# Incomputability of Simply Connected Planar Continua

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**Abstract**

Le Roux and Ziegler asked whether every simply connected compact nonempty planar $\Pi^0_1$ set always contains a computable point. In this paper, we solve the problem of le Roux and Ziegler by showing that there exists a planar $\Pi^0_1$ dendroid without computable points. We also provide several pathological examples of tree-like $\Pi^0_1$ continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$-include a $\Pi^0_1$ tree; there is a $\Pi^0_1$ dendrite which does not $*$-include a computable dendrite; there is a computable dendroid which does not $*$-include a $\Pi^0_1$ dendrite. Here, a continuum $A$ *$*$-includes* a member of a class $\mathcal{P}$ of continua if, for every positive real $\varepsilon$, $A$ includes a continuum $B \in \mathcal{P}$ such that the Hausdorff distance between $A$ and $B$ is smaller than $\varepsilon$.

## 1 Background

Every nonempty open set in a computable metric space (such as Euclidean $n$-space $\mathbb{R}^n$) contains a computable point. In contrast, the Non-Basis Theorem asserts that a nonempty *co-c.e. closed* set (also called a $\Pi^0_1$ set) in Cantor space (hence, even in Euclidean 1-space) can avoid any computable points. Non-Basis Theorems can shed new light on connections between *local* and *global* properties by incorporating the notions of *measure* and *category*. For instance, Kreisel-Lacombe [6] and Tanaka [17] showed that there is a $\Pi^0_1$ set with positive measure that contains no computable point. Recent exciting progress in *Computable Analysis* [18] naturally raises the question whether Non-Basis Theorems exist for *connected* $\Pi^0_1$ sets. However, we observe that, if a nonempty $\Pi^0_1$ subset of $\mathbb{R}^1$ contains no computable points, then it must be totally disconnected. Then, in higher dimensional Euclidean space, can there exist a connected $\Pi^0_1$ set containing no computable points? It is easy to construct a nonempty connected $\Pi^0_1$ subset of $[0,1]^2$ without computable points, and a nonempty simply connected $\Pi^0_1$ subset of $[0,1]^3$ without computable points. An open problem, formulated by Le Roux and Ziegler [13] was whether every nonempty simply connected compact planar $\Pi^0_1$ set contains a computable point. As mentioned in Penrose's book "*Emperor's New Mind*" [12], *the Mandelbrot set* is an example of a simply connected compact planar $\Pi^0_1$ set which contains a computable point, and he conjectured that the Mandelbrot set is not computable *as a closed set*. Hertling [5] observed that the Penrose conjecture has an implication for a

*This work was supported by Grant-in-Aid for JSPS fellows.

famous open problem on local connectivity of the Mandelbrot set. Our interest is which topological assumption (especially, connectivity assumption) on a $\Pi^0_1$ set can force it to possess a given computability property. Miller [10] showed that every $\Pi^0_1$ sphere in $\mathbb{R}^n$ is computable, and so it contains a dense c.e. subset of computable points. He also showed that every $\Pi^0_1$ ball in $\mathbb{R}^n$ contains a dense subset of computable points. Iljazović [7] showed that chainable continua (e.g., arcs) in certain metric spaces are almost computable, and hence there always is a dense subset of computable points. In this paper, we show that *not* every $\Pi^0_1$ dendrite is almost computable, by using a tree-immune $\Pi^0_1$ class in Cantor space. This notion of immunity was introduced by Cenzer, Weber Wu, and the author [4]. We also provide pathological examples of tree-like $\Pi^0_1$ continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$-include a $\Pi^0_1$ tree; there is a computable dendroid which does not $*$-include a $\Pi^0_1$ dendrite. Finally, we solve the problem of Le Roux and Ziegler [13] by showing that there exists a planar $\Pi^0_1$ dendroid without computable points. Indeed, our planar dendroid is contractible. Hence, our dendroid is also the first example of a contractible Euclidean $\Pi^0_1$ set without computable points.

# 2 Preliminaries

**Basic Notation:** $2^{<\mathbb{N}}$ denotes the set of all finite binary strings. Let $X$ be a topological space. For a subset $Y \subseteq X$, $cl(Y)$ ($int(Y)$, resp.) denotes the closure (the interior, resp.) of $Y$. Let $(X; d)$ be a metric space. For any $x \in X$ and $r \in \mathbb{R}$, $B(x; r)$ denotes the open ball $B(x; r) = \{y \in X : d(x, y) < r\}$. Then $x$ is called *the center of* $B(x; r)$, and $r$ is called *the radius of* $B(x; r)$. For a given open ball $B = B(x; r)$, $\hat{B}$ denotes the corresponding closed ball $\hat{B} = \{y \in X : d(x, y) \leq r\}$. For $a, b \in \mathbb{R}$, $[a, b]$ denotes the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a, b)$ denotes the open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, and $\langle a, b\rangle$ denotes a point of Euclidean plane $\mathbb{R}^2$. For $X \subseteq \mathbb{R}^n$, $\mathrm{diam}(X)$ denotes $\max\{d(x, y) : x, y \in X\}$.

**Continuum Theory:** *A continuum* is a compact connected metric space. For basic terminology concerning *Continuum Theory*, see Nadler [11] and Illanes-Nadler [8].

Let $X$ be a topological space. The set $X$ is *a Peano continuum* if it is a locally connected continuum. The set $X$ is *a dendrite* if it is a Peano continuum which contains no Jordan curve. The set $X$ is *unicoherent* if $A \cap B$ is connected for every connected closed subsets $A, B \subseteq X$ with $A \cup B = X$. The set $X$ is *hereditarily unicoherent* if every subcontinuum of $X$ is unicoherent. The set $X$ is *a dendroid* if it is an arcwise connected hereditary unicoherent continuum. For a point $x$ of a dendroid $X$, $r_X(x)$ denotes the cardinality of the set of arc-components of $X \setminus \{x\}$. If $r_X(x) \geq 3$ then $x$ is said to be *a ramification point of* $X$. The set $X$ is *a tree* if it is dendrite with finitely many ramification points. Note that a topological space $X$ is a dendrite if and only if it is a locally connected dendroid. Hahn-Mazurkiewicz's Theorem states that a Hausdorff space $X$ is a Peano continuum if and only if $X$ is an image of a continuous curve.

**Example 1** (Planar Dendroids)**.**

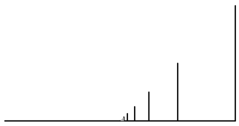

Figure 1: The basic dendrite

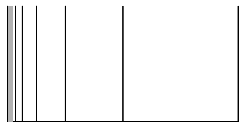

Figure 2: The harmonic comb

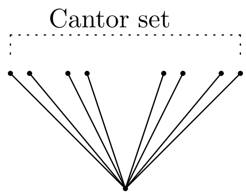


Figure 3: The Cantor fan

1. Put $\mathcal{B}_t = \{2^{-t}\} \times [0, 2^{-t}]$. Then the following set $\mathcal{B} \subseteq \mathbb{R}^2$ is dendrite.
$$\mathcal{B} = \bigcup_{t \in \mathbb{N}} \mathcal{B}_t \cup ([-1, 1] \times \{0\}).$$
We call $\mathcal{B}$ *the basic dendrite*. The set $\mathcal{B}_t$ is called *the $t$-th rising of $\mathcal{B}$*. See Fig. 1.

2. The set $\mathcal{H} = cl((\{1/n : n \in \mathbb{N}\} \times [0, 1]) \cup ([0, 1] \times \{0\}))$ is called *a harmonic comb*. Then $\mathcal{H}$ is a dendroid, but not a dendrite. The set $\{1/n\} \times [0, 1]$ is called *the $n$-th rising of the comb $\mathcal{H}$*, and the set $[0, 1] \times \{0\}$ is called *the grip of $\mathcal{H}$*. See Fig. 2.

3. Let $C \subseteq \mathbb{R}^1$ be the middle third Cantor set. Then the one-point compactification of $C \times (0, 1]$ is called *the Cantor fan*. (Equivalently, it is the quotient space $\mathrm{Cone}(C) = (C \times [0, 1])/(C \times \{0\})$.) The Cantor fan is a dendroid, but not a dendrite. See Fig. 3.

Let $X$ be a topological space. $X$ is *$n$-connected* if it is path-connected and $\pi_i(X) \equiv 0$ for any $1 \leq i \leq n$, where $\pi_i(X)$ is the $i$-th homotopy group of $X$. $X$ is *simply connected* if $X$ is 1-connected. $X$ is *contractible* if the identity map on $X$ is null-homotopic. Note that, if $X$ is contractible, then $X$ is $n$-connected for each $n \geq 1$. It is easy to see that the dendroids in Example 1 are contractible.

**Computability Theory:** We assume that the reader is familiar with Computability Theory on the natural numbers $\mathbb{N}$, Cantor space $2^{\mathbb{N}}$, and Baire space $\mathbb{N}^{\mathbb{N}}$ (see also Soare [16]). For basic terminology concerning *Computable Analysis*, see Weihrauch [18], Brattka-Weihrauch [3], and Brattka-Presser [2].

Hereafter, we fix a countable base for the Euclidean $n$-space $\mathbb{R}^n$ by $\rho = \{B(x; r) : x \in \mathbb{Q}^n \;\&\; r \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+$ denotes the set of all positive rationals. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be an effective enumeration of $\rho$. We say that a point $x \in \mathbb{R}^n$ is *computable* if the code of its principal filter $\mathcal{F}(x) = \{i \in \mathbb{N} : x \in \rho_i\}$ is computably enumerable (hereafter c.e.) A closed subset $F \subseteq \mathbb{R}^n$ is $\Pi^0_1$ if there is a c.e. set $W \subseteq \mathbb{N}$ such that $F = X \setminus \bigcup_{e \in W} \rho_e$. A closed subset $F \subseteq \mathbb{R}^n$ is *computably enumerable* (hereafter *c.e.*) if $\{e \in \mathbb{N} : F \cap \rho_e \neq \emptyset\}$ is c.e. A closed subset $F \subseteq \mathbb{R}^n$ is *computable* if it is $\Pi^0_1$ and c.e. on $\mathbb{R}^n$.

**Almost Computability:** Let $A_0, A_1$ be nonempty closed subsets of a metric space $(X, d)$. Then *the Hausdorff distance* between $A_0$ and $A_1$ is defined by
$$d_H(A_0, A_1) = \max_{i<2} \sup_{x \in A_i} \inf_{y \in A_{1-i}} d(x, y).$$
Let $\mathcal{P}$ be a class of continua. We say that a continuum $A$ *$*$-includes a member of $\mathcal{P}$* if $\inf\{d_H(A, B) : A \supseteq B \in \mathcal{P}\} = 0$.

**Proposition 2.** *Every Euclidean dendroid $*$-includes a tree.*

*Proof.* Fix a Euclidean dendroid $D \subseteq \mathbb{R}^n$, and a positive rational $\varepsilon \in \mathbb{Q}$. Then $D$ is covered by finitely many open rational balls $\{B_i\}_{i<n}$ of radius $\varepsilon/2$. Choose $d_i \in D \cap B_i$ for each $i < n$ if $B_i$ intersects with $D$. Note that $\{B(d_i;\varepsilon)\}_{i<n}$ covers $D$. Since $D$ is dendroid, there is a unique arc $\gamma_{i,j} \subseteq D$ connecting $d_i$ and $d_j$ for each $i,j < n$. Then, $E = \bigcup_{\{i,j\}\subseteq n} \gamma_{i,j}$ is connected and locally connected, since $E$ is a union of finitely many arcs (i.e., it is a graph, in the sense of Continuum Theory; see also Nadler [11]). It is easy to see that $E$ has no Jordan curve, since $E$ is a subset of the dendroid $D$. Consequently, $E$ is a tree. Moreover, clearly $d_H(E,D) < \varepsilon$, since $d_i \in E$ for each $i < n$. □

The class $\mathcal{P}$ has *the almost computability property* if every $A \in \mathcal{P}$ $*$-includes a computable member of $\mathcal{P}$ as a closed set. In this case, we simply say that *every $A \in \mathcal{P}$ is almost computable.* Iljazović [7] showed that every $\Pi^0_1$ chainable continuum is almost computable, hence every $\Pi^0_1$ arc is almost computable.

# 3 Incomputability of Dendrites

By Proposition 2, topologically, every planar dendrite $*$-includes a tree. However, if we try to effectivize this fact, we will find a counterexample.

**Theorem 3.** *Not every computable planar dendrite $*$-includes a $\Pi^0_1$ tree.*

*Proof.* Let $A \subseteq \mathbb{N}$ be an incomputable c.e. set. Thus, there is a total computable function $f_A : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{range}(f_A) = A$. We may assume $f_A(s) \leq s$ for every $s \in \mathbb{N}$. Let $A_s$ denote the finite set $\{f_A(u) : u \leq s\}$. Then $\mathrm{st}^A : \mathbb{N} \to \mathbb{N}$ is defined as $\mathrm{st}^A(n) = \min\{s \in \mathbb{N} : n \in A_s\}$. Note that $\mathrm{st}^A(n) \geq n$ by our assumption $f_A(s) \leq s$.

**Construction. .** Recall the definition of the basic dendrite from Example 1. We construct a computable dendrite by modifying the basic dendrite $\mathcal{B}$. For every $t \in \mathbb{N}$, we introduce *the width of the $t$-rising* $w(t)$ as follows:

$$w(t) = \begin{cases} 2^{-(2+\mathrm{st}^A(t))}, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $I_t$ be the closed interval $[2^{-t} - w(t), 2^t + w(t)]$. Since $\mathrm{st}^A(n) \geq n$, we have $I_t \cap I_s = \emptyset$ whenever $t \neq s$. We observe that $\{w(t)\}_{t\in\mathbb{N}}$ is a uniformly computable sequence of real numbers. Now we define a computable dendrite $D \subseteq \mathbb{R}^2$ by:

$$\begin{aligned} D^0_t &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-t}] \\ D^1_t &= [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} \\ D^2_t &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-t}) \\ D &= \Big(\bigcup_{t\in\mathbb{N}} (D^0_t \cup D^1_t)\Big) \cup \Big(([-1,1] \times \{0\}) \setminus \bigcup_{t\in\mathbb{N}} D^2_{t,m}\Big). \end{aligned}$$

We call $D_t = D^0_t \cup D^1_t$ *the $t$-th rising of $D$*. See Fig. 4.

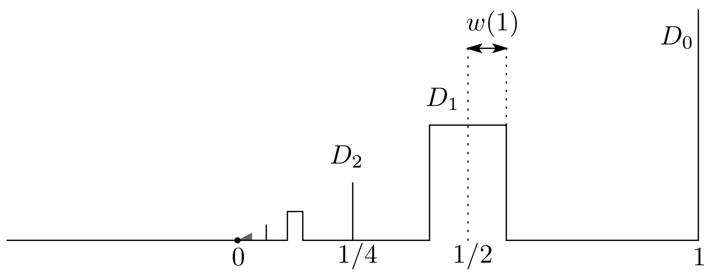


Figure 4: The dendrite $D$ for $0, 2, 4 \notin A$ and $1, 3 \in A$.

**Claim.** The set $D$ is a dendrite.

To prove $D$ is a Peano continuum, by the Hahn-Mazurkiewicz Theorem, it suffices to show that $D = \mathrm{Im}(h)$ for some continuous curve $h : [-1, 1] \to \mathbb{R}^2$. We divide the unit interval $[0, 1]$ into infinitely many parts $I_t = [2^{-(t+1)}, 2^{-t}]$. Furthermore, we also divide each interval $I_{2t}$ into three parts $I^0_{2t}$, $I^1_{2t}$, and $I^2_{2t}$, where $I^i_{2t} = [(5-i) \cdot 3^{-1} \cdot 2^{-(2t+1)}, (6-i) \cdot 3^{-1} \cdot 2^{-(2t+1)}]$ for each $i < 3$. Then we define a desired curve $h$ as follows.

$$h(x) \text{ moves in } \begin{cases} \{2^{-t} + w(t)\} \times [0, 2^{-t}] & \text{if } x \in I^0_{2t}, \\ [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} & \text{if } x \in I^1_{2t}, \\ \{2^{-t} - w(t)\} \times [0, 2^{-t}] & \text{if } x \in I^2_{2t}, \\ [2^{-(t+1)} + w(t+1), 2^{-t} - w(t)] \times \{0\} & \text{if } x \in I_{2t+1}, \\ [-1, 0] \times \{0\} & \text{if } x \in [-1, 0]. \end{cases}$$

Clearly, $h$ can be continuous, and indeed computable, since the map $w : \mathbb{R} \to \mathbb{R}$ is computable. It is easy to see that $D = \mathrm{Im}(h)$. Moreover, $\mathrm{Im}(h)$ contains no Jordan curve since $\pi_0(h(x)) \le \pi_0(h(y))$ whenever $x \le y$, where $\pi_0(p)$ denotes the first coordinate of $p \in \mathbb{R}^2$. Consequently, $D$ is a dendrite.

Moreover, by construction, it is easy to see that $D$ is computable.

**Claim.** The computable dendrite $D$ does not $*$-include a $\Pi^0_1$ tree.

Suppose that $D$ contains a $\Pi^0_1$ subtree $T \subseteq D$. We consider a rational open ball $B_t$ with center $\langle 2^{-t}, 2^{-t} \rangle$ and radius $2^{-(t+2)}$, for each $t \in \mathbb{N}$. Note that $B_t \cap D \subseteq D_t$ for every $t \in \mathbb{N}$. Since $T$ is $\Pi^0_1$ in $\mathbb{R}^2$, $B = \{t \in \mathbb{N} : \hat{B}_t \cap T = \emptyset\}$ is c.e. If $w(t) > 0$ (i.e., $t \in A$) then $D \setminus (D_t \cap B_t)$ is disconnected. Therefore, either $T \subseteq [-1, 2^{-t}] \times \mathbb{R}$ or $T \subseteq [2^{-t}, 1] \times \mathbb{R}$ holds whenever $\hat{B}_t \cap T = \emptyset$ (i.e., $t \in B$), since $T$ is connected. Thus, if the condition $\#(A \cap B) = \aleph_0$ is satisfied, then either $T \subseteq [-1, 0] \times \mathbb{R}$ or $T \subseteq [0, 1] \times \mathbb{R}$ holds. Consequently, we must have $d_H(T, D) \ge 1$.

Therefore, we may assume $\#A \cap B < \aleph_0$. Since $A$ is coinfinite, $D$ has infinitely many ramification points $\langle 2^{-t}, 0 \rangle$ for $t \notin A$. However, by the definition of tree, $T$ has only finitely many ramification points. Thus we must have $(D^0_t \cap T) \setminus \{\langle 2^{-t}, 0 \rangle\} = \emptyset$ for almost all $t \notin A$. Since $\hat{B}_t \cap T \subseteq (D^0_t \cap T) \setminus \{\langle 2^{-t}, 0 \rangle\}$, we have $t \in B$ for almost all $t \in \mathbb{N} \setminus A$. Consequently, we have $\#((\mathbb{N} \setminus A) \triangle B) < \aleph_0$. This implies that $\mathbb{N} \setminus A$ is also c.e., since $B$ is c.e. This contradicts that $A$ is incomputable. □

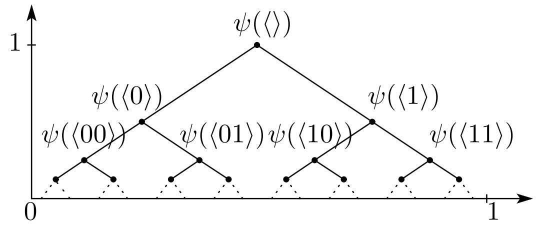


Figure 5: The plotted tree $\Psi(2^{<\mathbb{N}})$.

Note that a Hausdorff space (hence every metric space) is (locally) arcwise connected if and only if it is (locally) pathwise connected. However, Miller [10] pointed out that the effective versions of arcwise connectivity and pathwise connectivity do *not* coincide. Theorem 3 could give a result on effective connectivity properties. Note that *effectively pathwise connectivity* is defined by Brattka [1]. Clearly, the dendrite $D$ is effectively pathwise connected. We now introduce a new effective version of arcwise connectivity property by thinking arcs as closed sets. Let $\mathcal{A}_-(X)$ denote the hyperspace of closed subsets of $X$ with negative information (see also Brattka [1]).

**Definition 4.** A computable metric space $(X, d, \alpha)$ is *semi-effectively arcwise connected* if there exists a total computable multi-valued function $P : X^2 \rightrightarrows \mathcal{A}_-(X)$ such that $P(x, y)$ is the set of all arcs $A$ whose two end points are $x$ and $y$, for any $x, y \in X$.

Obviously $D$ is not semi-effectively arcwise connected. Indeed, for every $\varepsilon > 0$ there exists $x_0, x_1 \in [0, 1]$ with $d(x_0, x_1) < \varepsilon$ such that $\langle x_0, 0\rangle, \langle x_1, 0\rangle \in D$ cannot be connected by any $\Pi^0_1$ arc. Thus, we have the following corollary.

**Corollary 1.** *There exists an effectively pathwise connected Euclidean continuum $D$ such that $D$ is not semi-effectively arcwise connected.*

**Theorem 5.** *Not every $\Pi^0_1$ planar dendrite is almost computable.*

To prove Theorem 5, we need to prepare some tools. For a string $\sigma \in 2^{<\mathbb{N}}$, let $lh(\sigma)$ denote the length of $\sigma$. Then

$$\psi(\sigma) = \left\langle 2^{-1} \cdot 3^{-i} + 2 \sum_{i<lh(\sigma)\ \&\ \sigma(i)=1} 3^{-(i+1)}, 2^{-lh(\sigma)} \right\rangle \in \mathbb{R}^2.$$

For two points $\vec{x}, \vec{y} \in \mathbb{R}^2$, the closed line segment $L(\vec{x}, \vec{y})$ from $\vec{x}$ to $\vec{y}$ is defined by $L(\vec{x}, \vec{y}) = \{(1-t)\vec{x} + t\vec{y} : t \in [0, 1]\}$. For a (possibly infinite) tree $T \subseteq 2^{<\mathbb{N}}$, we plot an embedded tree $\Psi(T) \subseteq \mathbb{R}^2$ by

$$\Psi(T) = cl\left(\bigcup\{L(\psi(\sigma), \psi(\tau)) : \sigma, \tau \in T\ \&\ lh(\sigma) = lh(\tau) + 1\}\right).$$

Then $\Psi(T)$ is a dendrite (but not necessarily a tree, in the sense of Continuum Theory), for any (possibly infinite) tree $T \subseteq 2^{\mathbb{N}}$. See Fig. 5.

We can easily prove the following lemmata.

**Lemma 6.** *Let $T$ be a subtree of $2^{<\mathbb{N}}$, and $D$ be a planar subset such that $\psi(\langle\rangle) \in D \subseteq \Psi(T)$ for the root $\langle\rangle \in 2^{<\mathbb{N}}$. Then $D$ is a dendrite if and only if $D$ is homeomorphic to $\Psi(S)$ for a subtree $S \subseteq T$.*

*Proof.* The "if" part is obvious. Let $D$ be a dendrite. For a binary string $\sigma$ which is not a root, let $\sigma^-$ be an immediate predecessor of $\sigma$. We consider the set $S = \{\langle\rangle\} \cup \{\sigma \in 2^{<\mathbb{N}} : \sigma \neq \langle\rangle \ \& \ D \cap (L(\psi(\sigma^-), \psi(\sigma)) \setminus \{\psi(\sigma^-)\}) \neq \emptyset\}$. Since $D$ is connected, $S$ is a subtree of $T$. It is easy to see that $D$ is homeomorphic to $\Psi(S)$. $\square$

**Lemma 7.** *Let $T$ be a subtree of $2^{<\mathbb{N}}$. Then $T$ is $\Pi^0_1$ (c.e., computable, resp.) if and only if $\Psi(T)$ is a $\Pi^0_1$ (c.e., computable, resp.) dendrite in $\mathbb{R}^2$.*

*Proof.* With our definition of $\Psi$, the dendrite $\Psi(2^{<\mathbb{N}})$ is clearly a computable closed subset of $\mathbb{R}^2$. So, if $T$ is $\Pi^0_1$, then it is easy to prove that $\Psi(T)$ is also $\Pi^0_1$. Assume that $T$ is a c.e. tree. At stage $s$, we compute whether $L(\psi(\sigma^-), \psi(\sigma))$ intersects with the $e$-th open rational ball $\rho_e$, for any $e < s$ and any string $\sigma$ which is already enumerated into $T$ by stage $s$. If so, we enumerate $e$ into $W_T$ at stage $s$. Then $\{e \in \mathbb{N} : \Psi(T) \cap \rho_e \neq \emptyset\} = W_T$ is c.e.

Assume that $\Psi(T)$ is $\Pi^0_1$. We consider an open rational ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$ for each $\sigma \in 2^{<\mathbb{N}}$. Note that $\hat{B}_-(\sigma) \cap \hat{B}_-(\tau) = \emptyset$ for $\sigma \neq \tau$. Since $\Psi(T)$ is $\Pi^0_1$, $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e., and it is easy to see that $T = 2^{<\mathbb{N}} \setminus T^*$. Thus, $T$ is a $\Pi^0_1$ tree of $2^{<\mathbb{N}}$. We next assume that $\Psi(T)$ is c.e. We can assume that $\Psi(T)$ contains the root $\psi(\langle\rangle)$, otherwise $T = \emptyset$, and clearly it is c.e. For a binary string $\sigma$ which is not a root, let $\sigma^-$ be an immediate predecessor of $\sigma$. Pick an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each $\sigma$. Since $\Psi(T)$ is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap B_+(\sigma) \neq \emptyset\}$ is c.e. If $\sigma$ is not a root and $\sigma \in T$ then $L(\psi(\sigma^-), \psi(\sigma)) \subseteq \Psi(T)$, so $\Psi(T) \cap B_+(\sigma) \neq \emptyset$. We observe that if $\sigma \notin T$ then $L(\psi(\sigma^-), \psi(\sigma)) \cap \Psi(T) = \emptyset$, so $\Psi(T) \cap B_+(\sigma) = \emptyset$. Thus, we have $T = T^*$. In the case that $\Psi(T)$ is computable, $\Psi(T)$ is c.e. and $\Pi^0_1$, hence $T$ is c.e. and $\Pi^0_1$, i.e., $T$ is computable. $\square$

**Lemma 8.** *Let $D$ be a computable subdendrite of $\Psi(2^{<\mathbb{N}})$. Then there exists a computable subtree $T^+ \subseteq 2^{<\mathbb{N}}$ such that $D \subseteq \Psi(T^+)$ and $([0,1] \times \{0\}) \cap D = ([0,1] \times \{0\}) \cap \Psi(T^+)$.*

*Proof.* We can assume $\psi(\langle\rangle) \in D$, otherwise we connect $\psi(\langle\rangle)$ and the root of $D$ by a subarc of $\Psi(2^{<\mathbb{N}})$. Again we consider an open rational ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$, and an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each $\sigma \in 2^{<\mathbb{N}}$. Since $D$ is $\Pi^0_1$, $U^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e. Since $D$ is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap B_+(\sigma) \neq \emptyset\}$ is c.e., and it is a tree by Lemma 6. For every $\sigma \in 2^{<\mathbb{N}}$, either $D \cap \hat{B}_-(\sigma) = \emptyset$ or $D \cap B_+(\sigma) \neq \emptyset$ holds. Therefore, we have $T^* \cup U^* = 2^{<\mathbb{N}}$. Moreover, for the set of *leaves of* $T^*$, $L^*_T = \{\rho \in T^* : (\forall i < 2)\ \rho^\frown\langle i\rangle \notin T^*\}$, we observe that $T^* \cap U^* \subseteq L^*_T$. Recall that the pointclass $\Sigma^0_1$ has the reduction property, that is, for two c.e. sets $T^*$ and $U^*$, there exist c.e. subsets $T \subseteq T^*$ and $U \subseteq U^*$ such that $T \cup U = T^* \cup U^*$ and $T \cap U = \emptyset$. This is because, for $\sigma \in T^* \cap U^*$, $\sigma$ is enumerated into $T$ when $\sigma$ is enumerated into $T^*$ before it is enumerated into $U^*$; $\sigma$ is enumerated into $U$ otherwise. Since $T^* \cap U^* \subseteq L^*_T$, $T$ must be tree. Furthermore, $T$ is c.e., and $U = 2^{<\mathbb{N}} \setminus T$ is also c.e. Thus, $T$ is a computable tree. Therefore, $T^+ = \{\sigma^\frown\langle i\rangle : \sigma \in T \ \& \ i < 2\}$ is also a computable tree. Then, $D \subseteq \Psi(T^+)$, and we have $([0,1] \times \{0\}) \cap D = ([0,1] \times \{0\}) \cap \Psi(T^+)$ since the set of all infinite paths of $T$ and that of $T^+$ coincide. $\square$

Cenzer, Weber and Wu, and the author [4] introduced the notion of *tree-immunity* for closed sets in Cantor space $2^{\mathbb{N}}$. For $\sigma \in 2^{<\mathbb{N}}$, define $I_\sigma$ as $\{f \in 2^{\mathbb{N}} : (\forall n < lh(\sigma))\ f(n) = \sigma(n)\}$. Note that $\{I_\sigma : \sigma \in 2^{<\mathbb{N}}\}$ is a countable base for Cantor space.

**Definition 9** (Cenzer-Kihara-Weber-Wu [4])**.** A nonempty closed set $F \subseteq 2^{\mathbb{N}}$ is said to be *tree-immune* if the tree $T_F = \{\sigma \in 2^{<\mathbb{N}} : F \cap I_\sigma \neq \emptyset\} \subseteq 2^{<\mathbb{N}}$ contains no infinite computable subtree.

For a nonempty $\Pi^0_1$ subset $P \subseteq 2^{\mathbb{N}}$, the corresponding tree $T_P$ is $\Pi^0_1$, and it has no dead ends. The set of *all complete consistent extensions of Peano Arithmetic* is an example of a tree-immune $\Pi^0_1$ subset of $2^{\mathbb{N}}$. Tree-immune $\Pi^0_1$ sets have the following remarkable property.

**Lemma 10.** *Let $P$ be a tree-immune $\Pi^0_1$ subset of $2^{\mathbb{N}}$ and let $D \subseteq \Psi(T_P)$ be any computable subdendrite. Then $([0,1] \times \{0\}) \cap D = \emptyset$ holds.*

*Proof.* By Lemma 8, there exists a computable subtree $T \subseteq 2^{<\mathbb{N}}$ such that $D \subseteq \Psi(T)$ and $\Psi(T)$ agrees with $D$ on $[0,1] \times \{0\}$. Since $D \subseteq \Psi(T_P)$, and since $T_P$ has no dead ends, $T \subseteq T_P$ holds. Since $P$ is tree-immune, $T$ must be finite. By using weak König's lemma (or, equivalently, compactness of Cantor space), $T \subseteq 2^l$ holds for some $l \in \mathbb{N}$. Thus, $D \subseteq \Psi(T) \subseteq [0,1] \times [2^{-l}, 1]$ as desired. □

Note that if $P$ is a nonempty $\Pi^0_1$ set in Cantor space $2^{\mathbb{N}}$, then for every $\delta > 0$ it holds that $((0,1) \times (0,\delta)) \cap \Psi(T_P) \neq \emptyset$. Finally, we are ready to prove Theorem 5.

*Proof of Theorem 5.* Again, we adapt the construction in the proof of Theorem 3. We fix a nonempty tree-immune $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$. For $\sigma \in 2^{<\mathbb{N}}$, put $E(\sigma) = \{\tau \in 2^{<\mathbb{N}} : \tau \supseteq \sigma\}$. For a $\Pi^0_1$ tree $T_P \subseteq 2^{<\mathbb{N}}$, there exists a computable function $f_P : \mathbb{N} \to 2^{<\mathbb{N}}$ such that $T_P = 2^{<\mathbb{N}} \setminus \bigcup_n E(f_P(n))$ and such that for each $\sigma \in 2^{<\mathbb{N}}$ and $s \in \mathbb{N}$ we have $\sigma \in \bigcup_{t<s} E(f_P(t))$ whenever $\sigma^\frown 0, \sigma^\frown 1 \in \bigcup_{t<s} E(f_P(t))$. For such a computable function $f_P : \mathbb{N} \to 2^{<\mathbb{N}}$, we let $T_{P,s}$ denote $2^{<\mathbb{N}} \setminus \bigcup_{t<s} E(f_P(t))$. Then $T_{P,s}$ is a tree without dead ends, and $\{T_{P,s} : s \in \mathbb{N}\}$ is computable uniformly in $s$.

**Construction. .** Let $\vec{e}_1$ denote $\langle 1, 0\rangle \in \mathbb{R}^2$. For a tree $T \subseteq 2^{<\mathbb{N}}$ and $w \in \mathbb{Q}$, we define $\Psi(T; w)$, *the edge of the fat approximation of the tree $T$ of width $w$*, by

$$\Psi(T; w) = cl\Bigg(\bigcup \Big\{ L\left(\psi(\sigma) \pm (3^{-|\sigma|} \cdot w)\vec{e}_1, \psi(\tau) \pm (3^{-|\tau|} \cdot w)\vec{e}_1\right)$$
$$: \pm \in \{-,+\} \ \&\ \sigma, \tau \in T \ \&\ lh(\sigma) = lh(\tau) + 1 \Big\}\Bigg).$$

If $\lim_s w_s = 0$ then we have $\lim_s \Psi(T; w_s) = \Psi(T)$. Moreover, if $\{w_s : s \in \mathbb{N}\}$ is a uniformly computable sequence of rational numbers, then $\{\Psi(T; w_s) : s \in \mathbb{N}\}$ is also a uniformly computable sequence of computable closed sets. Additionally, the set $\Psi(T; w, c, t, q)$, for a tree $T \subseteq 2^{<\mathbb{N}}$, for $w, c, q \in \mathbb{Q}$, and for $t \in \mathbb{N}$, is defined by

$$\Psi(T; w, c, t, q) = \left\{ \left\langle c + q \cdot \left(x - \frac{1}{2}\right), \frac{2 - y}{2^{t+1}} \right\rangle \in \mathbb{R}^2 : \langle x, y\rangle \in \Psi(T; w) \right\}.$$

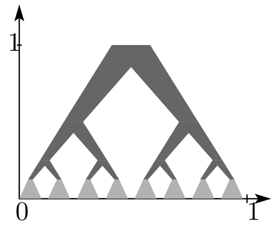


Figure 6: The fat approximation $\Psi(T;w)$.

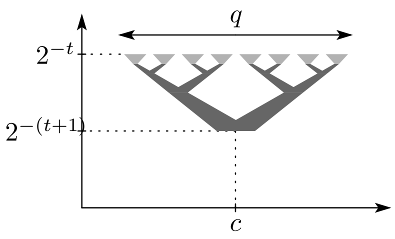


Figure 7: The basic object $\Psi(T;w,c,t,q)$.

Note that $\Psi(T;w,c,t,q) \subseteq [c-q/2, c+q/2] \times [2^{-(t+1)}, 2^{-t}]$ as in Fig. 7. For $t \in \mathbb{N}$, and for $\mathrm{st}^A(t) = \min\{s : t \in A_s\}$ in the proof of Theorem 3, let $l(t) \in 2^{\mathbb{N}}$ be the leftmost path of $T_{P,\mathrm{st}^A(t)}$. If $\mathrm{st}^A(t)$ is undefined (i.e., $t \notin A$) then $l(t)$ is also undefined. For each $t \in \mathbb{N}$ we define $F(t) = \{\sigma \in 2^{<\mathbb{N}} : \sigma \subseteq l(t)\}$ if $l(t)$ is defined; $F(t) = T_P$ otherwise. Then $\{F(t) : t \in \mathbb{N}\}$ is a computable sequence of $\Pi^0_1$ subsets of $2^{<\mathbb{N}}$. Furthermore, we have $\Psi(F(t)) \cap ([0,1] \times \{0\}) \neq \emptyset$, since $F(t)$ has a path for every $t \in \mathbb{N}$. For each $t \in \mathbb{N}$, $w(t)$ is defined again as in the proof of Theorem 3. Now we define a $\Pi^0_1$ dendrite $H \subseteq \mathbb{R}^2$ as follows:

$$
\begin{aligned}
H_t^* &= \Psi(F(t); w(t), 2^{-t}, t, 2^{-(t+2)}) \\
H_t^0 &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-(t+1)}] \\
H_t^{**} &= (2^{-t} - w(t), 2^{-t} + w(t)) \times \{2^{-(t+1)}\} \\
H_t^2 &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-(t+1)}) \\
H &= \Big( \bigcup_{t \in \mathbb{N}} (H_t^* \cup H_t^0 \setminus (H_t^{**} \cup int H_t^*)) \Big) \cup \Big( ([-1,1] \times \{0\}) \setminus \bigcup_{t \in \mathbb{N}} H_t^2 \Big).
\end{aligned}
$$

Put $H_t = H_t^* \setminus (H_t^{**} \cup int H_t^*)$ (see Fig. 8). We can also show that $H$ is a $\Pi^0_1$ dendrite in the same way as for Theorem 3.

**Claim.** The $\Pi^0_1$ dendrite $H$ does not $*$-include a computable dendrite.

Let $J$ be a computable subdendrite of $H$. Put $S(t) = [3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-(t+1)}, 2^{-t}]$. Then, we note that $J(t) = J \cap S(t)$ is also a computable dendrite, since $H_t \subseteq S(t)$ and it is a dendrite. However, by Lemma 10, if $t \notin A$ then we have $J(t) \cap (\mathbb{R} \times \{2^{-t}\}) = \emptyset$. So we consider the following set:

$$C = \{t \in \mathbb{N} : J(t) \cap \Big( [3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-t}, 1] \Big) = \emptyset\}.$$

Since $J(t)$ is uniformly computable in $t$, the set $C$ is clearly c.e., and we have $\mathbb{N} \setminus A \subseteq C$. However, if $\mathbb{N} \setminus A = C$, then this contradicts the incomputability of $A$. Thus, there must be infinitely many $t \in A$ such that $t$ is enumerated into $C$. However, if $t \in A$ is enumerated into $C$, it *cuts* the dendrite $H$. In other words, since $J \subseteq H$ is connected, either $J \subseteq [-1, 5 \cdot 2^{-(t+2)}] \times \mathbb{R}$ or $J \subseteq [3 \cdot 2^{-(t+2)}, 1] \times \mathbb{R}$. Hence we must have $d_H(J, H) \geq 1$. □

**Corollary 2.** *There exists a nonempty $\Pi^0_1$ subset of $[0,1]^2$ which is contractible, locally contractible, and $*$-includes no connected computable closed subset.*

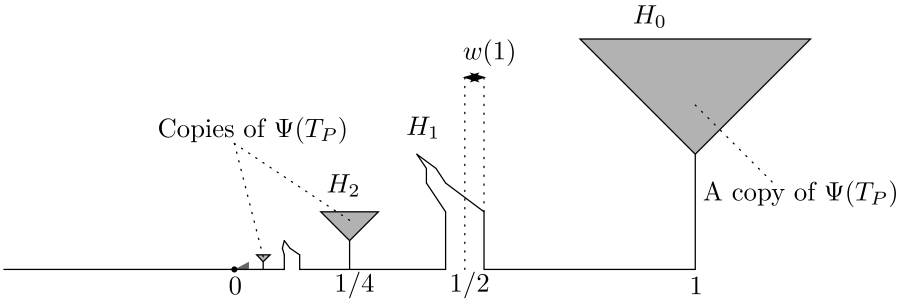


Figure 8: The dendrite $H$ for $0, 2, 4 \notin A$ and $1, 3 \in A$.

# 4 Incomputability of Dendroids

**Theorem 11.** *Not every computable planar dendroid $*$-includes a $\Pi^0_1$ dendrite.*

**Lemma 12.** *There exists a limit computable function $f$ such that, for every uniformly c.e. sequence $\{U_n : n \in \mathbb{N}\}$ of cofinite c.e. sets, we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$.*

*Proof.* Let $\{V_e : e \in \mathbb{N}\}$ be an effective enumeration of all uniformly c.e. non-increasing sequences $\{U_n : n \in \mathbb{N}\}$ of c.e. sets such that $\min U_n \geq n$, where $(V_e)_n = U_n = \{x \in \mathbb{N} : (n, x) \in V_e\}$. *The $e$-state of $y$* is defined by $\sigma(e, y) = \{i \leq e : y \in (V_i)_e\}$, and *the maximal $e$-state* is defined by $\sigma(e) = \max_z \sigma(e, z)$. The construction of $f : \mathbb{N} \to \mathbb{N}$ is to maximize the $e$-state. For each $e \in \mathbb{N}$, $f(e)$ chooses the least $y \in \mathbb{N}$ having the maximal $e$-state $\sigma(e, y) = \sigma(e)$. Since $\{\sigma(e, y) : e, y \in \mathbb{N}\}$ is uniformly c.e., and $\sigma(e, y) \in 2^e$, the function $e \mapsto \sigma(e) = \max_z \sigma(e, z)$ is total limit computable. Thus, $f$ is limit computable. It is easy to see that $\lim_e \sigma(e)(n)$ exists for each $n \in \mathbb{N}$. Let $U = \{U_n : n \in \mathbb{N}\}$ be a uniformly c.e. sequence of cofinite c.e. sets. Then $V = \{\bigcap_{m \leq n} U_m : n \in \mathbb{N}\}$ is a uniformly c.e. non-increasing sequence of cofinite c.e. sets. Thus, $V_i = V$ for some index $i$. Then $i \in \sigma(e, y)$ for almost all $e, y \in \mathbb{N}$. This ensures that $i \in \sigma(e)$ for almost all $e \in \mathbb{N}$ by our assumption $\min U_n \geq n$. Hence we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$. □

**Remark.** The proof of Lemma 12 is similar to the standard construction of a maximal c.e. set (see Soare [16]). Recall that the principal function of the complement of a maximal c.e. set is *dominant*, i.e., it dominates all total computable functions. The limit computable function $f$ in Lemma 12 is also dominant. Indeed, for any total computable function $g$, if we set $U^g_n = \{y \in \mathbb{N} : y \geq g(n)\}$ then $\{U^g_n : n \in \mathbb{N}\}$ is a uniformly c.e. sequence of cofinite c.e. sets, and if $f(n) \in U^g_n$ holds then we have $f(n) \geq g(n)$.

*Proof of Theorem 11.* Pick a limit computable function $f = \lim_s f_s$ in Lemma 12. For every $t, u \in \mathbb{N}$, put $v(t, u) = 2^{-s}$ for the least $s$ such that $f_s(t) = u$ if such $s$ exists; $v(t, u) = 0$ otherwise. Since $\{f_s : s \in \mathbb{N}\}$ is uniformly computable, $v : \mathbb{N}^2 \to \mathbb{R}$ is computable.

**Construction. .** For each $t \in \mathbb{N}$, *the center position of the $u$-th rising of the $t$-th comb* is defined as $c_*(t, u) = 2^{-(2t+1)} + 2^{-(2t+u+1)}$, and *the width of the*

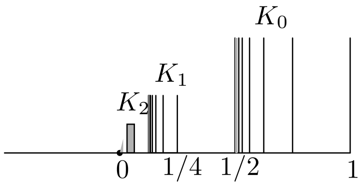


Figure 9: The dendroid $K$.

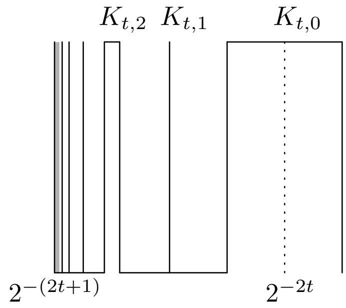


Figure 10: The harmonic comb $K_t$ for $f_0(t) = 0$, $f_1(t) = 0$, $f_2(t) = 2$, ...

*$u$-th rising of the $t$-th comb* is defined as $v_*(t,u) = v(t,u) \cdot 2^{-(2t+u+3)}$. Then, we define *the $t$-th harmonic comb $K_t$* for each $t \in \mathbb{N}$ as follows:

$$\begin{aligned}
K_t^* &= \{2^{-(2t+1)}\} \times [0, 2^{-t}] \\
K_{t,u}^0 &= \{c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)\} \times [0, 2^{-t}] \\
K_{t,u}^1 &= [c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)] \times \{2^{-t}\} \\
K_{t,u}^2 &= (c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)) \times (-1, 2^{-t}) \\
K_t &= \left( K_t^* \cup \bigcup_{i<2} \bigcup_{u \in \mathbb{N}} K_{t,u}^i \right) \cup \left( ([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{u \in \mathbb{N}} K_{t,u}^2 \right).
\end{aligned}$$

Note that $K_t$ is homeomorphic to the harmonic comb $\mathcal{H}$ for each $t \in \mathbb{N}$. This is because, for fixed $t \in \mathbb{N}$, since $\lim_s f_s(t)$ exists we have $v(t,u) = 0$ for almost all $u \in \mathbb{N}$. Then the desired computable dendroid is defined as follows.

$$K = ([-1, 0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \left( ([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t \right).$$

**Claim.** The set $K$ is a computable dendroid.

Clearly $K$ is a computable closed set. To show that $K$ is pathwise connected, we consider the following subcontinuum $K_t^-$, *the grip of the comb $K_{t,m}$*, for each $t \in \mathbb{N}$.

$$K_t^- = (\bigcup_{i<2} \bigcup_{v(t,u)>0} K_{t,u}^i) \cup \big(([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{v(t,u)>0} K_{t,u}^2\big).$$

Then $K^- = ([-1,0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \big(([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t^-\big)$ has no ramification points. We claim that $K^-$ is connected and $K^-$ is even an arc. To show this claim, we first observe that $K_t^-$ is an arc for any $t \in \mathbb{N}$, since $v(t,u) > 0$ occurs for finitely many $u \in \mathbb{N}$. Moreover $K_t^- \subseteq S(t)$, and $\lim_t \operatorname{diam}(S(t)) = 0$ holds. Therefore, we see that $K^-$ is locally connected and, hence, an arc. For points $p, q \in K$, if $p, q \in K^-$ then $p$ and $q$ are connected by a subarc of $K^-$. In the case $p \in K \setminus K^-$, the point $p$ lies on $K_{t,u}^0$ for some $t, u$ such that $v(t,u) = 0$. If $q \in K^-$ then there is a subarc $A \subseteq K^-$ (one of whose endpoints must be $\langle c_*(t,u), 0 \rangle$) such that $A \cup K_{t,u}^0$ is an arc containing $p$ and $q$. In the case $q \in K \setminus K^-$, similarly we can connect $p$ and $q$ by an arc in $K$. Hence, $K$ is pathwise connected. $K$ is hereditarily unicoherent, since the harmonic comb is hereditarily unicoherent. Thus, $K$ is a dendroid.

**Claim.** The computable dendroid $K$ does not $*$-include a $\Pi^0_1$ dendrite.

What remains to show is that every $\Pi^0_1$ subdendrite $R \subseteq K$ satisfies $d_H(R, K) \geq 1$. Let $R \subseteq K$ be a $\Pi^0_1$ dendrite. Put $S(t) = [2^{-(2t+1)}, 2^{-2t}] \times [0, 2^{-t}]$. Since $R$ is locally connected, $R \cap S(t) = R \cap K_t$ is also locally connected for each $t \in \mathbb{N}$ and $m < 2^t$. Thus, for fixed $t \in \mathbb{N}$, let $K^{1*}_{t,u} = [c_*(t, u) - 2^{-(2t+u+3)}, c_*(t, u) + 2^{-(2t+u+3)}] \times \{2^{-t}\}$. For any continuum $R^* \subset K_t$, if $R^* \cap K^{1*}_{t,u} \neq \emptyset$ for infinitely many $u \in \mathbb{N}$, then $R^*$ must be homeomorphic to the harmonic comb $\mathcal{H}$. Hence, $R^*$ is not locally connected. Therefore, we have $R \cap K^{1*}_{t,u} = \emptyset$ for almost all $u \in \mathbb{N}$. Since $K^{1*}_{t,u}$ and $K^{1*}_{s,v}$ is disjoint whenever $\langle t, u\rangle \neq \langle s, v\rangle$, and since $R$ is $\Pi^0_1$, we can effectively enumerate $U_t = \{u \in \mathbb{N} : R \cap K^{1*}_{t,u} = \emptyset\}$, i.e., $\{U_t : t \in \mathbb{N}\}$ is uniformly c.e. Moreover, $U_t$ is cofinite for every $t \in \mathbb{N}$. Then, by our definition of $f = \lim_s f_s$ in Lemma 12, there exists $t^* \in \mathbb{N}$ such that $f(t) \in U_t$ for all $t \geq t^*$. Note that $v(t, f(t)) > 0$ and thus the condition $f(t) \in U_t$ (i.e., $R \cap K^{1*}_{t,f(t)} = \emptyset$) implies that, for every $t \geq t^*$, either $R \subseteq [-1, c_*(t, u) + v_*(t, u)] \times [0, 1]$ or $R \subseteq [c_*(t, u) - v_*(t, u), 1] \times [0, 1]$ holds. Thus we obtain the desired condition $d_H(R, K) \geq 1$. ☐

**Remark.** It is easy to see that the dendroid constructed in the proof of Theorem 11 is contractible.

**Corollary 3.** *There exists a nonempty contractible planar computable closed subset of* $[0, 1]^2$ *which $*$-includes no* $\Pi^0_1$ *subset which is connected and locally connected.*

**Theorem 13.** *Not every nonempty* $\Pi^0_1$ *planar dendroid contains a computable point.*

*Proof.* One can easily construct a $\Pi^0_1$ Cantor fan $F$ containing at most one computable point $p \in F$, and such $p$ is the unique ramification point of $F$. Our basic idea is to construct a topological copy of the Cantor fan $F$ along a pathological located arc $A$ constructed by Miller [10, Example 4.1]. We can guarantee that moving the fan $F$ along the arc $A$ cannot introduce new computable points. Additionally, this moving will make a ramification point $p^*$ in a copy of $F$ incomputable.

**Fat Approximation.** To archive this construction, we consider a fat approximation of a subset $P$ of the middle third Cantor set $C \subseteq \mathbb{R}^1$, by modifying the standard construction of $C$. For a tree $T \subseteq 2^{<\mathbb{N}}$, put $\pi(\sigma) = 3^{-1} + 2\sum_{i<lh(\sigma)\ \&\ \sigma(i)=1} 3^{-(i+2)}$ for $\sigma \in T$, and $J(\sigma) = [\pi(\sigma) - 3^{-(lh(\sigma)+1)}, \pi(\sigma) + 2 \cdot 3^{-(lh(\sigma)+1)}]$. If a binary string $\sigma$ is incomparable with a binary string $\tau$ then $J(\sigma) \cap J(\tau) = \emptyset$. We extend $\pi$ to a homeomorphism $\pi_*$ between Cantor space $2^{\mathbb{N}}$ and $C \cap [1/3, 2/3]$ by defining $\pi_*(f) = 3^{-1} + 2\sum_{f(i)=1} 3^{-(i+2)}$ for $f \in 2^{\mathbb{N}}$. Let $P^* \subseteq 2^{\mathbb{N}}$ be a nonempty $\Pi^0_1$ set without computable elements. Then there exists a computable tree $T_P$ such that $P^*$ is the set of all paths of $T_P$, since $P^*$ is $\Pi^0_1$. *A fat approximation* $\{P_s : s \in \mathbb{N}\}$ *of* $P = \pi_*(P^*)$ is defined as $P_s = \bigcup\{J(\sigma) : lh(\sigma) = s\ \&\ \sigma \in T_P\}$. Then $\{P_s : s \in \mathbb{N}\}$ is a computable decreasing sequence of computable closed sets, and we have $P = \bigcap_s P_s$. Since $P$ is a nonempty bounded closed subset of a real line $\mathbb{R}^1$, both $\min P$

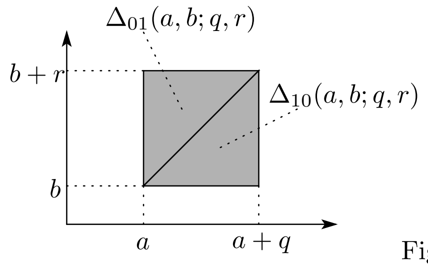


Figure 11: The cubes $\Delta_{ij}(a,b,q,r)$.

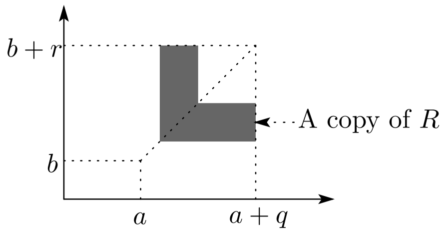


Figure 12: $[\llcorner](R;a,b;q,r)$ for $R = [1/6,1/2]$

and $\max P$ exist. By the same reason, both $l_s^- = \min P_s$ and $r_s^+ = \max P_s$ also exist, for each $s \in \mathbb{N}$, and $\lim_s l_s^- = \min P$ and $\lim_s r_s^+ = \max P$, where $\{l_s : s \in \mathbb{N}\}$ is increasing, and $\{r_s : s \in \mathbb{N}\}$ is decreasing. Let $l_s = l_s^- + 3^{-(s+1)}$ and $r_s = r_s^+ - 3^{-(s+1)}$. We also set $l_s^* = l_s^- + 3^{-(s+2)}$ and $r_s = r_s^+ - 3^{-(s+2)}$. Note that $l_s < r_s$, $\lim_s l_s = \min P$, and $\lim_s r_s = \max P$. Since $\min P, \max P \in P$ and $P$ contains no computable points, $\min P$ and $\max P$ are non-computable, and so $l_s < \min P < \max P < r_s$ holds for any $s \in \mathbb{N}$. The fat approximation of $P$ has the following remarkable property:

$$[l_s^-, l_s] \subseteq P_s,\ [l_s^-, l_s] \cap P = \emptyset,\ [r_s, r_s^+] \subseteq P_s, \text{ and } [r_s, r_s^+] \cap P = \emptyset.$$

To simplify the construction, we may also assume that $P$ has the following property:

$$P = \{1 - x \in \mathbb{R} : x \in P\}$$

Because, for any $\Pi^0_1$ subset $A \subseteq C$, the $\Pi^0_1$ set $A^* = \{x/3 : x \in A\} \cup \{1 - x/3 : x \in A\} \subseteq C$ has that property.

**Basic Notation.** For each $i, j < 2$, for each $a, b \in \mathbb{R}^2$, and for each $q, r \in \mathbb{R}$, *the* 2*-cube* $\Delta_{ij}(a,b;q,r) \subseteq [a, a+q] \times [b, b+r]$ is defined as the smallest convex set containing the three points $\{(a,b), (a+q,b), (a,b+r), (a+q,b+r)\} \setminus \{(a+(1-i)q, b+(1-j)r)\}$. Namely,

$$\begin{aligned}\Delta_{ij}(a,b;q,r) = \{&\langle(-1)^i x + a + iq, (-1)^j y + b + jr\rangle \in \mathbb{R}^2 \\ &: x, y \geq 0 \ \&\ rx + qy \leq qr\}.\end{aligned}$$

For a set $R \subseteq \mathbb{R}^1$ and real numbers $r, y \in \mathbb{R}$, put $\Theta(R; r, y) = \{rx + y \in \mathbb{R} : x \in R\}$. Clearly $\Theta(R;r,y)$ is computably homeomorphic to $R$. Let four symbols $\llcorner$, $\urcorner$, $\lrcorner$, and $\ulcorner$ denote $\langle 10, 01\rangle$, $\langle 01, 10\rangle$, $\langle 00, 11\rangle$, and $\langle 11, 00\rangle$, respectively. For $v \in \{\llcorner, \urcorner, \lrcorner, \ulcorner\}$ and for any $R \subseteq [0,1]$, $a, b \in \mathbb{R}^2$, and $q, r \in \mathbb{R}$, we define $[v](R;a,b;q,r) \subseteq [a,a+q] \times [b,b+r]$ as follows:

$$\begin{aligned}[v](R;a,b;q,r) = &\big(([a,a+q] \times \Theta(R;r,b)) \cap \Delta_{v(0)}(a,b;q,r)\big) \\ &\cup\big((\Theta(R;q,a) \times [b,b+r]) \cap \Delta_{v(1)}(a,b;q,r)\big).\end{aligned}$$

**Sublemma 1.** $[v](P;a,b;q,r)$ *is computably homeomorphic to* $P \times [0,1]$*. In particular,* $[v](P;a,b;q,r)$ *contains no computable points.*

To simplify our argument, we use the normalization $\tilde{P}_t^s$ of $P_t$ for $t \geq s$, that is defined by $\tilde{P}_t^s = \{(x - l_s^-)/(r_s^+ - l_s^-) \in \mathbb{R} : x \in P_t\}$, for each $s \in \mathbb{N}$.

Note that $\tilde{P}_t^s \subseteq [0,1]$ for $t \geq s$, and $0, 1 \in \tilde{P}_s^s$ holds for each $s \in \mathbb{N}$. Put $[v]_t^s([a, a+q] \times [b, b+r]) = [v](\tilde{P}_t^s; a, b; q, r)$ for $t \geq s$. We also introduce the following two notions:

$$[-]_t^s([a, a+q] \times [b, b+r]) = [a, a+q] \times \Theta(\tilde{P}_t^s; r, b);$$
$$[\,|\,]_t^s([a, a+q] \times [b, b+r]) = \Theta(\tilde{P}_t^s; q, a) \times [b, b+r].$$

Here we code two symbols $-$ and $|$ as 0 and 1 respectively.

**Sublemma 2.** *$[v]_t^s([a, a+q] \times [b, b+r]) \subseteq [a, a+q] \times [b, b+r]$, and $[v]_t^s([a, a+q] \times [b, b+r])$ intersects with the boundary of $[a, a+q] \times [b, b+r]$.*

**Sublemma 3.** *There is a computable homeomorphism between $[v]_t^s(a, b; q, r)$ and $P_t \times [0,1]$ for any $t \in \mathbb{N}$. Therefore, $\bigcap_t [v]_t^s(a, b; q, r)$ is computably homeomorphic to $P \times [0,1]$.*

**Blocks.** A *block* is a set $Z \subseteq \mathbb{R}^2$ with *a bounding box* $\mathrm{Box}(Z) = [a, a+q] \times [b, b+r]$. Each $\delta \in 2^2$ is called a *direction*, and directions $\langle 00 \rangle$, $\langle 01 \rangle$, $\langle 10 \rangle$, and $\langle 11 \rangle$ are also denoted by $[\leftarrow]$, $[\rightarrow]$, $[\downarrow]$, and $[\uparrow]$, respectively. For $\delta \in 2^2$, $\delta^\circ = \langle \delta(0), 1 - \delta(0) \rangle$ is called *the reverse direction of* $\delta$. Put $\mathrm{Line}(Z; [\leftarrow]) = \{a\} \times [b, b+r]$; $\mathrm{Line}(Z; [\rightarrow]) = \{a+q\} \times [b, b+r]$; $\mathrm{Line}(Z; [\downarrow]) = [a, a+q] \times \{b\}$; $\mathrm{Line}(Z; [\uparrow]) = [a, a+q] \times \{b+r\}$. Assume that a class $\mathcal{Z}$ of blocks is given. We introduce a relation $\overset{\delta}{\dashrightarrow}$ on $\mathcal{Z}$ for each direction $\delta$. Fix a block $Z_{\mathrm{first}} \in \mathcal{Z}$, and we call it *the first block*. Then we declare that $\overset{[\leftarrow]}{\dashrightarrow} Z_{\mathrm{first}}$ holds. We inductively define the relation $\overset{\delta}{\dashrightarrow}$ on $\mathcal{Z}$. If $Z \overset{\delta}{\dashrightarrow} Z_0$ (resp. $Z_0 \overset{\delta}{\dashrightarrow} Z$) for some $Z$ and $\delta$, then we also write it as $\overset{\delta}{\dashrightarrow} Z_0$ (resp. $Z_0 \overset{\delta}{\dashrightarrow}$). For any two blocks $Z_0$ and $Z_1$, the relation $Z_0 \overset{\delta}{\dashrightarrow} Z_1$ holds if the following three conditions are satisfied:

1. $Z_0 \cap Z_1 = \mathrm{Line}(Z_0; \delta) \cap Z_0 = \mathrm{Line}(Z_1; \delta^\circ) \cap Z_1 \neq \emptyset$.
2. $\overset{\varepsilon}{\dashrightarrow} Z_0$ has been already satisfied for some direction $\varepsilon$.
3. $Z_1 \overset{\varepsilon}{\dashrightarrow} Z_0$ does not satisfied for any direction $\varepsilon$

If $Z_0 \overset{\delta}{\dashrightarrow} Z_1$ for some $\delta$, then we say that $Z_1$ *is a successor of* $Z_0$ ($Z_0$ is a predecessor of $Z_1$), and we also write it as $Z_0 \dashrightarrow Z_1$.

We will construct a partial computable function $\mathcal{Z} : \mathbb{N}^3 \to \mathcal{A}(\mathbb{R}^2)$ with a computable function $k : \mathbb{N} \to \mathbb{N}$ and $\mathrm{dom}(\mathcal{Z}) = \{(u, i, t) \in \mathbb{N}^3 : u \leq t \ \&\ i < k(u)\}$ such that $\mathcal{Z}(u, i, t)$ is a block with a bounding box for any $(u, i, t) \in \mathrm{dom}(\mathcal{Z})$, and the block $\mathcal{Z}(u, i, t)$ is computably homeomorphic to $P_t \times [0,1]$ uniformly in $(u, i, t)$. Here $\mathcal{A}(\mathbb{R}^2)$ is the hyperspace of all closed subsets in $\mathbb{R}^2$ with positive and negative information. For each stage $t$, $\mathcal{Z}_t(u) = \{\mathcal{Z}(t, u, i) : i < k(u)\}$ for each $u \leq t$ is defined. Let $\mathcal{Z}(u)$ denote the finite set $\{\lambda t.\mathcal{Z}(t, u, i) : i < k(u)\}$ of functions, for each $u \in \mathbb{N}$. The relation $\dashrightarrow$ induces a pre-ordering $\prec$ on $\bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$ as follows: $Z_0 \prec Z_1$ if there is a finite path from $Z_0(t)$ to $Z_1(t)$ on the finite directed graph $(\bigcup_{u \leq t} \mathcal{Z}_t(u), \dashrightarrow)$ at some stage $t \in \mathbb{N}$. We will assure that $\prec$ is a well-ordering of order type $\omega$, and $Z_0 \prec Z_1$ whenever $Z_0 \in \mathcal{Z}(u)$, $Z_1 \in \mathcal{Z}(v)$, and $u < v$. In particular, for every $Z \in \bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$, the predecessor $Z_{\mathrm{pre}}$ of $Z$ and the successor $Z_{\mathrm{suc}}$ of $Z$ under $\prec$ are uniquely determined. If $Z_{\mathrm{pre}}(t) \overset{\delta}{\dashrightarrow} Z(t) \overset{\varepsilon}{\dashrightarrow} Z_{\mathrm{suc}}(t)$, then we say that $Z$ *moves from $\delta$ to $\varepsilon$*, and that $\langle \delta, \varepsilon \rangle$ is *the direction of $Z$*.

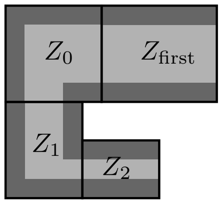


Figure 13: Example 14.

**Example 14.** Fig. 13 is an example satisfying $\overset{[\leftarrow]}{\dashrightarrow} Z_{\mathrm{first}} \overset{[\leftarrow]}{\dashrightarrow} Z_0 \overset{[\downarrow]}{\dashrightarrow} Z_1 \overset{[\rightarrow]}{\dashrightarrow} Z_2$.

**Destination Point.** Basically, our construction is similar as the construction by Miller [10]. Pick the standard homeomorphism $\rho$ between $2^{\mathbb{N}}$ and the middle third Cantor set, i.e., $\rho(M) = 2\sum_{i\in M}(1/3)^{i+1}$ for $M \subseteq \mathbb{N}$, and pick a non-computable c.e. set $B \subseteq \mathbb{N}$ and put $\gamma = \rho(B)$. We will construct a Cantor fan so that the first coordinate of the unique ramification point is $\gamma$, hence the fan will have a non-computable ramification point. Let $\{B_s : s \in \mathbb{N}\}$ be a computable enumeration of $B$, and let $n_s$ denote the element enumerated into $B$ at stage $s$, where we may assume just one element is enumerated into $B$ at each stage. Put $\gamma_s^{\min} = \rho(B_s)$ and $\gamma_s^{\max} = \rho(B_s \cup \{i \in \mathbb{N} : i \geq n_s\})$. Note that $\gamma$ is not computable, and so $\gamma_s^{\min} \neq \gamma$ and $\gamma_s^{\max} \neq \gamma$ for any $s \in \mathbb{N}$. This means that for every $s \in \mathbb{N}$ there exists $t > s$ such that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$. By this observation, without loss of generality, we can assume that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$ whenever $s \neq t$. We can also assume $1/3 \leq \gamma_s^{\min} \leq \gamma_s^{\max} \leq 2/3$ for any $s \in \mathbb{N}$.

**Stage** 0. We now start to construct a $\Pi^0_1$ Cantor fan $Q = \bigcap_{s\in\mathbb{N}} Q_s$. At the first stage 0, and for each $t \geq 0$, we define the following sets:

$$Z^{\mathrm{st}}_{0,t} = [-]^s_t([\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]);\ Z_0^{\mathrm{end}} = [\gamma_0^{\min} - 1/3, \gamma_0^{\min}] \times [l_0^-, r_0^+].$$

Moreover, we set $Q_0 = Z^{\mathrm{st}}_{0,0} \cup Z_0^{\mathrm{end}}$. By our choice of $P_0$, actually $Q_0 = [\gamma_0^{\min} - 1/3, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. $Z^{\mathrm{st}}_{0,0}$ is called *the straight block from* 2/3 *to* 1/3 *at stage* 0, and $Z_0^{\mathrm{end}}$ is called *the end box at stage* 0. The *bounding box* of the block $Z_0^{\mathrm{st}}$ is defined by $[\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. *The collection of* 0*-blocks at stage* $t$ is $\mathcal{Z}_t(0) = \{Z^{\mathrm{st}}_{0,t}\}$. We declare that $Z_0^{\mathrm{st}}$ is the first block, and that $\overset{[\leftarrow]}{\dashrightarrow} Z_0^{\mathrm{st}}$.

**Stage** $s+1$. Inductively assume that we have already constructed the collection of $u$-blocks $\mathcal{Z}_t(u)$ at stage $t \geq u$ is already defined for every $u \leq s$. For any $u$, we think of the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ as a finite set $\{Z_i^u\}_{i<\#\mathcal{Z}_u(u)}$ of computable functions $Z_i^u : \{t \in \mathbb{N} : t \geq u\} \to \bigcup_t \mathcal{Z}_t(u)$ such that $\mathcal{Z}_t(u) = \{Z_i^u(t) : i < \#\mathcal{Z}_u(u)\}$ for each $t \geq u$. We inductively assume that the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ satisfies the following conditions:

(IH1) For each $Z \in \mathcal{Z}(u)$ and for each $t \geq v \geq u$, $Z(t) \subseteq Z(v)$.

(IH2) There is a computable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \upharpoonright \bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ is a homeomorphism between $\bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ and $P_t \times [0, 1]$ for any $t \geq s$.

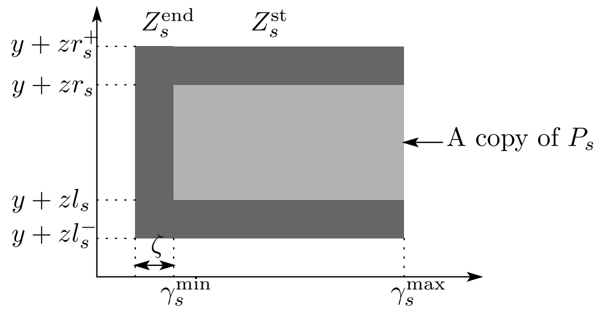


Figure 14: The active block $Z_s^{\text{st}} \cup Z_s^{\text{end}}$ at stage $s$.

(IH3) There are $y, z, \zeta \in \mathbb{Q}$ such that the blocks $Z_{s,t}^{\text{st}}$ and $Z_s^{\text{end}}$ are constructed as follows:

$$\begin{aligned} Z_{s,t}^{\text{st}} &= [-]_t^s([\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zl_s^-, y + zr_s^+]); \\ Z_s^{\text{end}} &= [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^+]. \end{aligned}$$

Here, a computable closed set $Q_s$, *an approximation of our* $\Pi_1^0$ *Cantor fan* $Q$ *at stage* $s$, is defined by $Q_s = Z_s^{\text{end}} \cup \bigcup \bigcup_{u \le s} \mathcal{Z}_s(u)$.

**Non-injured Case.** First we consider the case $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \subseteq [\gamma_s^{\min}, \gamma_s^{\max}]$, i.e., this is the case that our construction is *not injured* at stage $s+1$. In this case, we construct $(s+1)$-blocks in the active block $Z_s^{\text{st}} \cup Z_s^{\text{end}}$. We will define $Z_t(s, i, j)$ and $\text{Box}(s, i, j) = \text{Box}(Z_t(s, i, j))$ for each $j < 6$. *The first two corner blocks* at stage $t \ge s + 1$ are defined by:

$$\begin{aligned} \text{Box}(s, 0) =& [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^*], \\ Z_t(s, 0) =& [\llcorner]_t^s([\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^+]) \cap \text{Box}(s, 0), \\ \text{Box}(s, 1) =& [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zr_s^*, y + zr_s^+], \\ Z_t(s, 1) =& [\ulcorner]_t^s(\text{Box}(s, 1)). \end{aligned}$$

**Sublemma 4.** $Z_t(s, 0) \cup Z_t(s, 1) \subseteq Z_s^{\text{end}}$ *for any* $t \ge s + 1$.

**Sublemma 5.** $Z_{s,t}^{\text{st}} \overset{[\leftarrow]}{\dashrightarrow} Z_t(s, 0) \overset{[\uparrow]}{\dashrightarrow} Z_t(s, 1)$, *for any* $t \ge s + 1$.

The next block is *a straight block from* $\gamma_s^{\min}$ *to* $\gamma_{s+1}^{\max}$ which is defined as follows:

$$\begin{aligned} \text{Box}(s, 2) =& [\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zr_s^*, y + zr_s^+]. \\ Z_t(s, 2) =& [-](\text{Box}_t(s, 2)). \end{aligned}$$

For given $a, b, \alpha, \beta \in \mathbb{Q}$, we can calculate $N_{0,s}(a, b; \alpha, \beta)$ and $N_{1,s}(a, b; \alpha, \beta)$ satisfying $N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot l_s^- = a + b\alpha$, and $N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot r_s^+ = a + b\beta$. Put $y^\star = N_{0,s}(y, z; r_s^*, r_s^+)$, and $z^\star = N_{1,s}(y, z; r_s^*, r_s^+)$.

**Sublemma 6.** $\text{Box}(s, 2) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^+]$.

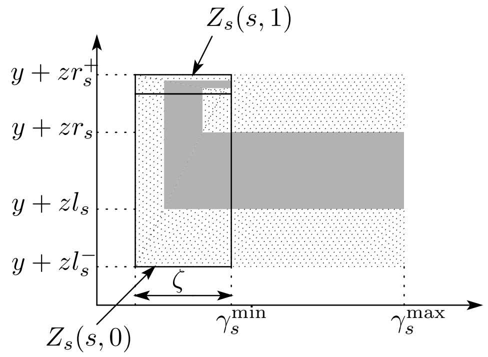


Figure 15: The first two corner blocks $Z_s(s,0)$ and $Z_s(s,1)$.

Put $\zeta^\star = (\gamma_s^{\max} - \gamma_{s+1}^{\max})/3^s$. Note that $\zeta^\star > 0$ since $\gamma_s^{\max} > \gamma_{s+1}^{\max}$. We then again define *corner blocks*.

$$\begin{aligned}
\mathrm{Box}(s,3) =& [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^*],\\
Z_t(s,3) =& [\lrcorner]_t^s([\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^+]) \cap \mathrm{Box}(s,3),\\
\mathrm{Box}(s,4) =& [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star r_s^*, y^\star + z^\star r_s^+],\\
Z_t(s,4) =& [\urcorner]_t^s(\mathrm{Box}(s,4)).
\end{aligned}$$

Next, *a straight block from $\gamma_s^{\min}$ to $\gamma_{s+1}^{\max}$* is defined as follows:

$$\begin{aligned}
\mathrm{Box}(s,5) &= [\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \times [y^\star + z^\star r_s^*, y^\star + z^\star r_s^+],\\
Z_t(s,5) &= [-]_t^s[\mathrm{Box}(s,5)].
\end{aligned}$$

Put $y^{\star\star} = N_{0,s}(y^\star, z^\star; r_s^*, r_s^+)$, and $z^{\star\star} = N_{1,s}(y^\star, z^\star; r_s^*, r_s^+)$.

**Sublemma 7.** $\mathrm{Box}(s,5) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^{\star\star} + z^{\star\star} l_s^-, y^{\star\star} + z^{\star\star} r_s^+]$.

Put $\zeta^{\star\star} = (\gamma_{s+1}^{\min} - \gamma_s^{\min})/3^s$. Note that $\zeta^{\star\star} > 0$ since $\gamma_{s+1}^{\min} > \gamma_s^{\max}$. *The end box at stage $s+1$* is:

$$Z(s,6) = [\gamma_{s+1}^{\min} - \zeta^{\star\star}, \gamma_{s+1}^{\min}] \times [y^{\star\star} + z^{\star\star} l_s^-, y^{\star\star} + z^{\star\star} r_s^+].$$

Then put $Z_{s+1,t}^{\mathrm{st}} = Z_t(s,5)$, $Z_{s+1}^{\mathrm{st}} = Z_{s+1,s+1}^{\mathrm{st}}$, and $Z_{s+1}^{\mathrm{end}} = Z(s,6)$. *The active block at stage $s+1$* is the set $Z_{s+1,s+1}^{\mathrm{st}} \cup Z_{s+1}^{\mathrm{end}}$, and *the collection of $(s+1)$-blocks at stage $t$* is defined by $\mathcal{Z}_t(s+1) = \{Z_t(s,i) : i \le 5\}$. Clearly, our definition satisfies the induction hypothesis (IH3) at stage $s+1$.

**Sublemma 8.** *$Z_t(s,i) \subseteq Z_v(s,i)$ for each $t \ge v \ge s+1$ and $i \le 5$.*

**Sublemma 9.** *For any $t \ge s+1$,*

$$Z_{s,t}^{\mathrm{st}} \overset{[\leftarrow]}{\dashrightarrow} Z_t(s,0) \overset{[\uparrow]}{\dashrightarrow} Z_t(s,1) \overset{[\rightarrow]}{\dashrightarrow} Z_t(s,2) \overset{[\rightarrow]}{\dashrightarrow} Z_t(s,3) \overset{[\uparrow]}{\dashrightarrow} Z_t(s,4) \overset{[\leftarrow]}{\dashrightarrow} Z_t(s,5).$$

*Proof.* It follows straightforwardly from the definition of these blocks $Z_t(s,i)$, and Sublemma 6 and 7. $\square$

**Sublemma 10.** $\bigcup_{2\le i\le 6} Z_t(s,i) \subseteq Z_s^{\mathrm{st}} \cap [\gamma_s^{\min}, \gamma_s^{\max}] \times (y + zr_s, y + zr_s^+]$. *Hence,* $\left(\bigcup_{2\le i\le 6} Z_t(s,i)\right) \cap Z_{s,s+1}^{\mathrm{st}} = \emptyset$

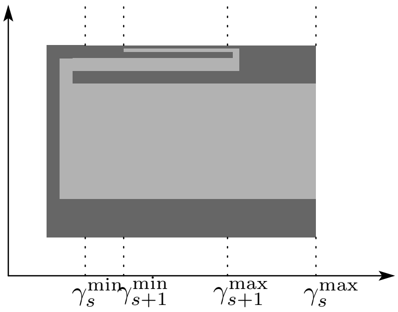


Figure 16: $Z_s(s-1,5) \cup \bigcup \mathcal{Z}_s(s+1)$.

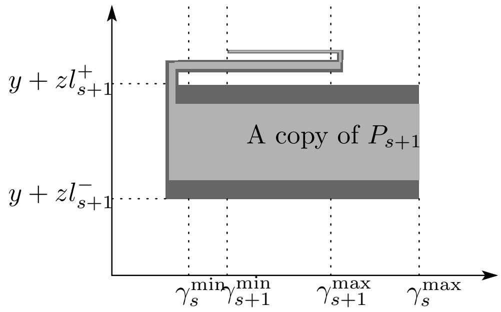


Figure 17: $Z_{s+1}(s-1,5) \cup \bigcup \mathcal{Z}_{s+1}(s+1)$.

Consequently, we can show the following extension property.

**Sublemma 11.** *Assume that we have a computable function $f_s : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_s \upharpoonright \bigcup\bigcup_{u \leq s} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+2), 1]$ for any $t \geq s$. Then we can effectively find a computable function $f_{s+1} : \mathbb{R}^2 \to \mathbb{R}^2$ extending $f_s \upharpoonright \bigcup\bigcup_{u\leq s} \mathcal{Z}_{s+1}(u)$ such that $f_{s+1} \upharpoonright \bigcup\bigcup_{u\leq s+1} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup\bigcup_{u\leq s+1} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+3), 1]$ for any $t \geq s+1$.*

*Proof.* By Sublemma 5, 9, and 10. □

By Sublemma 8 and 11, induction hypothesis (IH1) and (IH2) are satisfied. Since $Z_{s+1}^{\mathrm{end}} \cup \bigcup \mathcal{Z}_{s+1}(s+1) \subseteq Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}}$ by Sublemma 4 and 10, and $\bigcup \mathcal{Z}_{s+1}(u) \subseteq \bigcup \mathcal{Z}_s(u)$ for each $u \leq s$, by induction hypothesis (IH1), we have the following:

$$Q_{s+1} = Z_{s+1}^{\mathrm{end}} \cup \bigcup \bigcup_{u\leq s+1} \mathcal{Z}_{s+1}(u) \subseteq Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}} \cup \bigcup \bigcup_{u \leq s} \mathcal{Z}_s(u) \subseteq Q_s.$$

**Injured Case.** Secondly we consider the case that our construction *is injured*. This means that $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \not\subseteq [\gamma_s^{\min}, \gamma_s^{\max}]$. In this case, indeed, we have $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \cap [\gamma_s^{\min}, \gamma_s^{\max}] = \emptyset$. Fix the greatest stage $p \leq s$ such that $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \subseteq [\gamma_p^{\min}, \gamma_p^{\max}]$ occurs. We again, inside the end box $Z_s^{\mathrm{end}}$ at stage $s$, define corner blocks $Z_t(s,0)$ and $Z_t(s,1)$ as non-injuring stage, whereas the construction of $Z_t(s,i)$ for $i \geq 2$ differs from non-injuring stage. The end box of our construction at stage $s+1$ will turn back along all blocks belonging $\mathcal{Z}_s(u)$ for $p < u \leq s$ in the reverse ordering of $\prec$. Let $\{Z_i : i < k_{s+1}\}$ be an enumeration of all blocks in $\mathcal{Z}_s(u)$ for $p < u \leq s$, under the reverse ordering of $\prec$. In other words, $Z_i$ is the successor block of $Z_{i+1}$ under $\dashrightarrow$, for each $i < k_{s+1} - 1$. There are two kind of blocks; one is *a straight block*, and another is *a corner block*. We will define blocks $Z_t(s,i,j)$ for $i < k_{s+1}$ and $j < 3$. Now we check the direction $\langle \delta_i, \varepsilon_i \rangle$ of $Z_i$. Here, we may consistently assume that the condition $Z_0 \overset{[\leftarrow]}{\dashrightarrow}$ holds.

**Subcase 1.** If $\delta_i(0) = \varepsilon_i(0)$ then $Z_i$ is a straight block. In this case, we only construct $Z_t(s,i,0)$. Since $Z_i$ is straight, there are $y_i, z_i, \alpha, \beta \in \mathbb{Q}$ and $u \leq s$ such that, for $B_i(0) = [\alpha, \beta]$ and $B_i(1) = [y_i + z_i l_u^-, y_i + z_i r_u^+]$ such that $\mathrm{Box}(Z_i) = B_i(\delta_i(0)) \times B_i(1-\delta_i(0))$. If $\delta_i(1) = 0$, then set $y_i^\star = N_{0,s}(y_i, z_i; l_s^-, l_s^*)$ and $z_i^\star = N_{1,s}(y_i, z_i; l_s^-, l_s^*)$. If $\delta_i(1) = 1$, then set $y_i^\star = N_{0,s}(y_i, z_i; r_s^-, r_s^+)$ and

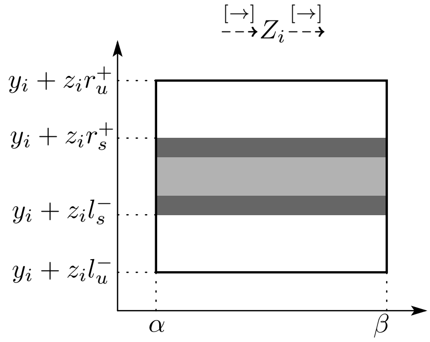


Figure 18: The block $Z_i$.

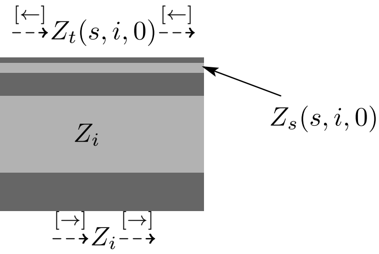


Figure 19: The block $Z_t(s,i,0)$.

$z_i^\star = N_{1,s}(y_i, z_i; r_s^*, r_s^+)$. Then, we define $Z_t(s,i,0)$ as the following straight block:

$$B_i^\star(0) = B_i(0); \quad B_i^\star(1) = [y_i^\star + z_i^\star l_s^-, y_i^\star + z_i^\star r_s^+];$$
$$Z_t(s,i,0) = [\delta_i(0)]_t^s(B_i^\star(\delta_i(0)) \times B_i^\star(1-\delta_i(0))).$$

Here, $\mathrm{Box}(Z_t(s,i,0))$ is defined by $B_i^\star(\delta_i(0)) \times B_i^\star(1-\delta_i(0))$.

**Sublemma 12.** $Z_t(s,i,0) \subseteq Z_i$.

*Proof.* By our definition of $N_{0,s}$ and $N_{1,s}$, we have $B_i^\star(1) = [y_i + z_i l_s^-, y_i + z_i l_s^*]$ or $B_i^\star(1) = [y_i + z_i r_s^*, y_i + z_i r_s^+]$. □

**Subcase 2.** If $\delta_i(0) \neq \delta_i(2)$ then $Z_i$ is a corner block. We will construct three blocks; one corner block $Z_t(s,i,0)$, and two straight blocks $Z_t(s,i,1)$ and $Z_t(s,i,2)$. We may assume that $Z_i$ is of the following form:

$$\begin{aligned} Z_i &= [e]_s^u([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+]), \\ \text{or } Z_i &= [e]_s^u([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+]) \\ &\qquad \cap ([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^*]) \end{aligned}$$

Set $z = 0$ if the former case occurs; otherwise, set $z = 1$. Let $\{p_n : n < 6\}$ be an enumeration of $\{l_u^-, l_s^-, l_s^*, r_s^*, r_s^+, r_u^+\}$ in increasing order, and let $p_6$ be $r_u^*$. First we compute the value $\mathtt{rot} = 2|\varepsilon_i(0) - |\delta_i(1) - \varepsilon_i(1)|| + 1$. Note that $\mathtt{rot} \in \{1,3\}$, and, if $Z_i$ rotates clockwise then $\mathtt{rot} = 1$; and if $Z_i$ rotates counterclockwise then $\mathtt{rot} = 3$. If $\overset{[\rightarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\rightarrow]}{\dashrightarrow}$, then put $D(0) = 1$; otherwise put $D(0) = 3$. If $\overset{[\downarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\downarrow]}{\dashrightarrow}$, then put $D(1) = 1$; otherwise put $D(1) = 3$. If $\overset{[\rightarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\leftarrow]}{\dashrightarrow}$, then put $E(0) = 0$; otherwise put $E(0) = 5 - \mathrm{rot}$. If $\overset{[\uparrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\downarrow]}{\dashrightarrow}$, then put $E(1) = 0$; otherwise put $E(1) = 5 - \mathrm{rot}$. Then we now define $Z_t(s,i,j)$ for $j < 3$ as follows:

$$\begin{aligned} \mathrm{Box}(s,i,0) &= [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}], \\ \mathrm{Box}(s,i,1) &= [x_i + \zeta_i p_{E(0)}, x_i + \zeta_i p_{E(0)+\mathtt{rot}}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}], \\ \mathrm{Box}(s,i,2) &= [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{E(1)}, y_i + z_i p_{E(1)+\mathtt{rot}+z}], \\ Z_t(s,i,0) &= [e]_t^s(\mathrm{Box}(s,i,0)), \\ Z_t(s,i,1) &= [-]_t^s(\mathrm{Box}(s,i,1)), \\ Z_t(s,i,2) &= [\,|\,]_t^s(\mathrm{Box}(s,i,2)). \end{aligned}$$

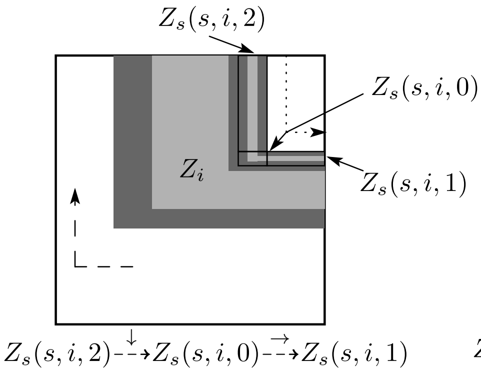

Figure 20: rot = 1.

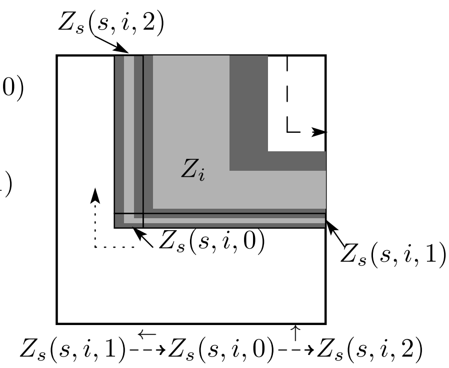

Figure 21: rot = 3.

Intuitively, $D(0) = 1$ (resp. $D(0) = 3$) indicates that $Z_t(s,i,0)$ passes the west (resp. the east) of $Z_i$; $D(1) = 1$ (resp. $D(1) = 3$) indicates that $Z_t(s,i,0)$ passes the south (resp. the north) of $Z_i$; $E(0) = 0$ (resp. $E(0) = 5 - \mathtt{rot}$) indicates that $Z_t(s,i,1)$ passes the west (resp. the east) border of the bounding box of $Z_i$; and $E(1) = 0$ (resp. $E(1) = 5 - \mathtt{rot}$) indicates that $Z_t(s,i,2)$ passes the south (resp. the north) border of the bounding box of $Z_i$. Note that the corner block $Z_t(s,i,0)$ leaves $Z_i$ on his right, and $Z_t(s,i,0)$ has the reverse direction to $Z_i$.

**Sublemma 13.** $Z_t(s,i,2-\delta_i(0)) \overset{\varepsilon^\circ}{\dashrightarrow} Z_t(s,i,0) \overset{\delta^\circ}{\dashrightarrow} Z_t(s,t,1+\delta_i(0))$.

**Sublemma 14.** $Z_t(s,i,j) \subseteq Z_i$.

For each $i < k_{s+1}$, we have already constructed $\mathcal{Z}_t(s+1;i) = \{Z_t(s,i,j) : j < 3\}$. All of these blocks constructed at the current stage are included in $Z_s^{\rm end} \cup \bigcup\bigcup_{p<u\leq s} \mathcal{Z}_s(u)$. Let $Z^0[i]$ (resp. $Z^1[i]$) be the $\prec$-least (resp. the $\prec$-greatest) element of $\{\lambda t.Z_t(s,i,j) : j < 3\}$. It is not hard to see that our construction ensures the following condition.

**Sublemma 15.** $Z_t^1[i] \dashrightarrow Z_t^0[i+1]$.

Thus, $\bigcup_{i<k_{s+1}} \mathcal{Z}_t(s+1;i)$ is computably homeomorphic to $P_t \times [0,1]$, uniformly in $t \geq s+1$. Therefore, we can connect blocks $Z_s(s,i)$ for $i < k_{s+1}$, and we succeed to return back on the current approximation of the $\prec$-greatest $p$-block $Z_s(p) = Z_{p,s}^{\rm st} \in \mathcal{Z}_s(p)$. Then we construct blocks $Z_t(s,k)$ for $2 \leq k \leq 6$ on the block $Z_s(p)$. The construction is essentially similar as the non-injuring case. By induction hypothesis (IH3), we note that $Z_s(p)$ must be of the following form for some $y_p, z_p \in \mathbb{Q}$:

$$Z_s(p) = [-]_s^p([\gamma_p^{\min}, \gamma_p^{\max}] \times [y_p + z_p l_p^-, y_p + z_p r_p^+]).$$

On $Z_s(p)$, we define *a straight block from $\gamma_p^{\min}$ to $\gamma_{s+1}^{\max}$* as follows:

$$Z_t(s,2) = [-]_s^p([\gamma_p^{\min}, \gamma_{s+1}^{\max}] \times [y_p + z_p r_s^*, y_p + z_p r_s^+]).$$

Here, by our assumption, $\gamma_{s+1}^{\max} < \gamma_p^{\max}$ holds since $\gamma_{s+1}^{\max} \leq \gamma_p^{\max}$. The blocks $Z_t(s,k)$ for $3 \leq k \leq 6$ are defined as in the same method as non-injuring case. The active block at stage $s+1$ is $Z_{s+1}(s,5)$, and the end box at stage $s+1$

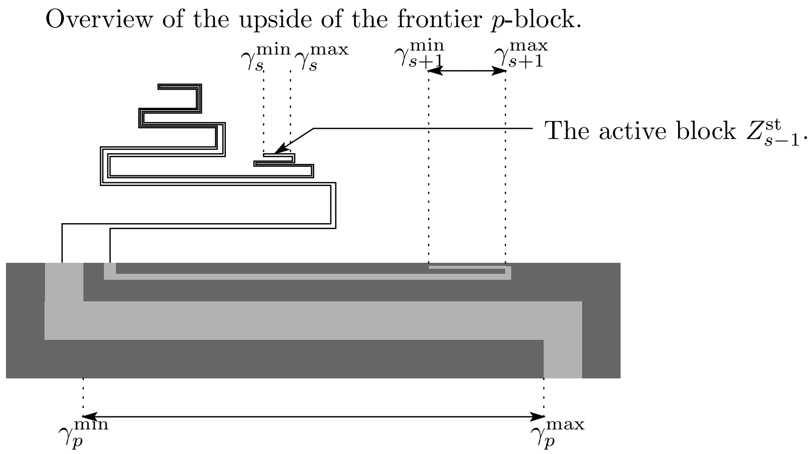


Figure 22: Outline of our construction of the injured case.

is $Z_{s+1}(s,6)$. $(s+1)$-*blocks at stage* $t$ are $Z_t(s,i)$ for $i<6$, and $Z_t(s,i,j)$ for $i<k_{s+1}$ and $j<3$ if it is constructed. $\mathcal{Z}_t(s+1)$ denotes *the collection of* $(s+1)$-*blocks at stage* $t$.

**Sublemma 16.** $Z_{s+1}^{\mathrm{end}} \cup \bigcup \mathcal{Z}_{s+1}(s+1) \subseteq Z_s^{\mathrm{end}} \cup \bigcup \bigcup_{p \le u \le s} \mathcal{Z}_s(u)$.

Thus we again have the following:

$$Q_{s+1} = Z_{s+1}^{\mathrm{end}} \cup \bigcup \bigcup_{u \le s+1} \mathcal{Z}_{s+1}(u) \subseteq Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}} \cup \bigcup \bigcup_{u \le s} \mathcal{Z}_s(u) \subseteq Q_s.$$

**Sublemma 17.** *Assume that we have a computable function* $f_s : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f_s \restriction \bigcup \bigcup_{u \le s} \mathcal{Z}_t(u)$ *is a computable homeomorphism between* $\bigcup \bigcup_{u \le s} \mathcal{Z}_t(u)$ *and* $P_t \times [1/(s+2), 1]$ *for any* $t \ge s$. *Then we can effectively find a computable function* $f_{s+1} : \mathbb{R}^2 \to \mathbb{R}^2$ *extending* $f_s \restriction \bigcup \bigcup_{u \le s} \mathcal{Z}_{s+1}(u)$ *such that* $f_{s+1} \restriction \bigcup \bigcup_{u \le s+1} \mathcal{Z}_t(u)$ *is a computable homeomorphism between* $\bigcup \bigcup_{u \le s+1} \mathcal{Z}_t(u)$ *and* $P_t \times [1/(s+3), 1]$ *for any* $t \ge s+1$.

Finally we put $Q = \bigcap_{s \in \mathbb{N}} Q_s$ and $\mathcal{Z}^* = \bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$. The construction is completed.

**Verification.** Now we start to verify our construction.

**Lemma 15.** $Q$ *is* $\Pi^0_1$.

**Sublemma 18.** $\bigcap_{t \in \mathbb{N}} \bigcup_{Z \in \mathcal{Z}^*} Z_t = \bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$.

*Proof.* The intersection $Z_s(p) \cap Z_s^i$ for $i < 2$ is included in some line segment $L_i \in \{[0,1] \times \{b\}, \{b\} \times [0,1] : b \in \mathbb{R}\}$, and $Z_s(p) \cap L_i = Z_s(p) \cap Z_s^i$ holds. □

**Sublemma 19.** $\bigcup_{Z \in \mathcal{Z}(u)} \bigcap_{t \in \mathbb{N}} Z_t$ *is computably homeomorphic to* $[0,1] \times P$, *for each* $u \in \mathbb{N}$.

*Proof.* By the induction hypothesis (IH2). □

**Sublemma 20.** $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ *is homeomorphic to* $(0,1] \times P$.

*Proof.* By Sublemma 11 and 17. □

**Lemma 16.** *$Q$ is homeomorphic to a Cantor fan.*

*Proof.* By Sublemma 18, there exists a real $y_0 \in \mathbb{R}$ such that the following holds:

$$Q = \left( \bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t \right) \cup \{\langle \gamma, y_0 \rangle\}.$$

Therefore, by Sublemma 20, $Q$ is homeomorphic to the one-point compactification of $(0, 1] \times P$. □

**Lemma 17.** *$Q$ contains no computable point.*

*Proof.* By Sublemma 19, $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ contains no computable point. □

By Lemmata 15, 16, and 17, $Q$ is the desired dendroid. □

**Remark.** Since dendroids are compact and simply connected, Theorem 13 is the solution to the question of Le Roux and Ziegler [13]. Indeed, the dendroid constructed in the proof of Theorem 13 is contractible.

**Corollary 4.** *Not every nonempty contractible $\Pi^0_1$ subset of $[0, 1]^2$ contains a computable point.*

**Question 18.** *Does every locally connected planar $\Pi^0_1$ set contain a computable point?*

# 5 Immediate Consequences

## 5.1 Effective Hausdorff Dimension

For basic definition and properties of the the effective Hausdorff dimension of a point of Euclidean plane, see Lutz-Weihrauch [9]. For any $I \subseteq [0, 2]$, let $\mathrm{DIM}^I$ denote the set of all points in $\mathbb{R}^2$ whose effective Hausdorff dimensions lie in $I$. Lutz-Weihrauch [9] showed that $\mathrm{DIM}^{[1,2]}$ is path-connected, but $\mathrm{DIM}^{(1,2]}$ is totally disconnected. In particular, $\mathrm{DIM}^{(1,2]}$ has no nondegenerate connected subset. It is easy to see that $\mathrm{DIM}^{(0,2]}$ has no nonempty $\Pi^0_1$ simple curve, since every $\Pi^0_1$ simple curve contains a computable point, and the effective Hausdorff dimension of each computable point is zero.

**Theorem 19.** $\mathrm{DIM}^{[1,2]}$ *has a nondegenerate contractible $\Pi^0_1$ subset.*

*Proof.* For any strictly increasing computable function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) = 0$ and any infinite binary sequence $\alpha \in 2^{\mathbb{N}}$, put $\iota_f(\alpha) = \prod_{i \in \mathbb{N}} \langle \alpha(i), \alpha(f(i)), \alpha(f(i)+1), \dots, \alpha(f(i+1)-1) \rangle$, where $\sigma \times \tau$ denotes the concatenation of binary strings $\sigma$ and $\tau$. Then, $r : 2^{\mathbb{N}} \to \mathbb{R}$ is defined as $r(\alpha) = \sum_{i \in \mathbb{N}} (\alpha(i) \cdot 2^{-(i+1)})$. Note that $\alpha \neq \beta$ and $r(\alpha) = r(\beta)$ hold if and only if there is a common initial segment $\sigma \in 2^{<\mathbb{N}}$ of $\alpha$ and $\beta$ such that $\sigma 0$ and $\sigma 1$ are initial segments of $\alpha$ and $\beta$ respectively, and that $\alpha(m) = 1$ and $\beta(m) = 0$ for any $m > lh(\sigma)$, where $lh(\sigma)$ denotes the length of $\sigma$. In this case, we say that $\alpha$ *sticks to* $\beta$ *on* $\sigma$. If $r(\alpha) \neq r(\beta)$, then clearly $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$. Assume that $\alpha$ sticks to $\beta$ on $\sigma$. Then there are $m_0 < m_1$ such that $\iota_f(\alpha)(m_0) = \iota_f(\alpha)(m_1) = \alpha(lh(\sigma)) = 0$ and $\iota_f(\beta)(m_0) = \iota_f(\beta)(m_1) = \beta(lh(\sigma)) = 1$ by our definition of $\iota_f$. Therefore,

$\iota_f(\alpha)$ does not stick to $\iota_f(\beta)$. Hence, $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$ whenever $\alpha \neq \beta$. Actually, $r \circ \iota_f : 2^{\mathbb{N}} \to \mathbb{R}$ is a computable embedding. For each $n \in \mathbb{N}$, put $k_f(n) = \#\{s : f(s) < n\}$. Then, there is a constant $c \in \mathbb{N}$ such that, for any $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $K(\iota_f(\alpha) \upharpoonright n + k_f(n) + 1) \geq K(\alpha \upharpoonright n) - c$, where $K$ denotes the prefix-free Kolmogorov complexity. Therefore, for any sufficiently fast-growing function $f : \mathbb{N} \to \mathbb{N}$ and any Martin-Löf random sequence $\alpha \in 2^{\mathbb{N}}$, the effective Hausdorff dimension of $r \circ \iota_f(\alpha)$ must be 1. Thus, for any nonempty $\Pi^0_1$ set $R \subseteq 2^{\mathbb{N}}$ consisting of Martin-Löf random sequences, $\{0\} \times (r \circ \iota_f(R))$ is a $\Pi^0_1$ subset of $\mathrm{DIM}^{\{1\}}$. Let $Q$ be the dendroid constructed from $P = r \circ \iota_f(R)$ as in the proof of Theorem 13, where we choose $\gamma = \rho(B)$ as Chaitin's halting probability $\Omega$. For every point $\langle x_0, x_1 \rangle \in Q$, the effective Hausdorff dimension of $x_i$ for some $i < 2$ is equivalent to that of an element of $P$ or that of $\Omega$. Consequently, $Q \subseteq \mathrm{DIM}^{[1,2]}$. □

## 5.2 Reverse Mathematics

**Theorem 20.** *For every $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$, there is a contractible planar $\Pi^0_1$ set $Q$ such that $Q$ is Turing-degree-isomorphic to $P$, i.e., $\{\deg_T(x) : x \in P\} = \{\deg_T(x) : x \in D\}$.*

*Proof.* We choose $B$ as a c.e. set of the same degree with the leftmost path of $P$. Then, the dendroid $Q$ constructed from $P$ and $B$ as in the proof of Theorem 13 is the desired one. □

A compact $\Pi^0_1$ subset $P$ of a computable topological space is *Muchnik complete* if every element of $P$ computes the set of all theorems of $T$ for some consistent complete theory $T$ containing Peano arithmetic. By Scott Basis Theorem (see Simpson [15]), $P$ is Muchnik complete if and only if $P$ is nonempty and every element of $P$ computes an element of any nonempty $\Pi^0_1$ set $Q \subseteq 2^{\mathbb{N}}$.

**Corollary 5.** *There is a Muchnik complete contractible planar $\Pi^0_1$ set.*

A compact $\Pi^0_1$ subset $P$ of a computable topological space is *Medvedev complete* (see also Simpson [15]) if there is a uniform computable procedure $\Phi$ such that, for any name $x \in \mathbb{N}^{\mathbb{N}}$ of an element of $P$, $\Phi(x)$ is the set of all theorems of $T$ for some consistent complete theory $T$ containing Peano arithmetic.

**Question 21.** *Does there exist a Medvedev complete simply connected planar $\Pi^0_1$ set? Does there exist a Medvedev complete contractible Euclidean $\Pi^0_1$ set?*

Our Theorem 13 also provides a reverse mathematical consequence. For basic notation for Reverse Mathematics, see Simpson [14]. Let $\mathsf{RCA}_0$ denote the subsystem of second order arithmetic consisting of $I\Sigma^0_1$ (Robinson arithmetic with induction for $\Sigma^0_1$ formulas) and $\Delta^0_1$-$\mathsf{CA}$ (comprehension for $\Delta^0_1$ formulas). Over $\mathsf{RCA}_0$, we say that a sequence $(B_i)_{i \in \mathbb{N}}$ of open rational balls is *flat* if there is a homeomorphism between $\bigcup_{i<n} B_i$ and the open square $(0,1)^2$ for any $n \in \mathbb{N}$. It is easy to see that $\mathsf{RCA}_0$ proves that every flat cover of $[0,1]$ has a finite subcover.

**Theorem 22.** *The following are equivalent over $\mathsf{RCA}_0$.*

1. *Weak König's Lemma: every infinite binary tree has an infinite path.*

2. *Every open cover of* $[0,1]$ *has a finite subcover.*

3. *Every flat open cover of* $[0,1]^2$ *has a finite subcover.*

*Proof.* The equivalence of the item (1) and (2) is well-known. It is not hard to see that $\mathsf{RCA}_0$ proves the existence of the sequence $\{Q_s\}_{s\in\mathbb{N}}$ as in our construction of the dendroid $Q$ in Theorem 13, by formalizing our proof in Theorem 13 in $\mathsf{RCA}_0$. Here we may assume that $\{Q_s\}_{s\in\mathbb{N}}$ is constructed from the set of all infinite paths of a given infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, and a c.e. complete set $B \subseteq \mathbb{N}$. Note that $\bigcup_{s<t}([0,1]^2 \setminus Q_s)$ does not cover $[0,1]^2$ for every $t \in \mathbb{N}$. Over $\mathsf{RCA}_0$, there is a flat sequence $\{[0,1]^2 \setminus Q_s^*\}_{s\in\mathbb{N}}$ of open rational balls such that $\bigcap_{s<t} Q_s^* \supseteq \bigcap_{s<t} Q_s$ for any $t \in \mathbb{N}$, and that an open rational ball $U$ is removed from some $Q_s^*$ if and only if an open rational ball $U$ is removed from some $Q_u$. However, if $T$ has no infinite path, then $Q$ has no element. In other words, $\{[0,1]^2 \setminus Q_s^*\}_{s\in\mathbb{N}}$ covers $[0,1]^2$. $\square$

**Acknowledgment. .** The author thank Douglas Cenzer, Kojiro Higuchi and Sam Sandars for valuable comments and helpful discussion.

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