ON THE SCATTERED FIELD GENERATED BY A BALL INHOMOGENEITY OF CONSTANT INDEX

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Abstract. We consider the solution of a scalar Helmholtz equation where the potential (or index) takes two positive values, one inside a disk of radius ε and another one outside. We derive sharp estimates of the size of the scattered field caused by this disk inhomogeneity, for any frequencies and any contrast. We also provide a broadband estimate, that is, a uniform bound for the scattered field for any contrast, and any frequencies outside of a set which tends to zero with ε.

1. Introduction

We consider a scalar field satisfying the Helmholtz equation with frequency $\omega > 0$ in $\mathbb{R}^2$. Given a prescribed incident field u^i, a non-singular solution of

$$\Delta u^i + \omega^2 q_0 u^i = 0 \text{ in } \mathbb{R}^2, \tag{1.1}$$

we are interested in the solution $u_\varepsilon \in H^1_{\text{loc}}\left(\mathbb{R}^2\right)$ of

$$\Delta u_\varepsilon + \omega^2 q_\varepsilon u_\varepsilon = 0 \text{ in } \mathbb{R}^2, \tag{1.2}$$

where, for $|x| > \varepsilon$, $u_\varepsilon = u^i + u^s_\varepsilon$, and q_ε equals q inside the inhomogeneity and q_0 outside. We take the inhomogeneity to be a disk of radius ε. The coordinate system is chosen so that the inhomogeneity is centered at the origin. In other words

$$q_\varepsilon(r) := \begin{cases} q & \text{if } r < \varepsilon \\ q_0 & \text{if } r > \varepsilon \end{cases}$$

We assume that both q_0 and q are real and positive. We assume that the scattered field satisfies the classical Silver-Müller [9, 10] outgoing radiation condition, given by

$$\frac{\partial}{\partial r} u^s_\varepsilon - i\omega\sqrt{q_0} u^s_\varepsilon = o\left(\frac{1}{\sqrt{r}}\right), \tag{1.3}$$

where, as usual $r := |x|$. Altogether, the conditions (1.1,1.2,1.3) imply that the incident field u^i, the scattered field u^s_ε and the transmitted field $u^t_\varepsilon = u_\varepsilon$ for $r < \varepsilon$, admit series expansions in terms of special functions, namely

$$u^i(x) \sim \sum_{n=-\infty}^{\infty} a_n J_n\left(\sqrt{q_0}\omega r\right) \exp\left(i\, n \arctan\left(\frac{x}{r}\right)\right), \tag{1.4}$$

$$u^s_\varepsilon(x) \sim \sum_{n=-\infty}^{\infty} a_n R_n\left(\omega_\varepsilon, \lambda\right) H^{(1)}_n\left(\sqrt{q_0}\omega r\right) \exp\left(i\, n \arctan\left(\frac{x}{r}\right)\right), \tag{1.5}$$

$$u^t_\varepsilon(x) \sim \sum_{n=-\infty}^{\infty} a_n T_n\left(\omega_\varepsilon, \lambda\right) J_n\left(\sqrt{q}\omega r\right) \exp\left(i\, n \arctan\left(\frac{x}{r}\right)\right). \tag{1.6}$$

In the above formulae, $J_n(x) = \Re(H^{(1)}_n(x))$, and $x \to H^{(1)}_n(x)$ is the Hankel function of the first kind of order n. The rescaled non-dimensional frequency ω_ε, and the contrast factor λ are given by

$$\omega_\varepsilon := \sqrt{q_0}\omega\varepsilon \text{ and } \lambda := \sqrt{\frac{q}{q_0}}. \tag{1.7}$$

The reflection and transmission coefficients R_n and T_n are given by the transmission problem on the boundary of the inhomogeneity, that is, at $r = \varepsilon$. They are the unique solutions of

$$\begin{aligned} T_n(\omega_\varepsilon,\lambda) J_n(\lambda\omega_\varepsilon) &= J_n(\omega_\varepsilon) + R_n(\omega_\varepsilon,\lambda) H_n^{(1)}(\omega_\varepsilon), \\ \lambda T_n(\omega_\varepsilon,\lambda) J_n'(\lambda\omega_\varepsilon) &= J_n'(\omega_\varepsilon) + R_n(\omega_\varepsilon,\lambda) H_n^{(1)\prime}(\omega_\varepsilon), \end{aligned}$$

which are

$$R_n(\omega_\varepsilon,\lambda) = -\frac{\Re\left(H_n^{(1)\prime}(\omega_\varepsilon) J_n(\lambda\omega_\varepsilon) - \lambda J_n'(\lambda\omega_\varepsilon) H_n^{(1)}(\omega_\varepsilon)\right)}{H_n^{(1)\prime}(\omega_\varepsilon) J_n(\lambda\omega_\varepsilon) - \lambda J_n'(\lambda\omega_\varepsilon) H_n^{(1)}(\omega_\varepsilon)}, \tag{1.8}$$

and, after a simplification using the Wronskian identity satisfied by $J_n(\cdot)$ and $H_n^{(1)}(\cdot)$,

$$T_n(\omega_\varepsilon,\lambda) = \frac{2i}{\pi}\frac{1}{H_n^{(1)\prime}(\omega_\varepsilon) J_n(\lambda\omega_\varepsilon) - \lambda J_n'(\lambda\omega_\varepsilon) H_n^{(1)}(\omega_\varepsilon)}. \tag{1.9}$$

It is well known that both R_n and T_n are well defined for all $\lambda > 0$ and $\omega_\varepsilon > 0$, see e.g. [4] for a proof. Note that $R_n = R_{-n}$, and $T_n = T_{-n}$ for all n.

In (1.4), (1.5) and (1.6), the $\sim$ symbol is an equality if the right-hand-side is replaced by its real part, the fields being real. By a common abuse of notations, in what follows we will identify u^i and u^s_ε with the full complex right-hand-side.

Such expansions have been known for almost two centuries. They allow in principle, with the help of modern computers and recent numerical methods, to compute the scattered field accurately, given the incident field ω, ε and q/q_0. Yet, they do not give any insight on the behavior of the scattered field when the frequency, the contrast, or the radius ε vary. When ε tends to zero, the behavior of the scattered field for this problem has been studied recently in [4]. The cases considered are either $a_n = 0$ for $n > N_0$, or $\Im(q) > 0$, or full reflection on the boundary of the inclusion, that is, $u_\varepsilon = 0$ at $r = \varepsilon$. In this work, we focus on non lossy inclusions, that is, when $\Im(q) = 0$, and we provide sharp estimates of the scattered field. These estimates are derived for any sequence (a_n), thus for any incident field. They are completely explicit, up to the numerical values of the constants involved. Such detailed results are possible because of the extensive studies of Hankel functions conducted by others. We will quote frequently the classical treatise of Watson [17], and we will indirectly refer to the book of Olver [12] by frequently citing the NIST Handbook of Mathematical Functions [13]. Other papers related to properties of Bessel functions [3, 6, 7, 8, 14, 15, 16] are also cited in the proofs. Some additional estimates that we could not find in the literature are provided in Appendix A. Some of them could be new, but we have not performed a comprehensive search of the vast literature on that topic. However with the exception of Section 4, our main results are stated in a form that does not require any knowledge of the literature related to Bessel functions, except possibly for some universal constants (approximate numerical values are provided).

Let us now discuss the norms we shall use. Given any $f \in C^0(\mathbb{R}^2)$, its restriction to the circle $|x| = R$ is a periodic function. We can therefore define its complex Fourier coefficients

$$c_n(f(|x| = R)) = \int_0^{2\pi} f(R,\theta)e^{-in\theta}d\theta,$$

and $f(|x| = R)$ can be measured in terms of the following Sobolev norm

$$\|f(|x| = R)\|_{H^\sigma} := \sqrt{2\pi}\sqrt{\sum_{n=-\infty}^{\infty} |c_n(f(|x| = R))|^2(1 + |n|)^{2\sigma}}, \tag{1.10}$$

for any real parameter σ. By density, this norm can be defined for less regular functions. If $f(|x| = R)$ is $L^2(0, 2\pi)$ for example it is bounded, for any $\sigma \leq 0$. To measure the oscillations of f only, we will use

$$\|f(|x| = R)\|_{H^\sigma_*} := \left\| f(|x| = R) - \frac{1}{2\pi}\int_0^{2\pi} f(|x| = R)\, d\theta \right\|_{H^\sigma}. \tag{1.11}$$

For radius independent estimates, we shall use the semi-norm

$$\mathcal{N}^{\sigma}(f) := \sqrt{2\pi}\sqrt{\sum_{n\neq 0}\sup_{R>0}|c_n\left(f(|x|=R)\right)|^2(1+|n|)^{2\sigma}}. \tag{1.12}$$

It is easy to see that this norm is finite for a smooth f with bounded radial variations. Finally, to document the sharpness of our estimates, we will provide lower bounds in terms of a semi-norm,

$$\mathbf{N}_p^{\sigma}\left(f\right) := \sqrt{2\pi}\sup_{|n|\geq p}\sup_{R>0}|c_n\left(f(|x|=R)\right)|(1+|n|)^{\sigma}, \tag{1.13}$$

where p is a positive parameter. These norms are satisfy the following inequality

$$\left\|f\left(|x|=R\right)\right\|_{H^{\sigma}} \leq \mathcal{N}^{\sigma}(f), \text{ and } \mathbf{N}_p^{\sigma}\left(f\right) \leq \mathcal{N}^{\sigma}(f),$$

and if for all R, $f\left(|x|=R\right)$ only has one non-zero Fourier coefficient,

$$\mathbf{N}_1^{\sigma}\left(f\right) = \mathcal{N}^{\sigma}(f) = \sup_{R>0}\left\|f\left(|x|=R\right)\right\|_{H_*^{\sigma}}.$$

We choose these three (semi-)norms $\|\cdot\|_{H^{\sigma}}$, $\mathcal{N}^{\sigma}$ and $\mathbf{N}_p^{\sigma}$ because they are compatible with expansions (1.4), (1.5) and (1.6). For example,

$$\left\|u_{\varepsilon}^{s}\left(|x|=R\right)\right\|_{H^{\sigma}} := \sqrt{2\pi}\left(\sum_{n=-\infty}^{\infty}\left|R_n\left(\omega_{\varepsilon},\lambda\right)a_n\right|^2\left(1+|n|\right)^{2\sigma}\left|H_n^{(1)}\left(\sqrt{q_0}\omega R\right)\right|^2\right)^{\frac{1}{2}}, \tag{1.14}$$

and

$$\mathcal{N}^{\sigma}\left(u^{i}\right) := \sqrt{2\pi}\left(\sum_{n\neq 0}\left|a_n\right|^2\sup_{x>0}\left|J_n\left(x\right)\right|^2\left(1+|n|\right)^{2\sigma}\right)^{1/2}. \tag{1.15}$$

Furthermore, it is known [7] that for all $n\neq 0$

$$\frac{4}{7}\frac{1}{(|n|+1)^{1/3}} \leq \sup_{x>0}\left|J_n\left(x\right)\right| \leq \frac{6}{7}\frac{1}{(|n|+1)^{1/3}}, \tag{1.16}$$

therefore $\mathcal{N}^{\sigma}\left(u^{i}\right)$ has upper and lower bounds depending on a_n only, namely,

$$\frac{8\pi}{7}\sum_{n\neq 0}\left|a_n\right|^2\left(1+|n|\right)^{2\sigma-2/3} \leq \left(\mathcal{N}^{\sigma}\left(u^{i}\right)\right)^2 \leq \frac{16\pi}{7}\sum_{n\neq 0}\left|a_n\right|^2\left(1+|n|\right)^{2\sigma-2/3}. \tag{1.17}$$

The motivation for this work comes from imaging. In electrostatics, the small volume asymptotic expansion for a diametrically bounded conductivity inclusion is now well established, and the first order expansion has been shown to be valid for any contrast [11]. It is natural to wonder whether such expansion could also hold for non-zero frequencies, even in a simple case.

Section 2 addresses the case $\lambda \leq 1$. In Theorem 2.1 we derive perturbation-type estimates when $\lambda\omega_{\varepsilon} < 1$, that is, proportional to $(\lambda-1)\omega_{\varepsilon}^2$ at first order, for all x such that $|x| \geq \varepsilon$. This can be seen as a generalization of the electrostatic case. We show that the range of frequencies for which this result applies is sharp. In Theorem 2.4, we provide an upper estimate for the scattered field valid for all frequencies and all $|x| \geq \varepsilon$, and we document its sharpness by providing a lower bound for the supremum of the scattered field for all frequencies. Section 3, 4, 5 address the case $\lambda \geq 1$. Theorem 3.1 is similar to Theorem 2.1 and applies when $\lambda\sqrt{\ln\lambda+1}\,\omega_{\varepsilon} < 1$, and when $\lambda\,\omega_{\varepsilon} < 1$ if there is no zero-order term. In Section 4, we provide a detailed study of quasi-resonances. These are frequencies located just after the perturbative range, at which the near-field becomes arbitrarily large. Theorem 4.3 provides lower bounds for the near field in this regime. We also provide numerical examples of quasi-resonant modes. Section 5 provides far field estimates, that is, for x such that $|x|\lambda > \varepsilon$, valid for all frequencies. As in Theorem 3.1, we show that the bounds provided are sharp.

Another inspiration for this work is recent results concerning the so-called cloaking-by-mapping method for the Helmholtz equation. In [5], the authors show that cloaks can be constructed using lossy layers, and that non-lossy media could not be made invisible to some particular frequencies (the quasi-resonant frequencies). In Section 6, we show in Lemma 6.2 that if an interval around these frequencies is removed, contrast independent estimates for the near-field can be obtained. When $\lambda = \varepsilon^{-1}$, the following proposition is proved as a corollary of Lemma 6.2.

Proposition. *Assume $\varepsilon < 1/7$, and $\lambda = \varepsilon^{-1}$. Then, for any $\alpha, \beta > 0$, there exists a set I_1 depending on ε, α and β and a set I_0 depending on ε and β which satisfies*

$$|I_1| \leq \varepsilon^{\beta} \left|\ln \varepsilon\right|, \quad |I_0| \leq \frac{\ln \left|\ln \varepsilon\right|}{\left(\left|\ln \varepsilon\right| + 1\right)^{\beta}},$$

such that for all $R \geq \varepsilon$,

$$\sup_{\sqrt{q_0}\omega \in (0,\infty) \setminus I_1} \left\| u_\varepsilon^s \left(|x| = R\right) \right\|_{H_*^\sigma}^2 \leq \frac{18}{\alpha} \sqrt{\frac{\varepsilon^{1-2\beta}}{R}} \mathcal{N}^{\sigma+2+\alpha}\left(u^i\right),$$

and

$$\sup_{\sqrt{q_0}\omega \in (0,\infty) \setminus I_0} \left| \frac{1}{2\pi} \int_0^{2\pi} u_\varepsilon^s \left(|x| = R\right) \right| \leq 12 \frac{1}{\sqrt{\left(\left|\ln \varepsilon\right| + 1\right)^{3/2 - 2\beta} R}}.$$

We do not prove that this result is sharp. Combining Lemma 6.2 with results of the previous sections, we show in Theorem 6.5 that broadband estimates uniform with respect to the contrast are possible. In particular we show that when observed at any fixed distance $|x| = R > 0$, the scattered field vanishes in the limit $\varepsilon = 0$ except in a set of frequencies of zero measure.

Section 7 is devoted to the proof of intermediate estimates stated without proofs in Section 2 and 3. Section 8 is devoted the proof of an intermediate estimate used in Lemma 6.2.

2. Inclusions with relative index smaller than one

This section is devoted to the case when $q < q_0$. We estimate the scattered field at a distance $R \geq \varepsilon$ from the center of the inclusion. Our first result addresses the case of moderate frequencies.

Theorem 2.1. *Let $y_{0,1}$ be the first positive solution of $Y_0(x) = 0$. When $\varepsilon \leq R$, $\lambda \leq 1$, and $\omega_\varepsilon < y_{0,1}$, there holds*

$$\left\| u_\varepsilon^s \left(|x| = R\right) \right\|_{H^\sigma} \leq (1-\lambda)\, \omega_\varepsilon \left(3\sqrt{\frac{\varepsilon}{R}} \left\| u^i \left(|x| = \varepsilon\right) \right\|_{H_*^{\sigma - 1/3}} + 9\, \omega_\varepsilon \left| u^i(0) \right| \left| H_0^{(1)}\left(\sqrt{q_0}\omega R\right) \right| \right).$$

Furthermore, if for some $p > 0$ the first p Fourier coefficients of $u^i\left(|x| = (\varepsilon\omega)^{-1}\right)$ are zero, then for all $\omega_\varepsilon < p$ there holds

$$\left\| u_\varepsilon^s \left(|x| = R\right) \right\|_{H^\sigma} \leq 3\,(1-\lambda)\, \omega_\varepsilon \sqrt{\frac{\varepsilon}{R}} \left\| u^i \left(|x| = \varepsilon\right) \right\|_{H_*^{\sigma - 1/3}}.$$

To compare Theorem 2.1 with known results, we derive the following variant.

Corollary 2.2. *When $\varepsilon \leq R$, $\lambda \leq 1$ and $\omega_\varepsilon < y_{0,1}$ we have*

$$\left\| u_\varepsilon^s \left(|x| = R\right) \right\|_{H^\sigma} \leq 9\,(1-\lambda)\, \omega_\varepsilon^2 \left(\left| u^i(0) \right| \left| H_0^{(1)}\left(\sqrt{q_0}\omega R\right) \right| + \sqrt{\frac{\varepsilon}{R}} \mathcal{N}^{\sigma - \frac{1}{3}}\left(u^i\right) \right).$$

Remark 2.3. Under this form, one can read for example that the first order term in ε is correct both with respect to ε and with respect to the contrast. First order asymptotic expansions for small volume or small contrast perturbations [1, 2] derived for a fixed frequency are of the order of ε^2. Note that this estimate holds up to frequencies of the order ε^{-1}: this shows that the inhomogeneity can be viewed as a perturbation up to frequencies of that order.

Our second result is an estimate valid for all frequencies.

Theorem 2.4. *When $\varepsilon \leq R$ and $\lambda \leq 1$ there holds*

$$\sup_{\omega > 0} \left\| u_\varepsilon^s \left(|x| = R\right) \right\|_{H^\sigma} \leq \frac{5}{2} \sqrt{\frac{\varepsilon}{R}} \mathcal{N}^{\sigma}\left(u^i\right) + \sqrt{2\pi} \left| u^i(0) \right| \left| H_0^{(1)}\left(y_{0,1} \frac{R}{\varepsilon} \right) \right|. \tag{2.1}$$

Furthermore,

$$\sup_{\omega>0} \|u_\varepsilon^s (|x|=\varepsilon)\|_{H_*^\sigma} \geq \frac{1}{\sqrt{10}} \mathbf{N}_{n_0}^\sigma \left(u^i\right), \tag{2.2}$$

where n_0 is the smallest positive number such that

$$\lambda^2 \leq 1 - \frac{49}{9n^{2/3}}.$$

for all $n \geq n_0$.

Remark 2.5. The lower bound (2.2) shows that the upper bound (2.1) is sharp in the case when $\mathbf{N}_{n_0}^\sigma$ and $\mathcal{N}^\sigma$ are equivalent norms. We give a more precise upper bound in remark 2.7.

The proof of these results is based on a careful study of $R_n(\omega_\varepsilon, \lambda)$ conducted in Section 7. We prove that the following proposition holds

Proposition 2.6. *Let $y_{n,1}$ be the first positive solution of $Y_n(x) = 0$. When $\lambda \leq 1$, there holds, for all $n \geq 1$*

- *For all $\omega_\varepsilon < y_{n,1}$,*
$$\left| R_n(\omega_\varepsilon, \lambda) H_n^{(1)}(\omega_\varepsilon) \right| \leq \frac{5}{2} |J_n(\omega_\varepsilon)|.$$
- *For all $\omega_\varepsilon < n$,*
$$\left| R_n(\omega_\varepsilon, \lambda) H_n^{(1)}(\omega_\varepsilon) \right| \leq 2(1-\lambda) \frac{\omega_\varepsilon}{n^{1/3}} |J_n(\omega_\varepsilon)|.$$
- *when $\lambda^2 < 1 - \left(\frac{7}{3\,n^{1/3}}\right)^2$, then*
$$\left| R_n(n, \lambda) H_n^{(1)}(n) \right| > \frac{1}{2} |J_n(n)|.$$

For $n = 0$, there holds

- *For all $\omega_\varepsilon > 0$,*
$$\left| R_0(\omega_\varepsilon, \lambda) H_0^{(1)}\left(\omega_\varepsilon \frac{R}{\varepsilon}\right) \right| \leq \left| H_0^{(1)}\left(y_{0,1} \frac{R}{\varepsilon}\right) \right|.$$
- *When $x < y_{0,1}$,*
$$\left| R_0(\omega_\varepsilon, \lambda) H_0^{(1)}\left(x \frac{R}{\varepsilon}\right) \right| \leq \frac{\pi^2}{2\sqrt{2}} (1-\lambda) \omega_\varepsilon^2 \left| H_0^{(1)}\left(\omega_\varepsilon \frac{R}{\varepsilon}\right) \right|.$$

Proof. This follows from Lemma 7.1 and Proposition 7.6. □

Proof of Theorem 2.1. From formula (1.14), we have

$$\begin{aligned} \|u_\varepsilon^s (|x|=R)\|_{H^\sigma}^2 = & 2\pi \sum_{|n|\neq 0} |a_n|^2 (1+|n|)^{2\sigma} \left| R_n(\omega_\varepsilon, \lambda) |H_n^{(1)}(\omega_\varepsilon) \right|^2 \left| \frac{H_n^{(1)}(\sqrt{q_0}\omega R)}{H_n^{(1)}(\omega_\varepsilon)} \right|^2 \\ & + 2\pi |a_0|^2 \left| R_0(\omega_\varepsilon, \lambda) H_0^{(1)}(\sqrt{q_0}\omega R) \right|^2. \end{aligned}$$

Note that $y_{0,1} < 1$. Proposition 2.6 shows that when $\omega_\varepsilon \leq n$ and $n \geq 1$,

$$\left| R_n(\omega_\varepsilon, \lambda) H_n^{(1)}(\omega_\varepsilon) \right| \leq 2(1-\lambda) \frac{\omega_\varepsilon}{n^{1/3}} |J_n(\omega_\varepsilon)| \leq 3(1-\lambda) \frac{\omega_\varepsilon}{(1+|n|)^{1/3}} |J_n(\omega_\varepsilon)|,$$

When $n \neq 0$, $\sqrt{x}\left|H_n^1(x)\right|$ is decreasing [17, 13.74], therefore

$$R \left| H_n^{(1)}(\sqrt{q_0}\omega R) \right|^2 \leq \varepsilon \left| H_n^{(1)}(\omega_\varepsilon) \right|^2,$$

On the other hand, $a_0 = u^i(0)$, and Proposition 2.6 shows that

$$\left| R_0(\omega_\varepsilon, \lambda) H_0^{(1)}(\sqrt{q_0}\omega R) \right| \leq (1-\lambda) \frac{\pi^2}{2\sqrt{2}} \omega_\varepsilon^2 \left| H_0^{(1)}(\sqrt{q_0}\omega R) \right|$$

Combining these estimates we have

$$\begin{aligned}\left\|u_\varepsilon^s\left(|x|=R\right)\right\|_{H^\sigma}^2 \leq & \left(3\left(1-\lambda\right)\omega_\varepsilon\sqrt{\frac{\varepsilon}{R}}\right)^2 2\pi\sum_{|n|\neq 0}\left|a_n\right|^2\left(1+|n|\right)^{2\sigma-2/3}\left|J_n\left(\omega_\varepsilon\right)\right|^2\\ & +\left(u^i(0)\left(1-\lambda\right)\omega_\varepsilon^2\right)^2\frac{\pi^5}{4}\left|H_0^{(1)}\left(\sqrt{q_0}\omega R\right)\right|^2\\ & =\left(3\left(1-\lambda\right)\omega_\varepsilon\right)^2\left(\frac{\varepsilon}{R}\left\|u^i\left(|x|=\varepsilon\right)\right\|_{H_*^{\sigma-\frac{1}{3}}}^2+9\,\omega_\varepsilon^2\left|u^i(0)\right|^2\left|H_0^{(1)}\left(\sqrt{q_0}\omega R\right)\right|^2\right),\end{aligned}$$

and the conclusion follows. If for some $p>0$ the first p Fourier coefficients of $u_\varepsilon^s\left(|x|=(\varepsilon\omega)^{-1}\right)$ are zero, $a_n=0$ for $n=0,\dots,p-1$, and the argument above proves our claim. □

Proof of Corollary 2.2. For all $n\geq 1$, it is known that [14] for $0<x<y<n$,

$$J_n\left(x\right)\leq\frac{x^n}{y^n}J_n\left(y\right)\exp\left(\frac{y^2-x^2}{2n+2}\right).$$

In particular,

$$J_n\left(\omega_\varepsilon\right)\leq\frac{\omega_\varepsilon}{y_{0,1}}J_n\left(y_{0,1}\right)\exp\left(\frac{y_{0,1}^2}{4}\right)\leq 3\,\omega_\varepsilon\,J_n\left(y_{0,1}\right).$$

This implies that

$$\left\|u^i\left(|x|=\omega_\varepsilon\right)\right\|_{H_*^{\sigma-\frac{1}{3}}}\leq 3\,\omega_\varepsilon\,\left\|u^i\left(|x|=y_{0,1}\right)\right\|_{H_*^{\sigma-\frac{1}{3}}}.$$

inserting this upper bound in the estimate provided by Theorem 2.1 proves our claim. □

Proof of Theorem 2.4. Starting from the formula (1.14), using the monotonicity of $x\left|H_n^{(1)}\left(x\right)\right|^2$ for $n\geq 1$ as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\left\|u_\varepsilon^s\left(|x|=R\right)\right\|_{H^\sigma}^2\leq\quad & 2\pi\sum_{|n|\neq 0}\left|a_n\right|^2\left(1+|n|\right)^{2\sigma}\left|R_n\left(\omega_\varepsilon,\lambda\right)\left|H_n^{(1)}\left(\omega_\varepsilon\right)\right|^2\frac{\varepsilon}{R}\\ & +2\pi\left|a_0\right|^2\left|R_0\left(\omega_\varepsilon,\lambda\right)H_0^{(1)}\left(\sqrt{q_0}\omega R\right)\right|^2.\end{aligned}$$

Thanks to Proposition 2.6,

$$\left|R_0\left(\omega_\varepsilon,\lambda\right)H_0^{(1)}\left(\sqrt{q_0}\omega R\right)\right|\leq\left|H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|. \tag{2.3}$$

and when $n\geq 1$, and $\omega_\varepsilon<y_{n,1}$,

$$\left|R_n\left(\omega_\varepsilon,\lambda\right)\left|H_n^{(1)}\left(\omega_\varepsilon\right)\right|\leq\frac{5}{2}\left|J_n\left(\omega_\varepsilon\right)\right|\leq\frac{5}{2}\sup_{x>0}\left|J_n\left(x\right)\right|. \tag{2.4}$$

On the other hand, the definition of R_n (1.8) shows that $\left|R_n\left(\omega_\varepsilon,\lambda\right)\right|\leq 1$. It is known [17, 13.74] that for $x>n\geq 0$,

$$x\to\sqrt{x^2-n^2}\left|H_n(x)\right|^2$$

is an increasing function of x, with limit $2/\pi$. Consequently, since $y_{n,1}>n+\frac{13}{14}n^{1/3}$, [17, 15.3], [13], for all $\omega_\varepsilon\geq y_{n,1}$ we have

$$\left|R_n\left(\omega_\varepsilon,\lambda\right)H_n^{(1)}\left(\omega_\varepsilon\right)\right|\leq\sqrt{\frac{2}{\pi}\frac{1}{\sqrt{\omega_\varepsilon^2-n^2}}}\leq\frac{4}{5}\frac{1}{(1+n)^{1/3}}\leq\frac{7}{5}\sup_{x>0}\left|J_n\left(x\right)\right|, \tag{2.5}$$

thanks to (1.16). Combining (2.4), and (2.5) we have obtained that for all $\omega_\varepsilon>0$,

$$2\pi\sum_{|n|\neq 0}\left|a_n\right|^2\left(1+|n|\right)^{2\sigma}\left|R_n\left(\omega_\varepsilon,\lambda\right)\left|H_n^{(1)}\left(\omega_\varepsilon\right)\right|^2\leq\frac{5}{2}\mathcal{N}^\sigma\left(u^i\right)^2,$$

which concludes our proof of the upper bound (2.1).

Turning to the lower bound, we have

$$\sup_{\omega>0}\|u_\varepsilon^s(|x|=\varepsilon)\|_{H_*^\sigma}\geq\sup_{|n|\geq n_0}\sup_{\omega>0}\sqrt{2\pi}\left|a_n(1+|n|)^\sigma R_n(\omega_\varepsilon,\lambda)H_n^{(1)}(\omega_\varepsilon)\right|.$$

We know from Proposition 2.6 that provided $\lambda^2<1-\frac{49}{9n^{2/3}}$,

$$\left|R_n(n,\lambda)\,|H_n^{(1)}(n)\right|\geq\frac{1}{2}J_n(n)$$

Since it is known [17, 8.54] that $n\to n^{\frac{1}{3}}J_n(n)$ is increasing,

$$\left|R_n(n,\lambda)H_n^{(1)}(n)\right|>\frac{J_1(1)}{2n^{\frac{1}{3}}}\geq\frac{1}{\sqrt{10}}\sup_{x>0}|J_n(x)|,$$

where in the last inequality we used a variant of (1.16), [7]. Choosing ω such that $\omega_\varepsilon=n$, we obtain

$$\sup_{\omega>0}\|u_\varepsilon^s(|x|=\varepsilon)\|_{H_*^\sigma}\geq\frac{1}{\sqrt{10}}\mathbf{N}_{n_0}^\sigma\left(u^i\right),$$

as announced. □

Remark 2.7. The lower bound was obtained for $\omega_\varepsilon=n$, that is, in the special case when the order and argument are equal. This is precisely the upper limit for ω_ε in Theorem 2.1. When the argument is much larger than the order, one should expect a decay gain of 1/2 and not 1/3. The bound given by Theorem 5.1 is of this form, and applies here also (when λ is replaced by 1).

3. Inclusions with relative index larger than one: the perturbative regime

In this section, we consider the case when $q>q_0$ in the case of moderate frequencies and moderate contrast. Our result is expressed in terms of a threshold m_λ which depends on the contrast, given by

$$m_\lambda:=\frac{1}{\lambda\sqrt{\ln\lambda+1}}\tag{3.1}$$

Theorem 3.1. *Suppose $R\geq\varepsilon$, and $\lambda\geq1$. When*

$$\omega_\varepsilon<\min\left(\frac{1}{2},m_\lambda\right)$$

we have

$$\|u_\varepsilon^s(|x|=R)\|_{H^\sigma}\leq(1-\lambda)\,\omega_\varepsilon\left(3\sqrt{\frac{\varepsilon}{R}}\left\|u^i(|x|=\varepsilon)\right\|_{H_*^{\sigma-\frac{1}{3}}}+23\,\omega_\varepsilon\lambda\left|u^i(0)\right|\left|H_0^{(1)}(\sqrt{q_0}\omega R)\right|\right).$$

Furthermore, if for some $p>0$ the first p Fourier coefficients of $u_\varepsilon^s\left(|x|=(\varepsilon\omega)^{-1}\right)$ are zero, then for all $\omega_\varepsilon<\lambda^{-1}p$ there holds,

$$\|u_\varepsilon^s(|x|=R)\|_{H^\sigma}\leq3(1-\lambda)\,\omega_\varepsilon\sqrt{\frac{\varepsilon}{R}}\left\|u^i(|x|=\varepsilon)\right\|_{H_*^{\sigma-\frac{1}{3}}}$$

The proof of Theorem 3.1 is, *mutatis mutandis*, the same as that of Theorem 2.1, using Proposition 3.2, proved in Section 7 in lieu of Proposition 2.6, and we omit it.

Proposition 3.2. *When $\lambda\geq1$, there holds, for all $n\geq1$*

- *For all $\omega_\varepsilon\leq\lambda^{-1}y_{n,1}^{(1)}$,*

$$\left|R_n(\omega_\varepsilon,\lambda)H_n^{(1)}(\omega_\varepsilon)\right|\leq\frac{5}{2}\left|J_n(\omega_\varepsilon)\right|.$$

- *For all $\omega_\varepsilon<\lambda^{-1}n$,*

$$\left|R_n(\omega_\varepsilon,\lambda)H_n^{(1)}(\omega_\varepsilon)\right|\leq2(1-\lambda)\frac{\omega_\varepsilon}{n^{1/3}}\left|J_n(\omega_\varepsilon)\right|.$$

For $n=0$, there holds

- *For all* ω_ε *such that* $\min\left(\frac{1}{2}, m_\lambda\right)\omega_\varepsilon \leq 1$,
$$\left|R_0\left(\omega_\varepsilon,\lambda\right)H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \leq \frac{5\pi^2}{4}(\lambda-1)\lambda\omega_\varepsilon^2\left|H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|.$$
- *For all* ω_ε*, we have*
$$\left|R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \leq \sqrt{5}\left|H_0^{(1)}\left(\min\left(\frac{1}{2}, m_\lambda\right)\frac{R}{\varepsilon}\right)\right|.$$

Proof. This is follows from Lemma 7.1 and Proposition 7.6. □

In contrast with the case $\lambda \leq 1$, the range of frequencies for which Theorem 3.1 is valid becomes increasingly small as the contrast increases. The two extreme contrast cases, $\lambda = 0$ and $\lambda = \infty$ are therefore of a very different nature. As we will see in Section 4, the range of frequencies for which Theorem 2.1 applies is sharp: the behavior of the near field is drastically different when ω_ε is larger.

4. Inclusions with relative index larger than one: quasi-resonances

In this section, we investigate the behavior of the scattered field when $q > q_0$, and the product of the effective frequency and the contrast $\lambda\omega_\varepsilon$ is bounded. In such a case, a quasi-resonance phenomenon occurs: near the inclusion, the scattered field becomes extremely large, for some frequencies. We refer to such frequencies as quasi-resonant frequencies. They are defined in Definition 4.1.

Before we proceed, we remind the readers of usual notations and known properties of zeros of Bessel functions.

When $n \geq 0$, the k-th positive root of $J_n(x) = 0$ is written $j_{n,k}$. The first positive root of $Y_n(x) = 0$ is written $y_{n,1}$. When $n \geq 1$, the k-th positive solution of $J_n'(x) = 0$ is written $j_{n,k}^{(1)}$. When $n = 0$, we count non-negative solutions, that is, $j_{0,1}^{(1)} = 0$. The first positive root of $Y_n'(x) = 0$ is noted $y_{n,1}^{(1)}$ It is known [17, 15.3] that

$$\begin{aligned}
&\text{when } n \geq 1, \quad & j_{n,1} &= n + a_{n,1}n^{1/3}, \text{ with } a_{n,1} > \lim_{n\to\infty} a_{n,1} \approx 1.86,\\
&& j_{n,1}^{(1)} &= n + a_{n,1}^{(1)}n^{1/3}, \text{ with } a_{n,1}^{(1)} > \lim_{n\to\infty} a_{n,1}^{(1)} \approx 0.81,\\
&& y_{n,1} &= n + b_{n,1}n^{1/3}, \text{ with } b_{n,1} > \lim_{n\to\infty} b_{n,1} \approx 0.93,\\
&& y_{n,1}^{(1)} &= n + b_{n,1}^{(1)}n^{1/3}, \text{ with } b_{n,1}^{(1)} > \lim_{n\to\infty} b_{n,1}^{(1)} \approx 1.82,\\
&\text{when } n = 0, \quad & j_{0,1} &\approx 2.40, \quad j_{0,1}^{(1)} = 0, \quad y_{0,1} \approx 0.894 \quad y_{0,1}^{(1)} \approx 2.20.
\end{aligned} \tag{4.1}$$

The zeros of $J_n(\cdot)$ and $Y_n(\cdot)$ are interlacing [13, 10.21] and we have

$$n \leq j_{n,1}^{(1)} < y_{n,1} < y_{n,1}^{(1)} < j_{n,1} < \ldots < j_{n,k}^{(1)} < j_{n,k} < \ldots \tag{4.2}$$

and the first inequality is strict when $n > 0$. Using the notations

$$M_n(x) = \sqrt{J_n(x)^2 + Y_n(x)^2}, \theta_n(x) = \arg(J_n(x) + iY_n(x)), \tag{4.3}$$

we have [13, 10.18]

$$\lim_{x\to 0+}\theta_n(x) = -\frac{\pi}{2}, \quad \frac{d}{dx}\theta_n(x) > 0, \quad \theta_n(x) \approx x - \frac{2n+1}{4}\pi \text{ for } x \text{ large.} \tag{4.4}$$

This shows in particular that for n fixed, the size of the intervals $(j_{n,1}^{(1)}, j_{n,1})$ is strictly decreasing, and tends to $\pi/2$.

Definition 4.1. *For any* $n \geq 0$*, the triplet* (n, x, λ) *is called quasi-resonant if*
$$0 < x < y_{n,1},$$
and if the reflection coefficient given by (1.8) *is of maximal amplitude, that is,*
$$R_n(x,\lambda) = -1.$$

When $(n,\omega_\varepsilon,\lambda)$ is quasi-resonant, problem (1.2) has the following particular solution

$$u_\varepsilon=\begin{cases}\frac{Y_n(\omega_\varepsilon)}{J_n(\lambda\omega_\varepsilon)}J_n\left(\lambda\omega_\varepsilon\frac{|x|}{\varepsilon}\right)\exp\left(i\,n\arctan\left(\frac{x}{|x|}\right)\right) & \text{when } |x|\le\varepsilon\\ Y_n\left(\omega_\varepsilon\frac{|x|}{\varepsilon}\right)\exp\left(i\,n\arctan\left(\frac{x}{|x|}\right)\right) & \text{when } |x|\ge\varepsilon.\end{cases}\tag{4.5}$$

Note that u_ε is not truly a resonance, since $Y_n(\cdot)$ does not satisfy the outgoing radiation condition. The solution u_ε contains an incident field given by

$$u^i=J_n\left(\omega_\varepsilon\frac{|x|}{\varepsilon}\right)\exp\left(i\,n\arctan\left(\frac{x}{|x|}\right)+\frac{\pi}{2}\right).$$

The almost resonant behavior of this solution is apparent in the near field. The amplitude of the incident field at $|x|=\varepsilon$ is $J_n(\omega_\varepsilon)$, whereas the amplitude of the scattered field is given by $|H_n^{(1)}(\omega_\varepsilon)|$. Suppose for example that $\omega_\varepsilon\approx nK$, with $K>1$ fixed - as Proposition 4.2 below shows, this is the generic case. Then at $|x|=\varepsilon$ the amplitude of u_ε grows geometrically with n, [13, 10.19]

$$\lim_{n\to\infty}|Y_n(\omega_\varepsilon)|^{\frac{1}{n}}=\left(K+\sqrt{K-1}\right)e^{\sqrt{1-\frac{1}{K}}},$$

whereas the amplitude of the incident field and its normal derivative at decays with the inverse rate,

$$\lim_{n\to\infty}|J_n(\omega_\varepsilon)|^{\frac{1}{n}}=\lim_{n\to\infty}|\frac{d}{dx}J_n(\omega_\varepsilon)|^{\frac{1}{n}}=\frac{1}{K+\sqrt{K-1}}e^{-\sqrt{1-\frac{1}{K}}}.$$

The size of the scattered field is therefore not controlled by the size of the incident field: this behavior can be compared to that of a resonant mode. Note that the amplitude of the scattered field is also large compared to the maximal amplitude of the incident field anywhere, as the uniform bound (1.17) indicates. The lower bound for the maximal value of the incident field is the motivation from the restriction $\omega_\varepsilon<y_{n,1}$ in the definition of quasi-resonances. Indeed, when $\omega_\varepsilon>y_{n,1}$, using the bound [17, 13.74]

$$\left|H_n^{(1)}(\omega_\varepsilon)\right|\le\sqrt{\frac{2}{\pi\sqrt{\omega_\varepsilon^2-n^2}}},$$

we obtain, for all $n\ge 0$

$$\left|H_n^{(1)}(\omega_\varepsilon)\right|\le\frac{7}{5}\sup_{x>0}|J_n(x)|,\tag{4.6}$$

therefore the scattered field, and in turn the full field, is comparable to the maximal amplitude of the incident field in this regime.

The following variant of Dixon's Theorem on interlacing zeros [17, 15.23] proves the existence of quasi-resonances.

Proposition 4.2. *For any $n\ge 0$ and $\lambda>j_{n,1}/y_{n,1}$, in every interval*

$$U_{n,k}=\left(\frac{j_{n,k}^{(1)}}{\lambda}\frac{j_{n,k}}{\lambda}\right)\text{ such that }U_{n,k}\subset\left(\frac{j_{n,1}^{(1)}}{\lambda},y_{n,1}\right)$$

there exists a unique frequency $\omega_{n,k}$ such that the triplet $(n,\omega_{n,k},\lambda)$ is quasi-resonant. There are no quasi-resonances in the interval $(0,j_{n,1}^{(1)}/\lambda)$ when $n\ge 1$, or when $\lambda<j_{n,1}/y_{n,1}$.

The proof is given at the end of this section. To illustrate this result, we consider the case when $\lambda=2$ and $n=30$. The quasi-resonances are to be found in the interval $(j_{30,1}^{(1)}/2,y_{30,1})\approx(16.28,32,98)$. There are 8 such frequencies. The first one is $\omega_{30,1}\approx 17.4211682$, and the last one is $\omega_{30,8}\approx 31.4683226$. Figure 4.1 shows two plots on a logarithmic scale. The red line shows the radial component of full field u_ε, corresponding to a relative index $\lambda=2$, an effective frequency $\omega_{30,1}$. The blue line shows the radial component of the incident field u^i, $J_{30}(\omega_{30,1}\cdot)$. Note that the blow-up region is concentrated around ε. At $|x|=\lambda\varepsilon=2\varepsilon$, the full field and the incident field are of the same order of magnitude. This is the far field regime discussed in Section 5.

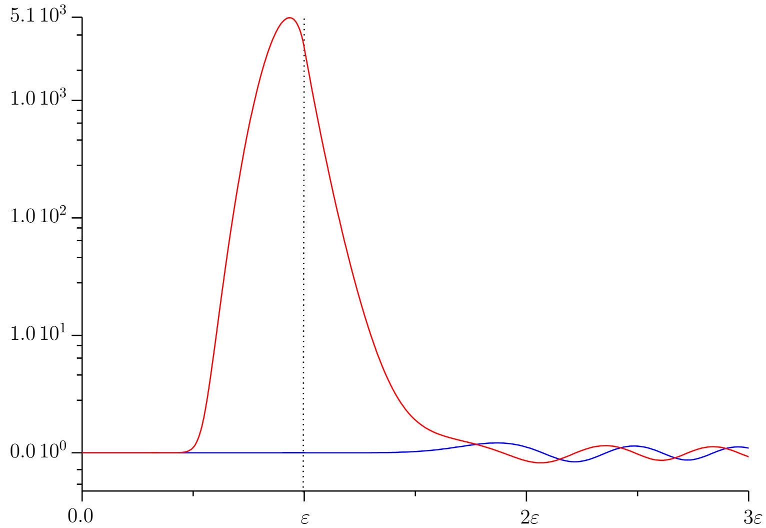

FIGURE 4.1. First quasi-resonant solution for $\lambda = 2$ and $n = 30$.

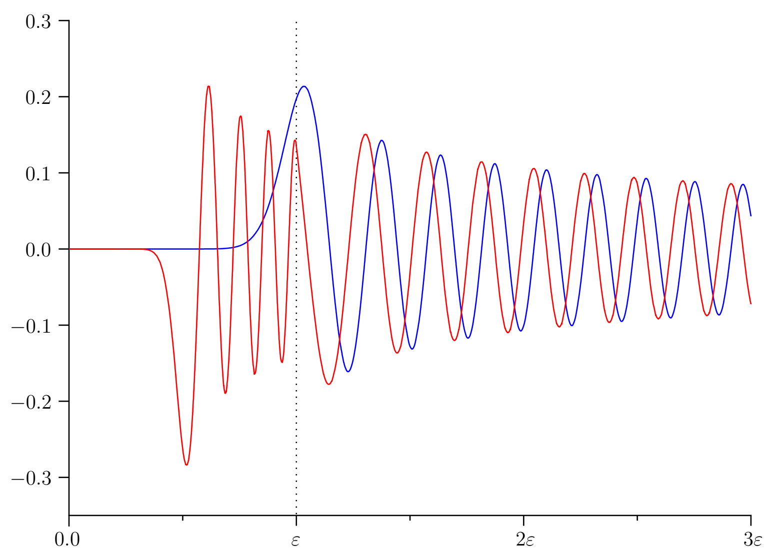

FIGURE 4.2. Last quasi-resonant solution for $\lambda = 2$ and $n = 30$.

Figure 4.2 shows a plot the radial component of full field u_ε in red, corresponding to a relative index $\lambda = 2$, an effective frequency $\omega_{30,8}$, and the radial component of the incident field u^i, $J_{30}(\omega_{30,8}\cdot)$, in blue.

This last quasi-resonance, situated close to the upper bound $y_{30,1}$, does not show a blow-up around $|x| = \varepsilon$. This vindicates the choice to limit the definition of quasi-resonances to the interval $(0, y_{n,1})$.

Quasi-resonances provide lower-bounds for frequency independent scattering estimates, as the following Theorem shows.

Theorem 4.3. *Given $\lambda > 1$, let n_0 be the smallest integer such that*

$$\lambda > \frac{j_{n_0,1}}{y_{n_0,1}}.$$

Then, for any $p \geq n_0$,

$$\sup\left\{\|u^s_\varepsilon(|x|=R)\|_{H^\sigma}, 0<\lambda\omega_\varepsilon \leq j_{p,1}\right\} \geq \sup_{n_0\leq n\leq p}\left(|a_{|n|}|(1+|n|)^\sigma H^{(1)}_n\left(\frac{j_{n,1}}{\lambda}\frac{R}{\varepsilon}\right)\right). \tag{4.7}$$

Furthermore, if $\lambda > \exp(2)$,

$$\sup\left\{\|u^s_\varepsilon(|x|=R)\|_{H^\sigma}, 0<\lambda\omega_\varepsilon < \frac{\sqrt{2}}{\sqrt{\ln\lambda}}\left(1+\frac{1}{2\sqrt{\ln\lambda}}\right)\right\}$$

$$\geq |a_0| H^{(1)}_0\left(\frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1+\frac{1}{2\sqrt{\ln\lambda}}\right)\frac{R}{\varepsilon}\right). \tag{4.8}$$

Remark 4.4. Note that $j_{n,1}/y_{n,1} = 1 + O(n^{-2/3})$ so for any $\lambda > 1$, n_0 exists and is finite. Since $j_{n,1}/n = 1 + O(n^{-2/3}) < 4$, the lower bound (4.7) also matches the end of the perturbative regime described in Theorem 3.1, which required $\omega_\varepsilon < n/\lambda$.

As we noted earlier, the lower bound blows-up geometrically with n. Thus, taking $p = \infty$, if the coefficients a_n decay only polynomially with n, then

$$\sup_{\omega>0}\|u^s_\varepsilon(|x|=\varepsilon)\|_{H^\sigma} = \infty$$

for any $s > -\infty$. This is the case for plane waves, for example, since

$$\exp(i\omega x\cdot\zeta) = \sum_{-\infty}^{\infty} J_n(\omega|\zeta||x|)\exp(in(\arg\left(\frac{x}{|x|}-\frac{\zeta}{|\zeta|}\right)+\frac{\pi}{2})).$$

Estimate (4.8) shows that even low frequencies are affected by quasi-resonances. However, the blow-up is milder. Indeed, $J_0(x)$ tends to 1 close to the origin, whereas

$$\left|H^{(1)}_0\left(\frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1+\frac{1}{2\sqrt{\ln\lambda}}\right)\right)\right| \approx \frac{2}{\pi}\ln(\lambda).$$

This quasi-resonance also occurs just after the perturbative regime, which applies when $\omega_\varepsilon < m_\lambda$. We may therefore argue that the estimates provided by Theorem 2.4 are optimal in terms of frequency range, up to a multiplicative factor of at most 4.

Proof of Theorem 4.3. Starting from formula (1.14), we have

$$\begin{aligned}\sup_{\omega_\varepsilon\leq j_{p,1}/\lambda}\|u^s_\varepsilon(|x|=R)\|_{H^\sigma} &\geq \sup_{\omega_\varepsilon\leq j_{p,1}/\lambda}\sup_{n_0\leq n\leq p}\sqrt{2\pi}\left|R_n(\omega_\varepsilon,\lambda)\,a_{|n|}\right|(1+|n|)^\sigma\left|H^{(1)}_n\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|\\ &\geq \sup_{n_0\leq n\leq p}\sqrt{2\pi}\left|a_{|n|}\right|(1+|n|)^\sigma\left|H^{(1)}_n\left(\omega_{n,1}\frac{R}{\varepsilon}\right)\right|.\end{aligned}$$

Where we used that $|R_n(\omega_{n,1},\lambda)| = 1$. Since $x \to \left|H^{(1)}_n(x)\right|$ is decreasing and from Proposition 4.2 $\omega_{n,1} < j_{n,1}/\lambda$, we obtain (4.7). The second bound (4.8) is proved similarly, using the monotonicity of $x \to \left|H^{(1)}_0(x)\right|$, and Lemma 4.5 below which shows that when $\lambda > \exp(2)$,

$$\omega_{0,1} < \frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1+\frac{1}{2\sqrt{\ln\lambda}}\right).$$

□

Let us now turn to the proof of Proposition 4.2. From the definition of R_n (1.8), it is clear that $R_n(x,\lambda) = -1$ if and only if

$$\Im\left(H^{(1)\prime}_n(x)\,J_n(\lambda x) - \lambda J'_n(\lambda x)\,H^{(1)}_n(x)\right) = Y'_n(x)\,J_n(\lambda x) - \lambda J'_n(\lambda x)\,Y_n(x) = 0$$

When $0 < x < y_{n,1}$, inequalities (4.2) show that $Y'_n(x) < 0$ and $Y'_n(x) > 0$. Dixon's Theorem [17, 15.23] shows that $J_n(x)$ and $J'_n(x)$ have no common zeros. Thus quasi-resonances cannot occur at any $j_{n,k}/\lambda$ or $j^{(1)}_{n,k}/\lambda$. Lastly, note that when $n > 1$, in the set $(0, j^{(1)}_{n,1}/\lambda)$, both $J_n(\lambda x)$ and

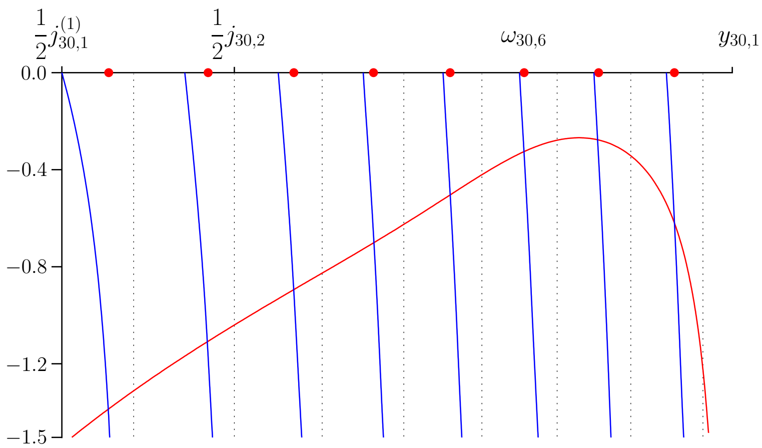

FIGURE 4.3. Quasi-resonant frequencies for $\lambda = 2$ and $n = 30$.

$J'_n(\lambda x)$ are positive, thus no quasi-resonance can occur. The quasi-resonances can only be in the sets $\left(\frac{j^{(1)}_{n,k}}{\lambda}\frac{j_{n,k}}{\lambda}\right)$, and are the solutions of

$$\lambda\frac{J'_n(\lambda x)}{J_n(\lambda x)} = \frac{Y'_n(x)}{Y_n(x)}. \tag{4.9}$$

Figure 4.3 shows a plot of $x \to 2J'_{30}(2x)/J_{30}(2x)$, in blue, and $x \to Y'_n(x)/Y_n(x)$, in red, in the interval $(j^{(1)}_{30,1}/2, y_{30,1}) \approx (16.28, 32, 98)$. The dashed lines represent the solutions of $J_{30}(2x) = 0$ in this interval. The eight red dots on the horizontal axis mark the quasi-resonant frequencies corresponding to $n = 30$ and $\lambda = 2$. To study the solutions of (4.9), we introduce, when $n \geq 1$

$$g_n := x \to \frac{x}{n}\frac{J'_n(x)}{J_n(x)} \text{ and } k_n := x \to -\frac{x}{n}\frac{Y'_n(x)}{Y_n(x)}, \tag{4.10}$$

when $n = 0$,

$$g_0(x) := x \to x\frac{J'_0(x)}{J_0(x)} \text{ and } k_0(x) := x \to -x\frac{Y'_0(x)}{Y_0(x)}. \tag{4.11}$$

and we rewrite (4.9) as

$$g_n(\lambda x) = -k_n(x). \tag{4.12}$$

Proof of Proposition 4.2. From the recurrence relation satisfied by Bessel functions, we derive that for all $n \geq 0$,

$$g'_n(x) = \frac{n}{x} - \frac{x}{n_+} - \frac{n_+}{x}g_n^2(x), \tag{4.13}$$

with the notation $n_+ = \max(n, 1)$. In particular, for $x > n$, g_n is decreasing. Thus, on each interval $U_{n,k}$, $x \to g_n(\lambda x)$ decreases from 0 to $-\infty$. When $n \geq 1$, on $[j^{(1)}_{n,1}/\lambda, y_{n,1})$, k_n is positive, thus there exists at least one solution to (4.12). When $n = 0$, since $(k_0(x) + g_0(x))/x$ tends to ∞ as x tends to zero, therefore at least one solution exists in $(0, j_{0,1}/\lambda)$.

To show uniqueness, we compute that the derivative of $g_n(\lambda\cdot) + k_n(\cdot)$ is

$$\lambda g'_n(\lambda x) + k'_n(x) = \frac{x}{n_+}\left(1 - \lambda^2\right) + \frac{n_+}{x}\left(k_n^2(x) - g_n^2(\lambda x)\right),$$

so at any point where $g_n(\lambda x) = -k_n(x)$, we have

$$\lambda g'_n(\lambda x) + k'_n(x) = \frac{x}{n_+}\left(1 - \lambda^2\right) < 0.$$

Thanks to the Intermediate Value Theorem, there can therefore only be one solution. Finally, note that when $\lambda < j_{n,1}/y_{n,1}$, $x^{-1}(k_n(x) + g_n(\lambda x))$ tends to $+\infty$ both at $x = 0$ and $x = y_{n,1}$, therefore there cannot be a unique solution of $k_n(x)) + g_n(\lambda x) = 0$ in this interval, and consequently there is none. $\square$

We conclude this section by an upper and lower estimate of $\omega_{0,1}$.

Lemma 4.5. *The quasi-resonant triplet* $(0, \omega_{0,1}, \lambda)$ *satisfies*

$$\frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1 - \frac{1}{2\sqrt{\ln(\lambda)}}\right) < \omega_{0,1} < \frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1 + \frac{1}{2\sqrt{\ln\lambda}}\right), \tag{4.14}$$

for all $\lambda \geq \exp(2)$.

Proof. Introducing the functions f_+ and f_- given by

$$f_\pm(\lambda) := \frac{\sqrt{2}}{\lambda\sqrt{\ln\lambda}}\left(1 \pm \frac{1}{2\sqrt{\ln(\lambda)}}\right),$$

thanks to Proposition 4.2, it is sufficient to check that

$$g_0(\lambda f_+(\lambda)) + k_0(f_+(\lambda)) < 0, \text{ and } g_0(\lambda f_-(\lambda)) + k_0(f_-(\lambda)) > 0.$$

The following bounds

$$-\frac{1}{2}x^2 - \frac{1}{12}x^4 \leq g_0(x) \leq -\frac{1}{2}x^2 - \frac{1}{16}x^4 \text{ for all } 0 \leq x \leq 1,$$

and

$$-\frac{1}{\gamma + \ln\left(\frac{x}{2}\right)} + \frac{1}{2}x^2 \leq k_0(x) \leq -\frac{1}{\gamma + \ln\left(\frac{x}{2}\right)} + x^2 \text{ for all } 0 \leq x \leq \frac{1}{4}.$$

can be derived using the asymptotic expansions of Bessel functions around $x = 0$ given in [13]. The proof (4.14) becomes a study of a function of one variable, λ. We omit this tedious but straightforward calculation. It is easy to visually confirm this result using a modern scientific computation software, using the built-in formulae of $J_0(x)$, $J_1(x)$, $Y_0(x)$ and $Y_1(x)$ to compute g_0 and k_0, and then verify for example that $\pm g_0(\lambda f_\pm(\lambda)) + \pm k_0(f_\pm(\lambda)) < 0$, and that both expressions are of order $(\ln\lambda)^{-3/2}$ for large λ. The lower bound $\lambda \geq \exp(2)$ is not optimal: it is convenient because of the form of the ansatz for k_0 given above. Numerically, it appears that (4.14) holds almost up threshold value $j_{0,1}/y_{0,1}$ (up to 1.003 times that value). $\square$

5. Inclusions with relative index larger than one: far-field estimates

As we could notice on Figure 4.1, the effect of quasi-resonances is localized close to $|x| = \varepsilon$. We now show that in the far field, that is, when $\lambda\varepsilon \leq |x|$, estimates valid for all frequencies can be derived in the spirit of Theorem 2.4.

Theorem 5.1. *When* $1 \leq \lambda$ *and* $\varepsilon\lambda < R$ *there holds*

$$\begin{aligned}\sup_{\omega>0}\|u^s_\varepsilon(|x| = R)\|_{H^\sigma} \leq{}& \frac{5}{2}\sqrt{\lambda\frac{\varepsilon}{R}}\left\|u^i(|x| = \varepsilon)\right\|_{H^\sigma_*} + 2\sqrt{\sum_{n\neq 0}|a_n|^2\frac{(1+|n|)^{2\sigma}}{\left(j^{(1)}_{n,1}\frac{R}{\varepsilon}\right)^2 - n^2}} \\ &+ \sqrt{10\pi}\left|u^i(0)\right|\left|H^{(1)}_0\left(\min\left(\frac{1}{2}, m_\lambda\right)\frac{R}{\varepsilon}\right)\right|,\end{aligned} \tag{5.1}$$

Furthermore,

$$\sup_{\omega>0}\|u^s_\varepsilon(|x| = R)\|_{H^\sigma_*} \geq \frac{2}{5}\sqrt{\lambda\frac{\varepsilon}{R}}\mathbf{N}_1^{\sigma-\frac{1}{6}}\left(u^i\right). \tag{5.2}$$

Remark 5.2. Just like in Theorem 2.4, the upper bound (5.1) can be replaced by the frequency independent bound

$$\|u^s_\varepsilon(|x| = R)\|_{H^\sigma} \leq \frac{5}{2}\sqrt{\lambda\frac{\varepsilon}{R}}\mathcal{N}^\sigma\left(u^i\right) + \sqrt{10\pi}\left|u^i(0)\right|\left|H^{(1)}_0\left(\min\left(\frac{1}{2}, m_\lambda\right)\frac{R}{\varepsilon}\right)\right|. \tag{5.3}$$

We chose the form (5.1) to obtain an optimal decay rate in n for $R \gg \varepsilon$. The dependence on the zero order term is sharp, as we have seen in Section 4. Note that the lower bound (5.2) and upper bound (5.1) have the same dependence on the contrast, and on ε/R.

Proof. Starting from the formula (1.14), using the monotonicity of $x\left|\frac{H_n^{(1)}(\sqrt{q_0}\omega R)}{H_n^{(1)}(\lambda\omega_\varepsilon)}\right|^2$ for $n \geq 1$ as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\|u_\varepsilon^s(|x|=R)\|_{H^\sigma}^2 \leq \quad & \lambda\frac{\varepsilon}{R}\left(2\pi\sum_{I_1}|a_n|^2(1+|n|)^{2\sigma}\left|R_n(\omega_\varepsilon,\lambda)\,|H_n^{(1)}(\lambda\omega_\varepsilon)\right|^2\right) \\ & 2\pi\sum_{I_2}|a_n|^2(1+|n|)^{2\sigma}\left|R_n(\omega_\varepsilon,\lambda)\,|H_n^{(1)}\left(\lambda\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|^2 \\ & +2\pi\,|a_0|^2\left|R_0(\omega_\varepsilon,\lambda)\,H_0^{(1)}(\sqrt{q_0}\omega R)\right|^2,\end{aligned}$$

where I_1 is the set of indices n for which $0<|n|$ and $\lambda\omega_\varepsilon < j_{|n|,1}^{(1)}$, and I_2 is the set of indices n for which $0<|n|$ and $\lambda\omega_\varepsilon > j_{n,1}^{(1)}$. Thanks to Proposition 3.2,

$$\left|R_0(\omega_\varepsilon,\lambda)\,H_0^{(1)}(\sqrt{q_0}\omega R)\right| \leq \sqrt{5}\left|H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|.$$

and when $n \in I_1$,

$$\left|R_n(\omega_\varepsilon,\lambda)\,H_n^{(1)}(\omega_\varepsilon)\right| \leq \frac{5}{2}\left|J_n(\omega_\varepsilon)\right|.$$

Alternatively, as in Theorem 2.4, when $n \in I_2$,

$$\left|R_n(\omega_\varepsilon,\lambda)\,H_n^{(1)}(\omega_\varepsilon)\right| \leq \sqrt{\frac{2}{\pi}\frac{1}{\sqrt{\left(j_{n,1}^{(1)}\frac{R}{\varepsilon}\right)^2-n^2}}}$$

Therefore, altogether, we have obtained that for all $\omega_\varepsilon > 0$,

$$\begin{aligned}& 2\pi\sum_{|n|\neq 0}|a_n|^2(1+|n|)^{2\sigma}\left|R_n(\omega_\varepsilon,\lambda)\,|H_n^{(1)}(\omega_\varepsilon)\right|^2 \\ \leq\quad & \frac{25}{4}2\pi\sum_{n\neq 0}|a_n|^2(1+|n|)^{2\sigma}\left|J_n(\omega_\varepsilon)\right|^2 \\ & +4\sum_{n\neq 0}|a_n|^2\frac{(1+|n|)^{2\sigma}}{\left(j_{n,1}^{(1)}\frac{R}{\varepsilon}\right)^2-n^2},\end{aligned}$$

which concludes our proof of the upper bound. Turning to the lower bound, we have

$$\begin{aligned}\|u_\varepsilon^s(|x|=R)\|_{H^\sigma} \quad & \geq \quad \sqrt{2\pi}\sup_{|n|\geq 1}\sup_{\omega_\varepsilon>0}|R_n^\varepsilon a_n|(1+|n|)^\sigma\left|H_n^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right| \\ & \geq \quad \sqrt{2\pi}\sup_{|n|\geq 1}|a_n|(1+|n|)^\sigma\left|H_n^{(1)}\left(\omega_{n,1}\frac{R}{\varepsilon}\right)\right|,\end{aligned}$$

where we used the first quasi-resonant frequency $\omega_{n,1}$ given by Proposition 4.2. Since $\sqrt{x}\left|H_n^1(x)\right|$ decreases to $\sqrt{2/\pi}$ [17, 13.74],

$$\left|H_n^{(1)}\left(\omega_{n,1}\frac{R}{\varepsilon}\right)\right| > \sqrt{\frac{2\varepsilon}{\pi R\omega_{n,1}}}$$

and, in turn, since from (1.16),

$$\omega_{n,1} < \frac{j_{n,1}}{\lambda} < \frac{4n}{\lambda} < 4\frac{(n+1)^{1/3}}{\lambda}\left(\frac{6}{7}\right)^2\left(\sup_{x>0}|J_n(x)|\right)^{-2},$$

we obtain

$$\|u_\varepsilon^s\left(|x|=R\right)\|_{H^\sigma_*} \geq \frac{2}{5}\sqrt{\lambda\frac{\varepsilon}{R}}\mathbf{N}_1^{\sigma-\frac{1}{6}}\left(u^i\right),$$

as announced. □

6. Broadband contrast and frequency independent estimates

In the previous sections, we have seen that we cannot hope for contrast independent estimates for all frequencies, because of the appearance of quasi-resonances. Combining Theorem 2.4 and Theorem 5.1, we see that scattered field tends to zero at a rate $\varepsilon^{1-\delta}$, when observed at a fixed distance, say $R=1$, provided the contrast λ does not grow faster than $\varepsilon^{-\delta}$ with $\delta<1$. On the other hand, when λ is of size ε^{-1} or larger, the lower bound provided by Theorem 5.1 shows that for some frequencies, the quasi-resonant frequencies, some components of the scattered field will be of size one or larger.

The following proposition shows that if an interval around quasi-resonant frequencies is excluded, the scattered field can be controlled by the incident field. It is proved in Section 8.

Proposition 6.1. *For any $0<\tau\leq\frac{1}{4}$, we define*

$$I_{n,k}(\tau) := \left\{x\in U_{n,k} \text{ such that } |g_n(\lambda x)+k_n(x)|\leq\tau\,|k_n(x)|\right\}, \tag{6.1}$$

If $\lambda>7$, $n\geq1$ and $\omega_\varepsilon\in(0,y_{n,1})\setminus(\cup_k I_{n,k}(\tau))$, then,

$$\left|R_n\left(\omega_\varepsilon,\lambda\right)H_n^{(1)}\left(\omega_\varepsilon\right)\right| \leq \frac{9}{2\,\tau}J_n\left(\omega_\varepsilon\right).$$

we have

$$(0,y_{n,1})\setminus(\cup_k I_{n,k}(\tau)) = (0,y_{n,1})\setminus\bigcup_{k\in K(\lambda,n)} I_{n,k}(\tau)$$

where $K(\lambda,n)$ is the set of all positive k such that $j_{n,k}^{(1)}<n\lambda$. Furthermore,

$$\left|\bigcup_{k\in K(\lambda,n)} I_{n,k}(\tau)\right| \leq 6\tau\frac{n\ln\lambda}{\lambda}.$$

If $\lambda>7$, $n=0$ and $\omega_\varepsilon\in(0,\zeta_0)\setminus(\cup_k I_{0,k}(\tau))$, where $\zeta_0\approx0.3135$ is defined Proposition A.2, then

$$\left|R_0\left(\omega_\varepsilon,\lambda\right)H_0^{(1)}\left(\omega_\varepsilon\right)\right| \leq \frac{5}{3\tau}J_0\left(\omega_\varepsilon\right) \leq \frac{5}{3\,\tau}.$$

We have

$$(0,\zeta_0)\setminus(\cup_k I_{0,k}(\tau)) = (0,\zeta_0)\setminus\bigcup_{k\in K(\lambda,0)} I_{0,k}(\tau)$$

where $K(\lambda,0)$ is the set of all positive k such that $j_{0,k}^{(1)}<\lambda\zeta_0$. Furthermore,

$$\left|\bigcup_{k\in K(\lambda,0)} I_{0,k}(\tau)\right| \leq 7\tau\frac{\ln(\ln\lambda)}{\lambda}.$$

Proof. This is the result of Proposition 8.3 (together with Lemma 7.1 for $\omega_\varepsilon<j_{n,1}^{(1)}/\lambda$ and Proposition 7.6 when $n=0$), and Proposition 8.2. □

This result allows us to prove the following.

Lemma 6.2. *Suppose $\lambda>7$. Let $\eta_{\max}$ be the following decreasing function of the contrast*

$$\eta_{\max} = \frac{3}{2}\frac{\ln\lambda}{\lambda}. \tag{6.2}$$

Given $\alpha>0$, for any $\eta>0$ such that

$$\eta\leq\frac{1}{\alpha}\eta_{\max}$$

there exists a set I_1 depending on $\eta, \alpha, \varepsilon$ and λ such that

$$|I_1| < \frac{\eta}{\varepsilon}$$

and, for any $R \geq \varepsilon$

$$\sup_{\sqrt{q_0}\omega \in (0,\infty) \setminus I_1} \|u^s_\varepsilon (|x| = R)\|_{H^\sigma_*} \leq 18\sqrt{\frac{\varepsilon}{R}} \frac{\eta_{\max}}{\eta\alpha} \mathcal{N}^{\sigma+2+\alpha}\left(u^i\right).$$

Let η_0 be given by

$$\eta_0 = \frac{7}{4}\frac{\ln(\ln\lambda)}{\lambda}.$$

for any $\eta > 0$ such that

$$\eta \leq \eta_0$$

there exists a set $I_0 \subset (0, \zeta_0)$ which depends on η, ε and λ such that

$$|I_0| < \frac{\eta}{\varepsilon}$$

and, for any $R \geq \varepsilon$

$$\sup_{\sqrt{q_0}\omega \in (0,\infty) \setminus I_0} \left| \frac{1}{2\pi} \int_0^{2\pi} u^s_\varepsilon (|x| = R) \right| \leq 7\frac{\eta_0}{\eta} \left| \frac{H^{(1)}_0\left(m_\lambda \frac{R}{\varepsilon}\right)}{H^{(1)}_0 (m_\lambda)} \right|.$$

Remark 6.3. Lemma 6.2 shows that by excluding some frequencies, around quasi-resonances, near-field estimates can be obtained, up to the boundary of the inclusion $|x| = \varepsilon$, at the cost of a little more than two powers of n when compared to the near field estimates given by Theorem 2.4 for $\lambda < 1$. We showed in Theorem 4.3 that if the quasi-resonances are not excluded, the blow up is geometric in n. The most striking feature of this result is that since $\eta_{\max}$ and η_0 tend to zero as λ tends to ∞, the size of the set of frequencies to exclude shrinks as the contrast increases.

Proof of Lemma 6.2. Starting from the formula (1.14), using the monotonicity of $x \left|\frac{H^{(1)}_n\left(\sqrt{q_0}\omega R\right)}{H^{(1)}_n(\omega_\varepsilon)}\right|^2$ for $n \geq 1$ as in the proof of Theorem 2.1, we obtain

$$\|u^s_\varepsilon (|x| = R)\|^2_{H^\sigma_*} \leq \frac{\varepsilon}{R}\left(2\pi \sum_{|n|\neq 0} |a_n|^2 \frac{1}{\tau_n^2} (1 + |n|)^{2\sigma} \left|R_n(\omega_\varepsilon, \lambda) |H^{(1)}_n(\lambda\omega_\varepsilon)\right|^2\right),$$

for any sequence of positive parameters $0 < \tau_n < \frac{1}{4}$ to be chosen later. Next we divide the non zero indices into three parts. The first set of indices is

$$N_1 := \left\{n \neq 0 \text{ such that } \lambda\omega_\varepsilon \leq j^{(1)}_{|n|,1} \text{ or } \omega_\varepsilon > n\right\}.$$

In N_1, thanks to Proposition 3.2 and Proposition 7.3, we have that either

$$\left|R_n(\omega_\varepsilon, \lambda) |H^{(1)}_n(\lambda\omega_\varepsilon)\right| \leq \frac{5}{2} J_n(\omega_\varepsilon) \leq \frac{5}{2}\sup_{x>0} |J_n(x)|$$

or $\omega_\varepsilon > y_{n,1}$, and (4.6) shows that

$$\left|R_n(\omega_\varepsilon, \lambda) |H^{(1)}_{|n|}(\lambda\omega_\varepsilon)\right|^2 \leq \frac{7}{5}\sup_{x>0} |J_n(x)|$$

We define the sets

$$N_2 := \left\{n \neq 0 \text{ such that } \left|g_{|n|}(\lambda x) + k_{|n|}(x)\right| \leq \tau_n \left|k_{|n|}(x)\right|\right\},$$

and

$$N_3 := \{n \neq 0, n \notin S_1 \text{ and } n \notin S_2\}.$$

Proposition 6.1 shows that for all $n \in N_3$

$$\left|R_n(\omega_\varepsilon, \lambda) H^{(1)}_n(\omega_\varepsilon)\right| \leq \frac{9}{2}\frac{1}{\tau_n} J_n(\omega_\varepsilon) \leq \frac{9}{2}\frac{1}{\tau_n}\sup_{x>0} |J_n(x)|$$

We have obtained that for all ω such that $N_2=\emptyset$, we have

$$\|u_\varepsilon^s(|x|=R)\|_{H_*^\sigma}^2<2\pi\left(\frac{9}{2}\right)^2\frac{\varepsilon}{R}\sum_{n\neq0}|a_n|\frac{1}{\tau_n^2}(1+|n|)^{2\sigma}\sup_{x>0}|J_n(x)|^2. \tag{6.3}$$

Thanks to Proposition 6.1, the forbidden values for $\sqrt{q_0}\omega$ lie in a collection of intervals $O_\varepsilon:=\varepsilon^{-1}\bigcup_{n,k}I_{n,k}(\tau_n)$ of total size of at most

$$|O_\varepsilon|\le\sum_{n=1}^{\infty}n\tau_n\frac{6\ln\lambda}{\varepsilon\lambda}=4\frac{\eta_{\max}}{\varepsilon}\sum_{n=1}^{\infty}n\tau_n.$$

This leads us to choose

$$\tau_n=\frac{\eta\alpha}{(1+|n|)^{2+\alpha}}\frac{1}{4\eta_{\max}}.$$

Then, an upper estimate of the total size of the forbidden intervals is

$$|O_\varepsilon|\le\frac{1}{\varepsilon}\sum_{n=1}^{\infty}\frac{\alpha}{(1+n)^{1+\alpha}}\le\frac{\eta}{\varepsilon},$$

and from (6.3) we obtain

$$\|u_\varepsilon^s(|x|=R)\|_{H_*^\sigma}\le18\sqrt{\frac{\varepsilon}{R}}\frac{\eta_{\max}}{\eta\alpha}\mathcal{N}^{\sigma+2+\alpha}\left(u^i\right),$$

as announced.

Let us now consider the other estimate. We have, from (1.5)

$$\left|\frac{1}{2\pi}\int_0^{2\pi}u_\varepsilon^s(|x|=R)\right|=|u^i(0)|\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|.$$

Proposition 7.7 shows that when $\omega_\varepsilon<m_\lambda$,

$$\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|\le4\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}(m_\lambda)}\right|\le\frac{1}{\tau}\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}(m_\lambda)}\right|.$$

Proposition 7.6 shows that when $\omega_\varepsilon>\zeta_0$,

$$\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|\le\left|H_0^{(1)}\left(\zeta_0\frac{R}{\varepsilon}\right)\right|\le\sqrt{\frac{\varepsilon}{R}}\sqrt{\frac{2}{\pi\zeta_0}}\le\frac{3}{2}\sqrt{\frac{\varepsilon}{R}}.$$

From Lemma A.3, we know that

$$\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}(m_\lambda)}\right|\ge\sqrt{\frac{\varepsilon}{R}},$$

therefore when $\omega_\varepsilon>\zeta_0$,

$$\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|\le\frac{1}{\tau}\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}(m_\lambda)}\right|.$$

On the other hand, Proposition 6.1 shows that if $\omega_\varepsilon\in(m_\lambda,\zeta_0)\setminus\bigcup_{k\in K(\lambda,0)}I_{0,k}(\tau)$ we have

$$\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}(\omega_\varepsilon)\right|\le\frac{5}{3}\frac{1}{\tau}.$$

Therefore, Lemma A.3 shows that $x\to\left|H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)/H_0^{(1)}(x)\right|$ is decreasing,

$$\begin{aligned}\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)\right|&=\left|R_0(\omega_\varepsilon,\lambda)H_0^{(1)}(\omega_\varepsilon)\right|\left|\frac{H_0^{(1)}\left(\omega_\varepsilon\frac{R}{\varepsilon}\right)}{H_0^{(1)}(\omega_\varepsilon)}\right|\\&\le\frac{5}{3}\frac{1}{\tau}\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}(m_\lambda)}\right|.\end{aligned}$$

We have obtained, for all $\omega_\varepsilon \in (0,\infty) \setminus \bigcup_{k\in K(\lambda,0)} I_{0,k}(\tau)$, and all $\tau \leq \frac{1}{4}$,

$$\left| R_0\left(\omega_\varepsilon, \lambda\right) H_0^{(1)}\left(\omega_\varepsilon \frac{R}{\varepsilon}\right)\right| \leq \frac{5}{3}\frac{1}{\tau}\left|\frac{H_0^{(1)}\left(m_\lambda \frac{R}{\varepsilon}\right)}{H_0^{(1)}\left(m_\lambda\right)}\right|$$

The total size of the set of forbidden values for $\sqrt{q_0}\omega$ is bounded by

$$|O_\varepsilon| \leq 7\frac{\ln\ln(\lambda)}{\lambda}\frac{\tau}{\varepsilon},$$

thus choosing $\tau = \frac{1}{4}\frac{\eta}{\eta_0}$ establishes our claim. □

As an application of Lemma 6.2, we provide a broadband estimate for the case $\lambda = \varepsilon^{-1}$.

Corollary 6.4. *Assume $\varepsilon < 1/7$, and $\lambda = \varepsilon^{-1}$. Then, for any $\alpha, \beta > 0$, there exists a set I_1 depending on ε, α and β and a set I_0 depending on ε and β which satisfies*

$$|I_1| \leq \varepsilon^\beta \left|\ln\varepsilon\right|, \quad |I_0| \leq \frac{\ln\left|\ln\varepsilon\right|}{\left(\left|\ln\varepsilon\right|+1\right)^\beta},$$

such that for all $R \geq \varepsilon$,

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_1} \left\|u_\varepsilon^s\left(|x|=R\right)\right\|_{H_*^\sigma}^2 \leq \frac{18}{\alpha}\sqrt{\frac{\varepsilon^{1-2\beta}}{R}}\mathcal{N}^{\sigma+2+\alpha}\left(u^i\right),$$

and

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_0}\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq 12\frac{1}{\sqrt{\left(\left|\ln\varepsilon\right|+1\right)^{3/2-2\beta}R}}.$$

Proof. This is an application of Lemma 6.2. In this case $\eta_{\max} = \varepsilon\left|\ln\varepsilon\right|$, and $\eta_0 = \varepsilon\ln\left|\ln\varepsilon\right|$. Choose $\eta = \varepsilon^\beta\eta_{\max}$. We have

$$|I_1| \leq \varepsilon^\beta\left|\ln\varepsilon\right|$$

and

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_1}\left\|u_\varepsilon^s\left(|x|=R\right)\right\|_{H_*^\sigma}^2 \leq \frac{21}{\alpha}\sqrt{\frac{\varepsilon^{1-2\beta}}{R}}.$$

Choose $\eta = \left(\left|\ln\varepsilon\right|+1\right)^{-\beta}\eta_0$. Then,

$$|I_0| \leq \frac{\ln\left|\ln\varepsilon\right|}{\left(\left|\ln\varepsilon\right|+1\right)^\beta}.$$

Using the bound $\left|H_0^{(1)}\left(m_\lambda\right)\right| > \frac{1}{2}(1+\left|\ln\varepsilon\right|)$, and the usual upper bound (7.13),

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_0}\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq 14\sqrt{\frac{2}{\pi}}\left(\left|\ln\varepsilon\right|+1\right)^{\beta-1+\frac{1}{4}}\frac{1}{\sqrt{R}} \leq 12\frac{1}{\sqrt{\left(\left|\ln\varepsilon\right|+1\right)^{3/2-2\beta}R}}.$$

□

Combining Lemma 6.2 with Theorem 2.4 and Theorem 5.1 we obtain the following broadband result, which provides a uniform estimate for all contrast, and almost all frequencies.

Theorem 6.5. *Suppose given $\lambda > 0$ and $\frac{1}{15} > \varepsilon > 0$. For any $\alpha > 0$, there exists a open set $I_1\left(\alpha,\lambda,\varepsilon\right) \subset \mathbb{R}$ satisfying*

$$\left|I_1\left(\alpha,\lambda,\varepsilon\right)\right| \leq \varepsilon^{1/8}\left|\ln\varepsilon\right|$$

and for any $R \geq \varepsilon^{1/4}$ we have

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_1}\left\|u_\varepsilon^s\left(|x|=R\right)\right\|_{H_*^\sigma}^2 \leq \frac{21}{\alpha}\sqrt{\frac{\varepsilon^{1/4}}{R}}\mathcal{N}^{\sigma+2+\alpha}\left(u^i\right). \tag{6.4}$$

When $\lambda \leq \varepsilon^{-3/4}$, (6.4) holds with $I_1 = \emptyset$. There exists another open set $I_0(\varepsilon,\lambda) \subset (0,\zeta_0)$ satisfying

$$|I_0(\lambda,\varepsilon)| \leq \frac{\ln\ln\varepsilon + 2}{\ln\varepsilon + 1}$$

and such that for any R such that $(|\ln\varepsilon| + 1)^{1/12} R \leq 1$, we have

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_0} \left| \frac{1}{2\pi}\int_0^{2\pi} u^s_\varepsilon(|x| = R) \right| \leq 21 \max\left(\sqrt{\frac{2}{(|\ln\varepsilon| + 1)^{1/12} R}}, \left| \frac{H_0^{(1)}(m_\lambda \frac{R}{\varepsilon})}{H_0^{(1)}(m_\lambda)} \right| \right), \tag{6.5}$$

where m_λ is defined in (3.1) *and equals*

$$m_\lambda = \frac{1}{\lambda\sqrt{\ln\lambda + 1}}.$$

When $\lambda^{-1} > \varepsilon(|\ln\varepsilon| + 1)^{7/12}$, (6.5) *holds with $I_0 = \emptyset$.*

Remark 6.6. Note that both I_1 and I_0 have zero measure when ε tends to zero: in this limit, the estimate is true almost everywhere. This is not a far-field result in the sense of 5.1, as we do not require $\lambda R > \varepsilon$. Any fixed positive R is possible for ε small enough. The decay of the size of I_0 is logarithmically slow, even though the quasi-resonances of the zero order term are at most logarithmic in the contrast, see Theorem 4.3. Note that the upper bound in (6.5) is always smaller than the numerical constant 21. Naturally, many variants of Theorem 6.5 can be derived by other combinations of Lemma 6.2, Theorem 2.4 and 5.1, and more precise broadband estimates can be obtained if λ is known to be in a particular range with respect to ε. Corollary 6.4 gives an improved estimate in the case $\lambda = \varepsilon^{-1}$. The dependence on n in (6.4) could probably be improved by a precise study of the distance between the roots of the Bessel Function $J_n(x)$ for $x \in (n/\varepsilon, n)$.

Proof. When $0 < \lambda \leq 1$, Theorem 2.4 implies (6.4), with $I_1 = \emptyset$. Similarly, when $1 < \lambda < \varepsilon^{-3/4}$, Theorem 5.1 implies (6.4), with $I_1 = \emptyset$. . Let us therefore suppose $\lambda > \varepsilon^{-3/4} > 7$. We apply Lemma 6.2 with

$$\eta = \frac{8}{9}\varepsilon^{3/8}\eta_{\max}.$$

Note that Since $\lambda \to \lambda^{-1}\ln\lambda$ is decreasing when $\lambda > 7$, therefore

$$\eta_{\max} \leq \frac{9}{8}|\ln\varepsilon|\,\varepsilon^{3/4},$$

and there exists a set I_1 such that

$$|I_1| \leq \varepsilon^{1/8}|\ln\varepsilon|$$

for which

$$\sup_{\sqrt{q_0}\omega\in(0,\infty)\setminus I_1} \|u^s_\varepsilon(|x| = R)\|^2_{H^\sigma_*} \leq \frac{21}{\alpha}\sqrt{\frac{\varepsilon^{1/4}}{R}}\mathcal{N}^{\sigma+2+\alpha}\left(u^i\right).$$

Let us now turn to the zero order term, and assume $u^i(0) = 1$ by linearity. The cases $\lambda < 1$ or $m_\lambda > 1/2$ are consequences of Theorem 2.4 and Theorem 5.1 with $I_0 = \emptyset$. When $m_\lambda < \frac{1}{2}$, and

$$\lambda \leq \lambda_0 := \frac{1}{\varepsilon(|\ln\varepsilon| + 1)^{7/12}},$$

Theorem 5.1 shows that with $I_0 = \emptyset$,

$$\left| \frac{1}{2\pi}\int_0^{2\pi} u^s_\varepsilon(|x| = R) \right| \leq \sqrt{10\pi}\left| H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right) \right|.$$

Since $H_0^{(1)}(\cdot)$ is decreasing, an upper bound is found by choosing $\lambda = \lambda_0$. Then, $\ln\lambda_0 \leq |\ln\varepsilon|$, and

$$\varepsilon^{-1}m_\lambda = (\ln\varepsilon + 1)^{7/12}\frac{1}{\sqrt{\ln\lambda_0 + 1}} \geq (|\ln\varepsilon| + 1)^{1/12}$$

which yields

$$\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq \sqrt{10\pi}\left|H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)\right| \leq 5\frac{1}{\sqrt{R\left(|\ln\varepsilon|+1\right)^{1/12}}}. \tag{6.6}$$

When

$$\lambda_0<\lambda<\frac{(|\ln\varepsilon|+1)^{1/12}}{\varepsilon},$$

we have

$$m_\lambda > m_1 := \frac{1}{2+e^{-2}}\frac{\varepsilon}{\left(1+|\ln\varepsilon|\right)^{7/12}}.$$

We turn now to Lemma 6.2. We have

$$\eta_0=\frac{7}{4}\frac{\ln\ln\lambda}{\lambda}\leq\frac{\ln\ln\lambda_0}{\lambda_0}\leq\varepsilon\ln(|\ln\varepsilon|)\left(1+|\ln\varepsilon|\right)^{7/12}$$

Choose

$$\eta=\frac{4}{7}\frac{1}{(1+|\ln\varepsilon|)^{2/3}}\eta_0.$$

Then,

$$|I_0|\leq\frac{\ln|\ln\varepsilon|}{(|\ln\varepsilon|+1)^{1/12}}$$

and since $x\to\left|H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|/\left|H_0^{(1)}\left(x\right)\right|$ is decreasing by Lemma A.3, outside of I_0 we have

$$\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq 7\left(\frac{7}{4}(1+|\ln\varepsilon|)^{2/3}\right)\frac{\left|H_0^{(1)}\left(m_1\frac{R}{\varepsilon}\right)\right|}{\left|H_0^{(1)}\left(m_1\right)\right|}. \tag{6.7}$$

We have $\left|H_0^{(1)}\left(m_1\right)\right|>\frac{1}{2}(1+|\ln\varepsilon|)$, and therefore

$$\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq \frac{49}{2}\sqrt{\frac{2(2+e^{-2})}{\pi}}(1+|\ln\varepsilon|)^{2/3-1+7/14}\frac{1}{\sqrt{R}}. \leq 29\frac{1}{\sqrt{R\left(1+|\ln\varepsilon|\right)^{1/12}}}.$$

Let us now assume $\lambda\geq\frac{(|\ln\varepsilon|+1)^{1/12}}{\varepsilon}$. In that case,

$$\eta_0=\frac{7}{4}\frac{\ln\left(\ln\lambda\right)}{\lambda}\leq\frac{11}{6}\varepsilon\frac{\ln|\ln\varepsilon|}{\left(1+|\ln\varepsilon|\right)^{1/12}}.$$

Choosing $\eta=\frac{6}{11}\eta_0$ and applying Lemma 6.2, we obtain

$$|I_0|\leq\frac{\ln|\ln\varepsilon|}{\left(1+|\ln\varepsilon|\right)^{1/12}},\ \text{ and }\ \sup_{\sqrt{q_0}\omega\in(0\infty)\setminus I_0}\left|\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon^s\left(|x|=R\right)\right| \leq 21\frac{\left|H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)\right|}{\left|H_0^{(1)}\left(m_\lambda\right)\right|}. \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) we obtain the (6.5). □

7. Estimates relating the scattered field and the incident field outside quasi-resonances

7.1. **The $n\neq 0$ case.** The main result of this section, Lemma 7.1 proves Proposition 2.6 and Proposition 3.2 when $n\neq 0$. Note that, for a given n, these results are focused on the case when ω_ε is bounded, namely $\omega_\varepsilon<y_{n,1}$. Since $y_{n,1}<j_{n,1}$ for all $n\geq 0$, we are thus only considering the case when $J_n\left(\omega_\varepsilon\right)>0$, and $\omega_\varepsilon<y_{n,1}$. To compare the scattered field with the incident field, it is convenient to introduce a new quantity, namely

$$S_n(\omega_\varepsilon):=-\frac{R_n\left(\omega_\varepsilon,\lambda\right)H_n^{(1)}\left(\omega_\varepsilon\right)}{J_n\left(\omega_\varepsilon\right)}. \tag{7.1}$$

A simple manipulation of the equation, together with the Wronskian identity satisfied by $J_n(x)$ and $Y_n(x)$ shows that S_n has two equivalent formulations, namely

$$S_n(x) = \frac{(g_n(\lambda x) - g_n(x))\,(1 + i\tan\theta_n(x))}{(g_n(\lambda x) - g_n(x)) + i\tan\theta_n(x)\,(g_n(\lambda x) + k_n(x))}, \tag{7.2}$$

$$= \frac{u_n(x)H_n^{(1)}(x)}{u_n(x)H_n^{(1)}(x) + i\frac{2}{\pi}J_n(\lambda x)}, \tag{7.3}$$

(7.4)

where θ_n is given by (4.3), g_n and k_n are defined by (4.10) and (4.11) and where u_n is given by

$$u_n(x) := \max(|n|, 1)J_n(x)\,J_n(\lambda x)\,(g_n(x) - g_n(\lambda x))\,.$$

Note that these formulae are not defined when $\lambda x = j_{n,1}$, that is, at the singular points of $g_n(\lambda x)$. However from Formula (7.3) we see that these singularity can be resolved, and we define S_n as the continuous limit of S_n towards these points (which is 1).

The main result of this section is the following Lemma.

Lemma 7.1. *For all $\lambda \in (0,\infty)$, and all $n \geq 1$, for $x \in \left[0, \min\left(j_{n,1}^{(1)}\lambda^{-1}, y_{n,1}\right)\right]$ we have the following bound*

$$|S_n(x)| \leq \frac{5}{2}.$$

For $x \in \left[0, \min\left(n\lambda^{-1}, n\right)\right]$, we also have

$$|S_n(x)| \leq 2\,|\lambda - 1|\,\frac{x}{n^{1/3}}.$$

and if $\lambda^2 < 1 - \left(\frac{7}{3n^{1/3}}\right)^2$, then

$$|S_n(n)| > \frac{1}{2}.$$

To prove Lemma 7.1, we shall use the following observation.

Proposition 7.2. *For all $0 < x \leq n$, and all $n \geq 1$, we have the following bounds*

$$\left(\frac{3}{5}\right)^2 < \frac{1 + Y_n^2(x)/J_n^2(x)}{1 + |Y_n'(x)|^2 / |J_n'(x)|^2} \leq \left(\frac{5}{2}\right)^2.$$

We do not use the lower bound in this paper. We include it to document the fact that one cannot hope for an upper bound tending to zero for n large, for example.

Proof. The proof is elementary from the inequalities (A.1) given in Appendix A. Since

$$\frac{2}{5} < \alpha_n = \frac{k_n}{g_n} = \left|\frac{Y_n'}{J_n'}\frac{J_n}{Y_n}\right| < \frac{5}{3},$$

it suffices to observe that

$$\frac{1 + Y_n^2(x)/J_n^2(x)}{1 + |Y_n'(x)|^2 / |J_n'(x)|^2} = \frac{1 + \tan^2\theta_n}{1 + \alpha_n^2\tan^2\theta_n}$$

therefore, since $\frac{2}{5} < 1 < \frac{5}{3}$,

$$\left(\frac{3}{5}\right)^2 < \frac{1 + Y_n^2(x)/J_n^2(x)}{1 + |Y_n'(x)|^2 / |J_n'(x)|^2} < \left(\frac{5}{2}\right)^2.$$

□

The final intermediate result we shall use is the following bound.

Proposition 7.3. *For all $x \in [n, y_{n,1}]$, we have*

$$|S_n| < \sqrt{5}.$$

Proof. Using that $|R_n(\omega_\varepsilon,\lambda)|\leq 1$, we have

$$|S_n|^2\leq 1+\tan^2\theta_n.$$

Remember that from (4.4), as x varies between 0 and $y_{n,1}$, $\theta_n(x)$ varies between $-\frac{\pi}{2}$ and 0. The map of $n\to-\theta_n(n)$ is decreasing to $\pi/3$, and is always close to its limit, as

$$\frac{9}{5}>\tan\left(-\theta_n(n)\right)>\sqrt{3}\text{ for }n\geq 1.$$

see [17, 15.8]. Consequently, for all $x\in[n,y_{n,1}]$, $\tan(-\theta_n(x))\leq\tan(-\theta_n(n))<\frac{9}{5}$, and

$$|S_n|^2\leq 1+\tan(\theta_n(x))^2<5,$$

as claimed. □

We can now conclude the proof of the estimate of this section.

Proof of Lemma 7.1. Let us first consider the case when $x\in\left[0,\min\left(n,j_{n,1}^{(1)}\lambda^{-1},y_{n,1}\right)\right]$. When $x>n$, the result follows from Proposition 7.3.

We have

$$|S_n(x)|^2=\frac{|g_n(\lambda x)-g_n(x)|^2\left(1+\tan^2\theta_n\right)}{|g_n(\lambda x)-g_n(x)|^2+\tan^2\theta_n\,|g_n(\lambda x)+k_n(x)|^2}.\tag{7.5}$$

From Proposition A.1, for $\lambda x\leq j_{n,1}^{(1)}$, $g_n(\lambda x)\geq 0$, and $k_n(x)>0$. The study of the function

$$u\to\frac{(u-a)^2\left(1+\tan^2\theta_n\right)}{(u-a)^2+\tan^2\theta_n(u+b)^2}$$

for $u>0,a>0$ and $b>0$ shows that it has a minimum for $u=a$, tends to 1 for $u\to\pm\infty$ and decreases between 0 and a. Therefore,

$$|S_n|^2\leq\max\left(1,\frac{|g_n|^2\left(1+\tan^2\theta_n\right)}{|g_n|^2+\tan^2\theta_n\,|k_n|^2}\right).$$

Now compute that

$$\frac{|g_n(x)|^2\left(1+\tan^2(x)\theta_n(x)\right)}{|g_n(x)|^2+\tan^2\theta_n(x)\,|k_n(x)|^2}=\frac{1+Y_n(x)^2/J_n(x)^2}{1+|Y_n'(x)|^2/|J_n'(x)|^2},$$

and thanks to the Proposition 7.2 this quotient is bounded by $\left(\frac{5}{2}\right)^2$. We have obtained that $|S_n|\leq 5/2$. From (7.5), we derive that

$$|S_n(x)|^2\leq\left|\frac{g_n(\lambda x)-g_n(x)}{k_n(x)+g_n(\lambda x)}\right|^2\frac{1+\tan^2\theta_n(x)}{\tan^2\theta_n(x)}.$$

As we will see in Proposition 7.3, $\tan(\theta_n)^{-2}+1\leq\frac{4}{3}$ when $x\leq n$. Thanks to Proposition A.1, $g_n(\lambda x)+k_n(x)\geq k_n(x)>\frac{3}{5}n^{-1/3}$, thus we have

$$|S_n(x)|^2\leq\frac{100}{27}n^{2/3}(\lambda-1)^2x^2\left(g_n'(n)\right)^2\leq 4(\lambda-1)^2\frac{x^2}{n^{2/3}},$$

where we used the bounds on g_n' given by Proposition A.1.

Let us now turn to the lower bounds. When $\lambda<1$, Consider the case $x=n$. Then Proposition A.1 shows that $g_n(\lambda n)>\sqrt{1-\lambda^2}$,

$$\frac{7}{6}\frac{1}{n^{1/3}}>k_n(n)\text{ and }\frac{13}{14n^{1/3}}>g_n(n)>\frac{1}{\sqrt{2}}\frac{1}{n^{1/3}}$$

and $\left|\tan^2\theta_n(n)\right| > 3$, therefore

$$\begin{aligned}
|S_n(x)|^2 &= \frac{|g_n(\lambda n)-g_n(n)|^2\left(1+\tan^2\theta_n(n)\right)}{|g_n(\lambda n)-g_n(n)|^2+\tan^2\theta_n(n)\,|g_n(\lambda n)+k_n(n)|^2}\\
&> \frac{4\,|g_n(\lambda n)-g_n(n)|^2}{|g_n(\lambda n)-g_n(n)|^2+3\left|g_n(\lambda n)+\frac{7}{6}n^{-1/3}\right|^2}\\
&> \frac{4\left|\sqrt{1-\lambda}-\frac{1}{n^{1/3}}\right|^2}{\left|\sqrt{1-\lambda}-n^{-1/3}\right|^2+3\left|\sqrt{1-\lambda}+\frac{7}{6}n^{-1/3}\right|^2}\\
&> \frac{1}{4},
\end{aligned}$$

when $\sqrt{1-\lambda^2} > 7/(3n^{1/3})$, or $\lambda < \sqrt{1-\left(\frac{7}{3n^{1/3}}\right)^2}$. $\square$

7.2. **The $n=0$ case.** We summarize here properties of g_0 and k_0. They are derived using methods similar to the ones used for k_n an g_n for $n \neq 0$, and can be checked by inspection with the help of a modern mathematical software. We therefore will omit the proof.

Proposition 7.4. *The function g_0 (resp. k_0) is defined on $(0,\infty)$ except at $j_{0,k}$ (resp. $y_{0,k}$), $k=1,\ldots,$ and cancels at each $j_{0,k}^{(1)}$ (resp. $y_{0,k}^{(1)}$).*

- *Where it is defined, g_0 is decreasing. On $(0, j_{0,1})$ g_0 is concave.*
$$\lim_{\substack{x\to j_{0,k}\\ x<j_{0,k}}} g_0(x) = -\infty, \qquad \lim_{\substack{x\to j_{0,k}\\ x>j_{0,k}}} g_0(x) = +\infty.$$
- *Where it is defined, k_0 is increasing, and*
$$\lim_{\substack{x\to y_{0,k}\\ x<y_{0,k}}} k_0(x) = +\infty, \qquad \lim_{\substack{x\to y_{0,k}\\ x>y_{0,k}}} k_0(x) = -\infty.$$
- *For $0 \le x \le 1$,*
$$-\frac{1}{2}x^2-\frac{1}{12}x^4 \le g_0(x) \le -\frac{1}{2}x^2-\frac{1}{16}x^4, \text{ and } |g_0'| \le \frac{4}{3}x.$$
- *For $0 \le x \le \frac{1}{4}$,*
$$-\frac{1}{\gamma+\ln\left(\frac{x}{2}\right)}+\frac{1}{2}x^2 \le k_0(x) \le -\frac{1}{\gamma+\ln\left(\frac{x}{2}\right)}+x^2.$$

Proposition 7.5. *For all $\lambda \le 1$, and all $0<x<y_{0,1}$,*

$$|S_0(x)| \le \frac{\pi}{2\sqrt{2}}\min(1,2-2\lambda)x^2\left|\ln\left(\frac{x}{2}\right)\right|.$$

For all $\lambda \ge 1$ and all $0 < x \le \min(\frac{1}{2}, m_\lambda)$,

$$|S_0(x)| \le \pi\min\left(1,\frac{5}{2}\frac{\lambda-1}{\lambda}\right)\lambda^2x^2\left|\ln\left(\frac{x}{2}\right)\right|.$$

Proof of Proposition 7.5. Let us first consider the case $\lambda < 1$. Since g_0 is decreasing, for all $x < j_{0,1}$, we have

$$g_0(x)-g_0(\lambda x) = (1-\lambda)xg_0'(\zeta_0) < 0,$$

Then, for all $x \le y_{0,1}$, we have $0 < J_0(\lambda x) \le 1$ and $0 \le u_0 Y_0$. Consequently,

$$\left|u_0(x)\left(J_0(x)+iY_0(x)\right)+i\frac{2}{\pi}J_0(\lambda x)\right| \ge \frac{2}{\pi}\left|J_0(\lambda x)\right|. \tag{7.6}$$

Thanks to Proposition 7.4, for all $0<x<y_{0,1}$,

$$|g_0(x)-g_0(\lambda x)| \le \min((1-\lambda)x\left|g_0'(x)\right|, \left|g_0'(x)\right| \le \min\left(\frac{4}{3}(1-\lambda),\frac{2}{3}\right)x^2. \tag{7.7}$$

Note that for $0 < x < 1$, there holds

$$|J_0(x) + iY_0(x)| < \frac{3}{2\sqrt{2}} \left|\ln\left(\frac{x}{2}\right)\right|. \tag{7.8}$$

Together with (7.7), this shows that

$$\begin{aligned} |u_0(x)\,(J_0(x) + iY_0(x))\,| &= |J_0\,(\lambda x)|\,|J_0\,(x)|\,|J_0(x) + iY_0(x)|\,|g_0(x) - g_0(\lambda x)| \\ &\leq \frac{1}{\sqrt{2}}\min(1, 2-2\lambda)x^2 \left|\ln\left(\frac{x}{2}\right)\right| |J_0\,(\lambda x)|\,. \end{aligned} \tag{7.9}$$

Inserting the estimates (7.6) and (7.9) in formula (7.2), we obtain

$$|S_0| \leq \frac{\pi}{2\sqrt{2}}\min(1, 2-2\lambda)x^2 \left|\ln\left(\frac{x}{2}\right)\right|.$$

Let us now suppose $1 \leq \lambda$. For all $x \leq \frac{1}{2\lambda}$, using the bounds on g_0 given by Proposition 7.4, we have

$$|g_0\,(x) - g_0\,(\lambda x)| \leq |g_0\,(\lambda x)| \leq \frac{\lambda^2 x^2}{2}\left(1 + \frac{\lambda^2}{6}x^2\right) \leq \frac{25}{48}\lambda^2\,x^2$$

Alternatively, note that for $x \leq m_\lambda$ and $\lambda \leq e^3$, that is, when $2 > \sqrt{\ln(\lambda) + 1}$, we can also bound

$$\begin{aligned} |g_0\,(x) - g_0\,(\lambda x)| &\leq \frac{\lambda^2 x^2}{2}\left(1 - \frac{1}{\lambda^2} + \left(\frac{\lambda^2}{6} - \frac{1}{8\lambda^2}\right)x^2\right) \\ &\leq \frac{\lambda^2 x^2}{2}\left(1 - \frac{1}{\lambda^2} + \left(\frac{\lambda^2}{6} - \frac{1}{8\lambda^2}\right)\frac{1}{\lambda^2\,(\ln(\lambda) + 1)}\right) \\ &< \frac{25}{48}\lambda^2\,x^2. \end{aligned}$$

Using that when $x < \frac{1}{2}$,

$$|J_0(x) + iY_0(x)| < \frac{3}{4}\left|\ln\left(\frac{x}{2}\right)\right|$$

and arguing as above, we obtain

$$|u_0(x)\,(J_0(x) + iY_0(x))\,| \leq \frac{2}{5}x^2\lambda^2 \left|\ln\left(\frac{x}{2}\right)\right| |J_0\,(\lambda x)|\,. \tag{7.10}$$

Alternatively, starting from the inequality

$$|g_0\,(x) - g_0\,(\lambda x)| \leq (\lambda - 1)x\,|g_0'\,(\lambda x)| \leq \frac{4}{3}\frac{\lambda - 1}{\lambda}\lambda^2 x^2,$$

we obtain

$$|u_0(x)\,(J_0(x) + iY_0(x))\,| \leq \frac{\lambda - 1}{\lambda}x^2\lambda^2 \left|\ln\left(\frac{x}{2}\right)\right| |J_0\,(\lambda x)|\,. \tag{7.11}$$

Next, using the inverse triangular inequality,

$$\left|u_0(x)\,(J_0(x) + iY_0(x)) + i\frac{2}{\pi}J_0\,(\lambda x)\right| \geq \left|\frac{2}{5}x^2\lambda^2\left|\ln\left(\frac{x}{2}\right)\right| - \frac{2}{\pi}\right| |J_0\,(\lambda x)| \geq \frac{2}{5\pi}|\,|J_0\,(\lambda x)|\,, \tag{7.12}$$

provided $x\sqrt{\left|\ln\left(\frac{x}{2}\right)\right|} < \sqrt{\frac{4}{\pi}\frac{1}{\lambda}}$. Inserting the estimates (7.11) and (7.12) in formula (7.2), we obtain

$$|S_0| \leq \pi\min\left(1, \frac{5}{2}\frac{\lambda - 1}{\lambda}\right)\lambda^2 x^2 \left|\ln\left(\frac{x}{2}\right)\right|$$

Then remark that when $x < m_\lambda$, then $x\sqrt{\left|\ln\left(\frac{x}{2}\right)\right|} < \sqrt{\frac{4}{\pi}\frac{1}{\lambda}}$, for all $\lambda \geq 1$. □

Proposition 7.6. *Let $R \geq \varepsilon$. For any $\lambda > 0$ and any $x > 0$,*

$$\left|R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \leq \frac{2}{\pi x}\frac{\varepsilon}{R}. \tag{7.13}$$

If $\lambda < 1$ then for all $x > 0$

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \le \left| H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|.$$

Furthermore, when $x < y_{0,1}$,

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \le \frac{\pi^2}{2\sqrt{2}}(1-\lambda)x^2\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|.$$

If $1 \le \lambda$ then for all $x > 0$

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \le \sqrt{5}\left| H_0^{(1)}\left(\min\left(\frac{1}{2}, m_\lambda\right)\frac{R}{\varepsilon}\right)\right|.$$

Furthermore, when $x < \min\left(\frac{1}{2}, m_\lambda\right)$,

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \le \frac{5\pi^2}{4}\frac{\lambda-1}{\lambda}x^2\lambda^2\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|.$$

Proof. Note that $x\left|H_0^{(1)}(x)\right|^2$ is an increasing function of x, with limit $\frac{2}{\pi}$. Since $|R_0^\varepsilon| \le 1$ for all $x > 0$, this implies (7.13).

Suppose now $\lambda < 1$. Note that $H_0^{(1)}(\cdot)$ is decreasing, therefore using the simple bound $|R_0^\varepsilon| \le 1$ we have for all $x \ge y_{0,1}$,

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \le \left| H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|^2$$

For $0 < x < 1$ it is easy to verify that,

$$1 + \left|\frac{Y_0(x)}{J_0(x)}\right|^2 > 1 + \frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2.$$

Therefore we obtain

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \le \frac{|S_0(x)|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2$$

Thanks to Proposition 7.5, we have, for all $0 < x < y_{0,1}$,

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \le \frac{\pi^2}{8}\min(1, 2-2\lambda)^2\frac{\varepsilon}{R}(1-\lambda)^2\frac{x^3\left|\ln\left(\frac{x}{2}\right)\right|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}\left(x\frac{R}{\varepsilon}\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2\right). \tag{7.14}$$

The function

$$x \to \frac{x^3\left|\ln\left(\frac{x}{2}\right)\right|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}$$

is increasing on $(0,1)$, and as we noted before, so is $x\left|H_0^{(1)}(x)\right|^2$. Therefore an upper bound is obtained by choosing $x = y_{0,1}$ in the right-hand-side of (7.14), which gives

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \le \frac{\pi^2}{8}\frac{y_{0,1}^4\ln\left(\frac{y_{0,1}}{2}\right)^2}{1+\frac{4}{\pi^2}\ln\left(\frac{y_{0,1}}{2}\right)^2}\left| H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|^2 \le \left| H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|^2,$$

where we used (7.13) in the second inequality. We have obtained that for all $x > 0$,

$$\left| R_0(\omega_\varepsilon, \lambda) H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \le \left| H_0^{(1)}\left(y_{0,1}\frac{R}{\varepsilon}\right)\right|.$$

Alternatively, note that the function $(\ln(x/2)^2)/(1+\frac{4}{\pi^2}\ln(x/2)^2)$ is decreasing on $(0,1)$, with a maximum of $\pi^2/4$, therefore (7.14) and (7.13) yield

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \le \frac{\pi^4}{8}(1-\lambda)^2x^4\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2,$$

for all $0 < x < y_{0,1}$.

Let us now consider the case $\lambda > 1$. We only consider the case when $m_\lambda \geq 2$, the proof in the other case is similar. Arguing as before, we have for all x such that $x \geq m_\lambda$

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \leq \left| H_0^{(1)}\left(m_\lambda \frac{R}{\varepsilon}\right)\right|.$$

Next, when $x < m_\lambda$, we have

$$\begin{aligned}\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 &\leq \frac{|S_0(x)|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \\ &\leq \min\left(1, \frac{5}{2}\frac{\lambda-1}{\lambda}\right)^2 \pi^2 \frac{x^4\lambda^4\left|\ln\left(\frac{x}{2}\right)\right|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2\end{aligned}$$

Arguing as in the case $\lambda < 1$, an upper bound is obtained by replacing x by its maximal value, namely

$$\begin{aligned}\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 &\leq \pi^2 \frac{\left(\ln\left(2\lambda\sqrt{\ln\lambda+1}\right)\right)^2}{(1+\ln(\lambda))^2\left(1+\frac{4}{\pi^2}\left(\ln\left(2\lambda\sqrt{\ln\lambda+1}\right)\right)^2\right)}\left| H_0^{(1)}\left(\frac{1}{\lambda\sqrt{\ln\lambda+1}}\frac{R}{\varepsilon}\right)\right|^2 \\ &\leq \frac{5}{1+\left(\frac{2}{\pi^2}\ln\lambda\right)^2}\left| H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)\right|^2. \qquad (7.15)\end{aligned}$$

We have obtained that for all $x > 0$,

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \leq 5\left| H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)\right|^2.$$

Alternatively, we also have, when $x < m_\lambda$,

$$\begin{aligned}\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 &\leq \pi^2\left(\frac{5}{2}\frac{\lambda-1}{\lambda}\right)^2 \frac{x^4\lambda^4\left|\ln\left(\frac{x}{2}\right)\right|^2}{1+\frac{4}{\pi^2}\left|\ln\left(\frac{x}{2}\right)\right|^2}\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2 \\ &\leq \left(\frac{5\pi^2}{4}\frac{\lambda-1}{\lambda}x^2\lambda^2\right)^2\left| H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right|^2.\end{aligned}$$

□

We conclude this section by an estimate which will prove useful for broadband estimations.

Proposition 7.7. *Let $R \geq \varepsilon$. For any $\lambda \geq 7$ and any $m_\lambda \geq x > 0$,*

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \leq 4\left|\frac{H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)}{H_0^{(1)}\left(m_\lambda\right)}\right|. \qquad (7.16)$$

Proof. Inequality (7.15) shows that for all $0 < x \leq m_\lambda$, we have

$$\left| R_0^\varepsilon H_0^{(1)}\left(x\frac{R}{\varepsilon}\right)\right| \leq \sqrt{\frac{5}{1+\left(\frac{2}{\pi^2}\ln\lambda\right)^2}}\left| H_0^{(1)}\left(m_\lambda\frac{R}{\varepsilon}\right)\right|.$$

We note (and can prove, it is a study of a function of one variable, $\ln\lambda$) that

$$\lambda \to \frac{\left|H_0^{(1)}\left(m_\lambda\right)\right|}{\sqrt{1+\left(\frac{2}{\pi^2}\ln\lambda\right)^2}}$$

is increasing for $\lambda > 7$, and its value at $\lambda = 7$ is greater than 3.2. This lower bound yields our estimate. □

8. Bounds near quasi-resonances

This section is devoted to the proof of Proposition 6.1.

We wish to bound the size of the "blow-up" regions, that is, the sets $I_{n,k}(\tau)$ defined by (6.1), centered on quasi-resonances. From (4.2) we know that $y^{(1)}_{n,1} > y_{n,1}$ for all $n \geq 0$, thus $k_n > 0$ on $(0, y_{n,1})$. Introducing

$$\begin{aligned} \phi_n := \quad (0, y_{n,1}) \setminus \cup_k \{j_{n,k}/\lambda\} \quad &\to \quad \mathbb{R} \\ x \quad &\to \quad \frac{g_n(\lambda x)}{k_n(x)}, \end{aligned} \tag{8.1}$$

we have

$$\phi_n(I_{n,k}(\tau)) = [-1-\tau, -1+\tau].$$

We first verify that ϕ_n is one-to-one on $I_{n,k}(\tau)$, for τ small enough and λ large enough.

Lemma 8.1. *Suppose $\tau \leq \frac{1}{4}$ and $7 \leq \lambda$. When $n \geq 1$ the function ϕ_n given by (8.1) satisfies*

$$\phi_n'(x) \leq \frac{1-\lambda^2}{2n_+ k_n(x)} < 0$$

where $n_+ = \max(n,1)$, for all $x \in I_{n,k}(\tau) \cap (0,n)$ when $n \geq 1$ and for all $x \in I_{0,k}(\tau) \cap (0, \zeta_0)$ when $n = 0$. Furthermore,

$$I_{0,k}(\tau) \subset (m_\lambda, y_{0,1}), \text{ and for } n \geq 1, \quad I_{n,k}(\tau) \subset \left(\frac{j^{(1)}_{n,1}}{\lambda}, y_{n,1}\right),$$

for all k.

Proof. We compute, using (4.13) and (A.3), for $n \geq 0$,

$$\phi_n'(x) = \frac{x}{n_+ k_n(x)}\left(1-\lambda^2\right) + \frac{k_n(x)+g_n(\lambda x)}{k_n(x)^2}\left(\frac{n}{x} - \frac{x}{n_+} - \frac{n_+}{x} g_n(\lambda x) k_n(x)\right), \tag{8.2}$$

Suppose first that $n \geq 1$. When $x \leq n$ and $-g_n > k_n$,

$$(k_n(x)+g_n(\lambda x))(\frac{n}{x} - \frac{x}{n} - \frac{n}{x} g_n(\lambda x) k_n(x)) < 0$$

therefore

$$\phi_n'(x) \leq \frac{x}{n k_n(x)}\left(1-\lambda^2\right) < 0.$$

On the other hand, when $g_n(\lambda x) + k_n(x) > 0$ and $x \in I_{n,k}(\tau)$, that is, when $0 < g_n(\lambda x) + k_n(x) \leq \tau k_n(x)$, we have

$$\frac{k_n(x)+g_n(\lambda x)}{k_n(x)}\left(\frac{n}{x} - \frac{x}{n} - \frac{n}{x} g_n(\lambda x) k_n(x)\right) \leq \frac{\tau}{n\,x}\left(n^2 - x^2 + n^2 k_n^2\right)$$

Using the upper bound on k_n given by Proposition A.1, we find that when $\frac{n}{\lambda} \leq x \leq \kappa_n$, we have

$$\frac{2\,\tau}{n\,x}\left(n^2 - x^2 + n^2 k_n^2\right) \leq 2\,\tau \frac{x}{n}\frac{\lambda^2-1}{\lambda} \leq \frac{1}{2}\frac{x}{n}\left(\lambda^2-1\right)$$

provided $\tau \leq \frac{1}{4}$. When $\kappa_n \leq x \leq n$,

$$\frac{2\,\tau}{n\,x}\left(n^2 - x^2 + n^2 k_n^2\right) \leq 2\tau \frac{x}{n}\frac{n^2 + 7/6n^{4/3}}{(n - 4/5n^{1/3})^2} \leq \frac{93\tau}{\lambda^2-1}\frac{x}{n}\frac{\lambda^2-1}{\lambda} < \frac{1}{2}\frac{x}{n}\left(\lambda^2-1\right),$$

when $\tau \leq \frac{1}{4}$ and $7 \leq \lambda$. We have obtained that when $x \in I_{n,k}(\tau)$,

$$\phi_n'(x) \leq \frac{x}{2n k_n(x)}\left(1-\lambda^2\right) < 0,$$

as announced. Finally, note that at $x = j^{(1)}_{n,1}/\lambda$ we have $g_n(\lambda x) = 0 > (\tau-1)k_n(x)$, thus $I_{n,k}(\tau)$ is a proper subset of $U_{n,k}$. Let us now consider the case $n = 0$. We have

$$\phi_0'(x) = \frac{x}{k_0(x)}\left(1-\lambda^2\right) + x\frac{k_0(x)+g_0(\lambda x)}{k_0(x)^2}\left(-1 - \frac{1}{x^2} g_0(\lambda x) k_0(x)\right). \tag{8.3}$$

When $-\tau k_0(x) \leq g_0(\lambda x) + k_0(x) \leq \tau k_0(x)$, thanks to Proposition A.2, we have

$$-\frac{1}{x^2} g_0(\lambda x) k_0(x) - 1 \geq (1-\tau)\frac{1}{(0.36)^2} - 1 > 0,$$

when $\tau \leq \frac{1}{4}$. Turning back to (8.3), this shows that when $k_0(x) + g_0(\lambda x) \leq 0$,

$$\phi_0'(x) \leq \frac{x}{k_0(x)}\left(1-\lambda^2\right).$$

Let us now assume $0 < k_0(x) + g_0(\lambda x) < \tau k_0(x)$. We claim that on $(0, m_\lambda)$, $-\phi_0 < 3/5$. Therefore no $I_{0,k}(\tau)$ lies in the interval $(0, m_\lambda)$, since $\tau \leq 1/4$. Using the bounds on g_0 and k_0 given by Proposition 7.4 we find, when $x \leq m_\lambda$ and $\lambda \geq 7$, maximizing in x first and then in λ,

$$\begin{aligned}
\frac{-g_0(\lambda x)}{k_0(x)} &\leq \frac{\lambda^2 x^2}{2} \ln\left(\frac{2}{x e^\gamma}\right)\left(1 + \frac{1}{12}\lambda^4 x^4\right) \\
&\leq \frac{1}{2\ln\lambda + 2} \ln\left(\frac{2\lambda\sqrt{\ln\lambda + 1}}{e^\gamma}\right)\left((1 + \frac{1}{12}\frac{1}{(\ln\lambda+1)^2}\right) \\
&\leq \frac{1}{2}\left(1 + e^{-3-2\gamma}\right)\left((1 + \frac{1}{12}\frac{1}{(\ln 7+1)^2}\right) < \frac{3}{5},
\end{aligned}$$

which is our claim. This is turn shows that when $x \in I_{0,k}(\tau)$, and $x \leq \zeta_0$, and $7 \leq \lambda$,

$$-\frac{1}{x^2} g_0(\lambda x) k_0(x) \leq \frac{1}{m_\lambda^2} k_0^2(m_\lambda) \leq \left(\frac{1}{m_\lambda\left(\ln\left(\frac{2}{e^\gamma}\right) - \ln(m_\lambda)\right)} + m_\lambda\right)^2 < \frac{\lambda^2}{\ln(\lambda)}. \tag{8.4}$$

We therefore have obtained that for all $x \in I_{0,k}(\tau) \cap (0, \zeta_0)$,

$$x\frac{k_0(x) + g_0(\lambda x)}{k_0(x)^2}\left(-1 - \frac{1}{x^2} g_0(\lambda x) k_0(x)\right) \leq \tau\frac{x}{k_0(x)}(\lambda^2 - 1),$$

which in turn shows that

$$\phi_0'(x) \leq \frac{x}{2k_0(x)}\left(1-\lambda^2\right).$$

∎

We are now ready to compute an upper bound on the sum of size of the intervals $I_{n,k}(\tau)$ for a given n.

Proposition 8.2. *Suppose $\lambda \geq 7$ and $\frac{1}{4} \geq \tau$. For all $n \geq 1$,*

$$\left|\bigcup_{k \in K(\lambda,n)} I_{n,k}(\tau)\right| \leq 6\tau\frac{n\ln\lambda}{\lambda}, \tag{8.5}$$

where $K(\lambda, n)$ is the set of all positive k such that $j_{n,k}^{(1)} < n\lambda$. We also have

$$\left|\bigcup_{k \in K(\lambda,0)} I_{0,k}(\tau)\right| \leq 7\tau\frac{\ln(\ln\lambda)}{\lambda}, \tag{8.6}$$

where $K(\lambda, 0)$ is the set of all positive k such that $j_{0,k}^{(1)} < \zeta_0\lambda$.

Proof. **When $n \geq 1$.** We know thanks to Lemma 8.1 that ϕ_n is a bijection on $I_{n,k}(\tau)$. We write $I_{n,k}(\tau) = [\alpha_{n,k}, \beta_{n,k}]$, that is $\beta_{n,k} \in U_{n,k}$ is such that $\phi(\beta_{n,k}) = -1-\tau$ and $\alpha_{n,k} \in U_{n,k}$ is such that $\phi(\beta_{n,k}) = -1+\tau$. We have

$$2\,\tau = \phi(\alpha_{n,k}) - \phi(\beta_{n,k}) = \int_{\alpha_{n,k}}^{\beta_{n,k}} -\phi'(u)du \geq \frac{\lambda^2-1}{2}\int_{\alpha_{n,k}}^{\beta_{n,k}} \frac{u}{nk_n(u)}du. \tag{8.7}$$

From Proposition A.2 we know that, introducing $\chi_n = n - 4/5n^{1/3}$, $u \to u/k_n(u)$ is increasing on $(0, \chi_n)$, and satisfies

$$\frac{x}{nk_n(x)} \geq \begin{cases} \left(\frac{n^2}{x^2} - 1\right)^{-1/2} & \text{when } 0 \leq x \leq \chi_n, \\ \left(\frac{n^2}{\chi_n^2} - 1\right)^{-1/2} & \text{when } \chi_n \leq x \leq n. \end{cases}$$

Let k_* be the largest indice such that $\alpha_{n,k} < \chi_n$. For all $k \leq k_*$ we have

$$\frac{4\tau}{\lambda^2 - 1} \geq \int_{\alpha_{n,k}}^{\beta_{n,k}} \left(\frac{n^2}{x^2} - 1\right)^{-1/2} dx \geq |I_{n,k}(\tau)| \left(\frac{n^2}{\alpha_{n,k}^2} - 1\right)^{-1/2}$$

and when $\alpha_{n,k} > \chi_n$,

$$\frac{4\tau}{\lambda^2 - 1} \geq \int_{\alpha_{n,k}}^{\beta_{n,k}} \left(\frac{n^2}{x^2} - 1\right)^{-1/2} dx \geq |I_{n,k}(\tau)| \left(\frac{n^2}{\chi_{n,k}^2} - 1\right)^{-1/2}$$

Observing that $I_{n,k}(\tau) \subset U_{n,k}$, and therefore $n > j_{n,k}/\lambda > \alpha_{n,k} > j_{n,k}^{(1)}/\lambda > \alpha_{n,0} := n/\lambda$ we have obtained that

$$\begin{aligned}
\left|\bigcup_{k=1}^{k_*} I_{n,k}(\tau)\right| &\leq \frac{4\tau}{\lambda^2 - 1} \sum_{k=1}^{k_*} \sqrt{\left(\frac{n}{\alpha_{n,k}}\right)^2 - 1} \\
&\leq \frac{4\tau}{\lambda^2 - 1} \left(\min_{k \leq k*} \left(\frac{\alpha_{n,k}}{n} - \frac{\alpha_{n,k-1}}{n}\right)\right)^{-1} \sum_{k=1}^{k_*} \sqrt{\left(\frac{n}{\alpha_{n,k}}\right)^2 - 1} \left(\frac{\alpha_{n,k}}{n} - \frac{\alpha_{n,k-1}}{n}\right) \\
&\leq \frac{4n\tau}{\lambda^2 - 1} \left(\min_{k \leq k*} (\alpha_{n,k} - \alpha_{n,k-1})\right)^{-1} \int_{\lambda^{-1}}^{1} \sqrt{\frac{1}{x^2} - 1}\, dx. \\
&= \frac{4\tau n\lambda \ln \lambda}{\lambda^2 - 1} \max_{k \leq k*} \frac{1}{(\lambda\alpha_{n,k} - \lambda\alpha_{n,k-1})}.
\end{aligned}$$

For $k \geq 1$, the distance $\lambda\alpha_{n,k+1} - \lambda\alpha_{n,k}$ is at least $j_{n,k+1}^{(1)} - j_{n,k}$. We know from (4.4) that this distance decreases with k, and tends to $\pi/2$. On the other-hand, using the estimates (4.1), $\lambda\alpha_{n,1} - \lambda\alpha_{n,0} > j_{n,1}^{(1)} - n > \frac{4}{5}n^{1/3} > \frac{4}{5}$. Therefore, we have

$$\left|\bigcup_{k=1}^{k_*} I_{n,k}(\tau)\right| \leq \frac{5\tau n\lambda \ln \lambda}{\lambda^2 - 1}. \tag{8.8}$$

Using again the fact that $j_{n,k}^{(1)} - j_{n,k-1}^{(1)}$ is at least π, there can be at most $(n - \chi_n)\lambda/\pi$ intervals $U_{n,k}$ in (χ_n, n). Therefore

$$\begin{aligned}
\left|\bigcup_{k \geq k_*} I_{n,k}(\tau)\right| &\leq \frac{4\lambda\tau}{\pi(\lambda^2 - 1)} \sqrt{\frac{n^2}{\chi_n^2} - 1}(n - \chi_n) \\
&\leq \frac{4n\lambda\tau}{\pi(\lambda^2 - 1)}.
\end{aligned}$$

Altogether, we have

$$\left|\bigcup_{k \in K(\lambda,n)} I_{n,k}(\tau)\right| \leq \frac{n\tau \ln \lambda}{\lambda} \left(\frac{\lambda^2}{\lambda^2 - 1}\left(5 + \frac{4}{\pi \ln \lambda}\right)\right) \leq \frac{6n\tau \ln \lambda}{\lambda}$$

which completes the proof of estimate (8.5).

When $n = 0$. As above, with the same notations, we have

$$2\,\tau = \phi(\alpha_{0,k}) - \phi(\beta_{0,k}) = \int_{\alpha_{0,k}}^{\beta_{0,k}} -\phi'(u)du \geq \frac{\lambda^2 - 1}{2} \int_{\alpha_{0,k}}^{\beta_{0,k}} \frac{u}{k_0(u)} du. \tag{8.9}$$

Proposition A.2 shows that $x \to x/k_0(x)$ is increasing until ζ_0, and decreasing afterwards. Thus

$$2\,\tau \geq \frac{\lambda^2-1}{2}\left|I_{0,k}(\tau)\right|\frac{\alpha_{0,k}}{k_0(\alpha_{0,k})}.$$

As before, we know that for $k \geq 1$, $\alpha_{0,k}-\alpha_{0,k+1} > \lambda^{-1}(j_{0,k}-j^{(1)}_{0,k+1}) > \lambda^{-1}\pi/2$. We have

$$\begin{aligned}
\sum_{k\geq 2}\frac{k_0(\alpha_{0,k})}{\alpha_{0,k}} &\leq \max_{k\geq 1}\frac{1}{\alpha_{0,k}-\alpha_{0,k+1}}\sum_{k\geq 2}\frac{k_0(\alpha_{0,k})}{\alpha_{0,k}}\left(\alpha_{0,k-1}-\alpha_{0,k}\right)\\
&\leq \frac{2\lambda}{\pi}\int_{j_{0,1}/\lambda}^{\zeta_0}\frac{k_0(x)}{x}dx\\
&\leq \frac{2\lambda}{\pi}\ln\left|\frac{Y_0\left(j_{0,1}/\lambda\right)}{Y_0\left(\zeta_0\right)}\right| \leq \frac{2}{\pi}\lambda\ln\left(\ln\lambda\right).
\end{aligned}$$

Finally, using the bound (8.4)

$$\frac{\alpha_{0,1}}{k_0(\alpha_{0,1})} \leq \frac{\lambda}{\sqrt{\ln\lambda}}$$

and we have obtained that

$$\left|\bigcup_{k\in K(\lambda,0)} I_{0,k}(\tau)\right| \leq \frac{4\lambda\tau}{\lambda^2-1}\left(\frac{1}{\sqrt{\ln\lambda}}+\frac{2}{\pi}\ln\left(\ln\lambda\right)\right) \leq \tau\frac{7\ln\left(\ln\lambda\right)}{\lambda},$$

which concludes our proof. $\square$

Let us now check that away from $\omega_{n,k}$, we can produce a bound S_n similar to that of the perturbative regime.

Proposition 8.3. *If $n \geq 1$ and $x \in \left(\lambda^{-1}j^{(1)}_{n,1}, y_{n,1}\right) \setminus \left(\cup_k I_{n,k}(\tau)\right)$, there holds*

$$|S_n| \leq \frac{9}{2\,\tau}.$$

When $n = 0$, if $x \in (m_\lambda, \zeta_0) \setminus \left(\cup_k I_{0,k}(\tau)\right)$, we have

$$|S_0| \leq \frac{5}{3\,\tau}.$$

Proof. **Case $n \neq 0$.** When $x \in [n, y_{n,1}]$ Proposition 7.3 shows that $|S_n| \leq \sqrt{5}$, which establishes the bound. The proof is along the lines of that of Lemma 7.1. Starting from the formula

$$S_n(x) = \frac{\left(g_n(\lambda x)-g_n(x)\right)\left(1+i\tan\theta_n\right)}{\left(g_n(\lambda x)-g_n(x)\right)+i\tan\theta_n\left(g_n(\lambda x)+k_n(x)\right)},$$

we write $a = g_n(x)$ and $b = k_n(x)$, and the study of the function

$$u \to \frac{(u-a)^2\left(1+\tan^2\theta_n\right)}{(u-a)^2+\tan^2\theta_n(u+b)^2}$$

for $a > 0$ and $b > 0$, with $u \in (-\infty, -(1+\tau)b) \cup (-(1-\tau)b, +\infty)$, shows that it has a minimum for $u = a$, tends to 1 for $u \to \pm\infty$, increases until $-(1+\tau)b$, decreases on $(-(1-\tau)b, a)$ and increases to 1 afterwards. Therefore, the maximum of S_n is smaller than the maximum of the two values A and B given by

$$A = \frac{\left(1+\tan^2\theta_n\right)\left(\left(1+\tau\right)b+a\right)^2}{\left(\left(1+\tau\right)b+a\right)^2+\tan^2\theta_n\tau^2b^2} = \frac{1+\tan^2\theta_n}{1+\tau^2\tan^2\theta_n\frac{b^2}{a^2}\left(1+(1+\tau)\frac{b}{a}\right)^{-2}},$$

$$B = \frac{\left(1+\tan^2\theta_n\right)\left(\left(1-\tau\right)b+a\right)^2}{\left(\left(1-\tau\right)b+a\right)^2+\tan^2\theta_n\tau^2b^2} = \frac{1+\tan^2\theta_n}{1+\tau^2\tan^2\theta_n\dfrac{b^2}{a^2}\left(1+(1-\tau)\dfrac{b}{a}\right)^{-2}}.$$

It is clear that

$$\frac{\tau}{1+(1+\tau)\frac{b}{a}} < \frac{\tau}{1+(1-\tau)\frac{b}{a}} < 1$$

therefore the maximum is A. Noticing that, for $b > \alpha a$

$$\frac{\frac{b}{a}\tau}{1+(1+\tau)\frac{b}{a}} > \frac{\alpha\tau}{1+2\alpha}$$

we obtain

$$A < \left(\frac{1+2\alpha}{\alpha\tau}\right)^2 \frac{1+\tan^2\theta_n}{\frac{1+2\alpha}{\alpha\tau}+\tan^2\theta_n} < \left(\frac{1+2\alpha}{\alpha\tau}\right)^2$$

Thanks to Proposition A.1 we know that $\frac{2}{5} < \frac{b}{a}$. We can therefore conclude that

$$A \leq \left(\frac{9}{2}\frac{1}{\tau}\right)^2,$$

which concludes the proof.

Case $n = 0$**.** The proof is slightly different when $n = 0$, as k_0/g_0 is unbounded near $x = 0$. Note that when $x < \frac{1}{3}$, (and $\zeta_0 < 1/3$),

$$\frac{3}{5}\frac{1}{k_0(x)} \leq \tan\theta_0(x) \leq \frac{4}{5}\frac{1}{k_0(x)} \quad \frac{9}{10} \leq \frac{k_0(x)+g_0(x)}{k_0(x)} \leq 1.$$

Introducing $u = \frac{g_0(\lambda x)+k_0(x)}{k_0(x)}$, and $v = (k_0(x)+g_0(x))/k_0(x)$ we have

$$u \in (-\infty,-\tau)\cup(\tau,\infty) \text{ and } \frac{9}{10} \leq v \leq 1,$$

$$\begin{aligned}
|S_0(x)|^2 &= \frac{|g_0(\lambda x)-g_0(x)|^2\left(1+\tan^2\theta_0(x)\right)}{|g_0(\lambda x)-g_0(x)|^2+\tan^2\theta_0(x)\,|g_0(\lambda x)+k_n(x)|^2} \\
&\leq \frac{|u-v|^2\left(\frac{4}{5}+k_0(x)^2\right)}{|u-v|^2 k_0(x)^2+\frac{3}{5}u^2}.
\end{aligned}$$

Relaxing u to an independent variable, we see that

$$\frac{|u-v|^2\left(\frac{4}{5}+k_0(x)^2\right)}{|u-v|^2 k_0(x)^2+\frac{3}{5}u^2} \leq \max(A,B,1)$$

where

$$A = \frac{|\tau-v|^2\left((\frac{4}{5})^2+k_0(x)^2\right)}{|\tau-v|^2 k_0(x)^2+(\frac{3}{5})^2\tau^2} \text{ and } B = \frac{|\tau+v|^2\left((\frac{4}{5})^2+k_0(x)^2\right)}{|\tau+v|^2 k_0(x)^2+(\frac{3}{5})^2\tau^2}$$

It is clear that $A < B$. Taking the maximum value for v and τ, we find

$$B \leq \frac{1+(\frac{5}{4})^2 k_0(x)^2}{\frac{5}{4}k_0(x)^2+(\frac{3}{5})^2\tau^2} \leq \frac{25}{9}\frac{1+\left(\frac{5}{4}k_0(m_\lambda)\right)^2}{\left(\frac{25}{12}k_0(m_\lambda)\right)^2+\tau^2} \leq \frac{25}{9\,\tau^2}.$$

□

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Appendix A. Miscellaenous Properties of Bessel Functions

Proposition A.1 details properties of quotients of Bessel functions. This result is therefore independent from the rest of the paper. Properties ii., v. and vi. could be new. Since many authors have worked on Bessel Functions, it is quite possible that similar results were proved before, but we are not aware of it.

Proposition A.1. *For all real $n \geq 1$,*

i. *The function g_n is strictly decreasing on $[0,\infty) \setminus \cup_k j_{n,k}$, and cancels at each $j^{(1)}_{n,k}$.*
ii. *On $[0, j_{n,1})$, g_n is concave.*
iii. *For $0 \leq x \leq n$, $g_n(x)$ satisfies*

$$\sqrt{1-\left(\frac{x}{n}\right)^2} < g_n(x) < \sqrt{1-\left(\frac{x}{n}\right)^2 + \frac{c_n^2}{n^{2/3}}\frac{x}{n}},$$

where $c_n := n^{1/3} g_n(n)$, satisfies

$$\frac{1}{\sqrt{2}} < c_n < \frac{13}{14}.$$

iv. *For* $0 < x < n$, $0 < -g_n'(x) \leq g_n^2(n)$. *On* $(0, y_{n,1})$, *the function* k_n *is positive. It decreases until* $\kappa_n \in (n - \frac{4}{5}n^{1/3}, n)$, *defined as the unique solution of*

$$k_n(\kappa_n) = \sqrt{1 - \frac{1}{n^2}\kappa_n^2}.$$

v. *For* $0 < x < n$, k_n *satisfies,*

$$\frac{3}{5}\frac{1}{n^{1/3}} \leq k_n(x) \leq \max\left(\sqrt{1 - \left(\frac{x}{n}\right)^2}, \frac{7}{6}\frac{1}{n^{1/3}}\right).$$

More precisely, for all $x \leq \kappa_n$,

$$\kappa^+\sqrt{1 - \left(\frac{x}{n}\right)^2} - g_n(x) \leq k_n(x) \leq \sqrt{1 - \left(\frac{x}{n}\right)^2},$$

with $\kappa^+ > 1.91$, *whereas for all* $\kappa_n \leq x \leq n$,

$$\frac{3}{5}\frac{1}{n^{1/3}} \leq k_n(x) \leq \frac{7}{6}\frac{1}{n^{1/3}}$$

vi. *Finally,*

$$\frac{2}{5} < \frac{k_n(x)}{g_n(x)} < \frac{5}{3}. \tag{A.1}$$

Proof. Property (i) is well-known, see e.g. [16, 6]. To obtain property (ii), notice that

$$\begin{aligned} -ng_n'(x) &= \frac{(J_n(x))^2 - J_{n+1}(x)\,J_{n-1}(x)}{(J_n(x))^2} \\ &= \frac{1}{n+1} + \frac{2}{n+2}\left(\frac{J_{n+1}(x)}{J_n(x)}\right)^2 \\ &\quad +2n\sum_{p=2}^{\infty}\left(\frac{J_{n+p}(x)}{J_n(x)}\right)^2 \frac{1}{n+p-1}\frac{1}{n+p+1}. \end{aligned}$$

The second identity is proved in [15, 16]. Since for $x > 0$, $x \to J_{n+1}(x)/J_n(x)$ is an increasing function as it can be readily observed from its continued fraction expansion, see [13, 10.10], we deduce that g_n is concave on $(0, j_{n,1})$. For property (iii), using the recurrence relations for Bessel functions, we notice that g_n satisfies the differential equation (4.13). Since g_n is decreasing, we deduce that

$$g_n(x) \geq \sqrt{1 - \frac{x^2}{n^2}}.$$

for $x \leq n$. We know [17, 8.55] that $n \to n^{1/3}g_n(n)$ is an increasing function of n, therefore

$$\frac{1}{\sqrt{2}} < g_1(1) \leq c_n \leq \lim_{n\to\infty} c_n < \frac{13}{14}.$$

Since g_n is concave, we have for all $x < n$ $g_n'(x) > g_n'(n) = -g_n^2(n)$, and inserting this inequality in (4.13), we obtain

$$g_n^2(x) \leq 1 - \frac{x^2}{n^2} + \frac{x}{n^{5/3}}c_n^2, \tag{A.2}$$

as announced. This also proves the first part of property (iv). On $(0, n)$, k_n is strictly positive, as $x \to Y_n(x)$ is negative and increasing until $y_{n,1}$, since from (4.2) $y_{n,1}^{(1)} > y_{n,1}$. The asymptotic development

$$k_n(x) = 1 - \frac{x^2}{2}\frac{1}{n(n-1)} + O(x^4)$$

shows that k_n initially decreases. Note that k_n satisfies

$$k_n'(x) = \frac{n}{x}\left(k_n^2(x) - 1 + \frac{x^2}{n^2}\right) \tag{A.3}$$

Since $k_n(n) > 0$, there exists $\kappa_n < n$ such that $k_n(\kappa_n) = \sqrt{1 - \frac{\kappa_n^2}{n^2}}$, and k_n increases on $(\kappa_n, y_{n,1})$. The lower bound on κ_n will be proved later. We now address property (v). The Wronskian identity can be written

$$k_n + g_n = \frac{2}{\pi n \left(-J_n\left(x\right) Y_n\left(x\right)\right)},$$

It is shown in [3] that for all $x \leq n$, and $n > 0$,

$$2\pi \left(-J_n\left(x\right) Y_n\left(x\right)\right) \sqrt{n^2 - x^2} \leq 2.09.$$

The lower bound on k_n, $k_n \geq \kappa^+ \sqrt{1 - \left(\frac{x}{n}\right)^2} - g_n(x)$ follows immediately. To derive an upper bound, we therefore need to estimate $k_n(n)$. Using the Wronskian identity, we can write

$$n^{1/3} k_n(n) = -n^{1/3} g_n(n) + \frac{2}{\pi \left(n^{1/3} J_n\left(n\right)\right)^2} \left(-\frac{J_n\left(n\right)}{Y_n\left(n\right)}\right)$$

Note that $n^{1/3} g_n(n), n^{1/3} J_n\left(n\right)$ and $-J_n\left(n\right)/Y_n\left(n\right)$ are bounded increasing functions of n, see [17, 8.54,855] and [8]. Therefore

$$\frac{4}{5} \leq \lim_{n\to\infty} \left(-n^{1/3} g_n(n) + \frac{2}{\pi \left(n^{1/3} J_n\left(n\right)\right)^2} \left(-\frac{J_1\left(1\right)}{Y_1\left(1\right)}\right)\right) \leq n^{1/3} k_n(n), \tag{A.4}$$

$$\frac{7}{6} \geq -g_1(1) + \frac{2}{\pi J_1\left(1\right)^2} \lim_{n\to\infty} \left(-\frac{J_n\left(n\right)}{Y_n\left(n\right)}\right) \geq n^{1/3} k_n(n). \tag{A.5}$$

We have obtained

$$k_n \leq \max\left(\sqrt{1 - \left(\frac{x}{n}\right)^2}, \frac{7}{6}\frac{1}{n^{1/3}}\right),$$

as announced. We can verify by inspection that $k_1 > 3/5$ on $(0, 1)$. Let us compute a lower bound for $n \geq 2$. Note that we have obtained that $k_n(n) < 1$. We compute

$$\begin{aligned}
k_n(n) - k_n(\kappa_n) = \int_{\kappa_n}^{n} k_n'(x) dx &= \int_{\kappa_n}^{n} \frac{n}{x}\left(k_n(x)^2 - 1 + \frac{x^2}{n^2}\right) dx, \\
&\leq \left[\frac{x^2}{2n} + n(kn(n)^2 - 1)\ln(x)\right]_{x_1}^{n} \\
&= \frac{n}{2}\left(k_n(\kappa_n)^2 + (1 - k_n(n)^2)\ln\left(1 - k_n(\kappa_n)^2\right)\right) \\
&\leq \frac{n k_n(n)^2}{2} k_n(\kappa_n)^2.
\end{aligned}$$

This implies, using the bounds $7/6 > k_n(n) n^{1/3} > 4/5$,

$$\begin{aligned}
\min_{0\leq x\leq 1}(k_n(x)) = k_n(\kappa_n) &\geq \frac{1}{n k_n^2(n)}\left(-1 + \sqrt{1 + 2n k_n^3(n)}\right) \\
&\geq \frac{1}{n^{1/3}} \min_{4/5\leq x\leq 7/6}\left(\frac{-1 + \sqrt{1 + 2x^3}}{x^2}\right) \\
&\geq \frac{3}{5n^{1/3}}.
\end{aligned}$$

Let us show now that $\kappa_n \geq n - \frac{4}{5}n^{1/3}$, to conclude the proof of Property (iv). Differentiating the identity (A.3) we obtain, when $k_n' \geq 0$,

$$\begin{aligned} k_n^{(2)}(x) &= -\frac{1}{x}k_n'(x) + \frac{n}{x}\left(2\frac{x}{n^2} + 2k_n'(x)k_n(x)\right) \\ &= \frac{2}{n} + \frac{k_n'(x)}{x}\left(2nk_n(x) - 1\right) \\ &\geq \frac{2}{n} + \frac{k_n'(x)}{x}\left(\frac{6}{5}n^{2/3} - 1\right) \\ &\geq \frac{2}{n}. \end{aligned}$$

Therefore, we can write using the upper and lower bound on k_n and the lower bound on $k_n^{(2)}$,

$$\left(\frac{7}{6} - \frac{3}{5}\right)\frac{1}{n^{1/3}} \geq k_n(n) - k_n(\kappa_n) = \int_{\kappa_n}^{n} k_n'(t)dt \geq \frac{2}{n}\int_{\kappa_n}^{n}(t - \kappa_n)dt = \frac{1}{n}(n - \kappa_n)^2.$$

Consequently,

$$n - \kappa_n \leq \sqrt{\frac{17}{30}n^{1/3}} \leq \frac{4}{5}n^{1/3},$$

which is the announced bound.

Finally, let us address property (vi.). Note that for $x \geq x_1$, k_n/g_n is increasing, as the quotient of an increasing function over a decreasing one, therefore the lower bound is in the interval $[0, x_1]$. Thus, the maximum is either at $x = 0$ or $x = x_1$, and

$$\max_{0 \leq x \leq x_1}(k_n/g_n) \leq \max(1, k_n(n)/g_n(n)) \leq \frac{7\sqrt{2}}{6} < \frac{5}{3}.$$

Using the differential equations (A.3) and (4.13), we obtain

$$\left(\frac{k_n}{g_n}\right)' = \frac{n}{xg_n}\left(1 - \frac{x^2}{n^2} - k_n g_n\right)\left(1 + \frac{k_n}{g_n}\right),$$

An expansion around zero shows that

$$1 - \frac{x^2}{n^2} - k_n g_n = \frac{1}{n^2 - 1}\frac{x^2}{n^2} + O(x^4),$$

therefore k_n/g_n initially decreases. Since $k_n(n)/g_n(n) > 0$, it decreases until $x_2 < x_1$ such that $k_n(x_2)g_n(x_2) = 1 - x_2^2/n^2$, and increases afterwards. Using the upper bound on g_n, we obtain, that, at that point,

$$\frac{k_n(x_2)}{g_n(x_2)} = \frac{1 - x_2^2/n^2}{g_n(x_2)^2} \geq \min_{0 \leq x \leq x_1}\left(1 + g_n^2(n)\frac{x}{n}\left(1 - \frac{x^2}{n^2}\right)^{-1}\right)^{-1} \geq \left(1 + \frac{g_n^2(n)}{k_n(\kappa_n)^2}\right)^{-1} \geq \frac{2}{5},$$

as claimed. ☐

Proposition A.2. *For any $n \neq 0$, let ζ_n be the first positive solution of*

$$(\ln|Y_n|)^{(2)}(\zeta_n) = 0.$$

On $(0, \zeta_n)$, $x \to x/k_n(x)$ is increasing, and ζ_n is the maximum of $x \to x/k_n(x)$ on $(0, n)$ (resp. on $(0, y_{0,1})$) when $n \geq 1$ (resp. $n = 0$).

When $n \geq 1$, we have $\zeta_n > \kappa_n$. Introducing $\chi_n := n - \frac{4}{5}n^{1/3}$ for $n \geq 1$, and $\chi_1 = 1/2$, we have

$$\frac{x}{nk_n(x)} \geq \begin{cases} \left(\frac{n^2}{x^2} - 1\right)^{-1/2} & \text{when } 0 \leq x \leq \chi_n, \\ \left(\frac{n^2}{\chi_n^2} - 1\right)^{-1/2} & \text{when } \chi_n \leq x \leq n. \end{cases} \tag{A.6}$$

When $n = 0$,

$$\zeta_0 \approx 0.3135, \text{ and } \frac{\zeta_0}{k_0(\zeta_0)} \approx 0.3524$$

In terms of previously defined functions, it is the unique solution of

$$k_0(\zeta_0) = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\zeta_0^2}.$$

Proof. From (A.3) we deduce that $x \to x/k_n(x)$ is a solution of the differential equation

$$\frac{d}{dx}\left(\frac{x}{k_n}(x)\right) = \frac{nk_n(x)(1-nk_n(x)) + n^2 - x^2}{nk_n^2(x)}$$

Clearly, while $x \leq \kappa_n$, that is, while k_n is decreasing, $x \to x/k_n(x)$ is increasing. We note that $nk_n(x)(1-nk_n(x)) + n^2 - x^2$ only has one root greater than $1/n$. From the lower bound on $k_n > 3/5n^{-1/3}$ given by Proposition A.1 and by inspection for $n = 1, 2$, we verify that $(x^{-1}k_n(x))'$ cancels at most once on (κ_n, n). We find

$$\frac{d}{dx}\left(\frac{x}{k_n}\right)(\kappa_n) = \frac{1}{k_n(\kappa_n)} > 0.$$

Using the lower estimate on $k_n(n)$ given by (A.4) and by inspection for $n = 1, 2$, we find

$$\frac{d}{dx}\left(\frac{x}{k_n}\right)(n) = \frac{1-nk_n(n)}{k_n(n)} < 0.$$

Thus, there exists a unique maximum for $x/k_n(x)$ on $(0, n)$. Noting that $k_n(x)/x = -(\ln|Y_n|)'(x)$, we conclude that this maximum is ζ_n. For any $x \in [\kappa_n, n]$, we obtain that

$$\frac{x}{nk_n(x)} \geq \min\left(\frac{\kappa_n}{nk_n(\kappa_n)}, \frac{6n^{1/3}}{7}\right) \geq \left(\frac{n^2}{\chi_n^2} - 1\right)^{-1/2},$$

where we used the upper bound for $x \leq \kappa_n$ given by Proposition A.1

$$k_n(x) \leq \sqrt{1 - \frac{x^2}{n^2}},$$

and where $\chi_n = n - \frac{4}{5}n^{1/3}$ is the lower bound κ_n given by the same proposition. In the case $n = 1$, $\kappa_1 \approx 0.52 > \frac{1}{2}$. Altogether, we have obtained

$$\frac{x}{nk_n(x)} \geq \begin{cases} \left(\frac{n^2}{x^2} - 1\right)^{-1/2} & \text{when } 0 \leq x \leq \chi_n, \\ \left(\frac{n^2}{\chi_n^2} - 1\right)^{-1/2} & \text{when } \chi_n \leq x \leq n. \end{cases}$$

For $n = 0$, we compute that

$$\frac{d}{dx}\left(\frac{1}{x}k_0\right) = 1 + \frac{1}{x^2}k_0\left(k_0 - 1\right)$$

Therefore $x \to \frac{1}{x}k_0(x)$ is decreasing until ζ_0, given by $k_0(\zeta_0) = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\kappa_0^2}$, and increasing afterwards. □

We conclude this section by a property of $x \to |H_0^{(1)}(x)|$ which is useful for broadband estimates.

Lemma A.3. *For any $x > 0$, the function $x \to \ln|H_0^{(1)}(x)|$ is convex. Furthermore, for any $y > 1$*

$$x \to \frac{\left|H_0^{(1)}(xy)\right|}{\left|H_0^{(1)}(x)\right|}$$

is decreasing on $(0, \infty)$, and

$$1 \geq \frac{\left|H_0^{(1)}(xy)\right|}{\left|H_0^{(1)}(x)\right|} \geq \frac{1}{\sqrt{y}}.$$

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