in order to ensure integrability at $\tau = t$. From (4.6) and (4.7), it follows that $u(t) \in \dot{B}^s_{2,q}(\mathbb{R}^3)$ for t > 0, and then we have that $u \in C([0,\infty); \dot{B}^s_{2,q}(\mathbb{R}^3))$, as desired.

Part (ii): In view of (4.1), we have that

$$||T_{\Omega}(t)u_{0}||_{L^{\theta}(0,\infty;\dot{B}_{p,\infty}^{s})} \leq |\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} ||u_{0}||_{\mathcal{I}}, \text{ for all } |\Omega| \geq \Omega_{0}.$$

$$(4.8)$$

Now, for $|\Omega| \geq \Omega_0$ consider

$$\Gamma(u)(t) = T_{\Omega}(t)u_0 - \mathfrak{B}(u, u)(t) \tag{4.9}$$

and

$$Z = \left\{ u \in L^{\theta}(0, \infty; \dot{B}^{s}_{p, \infty}(\mathbb{R}^{3})) : \|u\|_{L^{\theta}(0, \infty; \dot{B}^{s}_{p, \infty})} \le 2|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \|u_{0}\|_{\mathcal{I}}, \ \nabla \cdot u = 0 \right\}.$$

Taking $\frac{1}{r}=\frac{2}{p}-\frac{s}{3}$, and proceeding similarly to Part (i), we obtain a constant $\tilde{C}_2=\tilde{C}_2(s,p,\theta)$ such that

$$\|\Gamma(u) - \Gamma(v)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})} \leq \tilde{C}_{2}|\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\mathcal{I}} \|u - v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})} \|\Gamma(u)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})} \leq \|u_{0}\|_{\mathcal{I}}|\Omega|^{-\frac{1}{\theta} + \frac{3}{4} - \frac{3}{2p}} \left(1 + 2\tilde{C}_{2}|\Omega|^{\frac{1}{4} - \frac{s}{2}} \|u_{0}\|_{\mathcal{I}}\right),$$

$$(4.10)$$

for all $u, v \in Z$. Thus, for Ω and u_0 satisfying

$$|\Omega| \ge \Omega_0$$
 and $\tilde{C}_2 |\Omega|^{\frac{1}{4} - \frac{s}{2}} ||u_0||_{\mathcal{I}} \le \frac{1}{2}$,

we get

$$\|\Gamma(u)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})} \leq 2|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}-\frac{3}{2p}}\|u_{0}\|_{\mathcal{I}} \text{ and } \|\Gamma(u)-\Gamma(v)\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})} \leq \frac{1}{2}\|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,\infty})}.$$

Again, we can apply the Banach fixed point theorem in order to obtain a unique mild solution $u \in Z$ to (1.1). Assume now that $u_0 \in \dot{B}^s_{2,\infty}(\mathbb{R}^3)$. Since (4.6) and (4.7) hold true for $q = \infty$, it follows that $u \in C_\omega([0,\infty); \dot{B}^s_{2,\infty}(\mathbb{R}^3))$.

 \Diamond

Before proceeding, for $\Omega_0 > 0$ and $1 \le q \le \infty$ we define the space

$$\mathcal{F} := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \colon \|f\|_{\mathcal{F}} := \sup_{|\Omega| \ge \Omega_0} \|T_{\Omega}(t)f\|_{L^4(0,\infty; \dot{B}_{3,q}^{1/2})} < \infty \right\},\tag{4.11}$$

where, for simplicity, we have omitted the dependence on Ω_0 and q in the notation \mathcal{F} . We also define

$$\mathcal{F}_{0} := \left\{ f \in \mathcal{F} \colon \limsup_{|\Omega| \to \infty} \|T_{\Omega}(t)f\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{1/2})} = 0 \right\}. \tag{4.12}$$

Both spaces \mathcal{F} and \mathcal{F}_0 are endowed with the norm $\|\cdot\|_{\mathcal{F}}$. The next theorem deals with the critical case s=1/2.

Theorem 4.3. Let $2 \le q \le \infty$ and $u_0 \in D$ with $\nabla \cdot u_0 = 0$ where D is a precompact set in \mathcal{F}_0 . Then, there exist $\widetilde{\Omega} = \widetilde{\Omega}(D) > 0$ and a unique global solution u to (1.1) in $L^4(0,\infty;\dot{B}^{1/2}_{3,q}(\mathbb{R}^3))$ provided that $|\Omega| \ge \widetilde{\Omega}$. Moreover, if in addition $u_0 \in \dot{B}^{1/2}_{2,q}(\mathbb{R}^3)$ with $q \ne \infty$, then $u \in C([0,\infty);\dot{B}^{1/2}_{2,q}(\mathbb{R}^3))$. In the case $q = \infty$, we obtain $C_{\omega}([0,\infty);\dot{B}^{1/2}_{2,\infty}(\mathbb{R}^3))$.

Proof. Let δ be a positive number that will be chosen later. Given that D is a precompact set in \mathcal{F}_0 , there exist $L = L(\delta, D) \in \mathbb{N}$ and $\{g_k\} \subset \mathcal{F}_0$ such that

$$D \subset \bigcup_{k=1}^{L} B(g_k, \delta),$$

where $B(g_k, \delta)$ denotes the ball in \mathcal{F}_0 with center g_k and radius δ . On the other hand, using the definition (4.12), there exists $\tilde{\Omega} = \tilde{\Omega}(\delta, D) \geq \Omega_0 > 0$ such that

$$\sup_{k=1,2,...,L} \|T_{\Omega}(t)g_k\|_{L^4(0,\infty; \dot{B}_{3,q}^{\frac{1}{2}})} \leq \delta$$

provided that $|\Omega| \geq \tilde{\Omega}$. Now, given $g \in D$ there exists $k \in \{1, 2, \dots, L\}$ such that $g \in B(g_k, \delta)$. Therefore, for $|\Omega| \geq \tilde{\Omega}$ we can estimate

$$\begin{aligned} \|T_{\Omega}(t)g\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} &\leq \|T_{\Omega}(t)(g_{k}-g)\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + \|T_{\Omega}(t)g_{k}\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \\ &\leq C\|g_{k}-g\|_{\mathcal{F}} + \delta \\ &\leq (C+1)\delta. \end{aligned}$$

Thus, there exists $C_1 > 0$ such that

$$\sup_{g \in D} \|T_{\Omega}(t)g\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} \le C_1 \delta, \text{ for all } |\Omega| \ge \tilde{\Omega}. \tag{4.13}$$

Now, we consider the complete metric space Z defined by

$$Z = \left\{ u \in L^4(0, \infty; \dot{B}_{3,q}^{\frac{1}{2}}) : \|u\|_{L^4(0, \infty; \dot{B}_{3,q}^{\frac{1}{2}})} \le 2C_1 \delta, \nabla \cdot u = 0 \right\}, \tag{4.14}$$

endowed with the metric $d(u,v)=\|u-v\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})}$. Also, we consider the operator Γ defined in the proof of Theorem 4.1. For $u,v\in Z$, using Lemma 3.7, (2.2) and Hölder inequality, we can estimate

$$\|\Gamma(u) - \Gamma(v)\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} = \left\| \int_{0}^{t} T_{\Omega}(t-\tau) \mathbb{P} \nabla \cdot (u \otimes (u-v) + (u-v) \otimes v)(\tau) \, d\tau \right\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}$$

$$\leq C \|u \otimes (u-v) + (u-v) \otimes v\|_{L^{2}(0,\infty;\dot{B}_{2,q}^{\frac{1}{2}})}$$

$$\leq C \left(\left\| \|u\|_{\dot{B}_{3,q}^{\frac{1}{2}}} \|u-v\|_{\dot{B}_{3,q}^{\frac{1}{2}}} \right\|_{L^{2}(0,\infty)} + \left\| \|v\|_{\dot{B}_{3,q}^{\frac{1}{2}}} \|u-v\|_{\dot{B}_{3,q}^{\frac{1}{2}}} \right\|_{L^{2}(0,\infty)} \right)$$

$$\leq C_{2} \left(\left\| u \right\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + \left\| v \right\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \right) \|u-v\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}.$$

$$(4.15)$$

Taking v = 0 in (4.15), for $u \in Z$ it follows that

$$\|\Gamma(u)\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \leq \|\Gamma(0)\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + \|\Gamma(u) - \Gamma(0)\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}$$

$$\leq \|T_{\Omega}(t)u_{0}\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + C_{2}\|u\|_{L^{4}(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}^{2}.$$

$$(4.16)$$

Choosing $0 < \delta < \frac{1}{8C_1C_2}$, estimates (4.13), (4.15) and (4.16) yield

$$\begin{split} &\|\Gamma(u)\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} \leq 2C_1\delta, \text{ for all } u \in Z, \\ &\|\Gamma(u) - \Gamma(v)\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} \leq \frac{1}{2}\|u - v\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})}, \text{ for all } u, v \in Z, \end{split}$$

provided that $|\Omega| \geq \tilde{\Omega}$. Therefore, we can apply the Banach fixed point theorem to obtain a unique global solution $u \in L^4(0,\infty;\dot{B}_{3,g}^{\frac{1}{2}})$.

Moreover, using Lemma 3.3, Lemma 3.7 and $u \in L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})$, we have that

$$||u(t)||_{\dot{B}_{2,q}^{\frac{1}{2}}} = ||\Gamma(u)(t)||_{\dot{B}_{2,q}^{\frac{1}{2}}} \le C||u_0||_{\dot{B}_{2,q}^{\frac{1}{2}}} + C||u||_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}^2 < \infty, \tag{4.17}$$

for a.e. t>0. Since $\|u\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}\leq 2C_1\delta$ and $\delta<\frac{1}{8C_1C_2}$, it follows that

$$||u(t)||_{\dot{B}^{\frac{1}{2}}_{2,q}} \le C(||u_0||_{\dot{B}^{\frac{1}{2}}_{2,q}} + 1) < \infty, \text{ for all } |\Omega| \ge \tilde{\Omega},$$
 (4.18)

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and so $u(t) \in \dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3)$ for a.e. t > 0. Using this and above estimates, standard arguments yield $u \in C([0,\infty); \dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3))$ for $q \neq \infty$ and $u \in C_{\omega}([0,\infty); \dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3))$ for $q = \infty$.

Theorem 4.4. Let $2 \le q \le \infty$ and $u_0 \in \mathcal{F}_0$ with $\nabla \cdot u_0 = 0$. Then, there exist $\tilde{\Omega} = \tilde{\Omega}(u_0)$ and a unique global solution $u \in L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q}(\mathbb{R}^3))$ to (1.1) provided that $|\Omega| \ge \tilde{\Omega}$.

Proof. It is sufficient to apply Theorem 4.3 to the set $D = \{u_0\}$.

Corollary 4.5. Let $2 \le q < 4$ and $u_0 \in D$ with $\nabla \cdot u_0 = 0$ where D is a precompact set in $\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3)$. Then, there exist $\tilde{\Omega}(D) > 0$ and a unique global solution u to (1.1) in the class $C([0,\infty); \dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^3)) \cap L^4(0,\infty; \dot{B}^{\frac{1}{3}}_{3,q}(\mathbb{R}^3))$ provided that $|\Omega| \ge \tilde{\Omega}(D)$.

Proof. In view of Lemma 3.5, we have that $\dot{B}_{2,q}^{\frac{1}{2}} \hookrightarrow \mathcal{F}_0$ for $1 \leq q < 4$. Now the result follows by applying Theorem 4.3.

5 Asymptotic behavior as $|\Omega| \to \infty$

In this section we study the asymptotic behavior of the mild solutions as $|\Omega| \to \infty$. For convenience, we denote

$$\alpha_0 = -\frac{1}{\theta} + \frac{1}{2} - \frac{3}{2p} + \frac{s}{2}$$
 and $\beta_0 = \frac{1}{\theta} - \frac{3}{4} + \frac{3}{2p}$.

First, we consider the case 1/2 < s < 3/4.

Theorem 5.1. (i) Let $0 \le \epsilon < \frac{1}{12}$ and $1 \le q < \infty$, and suppose that s, p and θ satisfy

$$\frac{1}{2} + 3\epsilon < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3},$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4} - \frac{\epsilon}{4}, \quad \frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < \min\left\{1 - \frac{2}{p}, \frac{1}{q}\right\}.$$

Let u and v be solutions of (1.1) with initial data u_0 and v_0 in $\dot{B}^s_{2,q}(\mathbb{R}^3)$, respectively. Then, for $\alpha < 2\beta_0$

$$\lim_{|\Omega|\to\infty} |\Omega|^{\alpha} ||u(t)-v(t)||_{\dot{B}^{s+\epsilon}_{2,q}} = 0 \quad \text{if and only if} \quad \lim_{|\Omega|\to\infty} |\Omega|^{\alpha} ||T_{\Omega}(t)(u_0-v_0)||_{\dot{B}^{s+\epsilon}_{2,q}} = 0, \text{ for each fixed } t > 0. \tag{5.1}$$

(ii) Let $0 \le \epsilon < \frac{1}{6}$ and $1 \le q < \infty$. Assume that s, p and θ satisfy

$$\frac{1}{2} + \frac{3\epsilon}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3},$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < \min\left\{1 - \frac{2}{p}, \frac{1}{q}\right\}.$$

Let $\alpha < \alpha_0 + 2\beta_0 - \frac{\epsilon}{2}$ and assume that u and v are solutions of (1.1) with initial data u_0 and v_0 in $\dot{B}^s_{2,q}(\mathbb{R}^3)$, respectively. Then, for each fixed t > 0,

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} ||u - v||_{L^{\theta}(0,\infty; \dot{B}^{s+\epsilon}_{p,q})} = 0 \quad \text{if and only if} \quad \lim_{|\Omega| \to \infty} |\Omega|^{\alpha} ||T_{\Omega}(t)(u_0 - v_0)||_{L^{\theta}(0,\infty; \dot{B}^{s+\epsilon}_{p,q})} = 0. \quad (5.2)$$

Proof. First we write

$$u - v = T_{\Omega}(t)(u_0 - v_0) + \mathfrak{B}(u, u)(t) - \mathfrak{B}(v, v)(t). \tag{5.3}$$

Considering $\frac{1}{r} = \frac{2}{p} - \frac{s}{3}$, we estimate the $\dot{B}_{2,q}^{s+\epsilon}$ -norm of the nonlinear term in (5.3) as follows

$$\begin{split} \|\mathfrak{B}(u,u)(t) - \mathfrak{B}(v,v)(t)\|_{\dot{B}^{s+\epsilon}_{2,q}} &\leq C \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2}\right)} \|e^{\frac{1}{2}(t-\tau)\Delta} (u \otimes (u-v) + (u-v) \otimes v)(\tau)\|_{\dot{B}^{s+\epsilon}_{p,q}} \, d\tau \\ &\leq C \int_{0}^{t} (t-\tau)^{-\frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2}\right) - \frac{\epsilon}{2}} (\|u(\tau)\|_{\dot{B}^{s}_{p,q}} + \|u(\tau)\|_{\dot{B}^{s}_{p,q}}) \|(u-v)(\tau)\|_{\dot{B}^{s}_{p,q}} \, d\tau \\ &\leq C t^{\frac{1}{2} - \frac{2}{\theta} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2}\right) - \frac{\epsilon}{2}} (\|u\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})} + \|v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}) \|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}, \end{split}$$

where we have the integrability at $\tau = t$ due to the condition

$$\frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4} - \frac{\epsilon}{4} \Longrightarrow \frac{1}{2} - \frac{2}{\theta} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{2} \right) - \frac{\epsilon}{2} > 0.$$

Thus,

$$\|\Omega\|^{\alpha}\|\mathfrak{B}(u,u)(t)-\mathfrak{B}(v,v)(t)\|_{\dot{B}^{s+\epsilon}_{2,a}}\leq Ct^{\frac{1}{2}-\frac{2}{\theta}-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{\epsilon}{2}}\|\Omega\|^{\alpha-2\beta_{0}}, \text{ for all } |\Omega|\geq\Omega_{0}.$$

Since $\alpha < 2\beta_0$, it follows that

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} \|\mathfrak{B}(u, u)(t) - \mathfrak{B}(v, v)(t)\|_{\dot{B}^{s+\epsilon}_{2,q}} = 0, \text{ for each } t > 0.$$
 (5.4)

In view of (5.3) and (5.4), we obtain the desired property.

For item (ii), we proceed similarly as in the proof of Lemma 3.6 by taking $f=u\otimes(u-v)+(u-v)\otimes v$ in the nonlinear term of (5.3). Since $\frac{1}{\theta}<\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{\epsilon}{2}$, we can estimate

$$\|\mathfrak{B}(u,u)-\mathfrak{B}(v,v)\|_{L^{\theta}(0,\infty;\dot{B}^{s+\epsilon}_{p,q})}\leq C|\Omega|^{-\alpha_{0}+\frac{\epsilon}{2}}(\|u\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}+\|v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})})\|u-v\|_{L^{\theta}(0,\infty;\dot{B}^{s}_{p,q})}$$

and then

$$|\Omega|^{\alpha} \|\mathfrak{B}(u,u) - \mathfrak{B}(v,v)\|_{L^{\theta}(0,\infty;\dot{B}^{s+\epsilon}_{p,q})} \le C|\Omega|^{\alpha-\alpha_0+\frac{\epsilon}{2}-2\beta_0}, \text{ for all } |\Omega| \ge \Omega_0.$$
(5.5)

Finally, we obtain (5.2) by letting $|\Omega| \to \infty$ and using (5.5) and (5.3).

Remark 5.2. Let $1 \le q < \infty$, and consider s, γ_2, p and θ such that

$$\frac{1}{2} < s < \frac{3}{4}, \quad 0 < \gamma_2 < \frac{1}{2} \left(1 - \frac{1}{\epsilon} \right), \quad \frac{1}{2\gamma_2} \left(\frac{1}{8} - \frac{s}{4} + \gamma_2 \right) \le \frac{1}{p} < \frac{2}{3} - \frac{s}{3}, \tag{5.6}$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < 1 - \frac{2}{p}. \tag{5.7}$$

 \Diamond

Since $\frac{1}{\theta} < \frac{1}{2} - \frac{3}{2} \left(\frac{1}{r} - \frac{1}{p} \right) - \gamma_2 \left(1 - \frac{2}{p} \right)$ and

$$\left(\frac{\log(e+|\Omega|t)}{1+|\Omega|t}\right)^{\frac{1}{2}\left(1-\frac{2}{p}\right)} = \frac{\log(e+|\Omega|t)^{\frac{1}{2}\left(1-\frac{2}{p}\right)}}{(1+|\Omega|t)^{\gamma_1\left(1-\frac{2}{p}\right)}} \frac{1}{(1+|\Omega|t)^{\gamma_2\left(1-\frac{2}{p}\right)}} \le (|\Omega|t)^{-\gamma_2\left(1-\frac{2}{p}\right)},$$

where $\gamma_1, \gamma_2 > 0$, $\gamma_1 + \gamma_2 = \frac{1}{2}$ and $\gamma_2 < \frac{1}{2} \left(1 - \frac{1}{e}\right)$, we can estimate (similarly to Lemma 3.6)

$$\|\mathfrak{B}(u,u) - \mathfrak{B}(v,v)\|_{L^{\theta}(0,\infty;\dot{B}^{s+\epsilon}_{p,q})} \le C|\Omega|^{-\alpha_0 - \gamma_2 \left(1 - \frac{2}{p}\right)} (\|u\|_{L^{\theta}(0,\infty;\dot{B}^s_{p,q})} + \|v\|_{L^{\theta}(0,\infty;\dot{B}^s_{p,q})}) \|u - v\|_{L^{\theta}(0,\infty;\dot{B}^s_{p,q})}$$

which implies

$$|\Omega|^{\alpha} \|\mathfrak{B}(u,u) - \mathfrak{B}(v,v)\|_{L^{\theta}(0,\infty;\dot{B}^{s+\epsilon}_{p,q})} \le C|\Omega|^{\alpha-\alpha_0-\gamma_2\left(1-\frac{2}{p}\right)-2\beta_0}.$$

Thus, for $\alpha < \alpha_0 + 2\beta_0 + \gamma_2 \left(1 - \frac{2}{p}\right)$, we obtain the property (5.2).

In what follows, we address the asymptotic behavior of solutions in the critical case (s = 1/2).

Theorem 5.3. Let $2 \le q \le \infty$ and let u and v be mild solutions of (1.1) with initial data u_0 and v_0 in \mathcal{F}_0 , respectively. Then, for all $\alpha \ge 0$

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} \|u - v\|_{L^{4}(0,\infty; \dot{B}^{\frac{1}{2}}_{3,q})} = 0 \quad \text{if and only if} \quad \lim_{|\Omega| \to \infty} |\Omega|^{\alpha} \|T_{\Omega}(t)(u_{0} - v_{0})\|_{L^{4}(0,\infty; \dot{B}^{\frac{1}{2}}_{3,q})} = 0. \tag{5.8}$$

Moreover, for each t > 0, we have that

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} ||u(t) - v(t)||_{\dot{B}_{2,a}^{\frac{1}{2}}} = 0$$
(5.9)

provided that

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} \left(\|T_{\Omega}(t)(u_0 - v_0)\|_{\dot{B}_{2,a}^{\frac{1}{2}}} + \|T_{\Omega}(t)(u_0 - v_0)\|_{L^4(0,\infty;\dot{B}_{3,a}^{\frac{1}{2}})} \right) = 0.$$
 (5.10)

Proof. By the proof of Theorem 4.3, we know that $u \in L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})$ with

$$||u||_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} \le 2C_1\delta$$
, for all $|\Omega| \ge \tilde{\Omega}$,

and similarly for v. Thus,

$$\sup_{|\Omega|>\tilde{\Omega}} \|u\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} < 2C_1\delta \text{ and } \sup_{|\Omega|>\tilde{\Omega}} \|v\|_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} < 2C_1\delta. \tag{5.11}$$

Next, we estimate

$$\|u-v\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \leq \|T_{\Omega}(t)(u_0-v_0)\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + C_2(\|u\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} + \|v\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})})\|u-v\|_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})}$$

which yields

$$(1 - 4C_1C_2\delta)|\Omega|^{\alpha}||u - v||_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})} \le |\Omega|^{\alpha}||T_{\Omega}(t)(u_0 - v_0)||_{L^4(0,\infty;\dot{B}_{3,q}^{\frac{1}{2}})},$$
(5.12)

where C_1, C_2 and δ are as in the proof of Theorem 4.3. Since $1 - 4C_1C_2\delta > 0$ and the term on the right side converges to zero (by hypothesis), it follows the "if" part in (5.8). For the reverse, we write (5.3) as

$$T_{\Omega}(t)(u_0 - v_0) = u - v - [\mathfrak{B}(u, u)(t) - \mathfrak{B}(v, v)(t)]$$
(5.13)

and proceed similarly.

Next, we turn to (5.9). Applying the $\dot{B}_{2,q}^{\frac{1}{2}}$ -norm and using Lemma 3.7, we obtain

$$||u(t) - v(t)||_{\dot{B}^{\frac{1}{2}}_{2,q}} \le ||T_{\Omega}(t)(u_0 - v_0)||_{\dot{B}^{\frac{1}{2}}_{2,q}} + C(||u||_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})} + ||v||_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})})||u - v||_{L^4(0,\infty;\dot{B}^{\frac{1}{2}}_{3,q})}, \quad (5.14)$$

for each t > 0. Multiplying (5.14) by $|\Omega|^{\alpha}$, letting $|\Omega| \to \infty$, and using (5.11), (5.10) and (5.8), we get (5.9).

 \Diamond

Remark 5.4. Notice that we can take $v_0 = 0$ and v = 0 in Theorems 5.1 and 5.3 and obtain asymptotic behavior properties for $u = u_{\Omega}$ as $|\Omega| \to \infty$. In particular, in Theorem 5.3, we have that

$$\lim_{|\Omega| \to \infty} |\Omega|^{\alpha} ||u_{\Omega}||_{L^{4}(0,\infty; \dot{B}^{\frac{1}{2}}_{3,q})} = 0 \text{ provided that } \lim_{|\Omega| \to \infty} |\Omega|^{\alpha} ||T_{\Omega}(t)u_{0}||_{L^{4}(0,\infty; \dot{B}^{\frac{1}{2}}_{3,q})} = 0. \tag{5.15}$$

In the case $\alpha = 0$, notice that the latter limit holds true for $u_0 \in \dot{B}^{1/2}_{2,q}(\mathbb{R}^3)$ with $1 \le q < 4$ (see Lemma 3.5) and for all $u_0 \in \mathcal{F}_0$.

References

- [1] M.F. de Almeida, L.C.F. Ferreira, L.S.M. Lima, Uniform global well-posedness of the Navier-Stokes-Coriolis system in a new critical space, Math. Z. 287 (3-4) (2017), 735–750.
- [2] A. Babin, A. Mahalov and B. Nicolaenko, Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids, Asymptot. Anal. 15 (2) (1997), 103–150.
- [3] A. Babin, A. Mahalov and B. Nicolaenko, Global regularity of 3D rotating Navier-Stokes equations for resonant domains, Indiana Univ. Math. J. 48 (3) (1999), 1133–1176.
- [4] A. Babin, A. Mahalov and B. Nicolaenko, 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity. Indiana Univ. Math. J. 50 (2001), Special Issue, 1–35.
- [5] H. Bahouri, J.- Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der mathematischen Wissenschaften, Vol. 343, Springer, 2011.
- [6] J. Berg and J. Löfström, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin-New York, 1976.
- [7] D. Chae, Local existence and blow-up criterion for the Euler equations in the Besov spaces, Asymptotic Analysis 38 (2004), 339–358.
- [8] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, Studies in Applied Mathematics, vol. 31, 171–192, North-Holland, Amsterdam, 2002.
- [9] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical Geophysics, Oxford Lecture Ser. Math. Appl., vol. 32, The Clarendon Press, Oxford University Press, Oxford, 2006.
- [10] Y. Giga, K. Inui, A. Mahalov and S. Matsui, Navier-Stokes equations in a rotating frame in \mathbb{R}^3 with initial data nondecreasing at infinity, Hokkaido Math. J. 35 (2) (2006), 321–364.

- [11] Y. Giga, K. Inui, A. Mahalov and J.Saal, Uniform global solvability of the rotation Navier-Stokes equations for nondecaying initial data, Indiana Univ. Math J. 57 (6) (2008), 2775–2791.
- [12] M. Hieber and Y. Shibata, The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework, Math. Z. 265 (2010), 481–491.
- [13] T. Iwabuchi and R. Takada, Dispersive effect of the Coriolis force for the Navier-Stokes equations in the rotational framework, RIMS Kôkyûroku Bessatsu B42 (2013), 137–152.
- [14] T. Iwabuchi and R. Takada, Global solutions for the Navier-Stokes equations in the rotational framework, Math. Ann. 357 (2) (2013), 727–741.
- [15] T. Iwabuchi and R. Takada, Global well-posedness and ill-posedness for the Navier-Stokes equations with the Coriolis force in function spaces of Besov type, J. Funct. Anal. 267 (5)(2014), 1321–1337.
- [16] T. Iwabuchi and R. Takada, Dispersive effect of the Coriolis force and the local well-posedness for the Navier-Stokes equations in the rotational framework, Funkcialaj Ekvacioj 58 (2015), 365–385.
- [17] T. Iwabuchi, A. Mahalov and R. Takada, Stability of time periodic solutions for the rotating Navier-Stokes equations, Adv. Math. Fluid Mech. (2016), 321–335.
- [18] P. Konieczny and T. Yoneda, On dispersive effect of the Coriolis force for the stationary Navier-tokes equations, J. Differential Equations 250 (2011), 3859–3873.
- [19] H. Kozono, T. Ogawa and Y. Tanuichi, Navier-Stokes equations in the Besov space near L^{∞} and BMO, Kyushu J. Math. 57 (2003), 303–324.
- [20] T. Yoneda, Long-time solvability of the Navier-Stokes equations in a rotating frame with spatially almost periodic large data, Arch. Rational Mech. Anal. 200 (2011), 225-237.