Rate of convergence of a two-scale expansion for some “weakly” stochastic homogenization problems

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Abstract

We establish a rate of convergence of the two scale expansion (in the sense of homogenization theory) of the solution to a highly oscillatory elliptic partial differential equation with random coefficients that are a perturbation of periodic coefficients.

1 Introduction and presentation of the main result

This article focuses on establishing a rate of convergence of the two scale expansion (in the sense of homogenization theory) of the solution to a highly oscillatory partial differential equation with random coefficients. We begin our exposition by briefly discussing the same question in a deterministic setting, before turning to the stochastic setting.

Consider the highly oscillatory problem

$$\left\{ \begin{array}{ll} -\mathrm{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u_0^\varepsilon\right] = f & \text{in } \mathcal{D}, \\ u_0^\varepsilon = 0 & \text{on } \partial\mathcal{D}, \end{array} \right. \tag{1}$$

where $\mathcal{D}$ is a regular bounded domain of $\mathbb{R}^d$, $f \in L^2(\mathcal{D})$, and A_{per} is a Q-periodic elliptic bounded matrix, with $Q = (-1/2, 1/2)^d$. For simplicity, we manipulate henceforth *symmetric* matrices, but the arguments carry over to non-symmetric matrices up to slight modifications. It is well known (see e.g. the classical textbooks [7, 11, 17], and also [14] for a general, numerically oriented presentation) that u_0^ε converges, weakly in $H^1(\mathcal{D})$ and strongly in $L^2(\mathcal{D})$, to the solution $u_0^\star$ to

$$\left\{ \begin{array}{ll} -\mathrm{div}\left[A_{per}^\star \nabla u_0^\star\right] = f & \text{in } \mathcal{D}, \\ u_0^\star = 0 & \text{on } \partial\mathcal{D}, \end{array} \right. \tag{2}$$

where the homogenized matrix is given by

$$(A_{per}^\star)_{ij} = \int_Q (e_i + \nabla w_{e_i}^0(y))^T A_{per}(y)(e_j + \nabla w_{e_j}^0(y))\, dy, \tag{3}$$

where, for any $p \in \mathbb{R}^d$, w_p^0 is the unique (up to the addition of a constant) solution to the corrector problem associated to the periodic matrix A_{per}:

$$\left\{ \begin{array}{l} -\mathrm{div}\left[A_{per}(p + \nabla w_p^0)\right] = 0, \\ w_p^0 \text{ is } Q\text{-periodic.} \end{array} \right. \tag{4}$$

The corrector function allows to compute the homogenized matrix, and it also allows to obtain a convergence result in the H^1 strong norm. Indeed, in dimension $d > 1$, under some regularity assumptions recalled below, we have

$$\left\| u_0^\varepsilon - \left[u_0^\star + \varepsilon \sum_{i=1}^d w_{e_i}^0 \left(\frac{\cdot}{\varepsilon}\right) \frac{\partial u_0^\star}{\partial x_i} \right] \right\|_{H^1(\mathcal{D})} \leq C\sqrt{\varepsilon} \tag{5}$$

for a constant C independent of ε (in dimension $d = 1$, the difference is of order ε rather than $\sqrt{\varepsilon}$).

Note that $v_0^\varepsilon = u_0^\star + \varepsilon \sum_{i=1}^d w_{e_i}^0 \left(\frac{\cdot}{\varepsilon}\right) \frac{\partial u_0^\star}{\partial x_i}$ is a function of order 1 in the H^1 norm. At first sight, one could thus expect that the difference between u_0^ε and v_0^ε is of order ε, rather than $\sqrt{\varepsilon}$. This lower order (in dimension $d > 1$) is due to an inconsistency of the boundary conditions. Note indeed that, by definition, $u_0^\varepsilon = 0$ on $\partial\mathcal{D}$, which is not the case of v_0^ε. Note also that the (lower than expected) rate in (5) is not specific to the choice of homogeneous Dirichlet boundary conditions in (1), and also holds for Neumann boundary conditions, as stated in [17, p. 29] (see also [22]).

The order of approximation improves if we ignore the difference between u_0^ε and v_0^ε at the boundary of the domain (see [1, Theorem 2.3]). Alternatively, one can build functions, the so-called boundary layers, that correct v_0^ε in the neighboorhood of $\partial\mathcal{D}$, to eventually improve the accuracy of the approximation of u_0^ε so obtained, in the complete domain $\mathcal{D}$. We refer to [1, 21] and to [13, Appendix B] (see also [2, Chap. 5] for the study of the same question in a time-dependent, parabolic setting). On another note, we refer to [26] for studies on the rate of convergence of u_0^ε to $u_0^\star$ in the $L^\infty(\mathcal{D})$ norm (see also [14] and references therein, and [10] for extensions to some nonlinear cases), and to [21, 22] for similar studies on the lowest eigenvalue λ_0^ε of the operator $L^\varepsilon = -\text{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\cdot\right]$.

The result (5) is interesting from the theoretical viewpoint. It is also helpful for proving numerical analysis results. In particular, this result is a key ingredient to prove error bounds for the Multiscale Finite Element Method (MsFEM). This numerical approach aims at approximating the solution u_0^ε to the highly oscillatory problem (1) (for a small, but non vanishing small scale ε), and does so by performing a variationnal approximation of (1) using pre-computed basis functions that are *adapted* to the problem. Consequently, the MsFEM approach yields an accurate approximation of u_0^ε using only a limited number of degrees of freedom, in contrast to a standard Finite Element Method approach. In addition, the MsFEM approach is applicable in general situations, and is not limited to the case when the highly oscillatory coefficient of the equation reads $A^\varepsilon(x) \equiv A_{per}\left(\frac{x}{\varepsilon}\right)$ for a fixed periodic matrix A_{per}. See [13] and references therein. As described below, our motivation for this work stems from our work [19], where we suggest a possible extension of the MsFEM approach to weakly stochastic settings. Again, a key ingredient for proving error bounds on the approach we propose there is to have a rate of convergence of the type (5).

Let us now turn to the stochastic case. As will be seen below, less precise results are known than in the deterministic, periodic case. The highly oscillatory problem reads

$$\left\{ \begin{array}{ll} -\text{div}\left[A_\eta\left(\frac{\cdot}{\varepsilon},\omega\right)\nabla u_\eta^\varepsilon(\cdot,\omega)\right] = f & \text{in } \mathcal{D}, \\ u_\eta^\varepsilon(\cdot,\omega) = 0 & \text{on } \partial\mathcal{D}, \end{array} \right. \tag{6}$$

where the matrix A_η is now a stationary symmetric matrix, uniformly elliptic and bounded (see (8) below for a precise definition of stationarity, which is the common assumption in stochastic homogenization). The role of the parameter η will be made precise in (9) below. It can momentarily be ignored. Again, as in the periodic case, it is well known (see for instance [17]) that u_η^ε converges, almost surely, weakly in $H^1(\mathcal{D})$ and strongly in

$L^2(\mathcal{D})$, to $u_\eta^\star$, solution to the homogenized equation

$$\begin{cases} -\mathrm{div}\left[A_\eta^\star \nabla u_\eta^\star\right] = f & \text{in } \mathcal{D}, \\ u_\eta^\star = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where the homogenized matrix is given by

$$\left(A_\eta^\star\right)_{ij} = \mathbb{E}\left(\int_Q (e_i + \nabla w_{e_i}^\eta(y,\cdot))^T A_\eta(y,\cdot)(e_j + \nabla w_{e_j}^\eta(y,\cdot))\, dy \right),$$

where, for any $p \in \mathbb{R}^d$, w_p^η is the unique (up to the addition of a random constant) solution to the stochastic corrector problem

$$\begin{cases} -\mathrm{div}\left[A_\eta\left(\cdot,\omega\right)\left(p + \nabla w_p^\eta(\cdot,\omega)\right)\right] = 0 \text{ in } \mathbb{R}^d, \\ \nabla w_p^\eta \text{ is stationary in the sense of (8) below,} \\ \mathbb{E}\left(\int_Q \nabla w_p^\eta(y,\cdot)\, dy\right) = 0. \end{cases}$$

As in the periodic case, the corrector function w_p^η allows to obtain a convergence result in the H^1 norm (see [24, Theorem 3]):

$$\mathbb{E}\left[\left\| u_\eta^\varepsilon(\cdot,\omega) - \left[u_\eta^\star + \varepsilon \sum_{i=1}^d w_{e_i}^\eta\left(\frac{\cdot}{\varepsilon},\omega\right) \frac{\partial u_\eta^\star}{\partial x_i} \right] \right\|_{H^1(\mathcal{D})}^2 \right] \quad \text{converges to 0 as } \varepsilon \to 0. \tag{7}$$

However, in contrast to the periodic case, the rate of convergence is generally not known, in dimensions higher than one. In the one-dimensional case, this question has been addressed in [9, 6]. It is shown there that the rate can be arbitrary small, depending on the rate with which the correlations of the random coefficient in (6) vanish. The only assumptions of stationarity and ergodicity do not allow for a precise rate. See also [20] for the study of a similar question for a variant of stochastic homogenization, again in the one-dimensional case, and [5] for results in the multi-dimensional case, for a different equation.

The aim of this article is to show that, in a *weakly* stochastic case (the precise sense of which is given below), a convergence rate for (7) can be obtained (in the same spirit as (5)). As in the deterministic case, this result is interesting from the theoretical viewpoint, and somewhat complements the one-dimensional results of [9, 6, 20]. It is also useful from a numerical analysis viewpoint. In [19], we propose an extension of the MsFEM approach to weakly stochastic settings, and we use there the homogenization result that we prove in this work (see Theorem 2 below) to obtain error bounds (see [19, Theorem 10]).

Before presenting our result, let us briefly recall the basic setting of stochastic homogenization. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a random variable $X \in L^1(\Omega, d\mathbb{P})$, we denote by $\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ its expectation value. We assume that the group $(\mathbb{Z}^d, +)$ acts on Ω. We denote by $(\tau_k)_{k\in\mathbb{Z}^d}$ this action, and assume that it preserves the measure $\mathbb{P}$, i.e.

$$\forall k \in \mathbb{Z}^d, \quad \forall A \in \mathcal{F}, \quad \mathbb{P}(\tau_k A) = \mathbb{P}(A).$$

We assume that τ is *ergodic*, that is,

$$\forall A \in \mathcal{F}, \quad \left(\forall k \in \mathbb{Z}^d, \quad \tau_k A = A\right) \Rightarrow (\mathbb{P}(A) = 0 \quad \text{or} \quad 1).$$

We define the following notion of stationarity: any $F \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d, L^1(\Omega)\right)$ is said to be *stationary* if

$$\forall k \in \mathbb{Z}^d, \quad F(x+k,\omega) = F(x,\tau_k\omega) \text{ almost everywhere, almost surely.} \tag{8}$$

Note that we have chosen to present the theory in a *discrete* stationary setting, which is more appropriate for our specific purpose, which is to consider a setting close to *periodic* homogenization. Random homogenization is more often presented in the *continuous* stationary setting. This is only a matter of small modifications. We refer to the bibliography for the latter.

We now precisely describe the weakly stochastic setting we consider. We assume that the matrix A_η in (6) reads

$$A_\eta(x,\omega) = A_{per}(x) + \eta A_1(x,\omega), \tag{9}$$

where $\eta \in \mathbb{R}$ is *small* deterministic parameter, A_{per} is a symmetric uniformy elliptic bounded Q-periodic matrix, and A_1 is a symmetric matrix, stationary in the sense of (8), and bounded: $|A_1(x,\omega)| \le C$ almost everywhere in $\mathbb{R}^d$, almost surely. We also assume that A_η is uniformly elliptic and bounded, in the sense that, for all $\eta \in \mathbb{R}$,

$$A_\eta(\cdot,\omega) \in (L^\infty(\mathbb{R}^d))^{d\times d} \quad \text{a.s.}$$

and there exists $c > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \quad \xi^T A_\eta(x,\omega)\xi \ge c\, \xi^T \xi \quad \text{a.s., a.e. on } \mathbb{R}^d.$$

We furthermore assume that A_1 is of the form

$$A_1(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) X_k(\omega)\, B_{per}(x), \tag{10}$$

where $(X_k(\omega))_{k\in\mathbb{Z}^d}$ is a sequence of i.i.d. scalar random variables such that

$$\exists C,\ \forall k \in \mathbb{Z}^d, \quad |X_k(\omega)| \le C \quad \text{almost surely,}$$

and $B_{per} \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}$ is a Q-periodic matrix. Finally, we assume that

$$A_{per} \text{ is Hölder continuous,} \tag{11}$$
$$B_{per} \text{ is Hölder continuous.} \tag{12}$$

As pointed out above, the symmetry assumption is not essential, and our arguments below carry over to non-symmetric matrices up to slight modifications. Likewise, the assumption (10) can be relaxed. What is important in (10) is that A_1 is a sum of *direct products* of a function depending on x with a random variable, depending only on ω.

In contrast, it is difficult to weaken assumptions (11) and (12), which are used to obtain some regularity on the correctors w_p^0 and ψ_p, solutions to (4) and (16) below, respectively. We indeed recall that, under assumption (11), we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15, Theorem 8.22 and Corollary 8.36]), and similarly for ψ_p, under assumption (12). In the sequel, we will use the fact that w_p^0 and ψ_p belong to $W^{1,\infty}(\mathbb{R}^d)$, which is a standard assumption when proving convergence rates of two-scale expansions (see e.g. [17, p. 28]).

We also note that, following [3], the assumption (11) is useful to characterize the asymptotic behavior of the Green function associated to the operator $L = -\mathrm{div}\,[A_{per}\nabla\cdot]$ on the domain $\mathcal{D}/\varepsilon$ (with homogeneous Dirichlet boundary conditions). This Green function will be used in the sequel.

Remark 1. *There are several ways to formalize a notion of "weakly" stochastic setting, and* (9) *is only* one *of them. We refer to [18, 4] for other examples.*

Our main result is the following.

Theorem 2. *Assume that the dimension d is strictly higher than 1. Let u_η^ε be the solution to (6), and assume that A_η satisfies (9)-(10)-(11)-(12). Let $A_{per}^\star$, w_p^0 and $u_0^\star$ be defined by (3), (4) and (2). Let $\overline{B} \in \mathbb{R}^{d\times d}$ and $\overline{u}_1^\star \in H_0^1(\mathcal{D})$ be defined by*

$$\forall 1 \leq i,j \leq d, \quad \overline{B}_{ij} = \int_Q (e_i + \nabla w_{e_i}^0)^T B_{per} (e_j + \nabla w_{e_j}^0) \tag{13}$$

and

$$\left\{ \begin{array}{ll} -div\left[A_{per}^\star \nabla \overline{u}_1^\star\right] = div\left[\overline{B}\nabla u_0^\star\right] & in\ \mathcal{D}, \\ \overline{u}_1^\star = 0 & on\ \partial\mathcal{D}. \end{array} \right. \tag{14}$$

Introduce v_η^ε defined by

$$\begin{aligned} v_\eta^\varepsilon(\cdot,\omega) = u_0^\star + \eta \mathbb{E}(X_0)\overline{u}_1^\star + \varepsilon \sum_{p=1}^d & \left[w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right) (\partial_p u_0^\star + \eta\mathbb{E}(X_0)\partial_p \overline{u}_1^\star) \right. \\ & \left. + \eta\mathbb{E}(X_0)\psi_{e_p}\left(\frac{\cdot}{\varepsilon}\right)\partial_p u_0^\star + \eta \sum_{k\in I_\varepsilon} (X_k(\omega) - \mathbb{E}(X_0))\ \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right) \partial_p u_0^\star \right], \end{aligned} \tag{15}$$

where $\partial_p u_0^\star$ denotes the partial derivative $\dfrac{\partial u_0^\star}{\partial x_p}$,

$$I_\varepsilon = \left\{ k \in \mathbb{Z}^d \ such\ that\ \varepsilon(Q+k)\cap\mathcal{D} \neq \emptyset \right\},$$

and where, for any $p \in \mathbb{R}^d$, ψ_p is the solution (unique up to the addition of a constant) to

$$\left\{ \begin{array}{l} -div[A_{per}\nabla\psi_p] = div\left[B_{per}\left(p + \nabla w_p^0\right)\right], \\ \psi_p\ is\ Q\text{-}periodic, \end{array} \right. \tag{16}$$

and χ_p is the unique solution to

$$\left\{ \begin{array}{ll} -div[A_{per}\nabla\chi_p] = div\left[\mathbf{1}_Q B_{per}(p+\nabla w_p^0)\right] & in\ \mathbb{R}^d, \\ \chi_p \in L^2_{loc}(\mathbb{R}^d), \quad \nabla\chi_p \in \left(L^2(\mathbb{R}^d)\right)^d, & \\ \lim_{|x|\to\infty} \chi_p(x) = 0. & \end{array} \right. \tag{17}$$

We assume that $u_0^\star \in W^{2,\infty}(\mathcal{D})$ and $\overline{u}_1^\star \in W^{2,\infty}(\mathcal{D})$. Then

$$\sqrt{\mathbb{E}\left[\|u_\eta^\varepsilon - v_\eta^\varepsilon\|^2_{H^1(\mathcal{D})}\right]} \leq C\left(\sqrt{\varepsilon} + \eta\sqrt{\varepsilon\ln(1/\varepsilon)} + \eta^2\right), \tag{18}$$

where C is a constant independent of ε and η.

We wish to point out that the assumption $u_0^\star \in W^{2,\infty}(\mathcal{D})$ (and subsequently $\overline{u}_1^\star \in W^{2,\infty}(\mathcal{D})$) is a standard assumption when proving convergence rates of two-scale expansions (see e.g. [1, Theorem 2.1] and [17, p. 28]). Note that, in view of (2), this assumption implies that the right hand side f in (6) belongs to $L^\infty(\mathcal{D})$. We also note that v_η^ε is not uniquely defined, since w_p^0 and ψ_p are only defined up to an additive constant. However, adding a constant to any of these functions does not change the order of convergence in (18) with respect to ε and η, but only the constant C. Choosing the best constants in w_p^0 and ψ_p is hence irrelevant here, although it is an important matter from the practical viewpoint. Lastly, the existence and uniqueness of a function χ_p satisfying (17) is shown in Lemma 10 below, in dimension $d > 1$. In dimension $d = 1$, the boundary conditions of (17) need to be modified for this problem to have a solution. The one-dimensional version of Theorem 2 is as follows:

Theorem 3. *Assume that the dimension d is equal to one. Let u_η^ε be the solution to (6) in the domain $\mathcal{D}$ with $f \in L^2(\mathcal{D})$, and assume that A_η satisfies (9)-(10). Let v_η^ε be defined by (15), where the definition (17) of the function χ is replaced by*

$$\begin{cases} -\left[A_{per}\chi'\right]' = \left[\mathbf{1}_{(0,1)} B_{per}(1+(w^0)')\right]' & \text{in } \mathbb{R}, \\ \chi \in L^2_{loc}(\mathbb{R}), \quad \chi' \in L^2(\mathbb{R}), \end{cases}$$

where w^0 solves (4). Then

$$\sqrt{\mathbb{E}\left[\|u_\eta^\varepsilon - v_\eta^\varepsilon\|^2_{H^1(\mathcal{D})}\right]} \leq C\left(\varepsilon + \eta\sqrt{\varepsilon} + \eta^2\right), \tag{19}$$

$$\sqrt{\mathbb{E}\left[\|u_\eta^\varepsilon - v_\eta^\varepsilon\|^2_{L^\infty(\mathcal{D})}\right]} \leq C\left(\varepsilon + \eta\sqrt{\varepsilon} + \eta^2\right), \tag{20}$$

where C is a constant independent of ε and η.

Note that, in dimension $d = 1$, we do not need to assume (11) and (12). In dimensions $d > 1$, as pointed out above, these assumptions are used to have that the correctors w_p^0 and ψ_p, solutions to (4) and (16) respectively, both belong to $W^{1,\infty}(\mathbb{R}^d)$. In dimension $d = 1$, the coercivity assumption on A_{per} and the boundedness assumption on B_{per} are enough to show that w^0 and ψ both belong to $W^{1,\infty}(\mathbb{R})$. Likewise, when $d > 1$, we assumed that $u_0^\star \in W^{2,\infty}(\mathcal{D})$ and $\overline{u}_1^\star \in W^{2,\infty}(\mathcal{D})$ (which implies that $f \in L^\infty(\mathcal{D})$). When $d = 1$, the assumption $f \in L^2(\mathcal{D})$ is enough.

On another note, we notice that χ is now only defined up to an additive constant. Again, changing χ by a constant does not change the order of convergence in (19)-(20) with respect to ε and η, but only changes the constant C.

In addition to its theoretical interest, Theorem 2 has also interesting numerical counterparts. Indeed, to compute v_η^ε, one needs to solve problems set on a *bounded* domain (either with Dirichlet or periodic boundary conditions), and to solve for χ_p, solution to the problem (17), set on the entire space. However, the right hand side in (17) is the divergence of a compactly supported function, and we will see that $\chi_p(x)$ quickly vanishes when x is sufficiently large (see Lemma 10 below). Hence, in practice, it is possible to approximate (17) by using Dirichlet boundary conditions on a domain of limited a size.

The proof of Theorem 2 consists of two steps. The first one is to expand u_η^ε with respect to η. This is performed in Section 2 below (see Lemma 4). Each term of the expansion of u_η^ε is found to be the unique solution of a partial differential equation with a *deterministic*, highly oscillating coefficient, to which is associated a homogenized equation. The second step of the proof consists in successively estimating, for each of the terms of the expansion in η, the rate of convergence of their two scale expansion in ε. Corresponding results are stated in Section 3 (and proved in Section 5). Collecting these results, we are then in position to prove our main result, Theorem 2 (see Section 4, where we also prove Theorem 3).

2 Expansion in powers of η

In this section, we expand the solution u_η^ε to (6) with respect to η.

Lemma 4. *Let u_η^ε be the solution to (6). Under the assumption (9), it can be expanded in powers of η as follows:*

$$u_\eta^\varepsilon = u_0^\varepsilon + \eta u_1^\varepsilon + \eta^2 r_\eta^\varepsilon, \tag{21}$$

where u_0^ε is solution to the deterministic problem (1), u_1^ε is solution to

$$\begin{cases} -div\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u_1^\varepsilon(\cdot,\omega)\right] = div\left[A_1\left(\frac{\cdot}{\varepsilon},\omega\right)\nabla u_0^\varepsilon\right] & \text{in } \mathcal{D}, \\ u_1^\varepsilon(\cdot,\omega) = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{22}$$

and r^ε_η is solution to

$$\left\{ \begin{array}{ll} -div\left[A_\eta\left(\frac{\cdot}{\varepsilon},\omega\right)\nabla r^\varepsilon_\eta(\cdot,\omega)\right] = div\left[A_1\left(\frac{\cdot}{\varepsilon},\omega\right)\nabla u^\varepsilon_1(\cdot,\omega)\right] & \textit{in } \mathcal{D}, \\ r^\varepsilon_\eta(\cdot,\omega) = 0 & \textit{on } \partial\mathcal{D}. \end{array} \right.$$

In addition, we have, almost surely,

$$\|u^\varepsilon_0\|_{H^1(\mathcal{D})} \le C, \quad \|u^\varepsilon_1(\cdot,\omega)\|_{H^1(\mathcal{D})} \le C, \quad \|r^\varepsilon_\eta(\cdot,\omega)\|_{H^1(\mathcal{D})} \le C, \tag{23}$$

where C is a deterministic constant independent of ε and η.

Proof. The relation (21) is a simple consequence of the linearity of the considered equation. The bounds (23) follow from the uniform ellipticity of the matrices A_η and A_{per}, and the boundedness of A_1. $\square$

For the sequel, it is useful to further decompose u^ε_1 in a deterministic part and a stochastic part of vanishing expectation.

Lemma 5. *Under assumptions* (9)-(10)*, the solution u^ε_1 to* (22) *writes*

$$u^\varepsilon_1 = \mathbb{E}(X_0)\overline{u}^\varepsilon_1 + \sum_{k\in\mathbb{Z}^d}(X_k(\omega) - \mathbb{E}(X_0))\phi^\varepsilon_k, \tag{24}$$

where $\overline{u}^\varepsilon_1$ is the unique solution to

$$\left\{ \begin{array}{ll} -div\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla \overline{u}^\varepsilon_1\right] = div\left[B_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon_0\right] & \textit{in } \mathcal{D}, \\ \overline{u}^\varepsilon_1 = 0 & \textit{on } \partial\mathcal{D}, \end{array} \right. \tag{25}$$

and ϕ^ε_k is the unique solution to

$$\left\{ \begin{array}{ll} -div\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla \phi^\varepsilon_k\right] = div\left[\mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right)B_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon_0\right] & \textit{in } \mathcal{D}, \\ \phi^\varepsilon_k = 0 & \textit{on } \partial\mathcal{D}. \end{array} \right. \tag{26}$$

In addition, there exists C, independent of ε, such that

$$\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}^d}[X_k - \mathbb{E}(X_0)]\phi^\varepsilon_k\right\|^2_{H^1(\mathcal{D})}\right] \le C. \tag{27}$$

Assume furthermore that (11) *holds, and that*

$$f \in L^q(\mathcal{D}) \quad \textit{for some} \quad q > d. \tag{28}$$

Then there exists C, independent of k and ε, such that

$$\|\phi^\varepsilon_k\|_{L^\infty(\mathcal{D})} \le C\varepsilon. \tag{29}$$

Proof. We note that, if $k \in \mathbb{Z}^d$ is such that $\varepsilon(Q+k)\cap\mathcal{D} = \emptyset$, then (26) writes

$$\left\{ \begin{array}{ll} -\text{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla \phi^\varepsilon_k\right] = 0 & \text{in } \mathcal{D}, \\ \phi^\varepsilon_k = 0 & \text{on } \partial\mathcal{D}, \end{array} \right.$$

the solution of which is obviously $\phi_k^\varepsilon \equiv 0$. The sum in (24) hence only contains a finite number of terms, and the proof of the decomposition (24) goes by linearity of the equation (22). Note however that the number of terms in (24) depends on ε, and diverges when $\varepsilon \to 0$.

We now prove the bound (27). Using that A_{per} is coercive, we infer from (26) that

$$\begin{aligned}
\alpha\|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} &\leq \int_{\mathcal{D}} (\nabla\phi_k^\varepsilon)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\phi_k^\varepsilon \\
&\leq \int_{\mathcal{D}} (\nabla\phi_k^\varepsilon)^T \mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right) B_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u_0^\varepsilon \\
&\leq \|B_{per}\|_{L^\infty}\|\phi_k^\varepsilon\|_{H^1(\mathcal{D})}\|u_0^\varepsilon\|_{H^1(\varepsilon(Q+k))},
\end{aligned}$$

where $\alpha > 0$ is some constant that only depends on the coercivity constant of A_{per} and the Poincaré constant of the domain $\mathcal{D}$. Thus

$$\|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \alpha^{-2}\|B_{per}\|^2_{L^\infty}\|u_0^\varepsilon\|^2_{H^1(\varepsilon(Q+k))}.$$

Using that $\phi_k^\varepsilon \equiv 0$ as soon as $\varepsilon(Q+k)\cap\mathcal{D} = \emptyset$, we obtain

$$\sum_{k\in\mathbb{Z}^d}\|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \alpha^{-2}\|B_{per}\|^2_{L^\infty}\|u_0^\varepsilon\|^2_{H^1(\mathcal{D})}.$$

We deduce from that bound and the assumption that the random variables X_k are i.i.d. that

$$\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}^d}(X_k - \mathbb{E}(X_0))\phi_k^\varepsilon\right\|^2_{H^1(\mathcal{D})}\right] = \mathbb{V}\mathrm{ar}(X_0)\sum_{k\in\mathbb{Z}^d}\|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \mathbb{V}\mathrm{ar}(X_0)\alpha^{-2}\|B_{per}\|^2_{L^\infty}\|u_0^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C,$$

where C is independent of ε (we have used (23) to bound u_0^ε). We thus have shown (27).

We finally turn to the proof of (29). Let us define $\overline{\phi}_k^\varepsilon(x) = \phi_k^\varepsilon(\varepsilon x)$ on $\mathcal{D}/\varepsilon$. In view of (26), we see that $\overline{\phi}_k^\varepsilon$ solves

$$\begin{cases} -\mathrm{div}\left[A_{per}\nabla\overline{\phi}_k^\varepsilon\right] = \varepsilon\mathrm{div}\left[\mathbf{1}_{Q+k}B_{per}\nabla u_0^\varepsilon(\varepsilon\cdot)\right] & \text{in } \mathcal{D}/\varepsilon, \\ \overline{\phi}_k^\varepsilon = 0 & \text{on } \partial(\mathcal{D}/\varepsilon). \end{cases}$$

Introduce now the Green function $\Gamma_\varepsilon(x,y)$ associated to the operator $L = -\mathrm{div}\left[A_{per}\nabla\cdot\right]$ on the domain $\mathcal{D}/\varepsilon$, with homogeneous Dirichlet boundary conditions. We recall that $\Gamma_\varepsilon^T(x,y) := \Gamma_\varepsilon(y,x)$ is the Green function associated to the adjoint operator $L^T = -\mathrm{div}\left[A_{per}^T\nabla\cdot\right]$ on the domain $\mathcal{D}/\varepsilon$, with homogeneous Dirichlet boundary conditions (a proof of this fact is given in [16, Theorem 1.3] and [12, Theorem 1] in the case $d \geq 3$, and this proof carries over to the case $d = 2$). Consequently, we have $\Gamma_\varepsilon(x,y) = 0$ as soon as x or y belongs to the boundary $\partial(\mathcal{D}/\varepsilon)$ We can thus write

$$\begin{aligned}
\overline{\phi}_k^\varepsilon(x) &= \varepsilon\int_{\mathcal{D}/\varepsilon}\Gamma_\varepsilon(x,y)\ \mathrm{div}_y\left[\mathbf{1}_{Q+k}(y)B_{per}(y)\nabla u_0^\varepsilon(\varepsilon y)\right]\,dy \\
&= -\varepsilon\int_{Q+k}\nabla_y\Gamma_\varepsilon(x,y)B_{per}(y)\nabla u_0^\varepsilon(\varepsilon y)\,dy.
\end{aligned}$$

Hence, for any $x \in \mathcal{D}$, we have

$$\phi_k^\varepsilon(x) = -\varepsilon^{1-d}\int_{\varepsilon(Q+k)}\nabla_y\Gamma_\varepsilon\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)B_{per}\left(\frac{y}{\varepsilon}\right)\nabla u_0^\varepsilon(y)\,dy.$$

Using the fact (see [3, Proposition 8]) that, under assumption (11), the Green function Γ_ε on the domain $\mathcal{D}/\varepsilon$ satisfies

$$\forall x \in \mathcal{D}/\varepsilon, \quad \forall y \in \mathcal{D}/\varepsilon, \quad |\nabla_x \Gamma_\varepsilon(x,y)| + |\nabla_y \Gamma_\varepsilon(x,y)| \leq \frac{C}{|x-y|^{d-1}} \tag{30}$$

for a constant C independent of ε, we have

$$|\phi_k^\varepsilon(x)| \leq C \|B_{per}\|_{L^\infty} \|\nabla u_0^\varepsilon\|_{L^\infty(\mathcal{D})} \int_{\varepsilon(Q+k)} \frac{1}{|x-y|^{d-1}} dy. \tag{31}$$

We will show in the sequel that (28) implies that there exists C such that, for all ε,

$$\|\nabla u_0^\varepsilon\|_{L^\infty(\mathcal{D})} \leq C. \tag{32}$$

We are thus left with bounding the integral in (31). To this aim, we distinguish two cases. If $|x - \varepsilon k| \leq \varepsilon$, then there exists a constant ρ_d than only depends on the dimension such that $\varepsilon(Q+k) \subset B(x, \rho_d \varepsilon)$ (for instance, in dimension $d=2$, $\rho_2 = 1 + \sqrt{2}/2$). We then have

$$\int_{\varepsilon(Q+k)} \frac{1}{|x-y|^{d-1}} dy \leq \int_{B(x,\rho_d\varepsilon)} \frac{1}{|x-y|^{d-1}} dy \leq C\varepsilon, \quad C \text{ independent of } \varepsilon. \tag{33}$$

Otherwise, if $|x - \varepsilon k| \geq \varepsilon$, then any $y \in \varepsilon(Q+k)$ satisfies $\overline{\rho}_d \varepsilon \leq |x-y|$ for a constant $\overline{\rho}_d$ that only depends on d (in dimension $d=2$, $\rho_2 = 1 - \sqrt{2}/2$). For those x, we have

$$\int_{\varepsilon(Q+k)} \frac{1}{|x-y|^{d-1}} \, dy \leq \frac{1}{(\overline{\rho}_d \varepsilon)^{d-1}} \int_{\varepsilon(Q+k)} dy \leq C\varepsilon, \quad C \text{ independent of } \varepsilon. \tag{34}$$

Thus, collecting (31), (32), (33) and (34), we obtain that

$$\|\phi_k^\varepsilon\|_{L^\infty(\mathcal{D})} \leq C\varepsilon, \quad C \text{ independent of } \varepsilon.$$

Proving (29) therefore amounts to now proving (32). Again using the Green function $\Gamma_\varepsilon(x,y)$ associated to the operator $L = -\mathrm{div}\,[A_{per}\nabla \cdot]$ on the domain $\mathcal{D}/\varepsilon$, with homogeneous Dirichlet boundary conditions, we write

$$\nabla u_0^\varepsilon(x) = \varepsilon^{1-d} \int_{\mathcal{D}} \nabla_x \Gamma_\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \, f(y) \, dy.$$

Using the bound (30), we deduce that there exists C independent of ε such that

$$\forall x \in \mathcal{D}, \quad |\nabla u_0^\varepsilon(x)| \leq C \int_{\mathcal{D}} \frac{|f(y)|}{|x-y|^{d-1}} \, dy.$$

In view of assumption (28), we have $f \in L^q(\mathcal{D})$ for some $q > d$. Using Hölder inequality, we write

$$\forall x \in \mathcal{D}, \quad |\nabla u_0^\varepsilon(x)| \leq C \|f\|_{L^q(\mathcal{D})} \left\| \frac{1}{|x - \cdot|^{d-1}} \right\|_{L^{q^\star}(\mathcal{D})}, \quad \frac{1}{q} + \frac{1}{q^\star} = 1.$$

The function $y \mapsto |x-y|^{1-d}$ belongs to $L^p(\mathcal{D})$ for any $p < d/(d-1)$. Since $q > d$, we have $q^\star < d/(d-1)$, and the norm in $L^{q^\star}$ of $y \mapsto |x-y|^{1-d}$ is independent of x. The above estimate thus yields (32). This concludes the proof of Lemma 5. $\square$

3 Two scale expansions in powers of ε

Collecting (21) and (24), we have obtained that

$$u_\eta^\varepsilon(x,\omega) = u_0^\varepsilon(x) + \eta \left[\mathbb{E}(X_0)\overline{u}_1^\varepsilon(x) + \sum_{k\in\mathbb{Z}^d} (X_k(\omega) - \mathbb{E}(X_0))\phi_k^\varepsilon(x) \right] + \eta^2 r_\eta^\varepsilon(x,\omega), \tag{35}$$

where r_η^ε is bounded in $H^1(\mathcal{D})$ uniformly in ε, η and ω (see (23)).

We now consider successively each term of the above series and show a rate of convergence on the difference between u_0^ε, $\overline{u}_1^\varepsilon$ and ϕ_k^ε and their respective two-scale expansions. For clarity, the proofs of our results are postponed until Section 5.

We start by u_0^ε solution to (1). Note that this problem is a classical periodic homogenization problem, the limit of which, when $\varepsilon \to 0$, is well-known. The following result, giving a *rate of convergence* of u_0^ε to its homogenized limit, is also classical (see e.g. [17, p. 28]).

Proposition 6. *Let u_0^ε and $u_0^\star$ be the solution to* (1) *and* (2)*, respectively. For any $p \in \mathbb{R}^d$, we assume that the solution w_p^0 to* (4) *satisfies $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$. We also assume that $u_0^\star \in W^{2,\infty}(\mathcal{D})$. We then have*

$$u_0^\varepsilon = u_0^\star + \varepsilon \sum_{i=1}^d w_{e_i}^0 \left(\frac{\cdot}{\varepsilon}\right) \partial_i u_0^\star + \varepsilon\theta_0^\varepsilon, \tag{36}$$

where θ_0^ε satisfies

$$\|\varepsilon\theta_0^\varepsilon\|_{H^1(\mathcal{D})} \le C\sqrt{\varepsilon} \tag{37}$$

for a constant C independent of ε.

We recall that, under assumption (11), we indeed have that $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15, Theorem 8.22 and Corollary 8.36]).

We now turn to $\overline{u}_1^\varepsilon$ solution to (25). This problem is not a classical homogenization problem, since its right-hand side also varies at the scale ε, and only *weakly* converges in $H^{-1}(\mathcal{D})$ when $\varepsilon \to 0$. We first proceed formally, using the two-scale ansatz approach, to identify the homogenized equation. We next state a precise homogenization result, and finally evaluate the rate of convergence of the two scale expansion.

To derive formally the homogenized equation associated to (25), we make the classical two-scale ansatz

$$\overline{u}_1^\varepsilon(x) = \overline{u}_1^\star\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \overline{u}_1^1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 \overline{u}_1^2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

where each term of the above expansion is assumed to be periodic with respect to the second variable. Inserting this ansatz in (25) and using the two scale expansion (36) of u_0^ε (where we neglect the remainder $\varepsilon\theta_0^\varepsilon$), we can easily derive a hierarchy of equations. We deduce from the equation of order ε^{-2} that $\overline{u}_1^\star$ is independent of its second variable: $\overline{u}_1^\star(x,y) \equiv \overline{u}_1^\star(x)$. The equation of order ε^{-1} reads

$$-\mathrm{div}_y \left[A_{per}(y) \left(\nabla_x \overline{u}_1^\star(x) + \nabla_y \overline{u}_1^1(x,y)\right)\right] = \sum_{i=1}^d \partial_i u_0^\star(x) \ \mathrm{div}_y \left[B_{per}(y) \left(e_i + \nabla_y w_{e_i}^0(y)\right)\right].$$

Using the functions w_p^0 and ψ_p defined by (4) and (16), we thus see that

$$\overline{u}_1^1(x,y) = \tau(x) + \sum_{i=1}^d \partial_i u_0^\star(x)\psi_{e_i}(y) + \partial_i \overline{u}_1^\star(x) w_{e_i}^0(y), \tag{38}$$

where τ is an undetermined function that only depends on x. We are now in position to use the equation of order ε^0, which reads (recall we have neglected the remainder $\varepsilon\theta_0^\varepsilon$ in (36))

$$- \operatorname{div}_x \left[A_{per}(y) \left(\nabla_x \overline{u}_1^\star(x) + \nabla_y \overline{u}_1^1(x,y)\right)\right] - \operatorname{div}_y \left[A_{per}(y) \left(\nabla_x \overline{u}_1^1(x,y) + \nabla_y \overline{u}_1^2(x,y)\right)\right]$$
$$= \sum_{i=1}^d \operatorname{div}_x \left[B_{per}(y) \left(e_i + \nabla_y w_{e_i}^0(y)\right) \partial_i u_0^\star\right] + \operatorname{div}_y \left[\sum_{i=1}^d w_{e_i}^0(y) B_{per}(y) \nabla_x \partial_i u_0^\star(x)\right].$$

We close the hierarchy by integrating the above equation over the variable $y \in Q$, using that $y \mapsto \overline{u}_1^2(x,y)$ is Q-periodic. Using (38) and the expression (3), we then obtain that $\overline{u}_1^\star$ satisfies

$$\begin{cases} -\operatorname{div}\left[A_{per}^\star \nabla \overline{u}_1^\star\right] = \operatorname{div}\left[\widetilde{B} \nabla u_0^\star\right] & \text{in } \mathcal{D}, \\ \overline{u}_1^\star = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{39}$$

with

$$\forall 1 \le i,j \le d, \quad \widetilde{B}_{ij} = \int_Q e_i^T A_{per} \nabla \psi_{e_j} + \int_Q e_i^T B_{per}(e_j + \nabla w_{e_j}^0). \tag{40}$$

Mutiplying (16) (for $p = e_j$) by $w_{e_i}^0$ and integrating over Q, we find that

$$\forall 1 \le i,j \le d, \quad \int_Q (\nabla w_{e_i}^0)^T A_{per} \nabla \psi_{e_j} = - \int_Q (\nabla w_{e_i}^0)^T B_{per}(e_j + \nabla w_{e_j}^0).$$

Inserting this relation in (40), we deduce that the matrix $\widetilde{B}$ is equal to the matrix $\overline{B}$ defined by (13). We hence deduce from (39) that $\overline{u}_1^\star$ indeed satisfies (14).

These formal computations are formalized in a rigorous way in the following Propositions:

Proposition 7. *Assume that, for any $p \in \mathbb{R}^d$, the corrector w_p^0 solution to (4) satisfies $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$, and that the solution $u_0^\star$ to (2) satisfies $u_0^\star \in W^{2,\infty}(\mathcal{D})$. Then the function $\overline{u}_1^\varepsilon$ solution to (25) converges, weakly in $H^1(\mathcal{D})$ and strongly in $L^2(\mathcal{D})$, to the unique solution $\overline{u}_1^\star$ to (14).*

The regularity assumptions on w_p^0 and $u_0^\star$ ensure that ∇u_0^ε in the right-hand side of (25) can be controlled in the appropriate norm.

Proposition 8. *Let $\overline{u}_1^\varepsilon$ be the solution to (25), $\overline{u}_1^\star$ be the solution to (14) and $u_0^\star$ be the solution to (2). For any $p \in \mathbb{R}^d$, let w_p^0 be the solution to (4) and ψ_p be the solution to (16).*

Introduce $\overline{v}_1^\varepsilon$ defined by

$$\overline{v}_1^\varepsilon = \overline{u}_1^\star + \varepsilon \sum_{i=1}^d \left(w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right) \partial_i \overline{u}_1^\star + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i u_0^\star \right),$$

and assume that $u_0^\star \in W^{2,\infty}(\mathcal{D})$, $\overline{u}_1^\star \in W^{2,\infty}(\mathcal{D})$, and that, for any $p \in \mathbb{R}^d$, we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi_p \in W^{1,\infty}(\mathbb{R}^d)$. We then have

$$\|\overline{u}_1^\varepsilon - \overline{v}_1^\varepsilon\|_{H^1(\mathcal{D})} \le C\sqrt{\varepsilon}$$

for a constant C independent of ε.

Again, under assumptions (11) and (12), we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi_p \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15, Theorem 8.22 and Corollary 8.36]).

We finally turn to ϕ_k^ε solution to (26), namely

$$\begin{cases} -\operatorname{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla \phi_k^\varepsilon\right] = \operatorname{div}\left[c_k^\varepsilon\right] & \text{in } \mathcal{D}, \\ \phi_k^\varepsilon = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

with

$$c_k^\varepsilon(x) = \mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right) B_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla u_0^\varepsilon.$$

Assume momentarily that the sequence ∇u_0^ε is bounded in $L^\infty(\mathcal{D})$ (we have proved such a bound above, see (32), under the strong assumptions (28) and (11)). Then, for any $k \in \mathbb{Z}^d$, c_k^ε converges to 0 in $L^2(\mathcal{D})$. Using the coercivity of A_{per}, this implies that ϕ_k^ε converges to 0 in $H^1(\mathcal{D})$. We thus have the following result, which will be rigourously proved in Section 5 below:

Proposition 9. *Let ϕ_k^ε be the solution to* (26)*, and let $u_0^\star$ and w_p^0 be the solutions to* (2) *and* (4)*. Assume that $u_0^\star \in W^{2,\infty}(\mathcal{D})$ and that, for any $p \in \mathbb{R}^d$, we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$. Then ϕ_k^ε converges to 0 in $H^1(\mathcal{D})$.*

To describe more precisely the behavior of ϕ_k^ε, we need to introduce the auxilliary function χ_p defined by (41) below. Recall first that $Q = (-1/2, 1/2)^d$. Following the same arguments as in [8, Lemma 4], we have the following result, which will be useful in the sequel.

Lemma 10. *For any $p \in \mathbb{R}^d$, the problem*

$$\left\{\begin{array}{l} -div\left[A_{per}\nabla\chi_p\right] = div\left[\mathbf{1}_Q B_{per}(p + \nabla w_p^0)\right] \quad in\ \mathbb{R}^d, \\ \chi_p \in L^2_{loc}(\mathbb{R}^d), \quad \nabla\chi_p \in \left(L^2(\mathbb{R}^d)\right)^d, \end{array}\right. \tag{41}$$

has a solution which is unique up to the addition of a constant. In addition, under assumption (11)*, there exists a solution of* (41) *and a constant $C > 0$ such that*

$$\forall x \in \mathbb{R}^d \text{ with } |x| \geq 1, \quad |\nabla\chi_p| \leq \frac{C}{|x|^d}, \tag{42}$$

$$\forall x \in \mathbb{R}^d, \quad |\chi_p| \leq \frac{C}{1 + |x|^{d-1}}. \tag{43}$$

In the sequel, we will always refer to that particular solution of (41)*.*

We are now in position to make precise the behavior of ϕ_k^ε in the H^1 norm. Let us first argue formally. Introduce the matrix $E_k = \mathbf{1}_{Q+k} B_{per}$. Using the periodicity of A_{per}, B_{per} and w_p^0, and after changing variables, we recast (41) as

$$-\mathrm{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\chi_p\left(\frac{\cdot}{\varepsilon} - k\right)\right] = \mathrm{div}\left[E_k\left(\frac{\cdot}{\varepsilon}\right)\left(p + \nabla w_p^0\left(\frac{\cdot}{\varepsilon}\right)\right)\right].$$

In turn, the problem (26) reads

$$-\mathrm{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\phi_k^\varepsilon\right] = \mathrm{div}\left[E_k\left(\frac{\cdot}{\varepsilon}\right)\nabla u_0^\varepsilon\right] \approx \sum_{i=1}^d \mathrm{div}\left[E_k\left(\frac{\cdot}{\varepsilon}\right)\partial_i u_0^\star\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\right],$$

where we have used the expansion (36) of u_0^ε (in which we have only kept the highest order terms). Assuming that, in the above equation, x and x/ε are independent variables, we thus see that $\nabla\phi_k^\varepsilon(x) \approx \sum_{i=1}^d \partial_i u_0^\star(x)\, \nabla\chi_{e_i}\left(\frac{x}{\varepsilon} - k\right)$, and thus $\phi_k^\varepsilon(x) \approx \varepsilon\sum_{i=1}^d \partial_i u_0^\star(x)\, \chi_{e_i}\left(\frac{x}{\varepsilon} - k\right)$. These formal manipulations motivate the following result, the rigorous proof of which is postponed until Section 5:

Proposition 11. *Let ϕ_k^ε be the solution to* (26) *and χ_{e_i} be the solution to* (41)*, for $1 \leq i \leq d$. Introduce*

$$I_\varepsilon = \left\{k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q+k) \cap \mathcal{D} \neq \emptyset\right\}, \quad Card(I_\varepsilon) \sim \varepsilon^{-d}, \tag{44}$$

and

$$\overline{v}_k^\varepsilon = \varepsilon \sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right) \partial_i u_0^\star,$$

where $u_0^\star$ is solution to (2)*. Assume that $u_0^\star \in W^{2,\infty}(\mathcal{D})$, and that* (11) *holds. We then have*

$$\sum_{k \in I_\varepsilon} \|\phi_k^\varepsilon - \overline{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C\varepsilon \ln(1/\varepsilon),$$

where C is a constant independent of ε.

4 Proofs of Theorems 2 and 3

Proof of Theorem 2. We have shown above (see (35)) that

$$u_\eta^\varepsilon(x,\omega) = u_0^\varepsilon(x) + \eta\mathbb{E}(X_0)\overline{u}_1^\varepsilon(x) + \eta \sum_{k \in I_\varepsilon} (X_k(\omega) - \mathbb{E}(X_0))\phi_k^\varepsilon(x) + \eta^2 r_\eta^\varepsilon(x,\omega),$$

where the set I_ε is defined by (44) (recall that $\phi_k^\varepsilon \equiv 0$ whenever $k \in \mathbb{Z}^d$ is such that $k \notin I_\varepsilon$). Using the fact that X_k are i.i.d. scalar random variables, we have

$$\mathbb{E}\left[\|u_\eta^\varepsilon - v_\eta^\varepsilon\|^2_{H^1(\mathcal{D})}\right] \leq C\left[D_0^2 + D_1^2 + D_2^2 + D_3^2\right],$$

where

$$\begin{aligned}
D_0 &= \left\| u_0^\varepsilon - u_0^\star - \varepsilon \sum_{p=1}^d w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_p u_0^\star \right\|_{H^1(\mathcal{D})}, \\
D_1 &= \eta|\mathbb{E}(X_0)| \left\| \overline{u}_1^\varepsilon - \overline{u}_1^\star - \varepsilon \sum_{p=1}^d \left(w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_p \overline{u}_1^\star + \psi_{e_p}\left(\frac{\cdot}{\varepsilon}\right)\partial_p u_0^\star \right) \right\|_{H^1(\mathcal{D})}, \\
D_2 &= \eta\sqrt{\mathbb{V}\mathrm{ar}(X_0)} \sqrt{ \sum_{k \in I_\varepsilon} \left\| \phi_k^\varepsilon - \varepsilon \sum_{p=1}^d \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\partial_p u_0^\star \right\|^2_{H^1(\mathcal{D})} }, \\
D_3 &= \eta^2 \sqrt{\mathbb{E}\left[\|r_\eta^\varepsilon\|^2_{H^1(\mathcal{D})}\right]}.
\end{aligned}$$

We have shown in Propositions 6, 8 and 11 that $D_0 \leq C\sqrt{\varepsilon}$, $D_1 \leq C\eta\sqrt{\varepsilon}$ and $D_2 \leq C\eta\sqrt{\varepsilon\ln(1/\varepsilon)}$ respectively, for a constant C independent of ε and η (note that all assumptions of these propositions are satisfied since, in view of (11) and (12), we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi_p \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$). Next, using Lemma 4, we see that $D_3 \leq C\eta^2$ for a constant C independent of ε and η. This concludes the proof of (18). □

Proof of Theorem 3. To fix the idea, we choose $\mathcal{D} = (0,1)$. We again argue on the basis of (35). Tedious but straightforward computations show that, in dimension one, the estimates of Propositions 6, 8 and 11 read

$$\left\|\varepsilon\frac{d\theta_0^\varepsilon}{dx}\right\|_{L^2(0,1)} \leq C\varepsilon, \qquad \left\|\frac{d\overline{u}_1^\varepsilon}{dx} - \frac{d\overline{v}_1^\varepsilon}{dx}\right\|_{L^2(0,1)} \leq C\varepsilon, \qquad \sum_{k\in I_\varepsilon}\left\|\frac{d\phi_k^\varepsilon}{dx} - \frac{d\overline{v}_k^\varepsilon}{dx}\right\|^2_{L^2(0,1)} \leq C\varepsilon.$$

We thus obtain

$$\sqrt{\mathbb{E}\left[\left\|\frac{du_\eta^\varepsilon}{dx}-\frac{dv_\eta^\varepsilon}{dx}\right\|^2_{L^2(0,1)}\right]}\leq C\left(\varepsilon+\eta\sqrt{\varepsilon}+\eta^2\right). \tag{45}$$

We next write that, almost surely,

$$\left\|u_\eta^\varepsilon(\cdot,\omega)-v_\eta^\varepsilon(\cdot,\omega)\right\|_{L^\infty(0,1)}\leq\left\|\frac{du_\eta^\varepsilon}{dx}(\cdot,\omega)-\frac{dv_\eta^\varepsilon}{dx}(\cdot,\omega)\right\|_{L^2(0,1)}+\left|u_\eta^\varepsilon(0,\omega)-v_\eta^\varepsilon(0,\omega)\right|. \tag{46}$$

Using that $u_\eta^\varepsilon(0,\omega)=u_0^\star(0)=\overline{u}_1^\star(0)=0$ and that w^0 and ψ belong to $L^\infty(\mathbb{R})$, we obtain that

$$\left|u_\eta^\varepsilon(0,\omega)-v_\eta^\varepsilon(0,\omega)\right|\leq C\varepsilon+C\varepsilon\eta\left|(u_0^\star)'(0)\sum_{k\in I_\varepsilon}(X_k(\omega)-\mathbb{E}(X_0))\chi\left(-k\right)\right|,$$

hence, using that $\chi\in L^\infty(\mathbb{R})$, we have

$$\mathbb{E}\left[\left|u_\eta^\varepsilon(0,\omega)-v_\eta^\varepsilon(0,\omega)\right|^2\right]\leq C\varepsilon^2+C\mathbb{V}\mathrm{ar}(X_0)\varepsilon^2\eta^2\sum_{k\in I_\varepsilon}\chi^2\left(-k\right)\leq C\varepsilon^2+C\eta^2\varepsilon.$$

Collecting this result with (45) and (46) yields the bound (20). Likewise, collecting (20) and (45), we obtain the bound (19). This concludes the proof of Theorem 3. □

5 Proofs of the two scale expansions

We collect in this section the proofs of the results stated in Section 3. The following technical result, already present in [17, p. 27], and that we recall here for the sake of completeness, will be useful.

Lemma 12. *Let $\mathcal{D}$ be a bounded open set of $\mathbb{R}^d$. Consider $Z\in(L^2_{loc}(\mathbb{R}^d))^d$ a Q-periodic vector field such that*

$$div\ (Z)=0\quad and\quad\int_Q Z=0.$$

Then, for any $v\in W^{1,\infty}(\mathcal{D})$, we have

$$\left\|div\left[Z\left(\frac{\cdot}{\varepsilon}\right)v\right]\right\|_{H^{-1}(\mathcal{D})}\leq C\varepsilon\left\|\nabla v\right\|_{L^\infty(\mathcal{D})},$$

where C is a constant independent of ε and v.

Note that, as Z is divergence free, we have

$$\mathrm{div}\left[Z\left(\frac{\cdot}{\varepsilon}\right)v\right]=Z\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla v.$$

Since Z is Q-periodic, this quantity converges weakly in $L^2(\mathcal{D})$ to $\langle Z\rangle\cdot\nabla v=0$, as the average of Z vanishes. The above result hence shows that, in the $H^{-1}(\mathcal{D})$ norm, the above quantity vanishes at the rate ε.

Proof. In view of the assumptions of Z, there exists (see [17, p. 6]) a skew symmetric matrix J such that,

$$\forall 1\leq j\leq d,\quad Z_j=\sum_{i=1}^d\frac{\partial J_{ij}}{\partial x_i}$$

and

$$\forall 1 \le i,j \le d, \quad J_{ij} \in H^1_{loc}(\mathbb{R}^d), \quad J_{ij} \text{ is } Q\text{-periodic}, \quad \int_Q J_{ij} = 0.$$

The j-th coordinate of the vector $Z\left(\frac{\cdot}{\varepsilon}\right) v$ reads

$$\begin{aligned}
\left[Z\left(\frac{x}{\varepsilon}\right) v(x)\right]_j &= \sum_{i=1}^d \frac{\partial J_{ij}}{\partial x_i}\left(\frac{x}{\varepsilon}\right) v(x) \\
&= \varepsilon \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(J_{ij}\left(\frac{x}{\varepsilon}\right) v(x)\right) - \varepsilon \sum_{i=1}^d J_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial v}{\partial x_i}(x) \\
&= \varepsilon \widetilde{B}_j(x) - \varepsilon B_j(x),
\end{aligned}$$

where

$$B_j(x) = \sum_{i=1}^d J_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial v}{\partial x_i}(x) \quad \text{and} \quad \widetilde{B}_j(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(J_{ij}\left(\frac{x}{\varepsilon}\right) v(x)\right).$$

The vector $\widetilde{B}(x)$ is divergence free as J is skew symmetric. For any $\phi \in H^1_0(\mathcal{D})$, we thus have

$$\begin{aligned}
\left\langle \operatorname{div}\left[Z\left(\frac{\cdot}{\varepsilon}\right) v\right], \phi \right\rangle &= -\varepsilon \left\langle \operatorname{div}[B], \phi \right\rangle \\
&= \varepsilon \int_{\mathcal{D}} B \cdot \nabla \phi \\
&= \varepsilon \sum_{i,j=1}^d \int_{\mathcal{D}} \partial_j \phi \, J_{ij}\left(\frac{\cdot}{\varepsilon}\right) \partial_i v,
\end{aligned}$$

hence

$$\begin{aligned}
\left|\left\langle \operatorname{div}\left[Z\left(\frac{\cdot}{\varepsilon}\right) v\right], \phi \right\rangle\right| &\le \varepsilon \|\nabla v\|_{L^\infty(\mathcal{D})} \|\phi\|_{H^1(\mathcal{D})} \sum_{i,j=1}^d \left\| J_{ij}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\mathcal{D})} \\
&\le \varepsilon \|\nabla v\|_{L^\infty(\mathcal{D})} \|\phi\|_{H^1(\mathcal{D})} \sum_{i,j=1}^d \|J_{ij}\|_{L^2(Q)}.
\end{aligned}$$

As the above bound holds for any $\phi \in H^1_0(\mathcal{D})$, we deduce that there exists C such that, for any $v \in W^{1,\infty}(\mathcal{D})$ and any ε, we have

$$\left\| \operatorname{div}\left[Z\left(\frac{\cdot}{\varepsilon}\right) v\right]\right\|_{H^{-1}(\mathcal{D})} \le C\varepsilon \|\nabla v\|_{L^\infty(\mathcal{D})}.$$

This concludes the proof. □

5.1 Two scale expansion of $\overline{u}_1^\varepsilon$

In this section, we prove Propositions 7 and 8.

Proof of Proposition 7. This homogenization result is proved using the method of oscillating test functions [23, 25]. The variational formulation of (25) reads

$$\forall v \in H^1_0(\mathcal{D}), \quad \mathcal{A}_\varepsilon(\overline{u}_1^\varepsilon, v) = -L_\varepsilon(v), \tag{47}$$

where, for any u and v in $H^1_0(\mathcal{D})$,

$$\mathcal{A}_\varepsilon(u,v) = \int_{\mathcal{D}} (\nabla v)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla u \quad \text{and} \quad L_\varepsilon(v) = \int_{\mathcal{D}} (\nabla v)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla u^\varepsilon_0.$$

Using the coercivity of A_{per}, the boundedness of B_{per} and (23), and taking $v = \overline{u}^\varepsilon_1$ as a function test in (47), we obtain that $\overline{u}^\varepsilon_1$ is bounded in $H^1_0(\mathcal{D})$. Thus, using the Rellich Theorem, we deduce that there exists $\overline{u}^\star_1 \in H^1_0(\mathcal{D})$ such that, up to the extraction of a subsequence,

$$\overline{u}^\varepsilon_1 \text{ converges to } \overline{u}^\star_1, \text{ weakly in } H^1_0(\mathcal{D}) \text{ and strongly in } L^2(\mathcal{D}).$$

For any function $\varphi \in \mathcal{C}^\infty_0(\mathcal{D})$, define the test function

$$v^\varepsilon = \varphi + \varepsilon \sum_{i=1}^d w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i \varphi,$$

which obviously belongs to $H^1_0(\mathcal{D})$. In view of (47), we have

$$\mathcal{A}_\varepsilon(\overline{u}^\varepsilon_1, v^\varepsilon) = -L_\varepsilon(v^\varepsilon). \tag{48}$$

We now expand both sides of (48) in powers of ε:

$$\begin{aligned} \mathcal{A}_\varepsilon(\overline{u}^\varepsilon_1, v^\varepsilon) &= \mathcal{A}^0_\varepsilon(\overline{u}^\varepsilon_1, \varphi) + \varepsilon \mathcal{A}^1_\varepsilon(\overline{u}^\varepsilon_1, \varphi), \qquad (49)\\ L_\varepsilon(v^\varepsilon) &= L^0_\varepsilon(\varphi) + \varepsilon L^1_\varepsilon(\varphi), \qquad (50) \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}^0_\varepsilon(\overline{u}^\varepsilon_1, \varphi) &= \int_{\mathcal{D}} \left(\nabla\varphi + \sum_{i=1}^d \nabla w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i\varphi\right)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\overline{u}^\varepsilon_1,\\ \mathcal{A}^1_\varepsilon(\overline{u}^\varepsilon_1, \varphi) &= \int_{\mathcal{D}} \sum_{i=1}^d w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right)(\nabla\partial_i\varphi)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\overline{u}^\varepsilon_1,\\ L^0_\varepsilon(\varphi) &= \int_{\mathcal{D}} \left(\nabla\varphi + \sum_{i=1}^d \nabla w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i\varphi\right)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon_0,\\ L^1_\varepsilon(\varphi) &= \int_{\mathcal{D}} \sum_{i=1}^d w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right)(\nabla\partial_i\varphi)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla u^\varepsilon_0. \end{aligned}$$

We now successively study the limit of these four quantities as $\varepsilon \to 0$. Using (23), the fact that $w^0_{e_i} \in W^{1,\infty}(\mathbb{R}^d)$, that $\overline{u}^\varepsilon_1$ is bounded in $H^1(\mathcal{D})$ and the boundedness of A_{per} and B_{per}, we obtain

$$|\mathcal{A}^1_\varepsilon(\overline{u}^\varepsilon_1, \varphi)| \leq C \quad \text{and} \quad |L^1_\varepsilon(\varphi)| \leq C, \quad C \text{ independent of } \varepsilon. \tag{51}$$

We now turn to L^0_ε. Using the two scale expansion (36) of u^ε_0, we see that

$$L^0_\varepsilon(\varphi) = L^{00}_\varepsilon(\varphi) + L^{01}_\varepsilon(\varphi) + L^{02}_\varepsilon(\varphi), \tag{52}$$

where

$$
\begin{aligned}
L_\varepsilon^{00}(\varphi) &= \sum_{i,j=1}^d \int_{\mathcal{D}} \left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(e_j + \nabla w_{e_j}^0\left(\frac{\cdot}{\varepsilon}\right)\right) \partial_i\varphi \; \partial_j u_0^\star, \\
L_\varepsilon^{01}(\varphi) &= \varepsilon \sum_{i,j=1}^d \int_{\mathcal{D}} \left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right) w_{e_j}^0\left(\frac{\cdot}{\varepsilon}\right)(\nabla \partial_j u_0^\star) \; \partial_i\varphi, \\
L_\varepsilon^{02}(\varphi) &= \sum_{i=1}^d \int_{\mathcal{D}} \left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right) \varepsilon \nabla\theta_0^\varepsilon \; \partial_i\varphi.
\end{aligned}
$$

Using (37), $w_{e_i}^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $u_0^\star \in W^{2,\infty}(\mathcal{D})$, we obtain that

$$
|L_\varepsilon^{02}(\varphi)| \leq C\|\varepsilon\theta_0^\varepsilon\|_{H^1(\mathcal{D})} \leq C\sqrt{\varepsilon} \quad \text{and} \quad |L_\varepsilon^{01}(\varphi)| \leq C\varepsilon, \tag{53}
$$

where C is a constant independent of ε. Turning to L_ε^{00}, we see, using that B_{per} and $w_{e_i}^0$ are Q-periodic, that

$$
\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)^T B_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(e_j + \nabla w_{e_j}^0\left(\frac{\cdot}{\varepsilon}\right)\right) \rightharpoonup \overline{B}_{ij} \quad \text{weakly-}\star \text{ in } L^\infty,
$$

where $\overline{B}$ is defined by (13). Thus

$$
L_\varepsilon^{00}(\varphi) \to \int_{\mathcal{D}} (\nabla\varphi)^T \overline{B} \nabla u_0^\star \quad \text{as } \varepsilon \to 0. \tag{54}
$$

Collecting (52), (53) and (54), we obtain that

$$
L_\varepsilon^{0}(\varphi) \to \int_{\mathcal{D}} (\nabla\varphi)^T \overline{B} \nabla u_0^\star \quad \text{as } \varepsilon \to 0. \tag{55}
$$

We next turn to $\mathcal{A}_\varepsilon^0$. Using that $\operatorname{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\right] = 0$ and that A_{per} is symmetric, we obtain that

$$
\mathcal{A}_\varepsilon^0(\overline{u}_1^\varepsilon, \varphi) = -\sum_{i=1}^d \int_{\mathcal{D}} \overline{u}_1^\varepsilon (\nabla\partial_i\varphi)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right). \tag{56}
$$

Recall now that $\overline{u}_1^\varepsilon \to \overline{u}_1^\star$ strongly in $L^2(\mathcal{D})$ and that, as A_{per} and $w_{e_i}^0$ are Q-periodic, we have

$$
A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right) \rightharpoonup \int_Q A_{per}\left(e_i + \nabla w_{e_i}^0\right) = A_{per}^\star e_i \quad \text{weakly-}\star \text{ in } L^\infty,
$$

where $A_{per}^\star$ is defined by (3). We thus deduce from (56) that

$$
\mathcal{A}_\varepsilon^0(\overline{u}_1^\varepsilon, \varphi) \to -\sum_{i=1}^d \int_{\mathcal{D}} \overline{u}_1^\star (\nabla\partial_i\varphi)^T A_{per}^\star e_i \quad \text{as } \varepsilon \to 0.
$$

Collecting (48), (49), (50), (51), the above limit and (55), we obtain that $\overline{u}_1^\star$ satisfies

$$
-\sum_{i=1}^d \int_{\mathcal{D}} \overline{u}_1^\star (\nabla\partial_i\varphi)^T A_{per}^\star e_i = -\int_{\mathcal{D}} (\nabla\varphi)^T \overline{B} \nabla u_0^\star
$$

for any $\varphi \in \mathcal{C}_0^\infty(\mathcal{D})$. This shows that $\overline{u}_1^\star$ solves (14) (which has a unique solution) and thus concludes the proof of Proposition 7. ☐

Proof of Proposition 8. The proof mostly goes by using the coercivity of A_{per} and showing that, in some appropriate norm, $-\mathrm{div}\left[A_{per}(\nabla \overline{u}_1^\varepsilon - \nabla \overline{v}_1^\varepsilon)\right]$ is small. However, a technical difficulty comes from the fact that $\overline{v}_1^\varepsilon \notin H_0^1(\mathcal{D})$, as it does not vanish on $\partial\mathcal{D}$. A preliminary step (Step 1 below) thus consists in approximating $\overline{v}_1^\varepsilon$ by a function (namely g_1^ε defined by (57) below) that is equal to $\overline{v}_1^\varepsilon$ away from the boundary $\partial\mathcal{D}$, but vanishes on the boundary. Step 2 consists in estimating the difference $\overline{u}_1^\varepsilon - g_1^\varepsilon$.

Step 1: Truncation of $\overline{v}_1^\varepsilon$

Let us define $\tau_\varepsilon \in \mathcal{C}_0^\infty(\mathcal{D})$ such that $0 \le \tau_\varepsilon(x) \le 1$ for all $x \in \mathcal{D}$, $\tau_\varepsilon(x) = 1$ when $\mathrm{dist}(\partial\mathcal{D}, x) \ge \varepsilon$ and $\varepsilon\|\nabla\tau_\varepsilon\|_{L^\infty(\mathcal{D})} \le C$, where C is a constant independent of ε. We denote by $\mathcal{D}_\varepsilon \subset \mathcal{D}$ the set of $\mathbb{R}^d$ defined by

$$\mathcal{D}_\varepsilon := \{x \in \mathcal{D} \text{ such that } \mathrm{dist}(\partial\mathcal{D}, x) \ge \varepsilon\}$$

and we note that

$$|\mathcal{D} \setminus \mathcal{D}_\varepsilon| \le C\varepsilon.$$

Introduce now $g_1^\varepsilon \in H_0^1(\mathcal{D})$ defined by

$$g_1^\varepsilon = \overline{u}_1^\star + \varepsilon\tau_\varepsilon \sum_{i=1}^d \left(w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_i\overline{u}_1^\star + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i u_0^\star \right), \tag{57}$$

where $\overline{u}_1^\star$ is the solution to (14), $u_0^\star$ is the solution to (2), and $w_{e_i}^0$ and ψ_{e_i} are solutions (with $p = e_i$) to (4) and (16), respectively. Note that $g_1^\varepsilon = \overline{v}_1^\varepsilon$ except in a neighboorhood of $\partial\mathcal{D}$. In the sequel, we estimate $\overline{v}_1^\varepsilon - g_1^\varepsilon$. In the next Step, we estimate $g_1^\varepsilon - \overline{u}_1^\varepsilon$.

By definition,

$$\nabla\overline{v}_1^\varepsilon - \nabla g_1^\varepsilon = e_0^\varepsilon - e_1^\varepsilon + \varepsilon e_2^\varepsilon, \tag{58}$$

where

$$\begin{aligned}
e_0^\varepsilon &= (1-\tau_\varepsilon)\sum_{i=1}^d \left(\nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_i\overline{u}_1^\star + \nabla\psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i u_0^\star \right),\\
e_1^\varepsilon &= \varepsilon\nabla\tau_\varepsilon \sum_{i=1}^d \left(w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_i\overline{u}_1^\star + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i u_0^\star \right),\\
e_2^\varepsilon &= (1-\tau_\varepsilon)\sum_{i=1}^d \left(w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\nabla(\partial_i\overline{u}_1^\star) + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\nabla(\partial_i u_0^\star) \right).
\end{aligned}$$

We now bound from above successively the L^2 norm of e_2^ε, e_1^ε and e_0^ε. First, as $u_0^\star \in W^{2,\infty}(\mathcal{D})$, $\overline{u}_1^\star \in W^{2,\infty}(\mathcal{D})$, $\psi_{e_i} \in W^{1,\infty}(\mathbb{R}^d)$, $w_{e_i}^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $0 \le \tau_\varepsilon \le 1$, we have

$$\|e_2^\varepsilon\|_{L^2(\mathcal{D})}^2 \le C, \quad C \text{ independent of } \varepsilon. \tag{59}$$

The same arguments lead to

$$\begin{aligned}
\|e_1^\varepsilon\|_{L^2(\mathcal{D})}^2 &= \int_{\mathcal{D}} \left[\sum_{i=1}^d \left(w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\partial_i\overline{u}_1^\star + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right)\partial_i u_0^\star \right)\right]^2 |\varepsilon\nabla\tau_\varepsilon|^2\\
&\le C\,|\mathcal{D}\setminus\mathcal{D}_\varepsilon|\\
&\le C\varepsilon,
\end{aligned} \tag{60}$$

for a constant C independent of ε. We next write

$$\|e_0^\varepsilon\|^2_{L^2(\mathcal{D})} \leq |\mathcal{D} \setminus \mathcal{D}_\varepsilon| \left\| \sum_{i=1}^d \left(\nabla w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i \overline{u}_1^\star + \nabla \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i u_0^\star \right) \right\|^2_{L^\infty(\mathcal{D})} \leq C\varepsilon. \tag{61}$$

Collecting (58), (59), (60) and (61), we have

$$\|\nabla \overline{v}_1^\varepsilon - \nabla g_1^\varepsilon\|^2_{L^2(\mathcal{D})} \leq C\varepsilon, \quad C \text{ independent of } \varepsilon.$$

Observing that

$$\|\overline{v}_1^\varepsilon - g_1^\varepsilon\|^2_{L^2(\mathcal{D})} \leq 2d\varepsilon^2 \sum_{i=1}^d \left(\|w^0_{e_i}\|^2_{L^\infty} \|\overline{u}_1^\star\|^2_{H^1(\mathcal{D})} + \|\psi_{e_i}\|^2_{L^\infty} \|u_0^\star\|^2_{H^1(\mathcal{D})} \right) \leq C\varepsilon^2,$$

we obtain that

$$\|\overline{v}_1^\varepsilon - g_1^\varepsilon\|_{H^1(\mathcal{D})} \leq C\sqrt{\varepsilon}, \quad C \text{ independent of } \varepsilon. \tag{62}$$

Step 2: We next turn to estimating $\overline{u}_1^\varepsilon - g_1^\varepsilon$. Using that A_{per} is coercive and the fact that $\overline{u}_1^\varepsilon - g_1^\varepsilon \in H^1_0(\mathcal{D})$, we have

$$\begin{aligned}
\alpha \|\overline{u}_1^\varepsilon - g_1^\varepsilon\|^2_{H^1(\mathcal{D})} &\leq \int_{\mathcal{D}} (\nabla \overline{u}_1^\varepsilon - \nabla g_1^\varepsilon)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{u}_1^\varepsilon - \nabla g_1^\varepsilon) \\
&\leq \int_{\mathcal{D}} (\nabla \overline{u}_1^\varepsilon - \nabla g_1^\varepsilon)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{u}_1^\varepsilon - \nabla \overline{v}_1^\varepsilon) + \int_{\mathcal{D}} (\nabla \overline{u}_1^\varepsilon - \nabla g_1^\varepsilon)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{v}_1^\varepsilon - \nabla g_1^\varepsilon) \\
&\leq \|\overline{u}_1^\varepsilon - g_1^\varepsilon\|_{H^1(\mathcal{D})} \left(\left\| \operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{u}_1^\varepsilon - \nabla \overline{v}_1^\varepsilon) \right] \right\|_{H^{-1}(\mathcal{D})} + \|A_{per}\|_{L^\infty} \|\overline{v}_1^\varepsilon - g_1^\varepsilon\|_{H^1(\mathcal{D})} \right)
\end{aligned} \tag{63}$$

where the constant $\alpha > 0$ only depends on the coercivity constant of A_{per} and the Poincaré constant of the domain $\mathcal{D}$. In the sequel, we bound from above $\left\| \operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{u}_1^\varepsilon - \nabla \overline{v}_1^\varepsilon) \right] \right\|_{H^{-1}(\mathcal{D})}$.

By definition of $\overline{v}_1^\varepsilon$, we have

$$\overline{v}_1^\varepsilon = \widehat{v}_1^\varepsilon + \widetilde{v}_1^\varepsilon,$$

with

$$\widehat{v}_1^\varepsilon = \overline{u}_1^\star + \varepsilon \sum_{i=1}^d w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i \overline{u}_1^\star \quad \text{and} \quad \widetilde{v}_1^\varepsilon = \varepsilon \sum_{i=1}^d \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \partial_i u_0^\star.$$

Using the equation (25) on $\overline{u}_1^\varepsilon$ and the relation (14) between $\overline{u}_1^\star$ and $u_0^\star$, we compute

$$\begin{aligned}
\operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) (\nabla \overline{v}_1^\varepsilon - \nabla \overline{u}_1^\varepsilon) \right] &= \operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla \widehat{v}_1^\varepsilon - A^\star_{per} \nabla \overline{u}_1^\star \right] + \operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla \widetilde{v}_1^\varepsilon + B_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla u_0^\varepsilon - \overline{B} \nabla u_0^\star \right] \\
&= D_0 + D_1 + \varepsilon D_2,
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
D_0 &= \sum_{i=1}^d \operatorname{div} \left(\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \left(e_i + \nabla w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \right) - A^\star_{per} e_i \right] \partial_i \overline{u}_1^\star \right), \\
D_1 &= \sum_{i=1}^d \operatorname{div} \left(\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) + B_{per}\left(\frac{\cdot}{\varepsilon}\right) \left(e_i + \nabla w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \right) - \overline{B} e_i \right] \partial_i u_0^\star \right), \\
D_2 &= \operatorname{div} \left[B_{per}\left(\frac{\cdot}{\varepsilon}\right) \nabla \theta_0^\varepsilon \right] + \sum_{i=1}^d \operatorname{div} \left[A_{per}\left(\frac{\cdot}{\varepsilon}\right) \left(w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \nabla \partial_i \overline{u}_1^\star + \psi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \nabla \partial_i u_0^\star \right) + B_{per}\left(\frac{\cdot}{\varepsilon}\right) w^0_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \nabla \partial_i u_0^\star \right].
\end{aligned}$$

We now bound from above these three quantities. As A_{per} and B_{per} are bounded, we see that

$$\|D_2\|_{H^{-1}(\mathcal{D})} \le C\|\theta_0^\varepsilon\|_{H^1(\mathcal{D})} + C\sum_{i=1}^d \left[\|w_{e_i}^0\|_{L^\infty}\|\overline{u}_1^\star\|_{H^2(\mathcal{D})} + \|\psi_{e_i}\|_{L^\infty}\|u_0^\star\|_{H^2(\mathcal{D})} + \|w_{e_i}^0\|_{L^\infty}\|u_0^\star\|_{H^2(\mathcal{D})}\right],$$

from which we infer, in view of (37), that

$$\varepsilon\|D_2\|_{H^{-1}(\mathcal{D})} \le C\sqrt{\varepsilon}, \quad C \text{ independent of } \varepsilon. \tag{65}$$

Let us now turn to D_0. Consider, for any $1 \le i \le d$, the vector-valued function

$$Z(y) = A_{per}(y)\left(e_i + \nabla w_{e_i}^0(y)\right) - A_{per}^\star e_i.$$

We observe that $Z \in (L^2_{loc}(\mathbb{R}^d))^d$ is divergence free, Q-periodic and of vanishing mean. Since $\partial_i \overline{u}_1^\star \in W^{1,\infty}(\mathcal{D})$, we can use Lemma 12, and we obtain

$$\|D_0\|_{H^{-1}(\mathcal{D})} \le C\varepsilon, \quad C \text{ independent of } \varepsilon. \tag{66}$$

Turning now to D_1, we likewise consider, for any $1 \le i \le d$, the vector-valued function

$$\overline{Z}(y) = A_{per}(y)\nabla\psi_{e_i}(y) + B_{per}(y)\left(e_i + \nabla w_{e_i}^0(y)\right) - \overline{B}e_i.$$

By construction, $\overline{Z} \in (L^2_{loc}(\mathbb{R}^d))^d$ is Q-periodic and divergence free, in view of the definition (16) of ψ_{e_i}. In addition, the mean of $\overline{Z}$ vanishes. Indeed, for any $1 \le j \le d$, using (13), (16), the symmetry of A_{per} and (4), we have

$$\begin{aligned}
\int_Q \overline{Z}\cdot e_j &= \int_Q e_j^T A_{per}\nabla\psi_{e_i} + \int_Q e_j^T B_{per}\left(e_i + \nabla w_{e_i}^0\right) - \int_Q (e_j + \nabla w_{e_j}^0)^T B_{per}\left(e_i + \nabla w_{e_i}^0\right) \\
&= \int_Q e_j^T A_{per}\nabla\psi_{e_i} - \int_Q (\nabla w_{e_j}^0)^T B_{per}\left(e_i + \nabla w_{e_i}^0\right) \\
&= \int_Q e_j^T A_{per}\nabla\psi_{e_i} + \int_Q (\nabla w_{e_j}^0)^T A_{per}\nabla\psi_{e_i} \\
&= \int_Q (\nabla\psi_{e_i})^T A_{per}(e_j + \nabla w_{e_j}^0) \\
&= 0.
\end{aligned}$$

Since $\partial_i u_0^\star \in W^{1,\infty}(\mathcal{D})$, we have that $\overline{Z}$ and $\partial_i u_0^\star$ satisfy the assumptions of Lemma 12, hence

$$\|D_1\|_{H^{-1}(\mathcal{D})} \le C\varepsilon, \quad C \text{ independent of } \varepsilon. \tag{67}$$

Collecting (64), (65), (66) and (67), we have

$$\left\|\operatorname{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)(\nabla\overline{u}_1^\varepsilon - \nabla\overline{v}_1^\varepsilon)\right]\right\|_{H^{-1}(\mathcal{D})} \le C\sqrt{\varepsilon}, \tag{68}$$

where C is a constant independent of ε. We now infer from (62), (63) and (68) that

$$\alpha\|\overline{u}_1^\varepsilon - g_1^\varepsilon\|^2_{H^1(\mathcal{D})} \le C\|\overline{u}_1^\varepsilon - g_1^\varepsilon\|_{H^1(\mathcal{D})}\sqrt{\varepsilon},$$

hence

$$\|\overline{u}_1^\varepsilon - g_1^\varepsilon\|_{H^1(\mathcal{D})} \le C\sqrt{\varepsilon}, \quad C \text{ independent of } \varepsilon.$$

Step 3: Conclusion

Collecting the above bound with (62), we deduce that

$$\|\overline{u}_1^\varepsilon - \overline{v}_1^\varepsilon\|_{H^1(\mathcal{D})} \le C\sqrt{\varepsilon}, \quad C \text{ independent of } \varepsilon.$$

We thus have proved the claimed bound, and this concludes the proof of Proposition 8. □

5.2 Two-scale expansion of ϕ_k^ε

In this section, we prove Propositions 9 and 11.

Proof of Proposition 9. Introducing

$$c_k^\varepsilon(x) = \mathbf{1}_{Q+k}\left(\frac{x}{\varepsilon}\right) B_{per}\left(\frac{x}{\varepsilon}\right) \nabla u_0^\varepsilon(x),$$

the problem (26) writes

$$\begin{cases} -\mathrm{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla\phi_k^\varepsilon\right] = \mathrm{div}\left[c_k^\varepsilon\right] & \text{in } \mathcal{D}, \\ \phi_k^\varepsilon = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

Multiplying this equation by ϕ_k^ε, integrating over $\mathcal{D}$, and using the coercivity of A_{per}, we obtain that there exists C independent of k and ε such that

$$\|\phi_k^\varepsilon\|_{H^1(\mathcal{D})} \le C\|c_k^\varepsilon\|_{L^2(\mathcal{D})}. \tag{69}$$

Let us now show that c_k^ε converges to 0 in $L^2(\mathcal{D})$. Using the expansion (36), we write

$$\nabla u_0^\varepsilon = T^\varepsilon + \varepsilon\nabla\theta_0^\varepsilon,$$

with

$$T^\varepsilon = \sum_{i=1}^d \partial_i u_0^\star\left(e_i + \nabla w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right)\right) + \varepsilon\sum_{i=1}^d \nabla(\partial_i u_0^\star) w_{e_i}^0\left(\frac{\cdot}{\varepsilon}\right).$$

Using the fact that $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $u_0^\star \in W^{2,\infty}(\mathcal{D})$, we see that T^ε is bounded in $L^\infty(\mathcal{D})$. We next write

$$\begin{aligned} \|c_k^\varepsilon\|_{L^2(\mathcal{D})}^2 &\le \|B_{per}\|_{L^\infty(\mathbb{R}^d)}^2 \int_{\varepsilon(Q+k)} |\nabla u_0^\varepsilon|^2 \\ &\le C\varepsilon^d + C\int_{\varepsilon(Q+k)} |\varepsilon\nabla\theta_0^\varepsilon|^2 \\ &\le C\varepsilon^d + C\|\varepsilon\theta_0^\varepsilon\|_{H^1(\mathcal{D})}^2. \end{aligned}$$

Using the bound (37), we deduce that c_k^ε converges to 0 in $L^2(\mathcal{D})$. In view of (69), this implies that ϕ_k^ε converges to 0 in $H_0^1(\mathcal{D})$. This concludes the proof. ☐

Proof of Proposition 11. As in the proof of Proposition 8, the proof falls in two steps. We first truncate $\overline{v}_k^\varepsilon$ in a function $\widetilde{v}_k^\varepsilon$ (defined by (70) below) that vanishes on $\partial\mathcal{D}$. We next estimate the difference between $\widetilde{v}_k^\varepsilon$ and ϕ_k^ε.

Step 1: Truncation of $\overline{v}_k^\varepsilon$

Let us define $\tau_\varepsilon \in \mathcal{C}_0^\infty(\mathcal{D})$ such that $0 \le \tau_\varepsilon(x) \le 1$ for all $x \in \mathcal{D}$, $\tau_\varepsilon(x) = 1$ when $\mathrm{dist}(\partial\mathcal{D}, x) \ge \varepsilon$ and $\varepsilon\|\nabla\tau_\varepsilon\|_{L^\infty(\mathcal{D})} \le C$, where C is a constant independent of ε. We introduce

$$\begin{aligned} \mathcal{D}_\varepsilon &:= \{x \in \mathcal{D} \text{ such that } \mathrm{dist}(\partial\mathcal{D}, x) \ge \varepsilon\}, \quad |\mathcal{D}\setminus\mathcal{D}_\varepsilon| \sim \varepsilon, \\ J_\varepsilon &:= \left\{k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q+k)\cap\mathcal{D}\setminus\mathcal{D}_\varepsilon \neq \emptyset\right\}, \quad \mathrm{Card}(J_\varepsilon) \sim \varepsilon^{1-d}, \end{aligned}$$

and the function $\widetilde{v}_k^\varepsilon \in H_0^1(\mathcal{D})$ defined by

$$\widetilde{v}_k^\varepsilon = \varepsilon\tau_\varepsilon \sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right)\partial_i u_0^\star, \tag{70}$$

where $u_0^\star$ is solution to (2) and χ_{e_i} is solution to (41). Note that $\widetilde{v}_k^\varepsilon = \overline{v}_k^\varepsilon$ except in the neighboorhood of the boundary of $\mathcal{D}$. In the sequel, we estimate $\sum_{k\in I_\varepsilon} \|\overline{v}_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})}$, and, in the next Step, we estimate $\sum_{k\in I_\varepsilon} \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})}$, where, we recall (see (44)),

$$I_\varepsilon = \left\{ k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q+k)\cap\mathcal{D} \neq \emptyset \right\}, \quad \mathrm{Card}(I_\varepsilon) \sim \varepsilon^{-d}.$$

Recall also that, whenever $k \notin I_\varepsilon$, we have $\phi_k^\varepsilon \equiv 0$.

By definition,

$$\nabla\overline{v}_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon = e_0^{k,\varepsilon} - e_1^{k,\varepsilon} + e_2^{k,\varepsilon}, \tag{71}$$

where

$$\begin{aligned}
e_0^{k,\varepsilon} &= (1-\tau_\varepsilon)\sum_{i=1}^d \nabla\chi_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right)\partial_i u_0^\star, \\
e_1^{k,\varepsilon} &= \varepsilon\nabla\tau_\varepsilon \sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right)\partial_i u_0^\star, \\
e_2^{k,\varepsilon} &= \varepsilon(1-\tau_\varepsilon)\sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right)\nabla(\partial_i u_0^\star).
\end{aligned}$$

We now bound from above successively the L^2 norm of $e_2^{k,\varepsilon}$, $e_1^{k,\varepsilon}$ and $e_0^{k,\varepsilon}$. To this aim, the following computation will be useful: for any $1 \le i \le d$, we have

$$\sum_{k\in I_\varepsilon}\int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \chi^2_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right) \le \sum_{k\in I_\varepsilon}\sum_{j\in J_\varepsilon}\varepsilon^d\int_{Q+j}\chi^2_{e_i}(\cdot - k) \le \sum_{j\in J_\varepsilon}\varepsilon^d\sum_{k\in I_\varepsilon}\int_{Q+j-k}\chi^2_{e_i}.$$

There exists ρ such that

$$\forall\varepsilon,\ \forall j\in J_\varepsilon,\ \forall k\in I_\varepsilon, \quad Q+j-k\subset B(0,\rho/\varepsilon).$$

We thus obtain that

$$\sum_{k\in I_\varepsilon}\int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \chi^2_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right) \le \sum_{j\in J_\varepsilon}\varepsilon^d\int_{B(0,\rho/\varepsilon)}\chi^2_{e_i} \le \varepsilon\int_{B(0,\rho/\varepsilon)}\chi^2_{e_i}. \tag{72}$$

We next infer from (43) that

$$\int_{B(0,\rho/\varepsilon)}\chi^2_{e_i} \le \int_{B(0,\rho/\varepsilon)}\frac{C}{(1+|y|^{d-1})^2}dy \le C + C\int_1^{\rho/\varepsilon}\frac{1}{r^{d-1}}dr \le CR_{d,\varepsilon}, \tag{73}$$

where C is a constant independent of ε and

$$R_{d,\varepsilon} := \left\{ \begin{array}{c} 1+\ln(1/\varepsilon) \text{ if } d=2, \\ 1 \text{ if } d>2. \end{array}\right. \tag{74}$$

Collecting (72) and (73), we deduce that

$$\sum_{k\in I_\varepsilon}\int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon}\chi^2_{e_i}\left(\frac{\cdot}{\varepsilon} - k\right) \le C\varepsilon R_{d,\varepsilon}. \tag{75}$$

We now bound $e_2^{k,\varepsilon}$. As $u_0^\star \in W^{2,\infty}(\mathcal{D})$, and using (75), we have

$$\begin{aligned}
\sum_{k\in I_\varepsilon} \|e_2^{k,\varepsilon}\|^2_{L^2(\mathcal{D})} &= \sum_{k\in I_\varepsilon} \varepsilon^2 \int_{\mathcal{D}} \left[(1-\tau_\varepsilon)\sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right)\nabla(\partial_i u_0^\star)\right]^2 \\
&\leq C\varepsilon^2 \|\nabla^2 u_0^\star\|^2_{L^\infty} \sum_{k\in I_\varepsilon}\sum_{i=1}^d \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \chi^2_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right) \\
&\leq C\varepsilon^3 R_{d,\varepsilon}. \qquad (76)
\end{aligned}$$

We next turn to $e_1^{k,\varepsilon}$. The same arguments and the fact that $\varepsilon\|\nabla\tau_\varepsilon\|_{L^\infty} \leq C$ lead to

$$\begin{aligned}
\sum_{k\in I_\varepsilon} \|e_1^{k,\varepsilon}\|^2_{L^2(\mathcal{D})} &\leq \|\varepsilon\nabla\tau_\varepsilon\|^2_{L^\infty} \sum_{k\in I_\varepsilon} \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \left[\sum_{i=1}^d \chi_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right)\partial_i u_0^\star\right]^2 \\
&\leq C\sum_{i=1}^d \sum_{k\in I_\varepsilon} \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \chi^2_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right) \\
&\leq C\varepsilon R_{d,\varepsilon}, \qquad (77)
\end{aligned}$$

where we have again used (75). Turning to $e_0^{k,\varepsilon}$, we have, using $\nabla\chi_{e_i} \in \left(L^2(\mathbb{R}^d)\right)^d$,

$$\begin{aligned}
\sum_{k\in I_\varepsilon} \|e_0^{k,\varepsilon}\|^2_{L^2(\mathcal{D})} &\leq C\|\nabla u_0^\star\|^2_{L^\infty} \sum_{i=1}^d \sum_{k\in I_\varepsilon} \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \left|\nabla\chi_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right)\right|^2 \\
&\leq C\sum_{i=1}^d \sum_{j\in J_\varepsilon} \varepsilon^d \sum_{k\in I_\varepsilon} \int_{Q+j-k} |\nabla\chi_{e_i}|^2 \\
&\leq C\sum_{i=1}^d \sum_{j\in J_\varepsilon} \varepsilon^d \|\nabla\chi_{e_i}\|^2_{L^2(\mathbb{R}^d)} \\
&\leq C\varepsilon. \qquad (78)
\end{aligned}$$

Collecting (71), (76), (77) and (78), we deduce that

$$\sum_{k\in I_\varepsilon} \|\nabla\overline{v}_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})} \leq C\left(\varepsilon + \varepsilon R_{d,\varepsilon} + \varepsilon^3 R_{d,\varepsilon}\right),$$

where C is a constant independent of ε. Observing that

$$\sum_{k\in I_\varepsilon} \|\overline{v}_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})} \leq C\varepsilon^2 \|\nabla u_0^\star\|^2_{L^\infty(\mathcal{D})} \sum_{k\in I_\varepsilon}\sum_{i=1}^d \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \chi^2_{e_i}\left(\frac{\cdot}{\varepsilon}-k\right) \leq C\varepsilon^3 R_{d,\varepsilon},$$

we obtain that

$$\sum_{k\in I_\varepsilon} \|\overline{v}_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C\left(\varepsilon + \varepsilon R_{d,\varepsilon} + \varepsilon^3 R_{d,\varepsilon}\right) \leq \left\{ \begin{array}{c} C\varepsilon\left[1+\ln(1/\varepsilon)\right] \text{ if } d=2, \\ C\varepsilon \text{ if } d>2, \end{array}\right. \qquad (79)$$

where C is a constant independent of ε.

Step 2: We next turn to estimating $\sum_{k\in I_\varepsilon} \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})}$. Using that A_{per} is coercive and the fact that $\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon \in H^1_0(\mathcal{D})$, we have

$$\alpha \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \int_{\mathcal{D}} \left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right) = D_0^{k,\varepsilon} + D_1^{k,\varepsilon}, \tag{80}$$

where the constant $\alpha > 0$ only depends on the coercivity constant of A_{per} and the Poincaré constant of the domain $\mathcal{D}$, and where

$$\begin{aligned} D_0^{k,\varepsilon} &= \int_{\mathcal{D}} \left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla\phi_k^\varepsilon - \nabla\overline{v}_k^\varepsilon\right), \\ D_1^{k,\varepsilon} &= \int_{\mathcal{D}} \left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla\overline{v}_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right). \end{aligned}$$

We successively bound $D_0^{k,\varepsilon}$ and $D_1^{k,\varepsilon}$ from above. We begin with $D_0^{k,\varepsilon}$. Observe that, in view of (26),

$$\begin{aligned} -\operatorname{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla\phi_k^\varepsilon - \nabla\overline{v}_k^\varepsilon\right)\right] &= \operatorname{div}\left[\mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right) B_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla u_0^\varepsilon - \sum_{p=1}^d \left(e_p + \nabla w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\partial_p u_0^\star\right)\right] \\ &\quad + \sum_{p=1}^d \operatorname{div}\left[Z_k\left(\frac{\cdot}{\varepsilon}\right)\partial_p u_0^\star\right] + \varepsilon \operatorname{div}\left[A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla(\partial_p u_0^\star)\chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right], \end{aligned}$$

where the vector-valued function Z_k is defined by

$$Z_k(y) = \mathbf{1}_{Q+k}(y) B_{per}(y)\left(e_p + \nabla w_{e_p}^0(y)\right) + A_{per}(y)\nabla\chi_{e_p}(y-k).$$

Note that, in view of (41), Z_k is a divergence free vector, hence $\operatorname{div}\left[Z_k\left(\frac{\cdot}{\varepsilon}\right)\partial_p u_0^\star\right] = Z_k\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_p u_0^\star$. We can thus rewrite $D_0^{k,\varepsilon}$ as

$$D_0^{k,\varepsilon} = D_{00}^{k,\varepsilon} + D_{01}^{k,\varepsilon} + D_{02}^{k,\varepsilon}, \tag{81}$$

where

$$\begin{aligned} D_{00}^{k,\varepsilon} &= -\int_{\mathcal{D}} \left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right)^T \mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right) B_{per}\left(\frac{\cdot}{\varepsilon}\right)\left(\nabla u_0^\varepsilon - \sum_{p=1}^d \left(e_p + \nabla w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\partial_p u_0^\star\right), \\ D_{01}^{k,\varepsilon} &= \sum_{p=1}^d \int_{\mathcal{D}} \left(\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\right) \; Z_k\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla(\partial_p u_0^\star), \\ D_{02}^{k,\varepsilon} &= -\varepsilon\sum_{p=1}^d \int_{\mathcal{D}} \left(\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\right)^T A_{per}\left(\frac{\cdot}{\varepsilon}\right)\nabla(\partial_p u_0^\star)\;\chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right). \end{aligned}$$

We successively bound these three quantities. Since $\chi_{e_p} \in L^\infty(\mathbb{R}^d)$ (see (43)) and $u_0^\star \in W^{1,\infty}(\mathcal{D})$, we have $\|\widetilde{v}_k^\varepsilon\|_{L^\infty(\mathcal{D})} \leq C\varepsilon$. We also have that $\|\phi_k^\varepsilon\|_{L^\infty(\mathcal{D})} \leq C\varepsilon$, in view of (29) (recall indeed that $u_0^\star \in W^{2,\infty}(\mathcal{D})$ implies that $f \in L^\infty(\mathcal{D})$, in view of (2); assumptions of Lemma 5 are thus satisfied). Using that $u_0^\star \in W^{2,\infty}(\mathcal{D})$, we now bound from above $D_{01}^{k,\varepsilon}$:

$$\begin{aligned}
|D_{01}^{k,\varepsilon}| &\leq \sum_{p=1}^{d} \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|_{L^\infty(\mathcal{D})} \|\nabla^2 u_0^\star\|_{L^\infty(\mathcal{D})} \left\| Z_k\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^1(\mathcal{D})} \\
&\leq C\varepsilon \sum_{p=1}^{d} \left[\|B_{per}\|_{L^\infty} \int_{\mathcal{D}} \mathbf{1}_{Q+k}\left(\frac{\cdot}{\varepsilon}\right) \left| e_p + \nabla w_{e_p}^0 \left(\frac{\cdot}{\varepsilon}\right)\right| + \|A_{per}\|_{L^\infty} \int_{\mathcal{D}} \left|\nabla \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right| \right] \\
&\leq C\varepsilon \sum_{p=1}^{d} \left[\varepsilon^d \|e_p + \nabla w_{e_p}^0\|_{L^2(Q)} + \varepsilon^d \int_{\mathcal{D}/\varepsilon - k} |\nabla \chi_{e_p}| \right] \\
&\leq C\varepsilon^{d+1} \sum_{p=1}^{d} \left[1 + \left(\int_{B(0,1)} |\nabla \chi_{e_p}| + \int_{B(0,\rho/\varepsilon)\setminus B(0,1)} |\nabla \chi_{e_p}| \right) \right].
\end{aligned}$$

Using that $\nabla \chi_{e_p} \in (L^2(\mathbb{R}^d))^d$ (see Lemma 10) and the bound (42), we deduce that

$$\begin{aligned}
|D_{01}^{k,\varepsilon}| &\leq C\varepsilon^{d+1} \sum_{p=1}^{d} \left[1 + \left(\|\nabla \chi_{e_p}\|_{L^2(\mathbb{R}^d)} + C \int_1^{1/\varepsilon} \frac{1}{r} dr \right) \right] \\
&\leq C\varepsilon^{d+1} \left[1 + \ln(1/\varepsilon)\right],
\end{aligned}$$

where C is a constant independent of ε. We thus get

$$\sum_{k \in I_\varepsilon} |D_{01}^{k,\varepsilon}| \leq C\varepsilon \left[1 + \ln(1/\varepsilon)\right]. \tag{82}$$

We now turn to $D_{02}^{k,\varepsilon}$. Using (73), we observe that, for any $k \in I_\varepsilon$,

$$\left\| \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right\|_{L^2(\mathcal{D})}^2 = \int_{\mathcal{D}} \left| \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right|^2 = \varepsilon^d \int_{\mathcal{D}/\varepsilon - k} |\chi_{e_p}|^2 \leq \varepsilon^d \int_{B(0,\bar{\rho}/\varepsilon)} |\chi_{e_p}|^2 \leq C\varepsilon^d R_{d,\varepsilon}.$$

We thus can bound from above $D_{02}^{k,\varepsilon}$, using that $u_0^\star \in W^{2,\infty}(\mathcal{D})$:

$$\begin{aligned}
\sum_{k \in I_\varepsilon} |D_{02}^{k,\varepsilon}| &\leq \varepsilon \|\nabla^2 u_0^\star\|_{L^\infty(\mathcal{D})} \|A_{per}\|_{L^\infty} \sum_{p=1}^{d} \sum_{k \in I_\varepsilon} \|\nabla \phi_k^\varepsilon - \nabla \widetilde{v}_k^\varepsilon\|_{L^2(\mathcal{D})} \left\| \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right\|_{L^2(\mathcal{D})} \\
&\leq C\varepsilon \sqrt{\sum_{k \in I_\varepsilon} \|\nabla \phi_k^\varepsilon - \nabla \widetilde{v}_k^\varepsilon\|_{L^2(\mathcal{D})}^2} \sqrt{\sum_{p=1}^{d} \sum_{k \in I_\varepsilon} \left\| \chi_{e_p}\left(\frac{\cdot}{\varepsilon} - k\right)\right\|_{L^2(\mathcal{D})}^2} \\
&\leq C\varepsilon \sqrt{\sum_{k \in I_\varepsilon} \|\nabla \phi_k^\varepsilon - \nabla \widetilde{v}_k^\varepsilon\|_{L^2(\mathcal{D})}^2} \sqrt{R_{d,\varepsilon}}
\end{aligned} \tag{83}$$

where C is a constant independent of ε. We next turn to $D_{00}^{k,\varepsilon}$. Using the bound (37) on the two scale expansion of u_0^ε, we have

$$\begin{aligned}
\sum_{k\in I_\varepsilon} |D_{00}^{k,\varepsilon}| &\leq \|B_{per}\|_{L^\infty} \sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|_{L^2(\mathcal{D})} \left\| \nabla u_0^\varepsilon - \sum_{p=1}^d \left(e_p + \nabla w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\partial_p u_0^\star \right\|_{L^2(\varepsilon(Q+k))} \\
&\leq C\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}} \sqrt{\sum_{k\in I_\varepsilon} \left\| \nabla u_0^\varepsilon - \sum_{p=1}^d \left(e_p + \nabla w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\partial_p u_0^\star \right\|^2_{L^2(\varepsilon(Q+k))}} \\
&\leq C\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}} \left\| \nabla u_0^\varepsilon - \sum_{p=1}^d \left(e_p + \nabla w_{e_p}^0\left(\frac{\cdot}{\varepsilon}\right)\right)\partial_p u_0^\star \right\|_{L^2(\mathcal{D})} \\
&\leq C\sqrt{\varepsilon}\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}}. \qquad (84)
\end{aligned}$$

Collecting (81), (82), (83) and (84), we obtain that

$$\sum_{k\in I_\varepsilon} |D_0^{k,\varepsilon}| \leq C\left(\left(\sqrt{\varepsilon} + \varepsilon\sqrt{R_{d,\varepsilon}}\right)\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}} + \varepsilon\ln(1/\varepsilon)\right). \qquad (85)$$

We now turn to $D_1^{k,\varepsilon}$. Using (79), we have

$$\begin{aligned}
\sum_{k\in I_\varepsilon} |D_1^{k,\varepsilon}| &\leq C\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}} \sqrt{\sum_{k\in I_\varepsilon} \|\nabla\overline{v}_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}} \\
&\leq C\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}}\sqrt{\varepsilon R_{d,\varepsilon}}. \qquad (86)
\end{aligned}$$

Collecting (80), (85) and (86), we obtain

$$\alpha\sum_{k\in I_\varepsilon} \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C\left(\varepsilon\ln(1/\varepsilon) + \left(\sqrt{\varepsilon R_{d,\varepsilon}} + \varepsilon\sqrt{R_{d,\varepsilon}}\right)\sqrt{\sum_{k\in I_\varepsilon} \|\nabla\phi_k^\varepsilon - \nabla\widetilde{v}_k^\varepsilon\|^2_{L^2(\mathcal{D})}}\right)$$

with, in view of (74), $R_{d,\varepsilon} = 1 + \ln(1/\varepsilon)$ if $d = 2$, and $R_{d,\varepsilon} = 1$ if $d > 2$. This implies that

$$\sum_{k\in I_\varepsilon} \|\phi_k^\varepsilon - \widetilde{v}_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C\varepsilon\ln(1/\varepsilon), \quad C \text{ independent of } \varepsilon.$$

Collecting this bound with (79), we obtain the claimed bound. This concludes the proof of Proposition 11. □

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