Stabilization of a piezoelectric system

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Abstract. We consider a stabilization problem for a piezoelectric system. We prove an exponential stability result under some Lions geometric condition. Our method is based on an identity with multipliers that allows to show an appropriate observability estimate.

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1 Introduction

We consider the dynamical behavior of a piezoelectric system (which means the ability of some materials, like ceramics and quartz, to generate an electric field in response to applied mechanical stress), where a proper modeling involves the displacement vector, the electric field and the magnetic field, which are governed by the elasticity system coupled with Maxwell's equations. This system plays an important role in various applications in structural mechanics and in mechatronics, for such a model we refer to [10, 15].

Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ. In that domain we consider the non-stationary piezoelectric system that consists in a coupling between the elasticity system with the Maxwell equation. More precisely we analyze the partial differential equations based on the following relations between the stress tensor, the electric displacement and the magnetic induction:

(1.1) $$\sigma_{ij}(u,E) = a_{ijkl}\gamma_{kl}(u) - e_{kij}E_k \; \forall\, i,j = 1,2,3,$$

(1.2) $$D_i = \varepsilon_{ij}E_j + e_{ikl}\gamma_{kl}(u) \,\forall\, i = 1,2,3$$

(1.3) $$B = \mu H.$$

The equations of equilibrium are

(1.4) $$\partial_t^2 u_i = \partial_j \sigma_{ji} \,\forall\, i = 1,2,3$$

for the elastic displacement and

(1.5) $$\partial_t D = \mathrm{curl} H,\; \partial_t B = -\mathrm{curl} E$$

for the electric/magnetic fields.

This system models the coupling between Maxwell's system and the elastic one, in which $E(x,t), H(x,t)$ are the electric and magnetic fields at the point $x \in \Omega$ at time t, $u(x,t)$ is

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the displacement field at the point $x \in \Omega$ at time t, and $\gamma_{ij}(u)_{i,j=1}^3$ is the strain tensor given by

$$\gamma_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Here $\sigma = (\sigma_{ij})_{i,j=1}^3, D = (D_1, D_2, D_3)$, and $B = (B_1, B_2, B_3)$ are the stress tensor, electric displacement, and magnetic induction, respectively. ε, μ are the electric permittivity and magnetic permeability, respectively, and we will assume that they are positive real numbers. The elasticity tensor $(a_{ijkl})_{i,j,k,l=1,2,3}$ is made of constant entries such that

$$a_{ijkl} = a_{jikl} = a_{klij}$$

and satisfies the ellipticity condition

$$a_{ijkl}\gamma_{ij}\gamma_{kl} \geq \alpha_0\gamma_{ij}\gamma_{ij}, \tag{1.6}$$

for every symmetric tensor (γ_{ij}) and some $\alpha_0 > 0$. The piezoelectric tensor e_{kij} is also made of constant entries such that

$$e_{kij} = e_{kji}.$$

For shortness in the remainder of the paper introduce the tensor $\sigma(u) = (a_{ijkl}\gamma_{kl}(u))_{i,j=1}^3$ and let $\nabla\sigma$ be the vector field defined by

$$\nabla\sigma = (\partial_j\sigma_{ij})_{i=1}^3,$$

while for a tensor $\gamma = (\gamma_{ij})_{i,j=1}^3$, and a vector $F = (F_1, F_2, F_3)$, we set

$$e\gamma = (e_{ikl}\gamma_{kl})_{i=1}^3 \quad e^\top F = (e_{ikl}F_i)_{k,l=1}^3.$$

These last notations mean that e corresponds to a linear mapping from $\mathbb{R}^{3\times3}$ into $\mathbb{R}^3$ and that $e^\top$ is its adjoint. With these notations, we see that (1.1) is equivalent to

$$\sigma(u, E) = \sigma(u) - e^\top E,$$

while (1.2) is equivalent to

$$D = \varepsilon E + e\gamma(u).$$

The system (1.1)-(1.3) is completed with the boundary and Cauchy conditions. This means that we are considering the following system

$$\begin{cases} \partial_t^2 u - \nabla\sigma(u, E) = 0 \text{ in } Q := \Omega\times]0, +\infty[, \\ \partial_t D - \operatorname{curl} H = 0 \text{ in } Q, \\ \mu\partial_t H + \operatorname{curl} E = 0 \text{ in } Q, \\ \operatorname{div}(D) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ H \times \nu - (Q^*\partial_t u) \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Sigma = \Gamma \times (0, +\infty), \\ \sigma(u, E) \cdot \nu + Q(E \times \nu) + Au + \partial_t u = 0 \text{ on } \Sigma, \\ u(0) = u_0,\ \partial_t u(0) = u_1 \text{ in } \Omega, \\ E(0) = E_0,\ H(0) = H_0 \text{ in } \Omega, \end{cases} \tag{1.7}$$

where ν is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω, A is a positive constant and Q is a function from Γ into the set of 3×3 matrices with the regularity $Q \in L^\infty(\Gamma, \mathbb{C}^{3\times3})$.

Remark 1.1 Note that the image of Q of normal vector fields plays no role in the boundary conditions appearing in (1.7). Indeed for $X \in \mathbb{C}^3$, let $X_\nu = (X \cdot \nu)\nu$ and $X_\tau = X - X_\nu$ be the normal and tangential components of X respectively, by writing

$$Q_\nu X = QX_\nu, \quad Q_\tau X = QX_\tau,$$

we get the splitting

$$Q = Q_\nu + Q_\tau, \quad Q^* = Q_\nu^* + Q_\tau^*.$$

But by definition $Q_\nu^* X$ is orthogonal to the tangent plane. Therefore $Q(E \times \nu) = Q_T(E \times \nu)$ and $(Q^* \partial_t u) \times \nu = (Q_\tau^* \partial_t u) \times \nu$, which means that the normal part Q_ν of Q does not contribute to the boundary conditions.

Boundary or internal stability of the second order elliptic systems, like the wave equation or the elasticity system, have been studied by many authors, let us quote [5, 2, 4, 12, 14, 21, 23, 24, 27, 28] among others. Similar results for Maxwell's system can be found in [16, 20, 22, 21, 9, 31, 25, 34]. The combination of these results to the piezoelectric system in some particular cases has been treated in [17, 18, 32]. For the quasi-static case (corresponding to the hypothesis that E is curl free, hence the gradient of a potentiel), we can refer to [19, 29, 26].

In [18] the authors consider the above problem in the case $Q = 0$ with eventually discontinuous coefficients and an additional memory term and prove the exponential decay rate of the energy if A is small enough and if Ω satisfies some geometrical conditions (star like shape). On the contrary in [32], the author treats the case $Q = I$ and some nonlinear feedback terms, but with the choice of e such that $\nabla(e^\top E) = \xi \mathrm{curl} E$ for some real number ξ (case excluding the natural condition $e_{kij} = e_{kji}$) and proves the exponential decay rate of the energy in the case of linear feedbacks if Ω is strictly star shaped with respect to a point. In that last paper the author combines the multiplier technique with the one from [4], where the authors uses some tangential integration by parts and a technique from [6]. Our goal is here to perform the same analysis for the general system (1.7). For $Q \in L^\infty(\Gamma, \mathbb{C}^{3\times 3})$ we prove that the system is well-posed using semigroup theory. On the other hand using the multiplier method (see [18]) and a technique inspired from [2, 6, 13, 32] to absorb a zero order boundary term, we show that the system is exponential stable if $Q = \alpha I$ for some scalar continuously differentiable function α such that $\nabla \alpha$ is small enough.

The paper is organized as follows. The second section deals with the well-posedness of the problem. In the last section we give the main result of this paper which is the exponential stability of the piezoelectric system and its proof.

2 Well-posedness of the problem

We start this section with the well-posedness of problem (1.7). At the end we will check the dissipativeness of (1.7).

Let us introduce the Hilbert spaces (see e.g. [25, 30])

$$J(\Omega) = \{E \in L^2(\Omega)^3 | \mathrm{div} E = 0 \text{ in } \Omega\}, \tag{2.8}$$

$$\mathcal{H} = H^1(\Omega)^3 \times L^2(\Omega)^3 \times L^2(\Omega)^3 \times J(\Omega), \tag{2.9}$$

equipped with the norm induced by the inner product

$$\begin{aligned}
&(E, E')_\varepsilon = \int_\Omega \varepsilon\, E(x) \cdot E'(x)\, dx, \forall E, E' \in J(\Omega),\\
&((u, v, E, H), (u', v', E', H'))_{\mathcal{H}} = (u, u')_1 + (v, v')_0\\
&+(E, E')_\varepsilon + (H, H')_\mu, \forall (u, v, E, H), (u', v', E', H') \in \mathcal{H},
\end{aligned}$$

where we have set

$$(u,u')_0 = \int_\Omega u(x)\cdot u'(x)\,dx,$$
$$(u,u')_1 = \int_\Omega \sigma(u)(x) : \gamma(u')(x)\,dx + A\int_\Gamma u(x)\cdot u'(x)\,dS,$$

with the notation

$$\sigma(v):\gamma(v') := \sigma_{ij}(v)\gamma_{ij}(v').$$

Now define the linear operator $\mathcal{A}$ from $\mathcal{H}$ into itself as follows:

$$D(\mathcal{A}) = \{(u,v,E,H)\in\mathcal{H} | \nabla\sigma(u,E), \mathrm{curl}E, \mathrm{curl}H \in L^2(\Omega)^3; v\in H^1(\Omega)^3; \tag{2.10}$$
$$E\times\nu, H\times\nu \in L^2(\Gamma)^3 \text{ satisfying}$$
$$H\times\nu - (Q^*v)\times\nu + E\times\nu\times\nu = 0 \text{ on } \Gamma, \tag{2.11}$$
$$\sigma(u,E)\cdot\nu + Au + v + Q(E\times\nu) = 0 \text{ on } \Gamma\}. \tag{2.12}$$

For all $(u,v,E,H)\in D(\mathcal{A})$ we take

$$\mathcal{A}(u,v,E,H) = \left(v, \nabla\sigma(u,E), \varepsilon^{-1}(\mathrm{curl}H - e\gamma(v)), -\mu^{-1}\mathrm{curl}E\right).$$

The boundary conditions (2.11) and (2.12) are meaningful since for $(u,v,E,H)\in D(\mathcal{A})$, from section 2 of [3] the property $\nabla\sigma(u,E)\in L^2(\Omega)^3$ implies that $\sigma(u,E)\cdot\nu$ belongs to $H^{-1/2}(\Gamma)^3$. Since the properties $u,v\in H^1(\Omega)^3$ imply that $Au+v$ belongs to $H^{1/2}(\Gamma)^3$, the boundary condition (2.12) has a meaning (in $H^{-1/2}(\Gamma)^3$) and furthermore yields $\sigma(u,E)\cdot\nu\in L^2(\Gamma)^3$ (because $Q(E\times\nu)\in L^2(\Gamma)^3$). Similarly the properties of H and v give a meaning to the boundary condition (2.11) (as an equality in $L^2(\Gamma)^3$). In summary both boundary conditions (2.11) and (2.12) have to be understood as an equality in $L^2(\Gamma)^3$.

We now see that formally problem (1.7) is equivalent to

$$\left\{\begin{array}{l} \frac{\partial U}{\partial t} = \mathcal{A}U, \\ U(0) = U_0, \end{array}\right. \tag{2.13}$$

when $U=(u,\partial_t u,E,H)$ and $U_0=(u_0,u_1,E_0,H_0)$.

We shall prove that this problem (2.13) has a unique solution using semigroup theory by showing that $\mathcal{A}$ is a maximal dissipative operator.

Lemma 2.1 *$\mathcal{A}$ is a maximal dissipative operator.*

Proof: We start with the dissipativeness:

$$(\mathcal{A}U,U)_\mathcal{H} \le 0, \forall U\in D(\mathcal{A}).$$

From the definition of $\mathcal{A}$ and the inner product in $\mathcal{H}$, we have

$$(\mathcal{A}U,U)_\mathcal{H} = (v,u)_1 + (\nabla\sigma(u,E),v)_0$$
$$+\int_\Omega\{E\cdot(\mathrm{curl}H - e\gamma(v)) - \mathrm{curl}E\cdot H\}\,dx,$$

for any $(u,v,E,H)\in D(\mathcal{A})$. Lemma 2.2 of [31] and Green's formula yield equivalently

$$(\mathcal{A}U,U)_\mathcal{H} = (v,u)_1 - \int_\Omega \sigma(u,E):\gamma(v)\,dx$$
$$-\int_\Omega e\gamma(v)\cdot E\,dx$$
$$+\int_\Gamma\{(\sigma(u,E)\cdot\nu)\cdot v + (E\times\nu)\cdot H\}\,dS,$$

for any $(u, v, E, H) \in D(\mathcal{A})$. Using the definition of the inner product $(\cdot,\cdot)_1$ and the boundary conditions (2.11) and (2.12), we arrive at

$$(\mathcal{A}U, U)_{\mathcal{H}} = -\int_\Gamma \{|v|^2 + |E \times \nu|^2\}\, dS \leq 0,$$

for any $(u, v, E, H) \in D(\mathcal{A})$.

Let us now pass to the maximality. This means that for at least one non negative real number λ, $\lambda I - \mathcal{A}$ has to be surjective. Let us show that indeed $I - \mathcal{A}$ is surjective. This means that for all (f, g, F, G) in $\mathcal{H}$, we are looking for (u, v, E, H) in $D(\mathcal{A})$ such that

$$(I - \mathcal{A})\,(u, v, E, H) = (f, g, F, G)\,. \tag{2.14}$$

From the definition of $\mathcal{A}$, this equivalently means

$$\begin{cases} u - v = f, \\ v - \nabla\sigma(u, E) = g, \\ E - \varepsilon^{-1}(\mathrm{curl} H - e\gamma(v)) = F, \\ H + \mu^{-1}\mathrm{curl} E = G. \end{cases} \tag{2.15}$$

The first and fourth equations allow to eliminate H and v, since they are respectively equivalent to

$$v = u - f, \tag{2.16}$$

$$H = G - \mu^{-1}\mathrm{curl} E. \tag{2.17}$$

Substituting these expressions in the second and third equations yields formally

$$u - \nabla\sigma(u, E) = f + g, \tag{2.18}$$

$$\varepsilon E + \mathrm{curl}(\mu^{-1}\mathrm{curl} E) + e\gamma(u) = \varepsilon F + \mathrm{curl} G + e\gamma(f). \tag{2.19}$$

This system in (u, E) will be uniquely defined by adding boundary conditions on u and E. Indeed using the identities (2.16) and (2.17), we see that (2.11) and (2.12) are formally equivalent to

$$-\mu^{-1}\mathrm{curl} E \times \nu + Q^* u \times \nu + (E \times \nu) \times \nu = -G \times \nu + Q^* f \times \nu \text{ on } \Gamma, \tag{2.20}$$

$$\sigma(u, E) \cdot \nu + Au + u + Q(E \times \nu) = f \text{ on } \Gamma. \tag{2.21}$$

By formal integration by parts we remark that the variational formulation of the system (2.18)-(2.19) with the boundary conditions (2.20)-(2.21) is the following one: Find $(u, E) \in V$ such that

$$a((u, E), (u', E')) = F(u', E'), \forall (u', E') \in V, \tag{2.22}$$

where the Hilbert space V is given by $V = H^1(\Omega)^3 \times W$ when W is defined by

$$W = \{E \in L^2(\Omega)^3 | \mathrm{curl} E \in L^2(\Omega)^3 \text{ and } E \times \nu \in L^2(\Gamma)^3\},$$

with the norm

$$||E||_W^2 = \int_\Omega (|E|^2 + |\mathrm{curl} E|^2) dx + \int_\Gamma |E \times \nu|^2\, dS,$$

the form a is defined by

$$\begin{aligned} a((u, E), (u', E')) \quad &= \quad \int_\Omega \{\sigma(u, E) : \gamma(u') + u \cdot u'\}\, dx \\ &+ \quad \int_\Omega \{\mu^{-1}\mathrm{curl} E \cdot \mathrm{curl} E' + \varepsilon E \cdot E' + e\gamma(u) \cdot E'\}\, dx \\ &+ \quad \int_\Gamma \{(E \times \nu) \cdot (E' \times \nu) + (A + 1)u \cdot u' + Q(E \times \nu) \cdot u' - Q^* u \cdot (E' \times \nu)\}\, dS, \end{aligned}$$

and finally the form F is defined by

$$F(u',E') = \int_\Omega \{(f+g)\cdot u' + (\varepsilon F + e\gamma(f))\cdot E' + G\cdot \mathrm{curl}E'\}\, dx + \int_\Gamma (f\cdot u' - (Q^* f\times\nu)\cdot E')\, dS.$$

We easily see that the bilinear form a is coercive on V since

$$\begin{aligned} a((u,E),(u,E)) &= \int_\Omega \{\sigma(u):\gamma(u) + |u|^2\}\, dx \\ &+ \int_\Omega \{\mu^{-1}|\mathrm{curl}E|^2 + \varepsilon|E|^2\}\, dx \\ &+ \int_\Gamma \{|E\times\nu|^2 + (A+1)|u|^2\}\, dS, \end{aligned}$$

which is clearly greater than $||u||^2_{H^1(\Omega)^3} + ||E||^2_W$ by the ellipticity assumption on the elasticity tensor. Hence by the Lax-Milgram lemma, problem (2.22) has a unique solution $(u,E)\in V$.

To end our proof we need to show that the solution $(u,E)\in V$ of (2.22) and v, H given respectively by (2.16), (2.17) are such that (u,v,E,H) belongs to $D(\mathcal{A})$ and satisfies (2.14) (or equivalently (2.15)). First taking test functions u' in $\mathcal{D}(\Omega)^3$ and $E'=0$, we get

$$\nabla\sigma(u,E) + v = g \text{ in } \mathcal{D}'(\Omega).$$

This implies the second identity in (2.15) as well as the regularity $\nabla\sigma(u,E)\in L^2(\Omega)^3$ (from the fact that $v, \mathrm{curl}E$ as well as g belongs to that space).

Second we take test functions $u'=0$ and $E'=\chi$ with $\chi\in\mathcal{D}(\Omega)^3$ by Lemma 2.3 of [31] we get

$$\varepsilon E - \mathrm{curl}H + e\gamma(u) = \varepsilon F \text{ in } \mathcal{D}'(\Omega).$$

This means that the third identity in (2.15) holds as well as the regularity $\mathrm{curl}H\in L^2(\Omega)^3$.

Thirdly taking test functions $v'\in H^1(\Omega)^3$ and $E'=\chi$ with $\chi\in C^\infty(\bar\Omega)^3$ and applying Green's formula (see section 2 of [3] and Lemma 2.2 of [31]), we get

$$\begin{aligned} &\langle\sigma(u,E)\cdot\nu, v'\rangle - \int_\Gamma (H\times\nu)\cdot E'\, dS + \int_\Gamma (Q(E\times\nu)\cdot u' - (Q^* u\times\nu)\cdot E'\, dS \\ &+ \int_\Gamma \{(E\times\nu)\cdot(E'\times\nu) + (A+1)u\cdot u'\}\, dS = 0. \end{aligned}$$

This leads to the boundary conditions (2.11) and (2.12) since u' (resp. χ) was arbitrary in $H^1(\Omega)^3$ (resp. in $C^\infty(\bar\Omega)^3$) whose trace belongs to a dense subspace of $L^2(\Gamma)^3$.

Finally from (2.17) and the fact that μG is divergence free, μH is also divergence free.

■

Semigroup theory [33, 36] allows to conclude the following existence results:

Corollary 2.2 *For all $(u_0,u_1,E_0,H_0)\in\mathcal{H}$, the problem (1.7) admits a unique (weak) solution (u,E,H) satisfying $(u,\partial_t u,E,H)\in C(\mathbb{R}_+,\mathcal{H})$, or equivalently $u\in C^1(\mathbb{R}_+,L^2(\Omega)^3)\cap C(\mathbb{R}_+,H^1(\Omega)^3)$, $E\in C(\mathbb{R}_+,L^2(\Omega)^3)$ and $H\in C(\mathbb{R}_+,J(\Omega))$. If moreover (u_0,u_1,E_0,H_0) belongs to $D(\mathcal{A})$ and satisfies*

$$\mathrm{div}(e\gamma(u_0)+\varepsilon E_0) = 0 \textit{ in } \Omega,$$

then the problem (1.7) admits a unique (strong) solution (u,E,H) satisfying $(u,\partial_t u,E,H)\in C^1(\mathbb{R}_+,\mathcal{H})\cap C(\mathbb{R}_+,D(\mathcal{A}))$, or equivalently satisfying $u\in C^2(\mathbb{R}_+,L^2(\Omega)^3)\cap C^1(\mathbb{R}_+,H^1(\Omega)^3)$, $E\in C^1(\mathbb{R}_+,J(\Omega))\cap C(\mathbb{R}_+,W)$, $H\in C^1(\mathbb{R}_+,J(\Omega))\cap C(\mathbb{R}_+,W)$, satisfying (2.11)-(2.12) for a.e. t (with $v=\partial_t u$), as well as

$$\nabla\sigma(u,E)\in C(\mathbb{R}_+,L^2(\Omega)^3).$$

Note that, in that last case, $D = e\gamma(u) + \varepsilon E$ satisfies in particular

$$\operatorname{div} D = 0 \text{ in } \Omega \times \mathbb{R}_+.$$

We finish this section by showing the dissipativeness of our system.

Lemma 2.3 *The energy*

$$\begin{aligned}(2.23)\quad \mathcal{E}(t) &= \frac{1}{2}\int_\Omega (|\partial_t u(x,t)|^2 + \sigma(u)(x,t) : \gamma(u)(x,t))\, dx + \frac{A}{2}\int_\Gamma |u(x,t)|^2\, dS(x) \\ &+ \frac{1}{2}\int_\Omega (\varepsilon|\mathcal{E}(x,t)|^2 + \mu|H(x,t)|^2)\, dx\end{aligned}$$

is non-increasing. Moreover for $(u_0, u_1, E_0, H_0) \in D(\mathcal{A})$, we have for all $0 \le S < T < \infty$

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_\Gamma \{|E(x,t)\times\nu|^2 + |\partial_t u(x,t)|^2\}\, dSdt, \tag{2.24}$$

and for all $t \ge 0$

$$\partial_t \mathcal{E}(t) = -\int_\Gamma \{|E(x,t)\times\nu|^2 + |\partial_t u(x,t)|^2\}\, dS. \tag{2.25}$$

Proof: Since $D(\mathcal{A})$ is dense in $\mathcal{H}$ it suffices to show (2.25). For $(u_0, u_1, E_0, H_0) \in D(\mathcal{A})$, from the regularity of u, E, H, we have

$$\begin{aligned}\partial_t \mathcal{E}(t) &= \int_\Omega \{\partial_t^2 u \cdot \partial_t u + \sigma(u) : \gamma(\partial_t u)\}\, dx + A\int_\Gamma \partial_t u \cdot u\, dS \\ &+ \int_\Omega \{\varepsilon E \cdot \partial_t E + \mu H \cdot \partial_t H\}\, dx.\end{aligned}$$

By (1.7), we get

$$\begin{aligned}\partial_t \mathcal{E}(t) &= \int_\Omega \{\partial_t u \cdot \nabla\sigma(u,E) + \sigma(u) : \gamma(\partial_t u)\}\, dx + A\int_\Gamma \partial_t u \cdot u\, dS \\ &+ \int_\Omega \{E \cdot (\operatorname{curl} H - E\gamma(\partial_t u)\}\, dx \\ &= \left(A\left(u(t), \partial_t u(t), E(t), H(t)\right), \left(u(t), \partial_t u(t), E(t), H(t)\right)\right)_{\mathcal{H}}.\end{aligned}$$

We conclude by Lemma 2.1. ■

3 Exponential stability

In this section we prove the main result of this paper, namely the exponential stability of our system (1.7) when Ω is strictly star-shaped with respect to a point x_0. This result is based on an identity with multipliers proved in [18] that allows to show the next observability estimate.

Theorem 3.1 *Assume that there exists $x_0 \in \mathbb{R}^n$ and $\delta > 0$ such that*

$$m(x)\cdot\nu(x) \ge \delta \quad \forall x \in \partial\Omega, \tag{3.26}$$

where $m(x) = x - x_0$. Assume also that $Q = \alpha I$ with a continuously differentiable function α from Γ to $\mathbb{C}$. Set $c_\alpha = \max_\Omega |\nabla\alpha|$. Let (u, E, H) be the strong solution of problem (1.7). Then there exists a positive constants C (independent of α) such that for all $T > 0$, and all

θ, there exists a constant $C(\theta)$ (independent of T) such that the next observability estimate holds:

$$T\mathcal{E}(T) \leq \big(C(\theta)(1+c_\alpha T)+\theta T\big)\mathcal{E}(0) + C\int_{\Sigma_T} (|\partial_t u|^2 + |E_\tau|^2)\, dSdt, \tag{3.27}$$

where $\Sigma_T = \Gamma \times (0,T)$.

Proof: First the identity (3.9) of [18] with $t_0 = 0$ and $\varphi(x) = |x-x_0|^2/2$ yields

$$T\mathcal{E}(T) = r + \int_{\Sigma_T} V(x,t)\, dS(x)dt, \tag{3.28}$$

where we have set

$$\begin{aligned}
r &= -2\int_\Omega \Big\{\partial_t u \cdot \{u + (m\cdot\nabla)u)\} + \mu(m\times H)\cdot\{\varepsilon E + e\gamma(u)\}\Big\}\, dx\Big|_0^T,\\
V &= 2\{t\partial_t u + (m\cdot\nabla)u + u\}\cdot\sigma(u,E)\nu + m\cdot\nu\{|\partial_t u|^2 - \sigma(u):\gamma(u) + \varepsilon|E|^2 + \mu|H|^2\}\\
&+ 2t(H\times E)\cdot\nu - 2\varepsilon E\cdot\nu E\cdot m - 2\mu H\cdot\nu H\cdot m\\
&- 2(m\times e\gamma(u))\cdot(E\times\nu).
\end{aligned}$$

Using the boundary conditions from (1.7), we see that

$$\begin{aligned}
V &= -2t\partial_t u(Q(E\times\nu) + Au + \partial_t u) + \Delta\\
&+ m\cdot\nu\{\varepsilon|E|^2 + \mu|H|^2\}\\
&- 2t(Q^*\partial_t u)\times\nu)\cdot E - 2t|E_\tau|^2 - 2\varepsilon E_\nu(E_\nu m\cdot\nu + E_\tau\cdot m_\tau) - 2\mu H_\nu(H_\nu m\cdot\nu + H_\tau\cdot m_\tau)\\
&- 2(m\times e\gamma(u))\cdot(E\times\nu),
\end{aligned}$$

where we recall that $E_\nu = E\cdot\nu$, $E_\tau = E - E_\nu\nu$ and

$$\Delta = 2\{(m\cdot\nabla)u + u\}\cdot\sigma(u,E)\nu + m\cdot\nu\{|\partial_t u|^2 - \sigma(u):\gamma(u)\}.$$

By Young's inequality, there exists $C>0$ such that for all $\beta_1,\beta_2>0$

$$\begin{aligned}
V &\leq -2Atu\partial_t u\\
&- (m\cdot\nu - \beta_2)(\varepsilon|E_\nu|^2 + \mu|H_\nu|^2)\\
&+ \Delta + C(1+\frac{1}{\beta_2}+\frac{1}{\beta_1})|E\times\nu|^2 + C(1+\frac{1}{\beta_2})|H\times\nu|^2 + \beta_1\gamma(u):\gamma(u).
\end{aligned}$$

By using again the first boundary condition from (1.7), we get for all $\beta_1,\beta_2>0$

$$\begin{aligned}
V &\leq -2Atu\partial_t u + C(1+\frac{1}{\beta_2})|\partial_t u|^2 + C(1+\frac{1}{\beta_2}+\frac{1}{\beta_1})|E\times\nu|^2\\
&- (m\cdot\nu-\beta_2)(\varepsilon|E_\nu|^2 + \mu|H_\nu|^2) + \Delta + \beta_1\gamma(u):\gamma(u).
\end{aligned} \tag{3.29}$$

Let us transform the first term of this right-hand side:

$$-2A\int_{\Sigma_T} tu\partial_t u\, dSdt = -A\int_{\Sigma_T} t\frac{d}{dt}u^2\, dSdt,$$

and by an integration by parts in time, we get

$$-2A\int_{\Sigma_T} tu\partial_t u\, dSdt = A\int_{\Sigma_T} u^2\, dSdt - A\int_\Omega tu^2\, dx\Big|_0^T.$$

This proves that

$$-2A\int_{\Sigma_T} tu\partial_t u\, dSdt \leq A\int_{\Sigma_T} u^2\, dSdt. \tag{3.30}$$

Let us now estimate the term Δ. First using the second boundary condition from (1.7), we see that

$$\Delta = -2\{(m\cdot\nabla)u + u\}\cdot(Q(E\times\nu) + Au + \partial_t u) + m\cdot\nu\{|\partial_t u|^2 - \sigma(u) : \gamma(u)\}.$$

Using the ellipticity assumption (1.6) and condition (3.26) we obtain

$$\begin{aligned}
\Delta &\le -2\{(m\cdot\nabla)u + u\}\cdot(Q(E\times\nu) + Au + \partial_t u) + m\cdot\nu|\partial_t u|^2 - \alpha_0\delta\gamma(u) : \gamma(u) \\
&\le -2u\cdot Q(E\times\nu) - 2A|u|^2 - 2u\cdot\partial_t u - 2(m\cdot\nabla)u\cdot Q(E\times\nu) \\
&- 2A(m\cdot\nabla)u\cdot u - 2(m\cdot\nabla)u\cdot\partial_t u + m\cdot\nu|\partial_t u|^2 - \alpha_0\delta\gamma(u) : \gamma(u).
\end{aligned} \tag{3.31}$$

We need to estimate some terms of this right-hand side. First as before an integration by parts in time yields

$$-2\int_{\Sigma_T} u\partial_t u\, dSdt \le A\int_\Gamma |u(x,t=0)|^2\, dS(x) \le 2\mathcal{E}(0).$$

As in [4, 13], one can show that

$$-2A\int_{\Sigma_T}(m\cdot\nabla)u\cdot u\, dSdt \le \frac{C}{\theta_1}\int_{\Sigma_T}|u|^2\, dSdt + \theta_1\int_{\Sigma_T}\gamma(u) : \gamma(u)\, dSdt, \tag{3.32}$$

as well as

$$\int_{\Sigma_T}(m\cdot\nabla)u\cdot\partial_t u\, dSdt \le C\mathcal{E}(0) + \frac{C}{\theta_2}\int_{\Sigma_T}(|u|^2 + |\partial_t u|^2)\, dSdt + \theta_2\int_{\Sigma_T}\gamma(u) : \gamma(u)\, dSdt, \forall\theta_1,\theta_2 > 0. \tag{3.33}$$

By Young's inequality we clearly have

$$\int_{\Sigma_T} u\cdot Q(E\times\nu)\, dSdt \le C\int_{\Sigma_T}(|u|^2 + |E_\tau|^2)\, dSdt. \tag{3.34}$$

Now we notice that

$$(m\cdot\nabla)u\cdot Q(E\times\nu) = (Q^*(m\cdot\nabla)u)\cdot(E\times\nu),$$

and for any $k = 1,2,3$, we may write

$$\begin{aligned}
(Q^*(m\cdot\nabla)u)_k &= Q^*_{kj} m_i\partial_i u_j \\
&= 2Q^*_{kj} m_i\gamma_{ij}(u) - Q^*_{kj} m_i\partial_j u_i \\
&= 2Q^*_{kj} m_i\gamma_{ij}(u) + Q^*_{kj} u_i\partial_j m_i - Q^*_{kj}\partial_j(m_i u_i).
\end{aligned}$$

The two first terms of this right-hand side will be estimated by Young's inequality and it therefore remains to estimate the last term, namely by the previous identities we have

$$\begin{aligned}
\int_{\Sigma_T}(m\cdot\nabla)u\cdot Q(E\times\nu)\, dSdt &\le \theta_3\int_{\Sigma_T}\gamma(u) : \gamma(u)\, dSdt \\
+ C\int_{\Sigma_T}(|u|^2 + (1 + \frac{1}{\theta_3})|E_\tau|^2)\, dSdt &- \int_{\Sigma_T}(Q^*\nabla(m\cdot u))\cdot(E\times\nu)\, dSdt, \forall\theta_3 > 0.
\end{aligned} \tag{3.35}$$

Now using Green's formula, we see that

$$\int_{\Sigma_T}(Q^*\nabla(m\cdot u))\cdot(E\times\nu)\, dSdt = \int_{Q_T}\{\mathrm{curl}(Q^*\nabla(m\cdot u))\cdot E - Q^*\nabla(m\cdot u)\cdot\mathrm{curl}E\}\, dSdt,$$

where $Q_T = \Omega\times(0,T)$. Now using the fact that $Q(x) = \alpha(x)\, I$, and that $\mathrm{curl}E = \mu\partial_t H$, we obtain

$$\int_{\Sigma_T}(Q^*\nabla(m\cdot u))\cdot(E\times\nu)\, dSdt = \int_{Q_T}\{(\nabla\alpha\times\nabla(m\cdot u))\cdot E - Q^*\nabla(m\cdot u)\cdot\mu\partial_t H\}\, dSdt.$$

For this last term, we first integrate by parts in time and get

$$\int_{Q_T} Q^*\nabla(m\cdot u)\cdot \mu\partial_t H\, dSdt = -\int_{Q_T} Q^*\nabla(m\cdot \partial_t u)\cdot \mu H\, dSdt + \int_{\Omega} Q^*\nabla(m\cdot u)\cdot \mu H\, dx\Big|_0^T.$$

An integration by parts in space leads to

$$\begin{aligned}\int_{Q_T} Q^*\nabla(m\cdot u)\cdot \mu\partial_t H\, dSdt &= \int_{Q_T} (Q^* m\cdot \partial_t u \operatorname{div}(\mu H) + m\cdot\partial_t u\nabla\alpha\cdot(\mu H))\, dSdt \\ &- \int_{\Sigma_T} Q^* m\cdot \partial_t u(\mu H)\cdot\nu\, dSdt + \int_{\Omega} Q^*\nabla(m\cdot u)\cdot \mu H\, dx\Big|_0^T.\end{aligned}$$

These two identities and reminding that $\operatorname{div}(\mu H) = 0$ lead to

$$\begin{aligned}\int_{\Sigma_T} (Q^*\nabla(m\cdot u))\cdot(E\times\nu)\, dSdt &= \int_{Q_T} \{(\nabla\alpha\times\nabla(m\cdot u))\cdot E - m\cdot\partial_t u\nabla\alpha\cdot(\mu H)\}\, dSdt \\ &+ \int_{\Sigma_T} Q^* m\cdot\partial_t u(\mu H)\cdot\nu\, dSdt - \int_{\Omega} Q^*\nabla(m\cdot u)\cdot\mu H\, dx\Big|_0^T.\end{aligned}$$

By Young's inequality we find that

$$\int_{\Sigma_T} (Q^*\nabla(m\cdot u))\cdot(E\times\nu)\, dSdt \leq C(1+c_\alpha T)\mathcal{E}(0) + \int_{\Sigma_T} \{\frac{C}{\theta_4}|\partial_t u|^2 + \theta_4|H_\nu|^2\}\, dSdt, \forall\theta_4>0.$$

This last estimate in (3.35) leads to

$$\begin{aligned}\int_{\Sigma_T} (m\cdot\nabla)u\cdot Q(E\times\nu)\, dSdt &\leq C(1+c_\alpha T)\mathcal{E}(0) + \theta_3\int_{\Sigma_T}\gamma(u):\gamma(u)\, dSdt \\ +C\int_{\Sigma_T} \{|u|^2 &+ \frac{1}{\theta_4}|\partial_t u|^2 + (1+\frac{1}{\theta_3})|E_\tau|^2) + \theta_4|H_\nu|^2\}\, dSdt, \forall\theta_3,\theta_4>0.\end{aligned} \tag{3.36}$$

Now using again Young's inequality and the estimates (3.32), (3.33), (3.34) and (3.36) into the identity (3.31), we obtain that

$$\begin{aligned}\int_{\Sigma_T}\Delta\, dSdt &\leq C(1+c_\alpha T)\mathcal{E}(0) + (-\alpha_0\delta+\theta_1+\theta_2+\theta_3)\int_{\Sigma_T}\gamma(u):\gamma(u)\, dSdt \\ &\quad +C\int_{\Sigma_T}\{(1+\frac{1}{\theta_2}+\frac{1}{\theta_4})|\partial_t u|^2 + (1+\frac{1}{\theta_3})|E_\tau|^2) + \theta_4|H_\nu|^2\}\, dSdt \\ &+ C\{(1+\frac{1}{\theta_1}+\frac{1}{\theta_2}\}\int_{\Sigma_T}|u|^2\, dSdt, \forall\,\theta_1,\theta_2,\theta_3,\theta_4>0,\end{aligned}$$

This estimate in (3.29) and using (3.30), we get finally

$$\begin{aligned}\int_{\Sigma_T} V\, dSdt &\leq C(1+c_\alpha T)\mathcal{E}(0) + (-\alpha_0\delta+\beta_1+\theta_1+\theta_2+\theta_3)\int_{\Sigma_T}\gamma(u):\gamma(u)\, dSdt \\ &\quad +C\int_{\Sigma_T}(1+\frac{1}{\beta_2}+\frac{1}{\theta_2}+\frac{1}{\theta_1}+\frac{1}{\theta_4})|\partial_t u|^2\, dSdt \\ &\quad +C\int_{\Sigma_T}(1+\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\theta_3})|E_\tau|^2\, dSdt \\ &\quad +C\int_{\Sigma_T}\{(-m\cdot\nu+\beta_2)\varepsilon|E_\nu|^2 + ((-m\cdot\nu+\beta_2)\mu+\theta_4)|H_\nu|^2\}\, dSdt \\ &+ C\{1+\frac{1}{\theta_2}+\frac{1}{\theta_1}\}\int_{\Sigma_T}|u|^2\, dSdt, \forall\beta_1,\beta_2,\theta_1,\theta_2,\theta_3,\theta_4>0.\end{aligned}$$

By choosing β_1, β_2, θ_1, θ_2, θ_3 and θ_4 small enough, we have found that

$$\int_{\Sigma_T} V\, dSdt \leq C(1+c_\alpha T)\mathcal{E}(0) + C\int_{\Sigma_T}(|u|^2+|\partial_t u|^2+|E_\tau|^2)\, dSdt. \tag{3.37}$$

Coming back to (3.28) and using again Young's and Korn's inequalities to estimate r, we obtain

$$T\mathcal{E}(T) \leq C(1+c_\alpha T)\mathcal{E}(0) + C\int_{\Sigma_T} (|u|^2 + |\partial_t u|^2 + |E_\tau|^2)\, dSdt. \tag{3.38}$$

Now invoking Lemma 3.7 below, we arrive at

$$\begin{aligned} T\mathcal{E}(T) &\leq C(\theta)(1+c_\alpha T)\mathcal{E}(0) + C\int_{\Sigma_T} (|\partial_t u|^2 + |E_\tau|^2)\, dSdt + \theta\int_0^T \mathcal{E}(t)\, dt \\ &\leq \big(C(\theta)(1+c_\alpha T) + \theta T\big)\mathcal{E}(0) + C\int_{\Sigma_T} (|\partial_t u|^2 + |E_\tau|^2)\, dSdt, \forall\theta > 0, \end{aligned}$$

reminding that the energy is non increasing. This is the requested estimate (3.27). ∎

Remark 3.2 Note that the last term of the estimate (3.35) is zero if $Q^* = Q^*_\nu$, but according to Remark 1.1, this assumption is meaningless.

Theorem 3.3 *Under the assumptions of the previous theorem and if c_α is small enough, there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{3.39}$$

for all strong solution (u, E, H) of (1.7).

Remark 3.4 The same method yields the same exponential stability result in the case where ε, μ are positive functions satisfying some regularity and technical conditions.

Proof: The estimate (3.27) and Lemma 2.3 yield

$$T\mathcal{E}(T) \leq \big(C(\theta)(1+c_\alpha T) + \theta T\big)\mathcal{E}(0) + C(\mathcal{E}(0) - \mathcal{E}(T)), \forall\theta > 0,$$

which may be equivalently written

$$\mathcal{E}(T) \leq \frac{C(\theta)(1+c_\alpha T) + \theta T}{C+T}\mathcal{E}(0), \forall\theta > 0.$$

Now we choose $\theta = \frac{1}{2}$ and $c_\alpha \leq \frac{1}{4\,C(\frac{1}{2})}$, with this choice $\frac{C(\theta)(1+c_\alpha T)+\theta T}{C+T}$ tends to $C(\frac{1}{2})c_\alpha + \frac{1}{2} \leq \frac{3}{4}$ as T goes to infinity. Therefore for T large enough, we have found $r \in (0,1)$ such that

$$\mathcal{E}(T) \leq r\mathcal{E}(0).$$

Since our system is invariant by translation, standard arguments about uniform stabilization of hyperbolic system (see for instance [35, 31]) yield the conclusion. ∎

The key point in the above proof is to estimate appropriately the term $\int_{\Sigma_T} |u|^2\, dSdt$ in (3.38). Indeed a rough idea is to use the definition (2.23) of the energy to get

$$C\int_{\Sigma_T} |u|^2\, dSdt \leq \frac{2C}{A}\int_0^T \mathcal{E}(t)\, dt \leq \frac{2CT}{A}\mathcal{E}(0).$$

Hence from the previous proof we obtain an exponential stability result only for A small enough (depending on a constant C that is not known explicitly, see nevertheless [2]). In order to prove the stability result for any positive A, we then need to estimate $\int_{\Sigma_T} |u|^2\, dSdt$ in a different way. Its proof is based on the use of a solution z of a stationary problem (see [6, 2, 13, 32] and below) such that $z = u$ on Γ. Multiplying the first identity of (1.7) by z, integrating by parts and using the second boundary condition in (1.7), the term $\int_{\Sigma_T} |u|^2\, dSdt$ naturally appears. For standard problems (see [6, 2, 13, 32]) this term is estimated using elliptic regularity results on z. Here the specificity of our piezoelectric system requires a more careful analysis. We start with the stationary problem mentioned before.

Lemma 3.5 *Let (u, E, H) be a strong solution of (1.7). Then there exists $(z, \chi) \in H^1(\Omega)^3 \times H_0^1(\Omega)$ (depending on t) weak solution of*

$$\left\{ \begin{array}{l} \nabla(\sigma(z) - e^\top \nabla\chi) = 0 \text{ in } \Omega, \\ \operatorname{div}(\varepsilon\nabla\chi + e\gamma(z)) = 0 \text{ in } \Omega, \\ z = u, \chi = 0 \text{ on } \Gamma. \end{array} \right. \tag{3.40}$$

Moreover there exists a positive constant C (independent of t) such that

$$\int_\Omega |z|^2\, dx \leq C \int_\Gamma |u|^2\, dS \leq \frac{2C}{A}\mathcal{E}(t), \tag{3.41}$$

$$\int_\Omega |\partial_t z|^2\, dx \leq C \int_\Gamma |\partial_t u|^2\, dS \leq -C\partial_t\mathcal{E}(t). \tag{3.42}$$

Proof: Inspired from [6, 2, 13, 32] for each $t \geq 0$ we consider the weak solution (z, χ) (depending on t) of (3.40). This solution is characterized by $z = w + u$ where $(w, \chi) \in \tilde{V} := H_0^1(\Omega)^3 \times H_0^1(\Omega)$ is the unique solution of

$$\tilde{a}((w, \chi), (w', \chi')) = -\tilde{a}((u, 0), (w', \chi'), \forall (w', \chi') \in \tilde{V}, \tag{3.43}$$

where

$$\tilde{a}((w, \chi), (w', \chi')) = \int_\Omega \{(\sigma(w) - e^\top \nabla\chi) : \gamma(w') + (\varepsilon\nabla\chi + e\gamma(w)) \cdot \nabla\chi'\}\, dx, \forall (w', \chi') \in V.$$

The above problem has a unique solution since the bilinear form $\tilde{a}$ is coercive on V (consequence of Korn's inequality).

A direct consequence of (3.43) is that

$$\tilde{a}((z, \chi), (w', \chi')) = 0, \forall (w', \chi') \in \tilde{V}.$$

By taking as test function $w' = w = z - u$ and $\chi' = \chi$, we find that

$$\tilde{a}((z, \chi), (z, \chi)) = \tilde{a}((z, \chi), (u, 0)),$$

which implies

$$\int_\Omega \{\sigma(z) : \gamma(u) - e^\top \nabla\chi : \gamma(u)\}\, dx = \tilde{a}((z, \chi), (z, \chi)) \geq 0. \tag{3.44}$$

Note further that the coerciveness of $\tilde{a}$ leads to

$$\|w\|_{1,\Omega} + \|\chi\|_{1,\Omega} \leq C\|u\|_{1,\Omega},$$

and then to

$$\|z\|_{1,\Omega} + \|\chi\|_{1,\Omega} \leq C\|u\|_{1,\Omega} \leq C\mathcal{E}(t)^{1/2}, \tag{3.45}$$

where $\|u\|_{s,\Omega} = \|u\|_{H^s(\Omega)}$.

Now we consider the adjoint problem: Find $(w^*, \chi^*) \in \tilde{V}$ solution of

$$\left\{ \begin{array}{l} \nabla(\sigma(w^*) + e^\top \nabla\chi^*) = z \text{ in } \Omega, \\ \operatorname{div}(\varepsilon\nabla\chi^* - e\gamma(w^*)) = 0 \text{ in } \Omega, \\ w^* = 0, \chi^* = 0 \text{ on } \Gamma, \end{array} \right. \tag{3.46}$$

which is the unique solution of

$$\tilde{a}^*((w^*,\chi^*),(w',\chi')) = \int_\Omega z\cdot w'\,dx, \forall (w',\chi')\in V, \tag{3.47}$$

where

$$\tilde{a}^*((w,\chi),(w',\chi')) = \int_\Omega \{(\sigma(w)+e^\top\nabla\chi):\gamma(w')+(\varepsilon\nabla\chi - e\gamma(w))\cdot\nabla\chi'\}\,dx, \forall (w',\chi')\in V.$$

Again this problem has a unique solution since the bilinear form $\tilde{a}^*$ is also coercive on $\tilde{V}$. Since the system (3.46) is strongly elliptic, we deduce that (w^*,χ^*) belongs to $H^2(\Omega)^3\times H^2(\Omega)$ with the estimate (see Theorem 10.5 of [1] or Theorem 4.5.3 of [8])

$$\|w^*\|_{2,\Omega}+\|\chi^*\|_{2,\Omega}\le C\|z\|_{0,\Omega}, \tag{3.48}$$

where here and below C is a positive constant that depends only on $a_{ijkl},\varepsilon,\mu,e_{ijk}$ and on Ω.

By using the differential equations from (3.46), we may write

$$\begin{aligned}\int_\Omega |z|^2\,dx &= \int_\Omega \nabla(\sigma(w^*)+e^\top\nabla\chi^*)\cdot z\,dx\\ &= \int_\Omega \{\nabla(\sigma(w^*)+e^\top\nabla\chi^*)\cdot z + \operatorname{div}(\varepsilon\nabla\chi^* - e\gamma(w^*))\chi\}\,dx.\end{aligned}$$

Applying Green's formula we get

$$\begin{aligned}\int_\Omega |z|^2\,dx &= -\int_\Omega \{(\sigma(w^*)+e^\top\nabla\chi^*):\gamma(z)+(\varepsilon\nabla\chi^*-e\gamma(w^*))\cdot\nabla\chi\}\,dx\\ &+ \int_\Gamma (\sigma(w^*)+e^\top\nabla\chi^*)\nu\cdot z\,dS\\ &= -\int_\Omega \{(\sigma(z)-e^\top\nabla\chi):\gamma(w^*)+(\varepsilon\nabla\chi+e\gamma(z))\cdot\nabla\chi^*\}\,dx\\ &+ \int_\Gamma (\sigma(w^*)+e^\top\nabla\chi^*)\nu\cdot z\,dS.\end{aligned}$$

Applying again Green's formula and reminding problem (3.40), we have found that

$$\int_\Omega |z|^2\,dx = \int_\Gamma (\sigma(w^*)+e^\top\nabla\chi^*)\nu\cdot u\,dS.$$

By Cauchy-Schwarz's inequality and the estimate (3.48) (with the help of a trace theorem), we obtain finally

$$\int_\Omega |z|^2\,dx \le C\int_\Gamma |u|^2\,dS.$$

This proves (3.41) because $\frac{A}{2}\int_\Gamma |u|^2\,dS\le\mathcal{E}(t)$.

By deriving the system (3.40) in time, the estimate (3.41) also shows that

$$\int_\Omega |\partial_t z|^2\,dx \le C\int_\Gamma |\partial_t u|^2\,dS.$$

This yields (3.42) owing to the identity (2.25). ∎

At this stage we need to exploit the fact that $\varepsilon\nabla\chi + e\gamma(z)$ is divergence free, hence it is the curl of $\psi\in X_T(\Omega)$, where

$$X_T(\Omega)=\{\phi\in H^1(\Omega)^3 : \operatorname{div}\psi = 0 \text{ in } \Omega, \text{ and } \psi\cdot\nu = 0 \text{ on } \Gamma\}.$$

More precisely we have the following result.

Lemma 3.6 *Let (u, E, H) be a strong solution of (1.7) and $(z, \chi) \in H^1(\Omega)^3 \times H_0^1(\Omega)$ the weak solution of (3.40). Then there exists $\psi \in X_T(\Omega)$ such that*

$$\varepsilon\nabla\chi + e\gamma(z) = \mathrm{curl}\psi, \tag{3.49}$$

with the estimates

$$\|\psi\|_{0,\Omega}^2 \le C\|u\|_{0,\Gamma}^2 \le \frac{2C}{A}\mathcal{E}(t), \tag{3.50}$$

$$\|\partial_t\psi\|_{0,\Omega}^2 \le C\|\partial_t u\|_{0,\Gamma}^2 \le -C\partial_t\mathcal{E}(t). \tag{3.51}$$

where C is a positive constant independent of t.

Proof: We remark that (see (3.40)) $\varepsilon\nabla\chi + e\gamma(z)$ is divergence free in Ω, hence as Ω is simply connected, we deduce (see Theorem I. 3.5 in [11]) that there exists $\psi \in X_T(\Omega)$ such that (3.49) holds with the estimate

$$\|\psi\|_{1,\Omega} \le C\,\|\varepsilon\nabla\chi + e\gamma(z)\|_{1,\Omega}.$$

Thanks to (3.45), we get

$$\|\psi\|_{1,\Omega} \le C\|u\|_{1,\Omega} \le C\mathcal{E}(t)^{1/2}. \tag{3.52}$$

Let us finally consider the problem: find $\tilde{\chi}$ solution of

$$\left\{\begin{array}{l} \mathrm{curlcurl}\tilde{\chi} = \psi \text{ in } \Omega,\\ \mathrm{div}\tilde{\chi} = 0 \text{ in } \Omega,\\ \tilde{\chi}\cdot\nu = 0, \mathrm{curl}\tilde{\chi}\times\nu = 0 \text{ on } \Gamma. \end{array}\right. \tag{3.53}$$

The variational formulation of this problem is: find $\tilde{\chi} \in X_T(\Omega)$ solution of

$$b(\tilde{\chi}, \theta) = \int_\Omega \psi\cdot\theta\,dx, \forall\theta \in H_T(\Omega), \tag{3.54}$$

where

$$b(\tilde{\chi}, \theta) = \int_\Omega \{\mathrm{curl}\tilde{\chi}\mathrm{curl}\theta + \mathrm{div}\tilde{\chi}\mathrm{div}\theta\}\,dx, \forall\tilde{\chi}, \theta \in H_T(\Omega),$$

and

$$H_T(\Omega) = \{\phi \in H^1(\Omega)^3 : \phi\cdot\nu = 0 \text{ on } \Gamma\}.$$

It is well known (see for instance [7]) that b is coercive on $H_T(\Omega)$ and therefore problem (3.54) is well posed, its solution $\tilde{\chi}$ furthermore satisfies (3.53) because ψ is divergence free. Moreover as the system curlcurl$-\nabla$div $= -\Delta$ is strongly elliptic and the boundary conditions in (3.53) cover this system, we get that $\tilde{\chi}$ belongs to $H^2(\Omega)^3$ with (see again Theorem 10.5 of [1] or Theorem 4.5.3 of [8])

$$\|\tilde{\chi}\|_{2,\Omega} \le C\|\psi\|_{0,\Omega}. \tag{3.55}$$

Now as before we can write by using Green's formula and the identity (3.49)

$$\begin{aligned} \|\psi\|_{0,\Omega}^2 &= \int_\Omega \psi\cdot\mathrm{curlcurl}\tilde{\chi}\,dx\\ &= \int_\Omega \mathrm{curl}\psi\cdot\mathrm{curl}\tilde{\chi}\,dx\\ &= \int_\Omega (\varepsilon\nabla\chi + e\gamma(z))\cdot\mathrm{curl}\tilde{\chi}\,dx\\ &= -\int_\Omega \nabla(e^\top\mathrm{curl}\tilde{\chi})\cdot z\,dx + \int_\Gamma (e^\top\mathrm{curl}\tilde{\chi})\nu\cdot z\,dS. \end{aligned}$$

By the estimate (3.55) and reminding that $z = u$ on Γ, we obtain

$$\|\psi\|_{0,\Omega} \leq C(\|z\|_{0,\Omega} + \|u\|_{0,\Gamma}).$$

By the estimate (3.41), we arrive at

$$\|\psi\|^2_{0,\Omega} \leq C\|u\|^2_{0,\Gamma},$$

and we conclude as in the previous Lemma. ■

Lemma 3.7 *Let (u, E, H) be a strong solution of (1.7). Then for all $\theta > 0$ there exists a constant $C(\theta) > 0$ (which does not depend on T but depends on θ, the domain and the coefficients $a_{ijkl}, \varepsilon, \mu, e_{ijk}, A$) such that*

$$\int_{\Sigma_T} |u|^2 \, dSdt \leq C(\theta)\mathcal{E}(0) + \theta \int_0^T \mathcal{E}(t)\, dt. \tag{3.56}$$

Proof: We multiply the first identity of (1.7) by $z \in H^1(\Omega)^3$ from Lemma 3.5 and integrate on Q_T to get

$$\int_{Q_T} z \cdot (\partial_t^2 u - \nabla\sigma(u, E))\, dxdt = 0.$$

By Green's formula we obtain

$$\int_{Q_T} (z \cdot \partial_t^2 u + \sigma(u, E) : \gamma(z))\, dxdt - \int_{\Sigma_T} z \cdot (\sigma(u, E) \cdot \nu)\, dSdt = 0.$$

Using the second boundary condition in (1.7) and the boundary condition in (3.40), we obtain

$$A \int_{\Sigma_T} |u|^2 \, dSdt = -\int_{\Sigma_T} u \cdot (\partial_t u + Q(E \times \nu))\, dSdt - \int_{Q_T} (z \cdot \partial_t^2 u + \sigma(u, E) : \gamma(z))\, dxdt.$$

Owing to (3.44) we arrive at

$$A \int_{\Sigma_T} |u|^2 \, dSdt \leq$$

$$-\int_{\Sigma_T} u \cdot (\partial_t u + Q(E \times \nu))\, dSdt - \int_{Q_T} (z \cdot \partial_t^2 u + e^\top \nabla\chi : \gamma(u) - e^\top E : \gamma(z))\, dxdt.$$

By using the identity $e\gamma(u) = D - \varepsilon E$, we get

$$\begin{aligned} A \int_{\Sigma_T} |u|^2 \, dSdt &\leq -\int_{\Sigma_T} u \cdot (\partial_t u + Q(E \times \nu))\, dSdt \\ &- \int_{Q_T} \Big(z \cdot \partial_t^2 u + \nabla\chi \cdot D - E \cdot (e\gamma(z) + \varepsilon\nabla\chi)\Big)\, dxdt. \end{aligned} \tag{3.57}$$

We now transform the two last terms of this identity, first by Green's formula in space, we see that

$$\int_{Q_T} \nabla\chi \cdot D \, dxdt = -\int_{Q_T} \chi \mathrm{div} D \, dxdt + \int_{\Sigma_T} \chi D \cdot \nu \, dSdt = 0,$$

since D is divergence free and $\chi = 0$ on Γ. On the other hand, by the identity (3.49) we have

$$\int_{Q_T} E \cdot (e\gamma(z) + \varepsilon\nabla\chi)\, dxdt = \int_{Q_T} E \cdot \mathrm{curl}\psi \, dxdt,$$

and by Green's formula in space

$$\int_{Q_T} E \cdot (e\gamma(z) + \varepsilon\nabla\chi)\, dxdt = \int_{Q_T} \mathrm{curl} E \cdot \psi\, dxdt + \int_{\Sigma_T} (E \times \nu) \cdot \psi\, dSdt.$$

Now reminding that $\mu\partial_t H = \mathrm{curl}E$ and using an integration by parts in time, we arrive at

$$\int_{Q_T} E \cdot (e\gamma(z) + \varepsilon\nabla\chi)\, dxdt = \int_{Q_T} \mu H \cdot \partial_t\psi\, dxdt + \int_{\Omega} \mu H \cdot \psi\, dx\Big|_0^T + \int_{\Sigma_T} (E \times \nu) \cdot \psi\, dSdt.$$

In the same manner an integration by parts in time yields

$$\int_{Q_T} z \cdot \partial_t^2 u\, dxdt = -\int_{Q_T} \partial_t z \cdot \partial_t u\, dxdt + \int_{\Omega} z \cdot \partial_t u\, dx\Big|_0^T$$

These identities in (3.57) lead to

$$\begin{aligned} A\int_{\Sigma_T} |u|^2\, dSdt \le &-\int_{\Sigma_T} (u \cdot (\partial_t u + Q(E \times \nu)) + (E \times \nu) \cdot \psi)\, dSdt \\ &+ \int_{Q_T} (\partial_t z \partial_t u + \mu H \cdot \partial_t \psi)\, dxdt - \int_{\Omega} z \cdot \partial_t u\, dx\Big|_0^T + \int_{\Omega} \mu H \cdot \psi\, dx\Big|_0^T. \end{aligned} \tag{3.58}$$

It remains to estimate each term of this right-hand side. For the first term applying successively Cauchy-Schwarz's inequality, Young's inequality and the identity (2.25) we may write

$$\begin{aligned} \left|\int_{\Sigma_T} u \cdot (\partial_t u + Q(E \times \nu))\, dSdt\right| &\le \frac{A}{2}\int_{\Sigma_T} |u|^2\, dSdt + \frac{C}{2A}\int_{\Sigma_T} \left(|\partial_t u|^2 + |E \times \nu|^2\right)\, dSdt \\ &\le \frac{A}{2}\int_{\Sigma_T} |u|^2\, dSdt - \frac{C}{2A}\int_0^T \partial_t \mathcal{E}(t)\, dt. \end{aligned}$$

Since the energy is non-negative, we arrive at

$$\left|\int_{\Sigma_T} u \cdot \partial_t u\, dSdt\right| \le \frac{A}{2}\int_{\Sigma_T} |u|^2\, dSdt + \frac{C}{2A}\mathcal{E}(0). \tag{3.59}$$

For the second term by using Cauchy-Schwarz's inequality, Young's inequality, a trace theorem, the estimate (3.52) and again the identity (2.25)

$$\begin{aligned} \left|\int_{\Sigma_T} (E \times \nu) \cdot \psi\, dSdt\right| &\le \theta\int_0^T \|\psi\|_{1,\Omega}^2\, dt + \frac{C}{\theta}\int_{\Sigma_T} |E \times \nu|^2\, dSdt \\ &\le \theta\int_0^T \mathcal{E}(t)\, dt + \frac{C}{\theta}\int_{\Sigma_T} |E \times \nu|^2\, dSdt \\ &\le \theta\int_0^T \mathcal{E}(t)\, dt - \frac{C}{\theta}\int_0^T \partial_t \mathcal{E}(t)\, dt. \end{aligned}$$

As before the energy being non-negative, we arrive at

$$\left|\int_{\Sigma_T} (E \times \nu) \cdot \psi\, dSdt\right| \le \theta\int_0^T \mathcal{E}(t)\, dt + \frac{C}{\theta}\mathcal{E}(0). \tag{3.60}$$

For the third term we use successively Cauchy-Schwarz's inequality, Young's inequality, the estimate (3.42) and the definition of the energy to get for all $\theta > 0$

$$\begin{aligned}\left|\int_{Q_T} \partial_t z \cdot \partial_t u \, dxdt\right| &\leq \frac{1}{2\theta}\int_{Q_T} |\partial_t z|^2 \, dxdt + \frac{\theta}{2}\int_{Q_T} |\partial_t u|^2 \, dxdt \\ &\leq -\frac{C}{2\theta}\int_0^T \partial_t \mathcal{E}(t)\, dt + \theta \int_0^T \mathcal{E}(t)\, dt.\end{aligned}$$

Again we get

$$\left|\int_{Q_T} \partial_t z \cdot \partial_t u \, dxdt\right| \leq \frac{C}{\theta}\mathcal{E}(0) + \theta \int_0^T \mathcal{E}(t)\, dt. \tag{3.61}$$

As for the third term replacing the estimate (3.42) by (3.51) we get for the fourth term

$$\left|\int_{Q_T} \mu H \cdot \partial_t \psi \, dxdt\right| \leq \frac{C}{\theta}\mathcal{E}(0) + \theta \int_0^T \mathcal{E}(t)\, dt. \tag{3.62}$$

For the fifth term the application of Cauchy-Schwarz's inequality, the estimate (3.41) and the definition of the energy directly gives

$$\left|\int_{\Omega} z \cdot \partial_t u \, dx\Big|_0^T\right| \leq C(\mathcal{E}(0) + \mathcal{E}(T)) \leq 2C\mathcal{E}(0) \tag{3.63}$$

since the energy is non-decreasing.

Similarly using (3.50) instead of (3.41), we have

$$\left|\int_{\Omega} \mu H \cdot \psi \, dx\Big|_0^T\right| \leq C\mathcal{E}(0). \tag{3.64}$$

The estimates (3.59) to (3.64) into the estimate (3.58) yield the conclusion. ■

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