Hyperprojective Hierarchy of qcb₀-Spaces

Matthias Schröder *
Kurt Gödel Research Center, University of Vienna
Austria
and
Victor Selivanov[†]
A.P. Ershov Institute of Informatics Systems SB RAS
Novosibirsk, Russia

Abstract

We extend the Luzin hierarchy of qcb_0 -spaces introduced in [ScS13] to all countable ordinals, obtaining in this way the hyperprojective hierarchy of qcb_0 -spaces. We generalize all main results of [ScS13] to this larger hierarchy. In particular, we extend the Kleene-Kreisel continuous functionals of finite types to the continuous functionals of countable types and relate them to the new hierarchy. We show that the category of hyperprojective qcb_0 -spaces has much better closure properties than the category of projective qcb_0 -space. As a result, there are natural examples of spaces that are hyperprojective but not projective.

Key words. Hyperprojective hierarchy, qcb₀-space, continuous functionals of countable types, cartesian closed category.

This is an extended version of the conference paper [ScS14].

1 Introduction

A basic notion of Computable Analysis [We00] is the notion of an admissible representation of a topological space X. This is a partial continuous surjection δ from the Baire space \mathcal{N} onto X satisfying a certain universality property (see Subsection 2.3 for some more details). Such a representation of X usually induces a reasonable computability theory on X, and the class of admissibly represented spaces is wide enough to include most spaces of interest for Analysis or Numerical Mathematics. As shown by the first author [Sch03], this class coincides with the class of the so-called qcb₀-spaces, i.e. T_0 -spaces which are quotients of countably based spaces, and it forms a cartesian closed category (with the continuous functions as morphisms). Thus, among qcb₀-spaces one meets many important function spaces including the continuous functionals of finite types [Kl59, Kr59] interesting for several branches of logic and computability theory.

Along with the mentioned nice properties of qcb_0 -spaces, this class seems to be too broad to admit a deep understanding. Hence, it makes sense to search for natural subclasses of this class which still include "practically" important spaces but are (hopefully) easier to study. Interesting examples of such subclasses are obtained if we consider, for each level Γ of the classical Borel or Luzin (projective) hierarchies of Descriptive Set Theory [Ke95], the class of spaces which have

^{*}Supported by the FWF project "Definability and computability".

[†]Supported by the DFG Mercator professorship at the University of Würzburg, by the RFBR-FWF project "Definability and computability", and by the RFBR project 13-01-00015.

an admissible representation of the complexity Γ (below we make this precise). A study of the resulting Borel and Luzin hierarchies of qcb₀-spaces was undertaken in [ScS13]. In particular, it was shown that the Luzin hierarchy of qcb₀-spaces is closely related to the Kleene-Kreisel continuous functionals of finite types, and that the category of projective qcb₀-spaces is cartesian closed.

However, the class of projective qcb₀-spaces is in a sense too restricted. In particular, it is not closed under some natural constructions (e.g., countable products and coproducts) and does not contain some spaces of interest for Computable Analysis.

In this paper we extend the Luzin hierarchy of qcb₀-spaces to all countable ordinals, obtaining in this way the hyperprojective hierarchy of qcb₀-spaces. We generalize to this larger hierarchy all main results of [ScS13] concerning the Luzin hierarchy. In particular, we extend the Kleene-Kreisel continuous functionals of finite types to the continuous functionals of countable types and relate them to the new hierarchy. We show that the category of hyperprojective qcb₀-spaces has much better closure properties than the category of projective qcb₀-space. As a result, there are natural examples of spaces that are hyperprojective but not projective.

After recalling some notions and known facts in the next section, we summarize some basic facts on the hyperprojective hierarchy of sets in Section 3. In Section 4 we study the hyperprojective hierarchy of qcb₀-spaces, in particular we show that the category of hyperprojective qcb₀-spaces is closed under countable limits and countable colimits and function spaces. In Section 5 we introduce and study the continuous functionals of countable types and relate them in Section 6 to the hyperprojective hierarchy of qcb₀-spaces. In Section 7 we establish some properties of categories of hyperprojective qcb₀-spaces. Finally, in Section 8 we provide additional natural examples of hyperprojective qcb₀-spaces.

2 Notation and preliminaries

2.1 Notation

We freely use the standard set-theoretic notation like dom(f), rng(f) and graph(f) for the domain, range and graph of a function f, respectively, $X \times Y$ for the Cartesian product, $X \oplus Y$ for the disjoint union of sets X and Y, Y^X for the set of functions $f \colon X \to Y$ (but in the case when X, Y are qcb₀-spaces we use the same notation to denote the set of continuous functions from X to Y), and P(X) for the set of all subsets of X. For $A \subseteq X$, \overline{A} denotes the complement $X \setminus A$ of A in X. We identify the set of natural numbers with the first infinite ordinal ω . The first uncountable ordinal is denoted by ω_1 . The notation $f \colon X \to Y$ means that f is a (total) function from a set X to a set Y.

2.2 Topological spaces

We assume the reader to be familiar with the basic notions of topology. The collection of all open subsets of a topological space X (i.e. the topology of X) is denoted by τ_X ; for the underlying set of X we will write X in abuse of notation. We will usually abbreviate "topological space" to "space". Remember that a space is zero-dimensional, if it has a basis of clopen sets. A basis for the topology on X is a set \mathcal{B} of open subsets of X such that for every $x \in X$ and open U containing x, there is $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$. A space is countably based, if it has a countable basis. By a cb₀-space we mean a countably based T_0 -space. The class of cb_0 -spaces is denoted by CB_0 . We write $X \cong Y$, if X and Y are homeomorphic.

A space Y is called a *(continuous) retract* of a space X if there are continuous functions $s: Y \to X$ and $r: X \to Y$ such that composition rs coincides with the identity function id_Y

on Y. Such a pair of functions (s, r) is called a section-retraction pair. Note that the section s is a homeomorphism between Y and the subspace $s(Y) = \{x \in X \mid sr(x) = x\}$ of X, and $s^{-1} = r|_{s(Y)}$.

Let ω be the space of non-negative integers with the discrete topology. Of course, the spaces $\omega \times \omega = \omega^2$, and $\omega \oplus \omega$ are homeomorphic to ω , the first homeomorphism is realized by the Cantor pairing function $\langle \cdot, \cdot \rangle$.

Let $\mathcal{N} = \omega^{\omega}$ be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi \colon \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω , including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ . By $\sigma \xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ , and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in \mathcal{N} . For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\dots$ where $x(i) \in \omega$ for each $i \in \omega$. For $x \in \mathcal{N}$ and $n \in \omega$, let $x^{< n} = x(0)\dots x(n-1)$ denote the initial segment of x of length x.

By endowing \mathcal{N} with the product of the discrete topologies on ω , we obtain the so-called *Baire space*. The product topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. The Baire space is of primary importance for Descriptive Set Theory and Computable Analysis. The importance stems from the fact that many countable objects are coded straightforwardly by elements of \mathcal{N} , and it has very specific topological properties. In particular, it is a perfect zero-dimensional space, and the spaces \mathcal{N}^2 , \mathcal{N}^ω , $\omega \times \mathcal{N} = \mathcal{N} \oplus \mathcal{N} \oplus \dots$ (endowed with the product topology) are all homeomorphic to \mathcal{N} . Let $(x,y) \mapsto \langle x,y \rangle$ be a homeomorphism between \mathcal{N}^2 and \mathcal{N} . Let $(x_0, x_1, \dots) \mapsto \langle x_0, x_1, \dots \rangle$ be the homeomorphism between \mathcal{N}^ω and \mathcal{N} defined by $\langle x_0, x_1, \dots \rangle \langle m, n \rangle = x_m(n)$.

The space $P\omega$ is formed by the set of subsets of ω equipped with the Scott topology, the basic open sets of which are the sets $\{A \subseteq \omega \mid F \subseteq A\}$, where F ranges over the finite subsets of ω . It has the following well-known universality property:

Proposition 2.1 A topological space X embeds into $P\omega$ iff X is a cb_0 -space.

Remember that a space X is Polish, if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space. Important examples of Polish spaces are ω , \mathcal{N} , the space of reals \mathbb{R} and its Cartesian powers \mathbb{R}^n $(n < \omega)$, the closed unit interval [0, 1], the Hilbert cube $[0, 1]^{\omega}$ and the Hilbert space \mathbb{R}^{ω} . Simple examples of non-Polish spaces are the *Sierpinski* $space \mathbb{S} = \{\bot, \top\}$, where the set $\{\top\}$ is open but not closed, and the space of rationals.

2.3 Admissible representations and qcb₀-spaces

A representation of a space X is a surjection of a subspace of the Baire space \mathcal{N} onto X. A representation δ of X is admissible, if it is continuous and any continuous function $\nu: Z \to X$ from a subspace Z of \mathcal{N} to X is continuously reducible to δ , i.e. $\nu = \delta g$ for some continuous function $g: Z \to \mathcal{N}$. A topological space is admissibly representable if it has an admissible representation.

The notion of admissibility was introduced in [KW85] for representations of countably based spaces (in a different but equivalent formulation) and was extensively studied by many authors. In [Sch02, Sch03] the notion was extended to non-countably based spaces and a nice characterization of the admissibly represented spaces was achieved. Namely, the admissibly represented sequential topological spaces coincide with the qcb₀-spaces. Spaces which arise as topological quotients of countably based spaces are called qcb-spaces, and qcb-spaces that have the T_0 -property are called qcb₀-spaces.

The category QCB of qcb-spaces as objects and continuous functions as morphisms is known to be cartesian closed (cf. [ELS04, Sch03]). The same is true for its full subcategory QCB₀ of

qcb₀-spaces. The exponential Y^X to qcb-spaces X,Y has the set of continuous functions from X to Y as the underlying set, and its topology is the sequentialization of the compact-open topology on Y^X . By the sequentialization of a topology τ we mean the family of all sequentially open sets pertaining to this topology. (Remember that sequentially open sets are defined to be the complements of the sets that are closed under forming limits of converging sequences.) The sequentialization of τ is finer than or equal to τ . The topology of the QCB-product to X and Y, which we denote by $X \times Y$, is the sequentialization of the well-known Tychonoff topology on the cartesian product of the underlying sets of X and Y. If X and Y are additionally T_0 , then $X \times Y$ and Y^X are T_0 as well. So products and exponentials in QCB and in QCB₀ are formed in the same way as in its supercategory Seq of sequential topological spaces.

We will also use the following well-known facts (see e.g. [Sch03, We00]).

Proposition 2.2 There is a partial continuous function $u :\subseteq \mathcal{N}^2 \to \mathcal{N}$ such that $dom(u) \in \Pi_2^0(\mathcal{N}^2)$, for any partial continuous function g on \mathcal{N} there is some $p \in \mathcal{N}$ such that $u_p := \lambda x.u(p,x)$ is an extension of g, and for any partial continuous function $G :\subseteq \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ there is a total continuous function g on \mathcal{N} such that u(g(p),q) = F(p,q) for all $(p,q) \in dom(G)$.

We describe the construction of canonical admissible representations for products and function spaces formed in QCB₀ (cf. [Sch03]).

Proposition 2.3 Let δ and γ be admissible representations for qcb₀-spaces X and Y, respectively. Then canonical admissible representations $[\delta \times \gamma]$ for the QCB₀-product $X \times Y$ and $[\gamma^{\delta}]$ for the QCB₀-exponential Y^X can be defined by:

$$[\delta \times \gamma](\langle p, q \rangle) = (x, y) \text{ iff } \delta(p) = x \wedge \gamma(q) = y \text{ and } [\gamma^{\delta}](p) = f \text{ iff } f \delta = \gamma u_p|_{dom(\delta)}.$$

for $p, q \in \mathcal{N}$, $x \in X$, $y \in Y$, and $f: X \to Y$.

Along with the binary product, the category QCB₀ is closed under countable product. The countable product of a sequence X_0, X_1, \ldots of qcb₀-spaces is denoted by $\prod_{n \in \omega} X_n$. If δ_n is an admissible representation of X_n for each n, the induced admissible representation of $\prod_{n \in \omega} X_n$ is denoted by $\prod_{n \in \omega} \delta_n$. The category QCB₀ is also closed under other constructions including countable limits and colimits. We will discuss some such constructions in Section 4 where we also recall some relevant notions of Category Theory.

For any qcb₀-space X, let $\mathcal{O}(X)$ be the hyperspace of open sets in X endowed with the Scott topology. The space $\mathcal{O}(X)$ is well known to be homeomorphic to the function space \mathbb{S}^X , where \mathbb{S} is the Sierpinski space.

3 Hyperprojective hierarchy of sets

Here we recall some facts on hierarchies in arbitrary spaces, with the emphasis on the hyperprojective hierarchy in the Baire space. Additional information on the hyperprojective hierarchy may be found in [Ke83].

A pointclass on X is simply a collection $\Gamma(X)$ of subsets of X. A family of pointclasses [Se13] is a family $\Gamma = \{\Gamma(X)\}$ indexed by arbitrary topological spaces X such that each $\Gamma(X)$ is a pointclass on X and Γ is closed under continuous preimages, i.e. $f^{-1}(A) \in \Gamma(X)$ for every $A \in \Gamma(Y)$ and every continuous function $f: X \to Y$. A basic example of a family of pointclasses is given by the family $\mathcal{O} = \{\tau_X\}$ of the topologies of all the spaces X.

We will use some operations on families of pointclasses. First, the usual set-theoretic operations will be applied to the families of pointclasses pointwise: for example, the union $\bigcup_i \Gamma_i$ of the families of pointclasses $\Gamma_0, \Gamma_1, \ldots$ is defined by $(\bigcup_i \Gamma_i)(X) = \bigcup_i \Gamma_i(X)$.

Second, a large class of such operations is induced by the set-theoretic operations of L.V. Kantorovich and E.M. Livenson (see e.g. [Se13] for the general definition). Among them are the operation $\Gamma \mapsto \Gamma_{\sigma}$, where $\Gamma(X)_{\sigma}$ is the set of all countable unions of sets in $\Gamma(X)$, the operation $\Gamma \mapsto \Gamma_{\delta}$, where $\Gamma(X)_{\delta}$ is the set of all countable intersections of sets in $\Gamma(X)$, the operation $\Gamma \mapsto \Gamma_{c}$, where $\Gamma(X)_{c}$ is the set of all complements of sets in $\Gamma(X)$, the operation $\Gamma \mapsto \Gamma_{d}$, where $\Gamma(X)_{d}$ is the set of all differences of sets in $\Gamma(X)$, the operation $\Gamma \mapsto \Gamma_{\exists}$ defined by $\Gamma_{\exists}(X) := \{\exists^{\mathcal{N}}(A) \mid A \in \Gamma(\mathcal{N} \times X)\}$, where $\exists^{\mathcal{N}}(A) := \{x \in X \mid \exists p \in \mathcal{N}.(p, x) \in A\}$ is the projection of $A \subseteq \mathcal{N} \times X$ along the axis \mathcal{N} , and finally the operation $\Gamma \mapsto \Gamma_{\forall}$ defined by $\Gamma_{\forall}(X) := \{\forall^{\mathcal{N}}(A) \mid A \in \Gamma(\mathcal{N} \times X)\}$, where $\forall^{\mathcal{N}}(A) := \{x \in X \mid \forall p \in \mathcal{N}.(p, x) \in A\}$.

The operations on families of pointclasses enable to provide short uniform descriptions of the classical hierarchies in arbitrary spaces. E.g., the Borel hierarchy is the family of pointclasses $\{\Sigma_{\alpha}^{0}\}_{\alpha<\omega_{1}}$ defined by induction on α as follows [Se06, Br13]:

$$\Sigma_0^0(X) := \{\emptyset\}, \Sigma_1^0 := \mathcal{O}, \Sigma_2^0 := (\Sigma_1^0)_{d\sigma}, \text{ and } \Sigma_{\alpha}^0(X) := (\bigcup_{\beta < \alpha} \Sigma_{\beta}^0(X))_{c\sigma}$$

for $\alpha > 2$. The sequence $\{\mathbf{\Sigma}_{\alpha}^{0}(X)\}_{\alpha < \omega_{1}}$ is called the Borel hierarchy in X. We also let $\mathbf{\Pi}_{\beta}^{0}(X) := (\mathbf{\Sigma}_{\beta}^{0}(X))_{c}$ and $\mathbf{\Delta}_{\alpha}^{0}(X) := \mathbf{\Sigma}_{\alpha}^{0}(X) \cap \mathbf{\Pi}_{\alpha}^{0}(X)$. The classes $\mathbf{\Sigma}_{\alpha}^{0}(X), \mathbf{\Pi}_{\alpha}^{0}(X), \mathbf{\Delta}_{\alpha}^{0}(X)$ are called the levels of the Borel hierarchy in X.

For this paper, the hyperprojective hierarchy is of main interest.

Definition 3.1 The hyperprojective hierarchy is the family of pointclasses $\{\Sigma_{\alpha}^1\}_{\alpha<\omega_1}$ defined by induction on α as follows: $\Sigma_0^1 = \Sigma_2^0$, $\Sigma_{\alpha+1}^1 = (\Sigma_{\alpha}^1)_{c\exists}$, $\Sigma_{\lambda}^1 = (\Sigma_{<\lambda}^1)_{\delta\exists}$, where $\alpha, \lambda < \omega_1$, λ is a limit ordinal, and $\Sigma_{<\lambda}^1(X) := \bigcup_{\alpha<\lambda} \Sigma_{\alpha}^1(X)$.

In this way, we obtain for any topological space X the sequence $\{\Sigma^1_{\alpha}(X)\}_{\alpha<\omega_1}$, which we call here the hyperprojective hierarchy in X. The pointclasses $\Sigma^1_{\alpha}(X)$, $\Pi^1_{\alpha}(X) := (\Sigma^1_{\alpha}(X))_c$ and $\Delta^1_{\alpha}(X) := \Sigma^1_{\alpha}(X) \cap \Pi^1_{\alpha}(X)$ are called levels of the hyperprojective hierarchy in X. The finite non-zero levels of the hyperprojective hierarchy coincide with the corresponding levels of the Luzin's projective hierarchy [Br13, ScS13]. The class of hyperprojective sets in X is defined as the union of all levels of the hyperprojective hierarchy in X.

- **Remarks 3.2** (1) If X is Polish then one can equivalently take $\Sigma_0^1 = \Sigma_1^0$ in the definition of the hyperprojective hierarchy and obtain the same non-zero levels as above. For non-Polish spaces our definition guarantees the "right" inclusions of the levels, as the first item of the next proposition states.
 - (2) In the case of Polish spaces our "hyperprojective hierarchy" is in fact an initial segment of the hyperprojective hierarchy from [Ke83], so " ω_1 -hyperprojective" would be more precise name for our hierarchy; nevertheless, we prefer to use the easier term "hyperprojective" for our hierarchy.
 - (3) In the literature one can find two slightly different definitions of hyperprojective hierarchy. Our definition corresponds to that in [Ke83]. The other one (see e.g. Exercise 39.18 in [Ke95]) differs from ours only for limit levels, namely it takes $(\Sigma^1_{<\lambda})_{\sigma}$ instead of our Σ^1_{λ} . Our choice simplifies several formulations in Sections 5 and 7. The classes $(\Sigma^1_{<\lambda})_{\sigma}$ and their duals are also of interest for this paper, they are used in Sections 4 and 5.

The next assertion collects some properties of the hyperprojective hierarchy. They are proved just in the same way as for the classical projective hierarchy in Polish spaces [Ke95], we omit the corresponding details.

Proposition 3.3 (1) For any $\alpha < \beta < \omega_1$, $\Sigma^1_{\alpha} \cup \Pi^1_{\alpha} \subseteq \Delta^1_{\beta}$.

- (2) For any limit countable ordinal λ , $\Sigma^1_{<\lambda} = \Pi^1_{<\lambda}$ and $(\Sigma^1_{<\lambda})_{\delta} = (\Pi^1_{<\lambda})_{\sigma c}$.
- (3) For any non-zero $\alpha < \omega_1$, $\Sigma_{\alpha}^1 = (\Sigma_{\alpha}^1)_{\sigma} = (\Sigma_{\alpha}^1)_{\delta} = (\Sigma_{\alpha}^1)_{\exists}$. In particular, the class $\Sigma_{\alpha}^1(\mathcal{N})$ is closed under countable unions, countable intersections, continuous images, and continuous preimages of functions with a Π_2^0 -domain.
- (4) For any non-zero $\alpha < \omega_1$, $\Pi^1_{\alpha} = (\Pi^1_{\alpha})_{\sigma} = (\Pi^1_{\alpha})_{\delta} = (\Pi^1_{\alpha})_{\forall}$. In particular, the class $\Pi^1_{\alpha}(\mathcal{N})$ is closed under countable unions and countable intersections, and continuous preimages of functions with a Π^0_2 -domain.
- (5) For any limit countable ordinal λ , the family of pintclasses $(\Sigma^1_{<\lambda})_{\sigma}$ is closed under continuous preimages and the operations σ of countable union, \exists of projection along $\mathcal N$ and finite intersection but not under the operation δ of countable intersection.
- (6) For any uncountable Polish space (and also for any uncountable quasi-Polish space [Br13]) X, the hyperprojective hierarchy in X does not collapse, i.e. $\Sigma^1_{\alpha}(X) \not\subseteq \Pi^1_{\alpha}(X)$ for each $\alpha < \omega_1$.

4 Hyperprojective hierarchy of qcb₀-spaces

Here we discuss the hyperprojective hierarchy of qcb₀-spaces. In particular we extend all results from [ScS13] concerning the Luzin's projective hierarchy of qcb₀-spaces.

For any representation δ of a space X, let $EQ(\delta) := \{(p,q) \in \mathcal{N}^2 \mid p,q \in dom(\delta) \land \delta(p) = \delta(q)\}$. Let Γ be a family of pointclasses. A topological space X is called Γ -representable, if X has an admissible representation δ with $EQ(\delta) \in \Gamma(\mathcal{N}^2)$. The class of all Γ -representable spaces is denoted $QCB_0(\Gamma)$. This notion from [ScS13] enables to transfer hierarchies of sets to the corresponding hierarchies of qcb₀-spaces. In particular, we arrive at the following definition.

Definition 4.1 The sequence $\{\mathsf{QCB}_0(\Sigma_\alpha^1)\}_{\alpha<\omega_1}$ is called the *hyperprojective hierarchy* of qcb₀-spaces. By *levels* of this hierarchy we mean the classes $\mathsf{QCB}_0(\Sigma_\alpha^1)$ as well as the classes $\mathsf{QCB}_0(\Pi_\alpha^1)$ and $\mathsf{QCB}_0(\Delta_\alpha^1)$.

The next assertion summarizes extensions of the corresponding results from [ScS13] about the Luzin hierarchy. They are proved just in the same way as in [ScS13].

Proposition 4.2 (1) Let $\Gamma \in \{\Sigma_{\alpha}^1, \Pi_{\alpha}^1 \mid 0 \leq \alpha < \omega_1\}$ and let X be a Hausdorff space. Then X is Γ -representable, if X has an admissible representation δ with $dom(\delta) \in \Gamma(\mathcal{N})$.

- (2) Let $\Gamma \in {\{\Sigma_{\alpha}^1, \Pi_{\alpha}^1 \mid 0 \leq \alpha < \omega_1\}}$. Then any continuous retract of a Γ -representable space is a Γ -representable space.
- (3) The hyperprojective hierarchy of qcb₀-spaces does not collapse, more precisely, QCB₀(Σ_{α}^{1}) \nsubseteq QCB₀(Π_{α}^{1}) for each $\alpha < \omega_{1}$.
- (4) For any $\Gamma \in {\{\Sigma_{\alpha}^1, \Pi_{\alpha}^1 \mid 1 \leq \alpha < \omega_1\}}$, we have $QCB_0(\Gamma) \cap CB_0 = CB_0(\Gamma)$, where $CB_0(\Gamma)$ is the class of spaces homeomorphic to a Γ -subspace of $P\omega$.

Now we establish some closure properties of the hyperprojective hierarchy of qcb₀-spaces. We begin with exponentiation in QCB_0 . The next proposition extends and improves Theorem 7.1 in [ScS13].

Proposition 4.3 Let $1 \leq \alpha < \omega_1$, $X \in \mathsf{QCB}_0(\Sigma^1_\alpha)$ and $Y \in \mathsf{QCB}_0(\Pi^1_\alpha)$. Then $Y^X \in \mathsf{QCB}_0(\Pi^1_\alpha)$.

Proof. Let δ be a quotient representation of a qcb-space X with $EQ(\delta) \in \Sigma^1_{\alpha}(\mathcal{N}^2)$, and let γ be an admissible representation of a qcb₀-space Y with $EQ(\gamma) \in \Pi^1_{\alpha}(\mathcal{N}^2)$. By Proposition 4.2.5 in [Sch03], γ^{δ} is an admissible representation of Y^X . Using the universal function u from Section 2.3, we define the subset $M \subseteq \mathcal{N}^4$ by

$$M:=\Big\{(p_1,p_2,q_1,q_2)\in\mathcal{N}^4\,\Big|\,(p_1,q_1),(p_1,q_2),(p_2,q_1),(p_2,q_2)\in dom(u)\ \text{ and }$$

$$(u(p_1,q_1),u(p_1,q_2)),(u(p_1,q_1),u(p_2,q_1)),(u(p_2,q_1),u(p_2,q_2))\in EQ(\gamma)\Big\}.$$

By the definition of γ^{δ} we have for all $p_1, p_2 \in \mathcal{N}$

$$(p_1, p_2) \in EQ(\gamma^{\delta}) \iff \forall (q_1, q_2) \in EQ(\delta). (p_1, p_2, q_1, q_2) \in M$$

 $\iff \forall (q_1, q_2) \in \mathcal{N}^2. ((q_1, q_2) \notin EQ(\delta) \lor (p_1, p_2, q_1, q_2) \in M).$

Since Π^1_{α} is closed under forming preimages of partial continuous functions with a Π^0_2 -domain, M is in $\Pi^1_{\alpha}(\mathcal{N}^4)$. Thus $EQ(\gamma^\delta) \in \Pi^1_{\alpha}(\mathcal{N}^2)$, because $\Pi^1_{\alpha} = (\Pi^1_{\alpha})_{\forall} = (\Pi^1_{\alpha})_{\sigma}$ (see Proposition 3.3). Hence $Y^X \in \mathsf{QCB}_0(\Pi^1_{\alpha})$.

The next proposition provides some complexity bounds on products and co-products formed in QCB_0 .

Proposition 4.4 (1) Any non-zero level of the hyperprojective hierarchy of qcb₀-spaces is closed under countable QCB₀-products and coproducts.

(2) Let λ be a countable limit ordinal. Let $\{X_k\}_{k\in\omega}$ be a sequence of qcb₀-spaces such that $X_k \in \mathsf{QCB}_0(\Sigma^1_{<\lambda})$ for all k. Then the QCB_0 -product $\prod_{k\in\omega} X_k$ is in $\mathsf{QCB}_0((\Sigma^1_{<\lambda})_\delta)$ and the co-product $\bigoplus_{k\in\omega} X_k$ is in $\mathsf{QCB}_0((\Sigma^1_{<\lambda})_\sigma)$.

Proof. Let $\gamma_k : D_k \to X_k$ be an admissible representation of X_k for every $k \in \omega$. By [Sch03], admissible representations $\gamma = \prod_k \gamma_k : D \to \prod_{k \in \omega} X_k$ and $\phi : E \to \bigoplus_{k \in \omega} X_k$ for the product and the co-product can be defined by

$$\gamma(p) := (\gamma_0(\pi_0(p)), \gamma_1(\pi_1(p)), \gamma_2(\pi_2(p)), \dots)$$
 and $\phi(q) := (q(0), \gamma_{q(0)}(t(q)))$

for all $p \in D := \bigcap_{k \in \omega} \{ p \in \mathcal{N} \mid \pi_k(p) \in D_k \}$ and $q \in E := \bigcup_{k \in \omega} \{ q \in \mathcal{N} \mid q(0) = k, t(q) \in D_k \}$. Here π_k denotes the continuous k-th projection of the inverse of the homeomorphism $\langle . \rangle : \mathcal{N}^{\omega} \to \mathcal{N}$ from Section 2.2, and t(q) denotes the sequence $q(1)q(2)q(3) \ldots \in \mathcal{N}$. We obtain:

$$EQ(\gamma) = \bigcap_{k \in \omega} (\pi_k \times \pi_k)^{-1} [EQ(\gamma_k)],$$

$$EQ(\phi) = \bigcup_{k \in \omega} ((t \times t)^{-1} [EQ(\gamma_k)] \cap \{(p, q) \in \mathcal{N}^2 \mid p(0) = q(0) = k\}).$$

The claims follow from these equations and Proposition 3.3.

For equalizers we have the following result.

Proposition 4.5 Let α be a non-zero countable ordinal. Then $QCB_0(\Sigma^1_{\alpha})$ and $QCB_0(\Pi^1_{\alpha})$ are closed under forming equalizers.

Proof. Let $\Gamma \in \{\Sigma_{\alpha}^1, \Pi_{\alpha}^1\}$. Let $A, X \in \mathsf{QCB}_0(\Gamma)$ and let $f_1, f_2 \colon X \to A$ be continuous. Then the sequential subspace Y of X with underlying set $\{x \in X \mid f_1(x) = f_2(x)\}$ together with the inclusion map $\iota \colon Y \to X$ form an equalizer for f_1, f_2 in QCB_0 . Choose admissible representations ϕ and δ for A and X, respectively, such that $EQ(\phi), EQ(\delta) \in \Gamma(\mathcal{N}^2)$. Then the co-restriction γ of δ to the underlying set of Y is known to be an admissible representation for Y. There are partial continuous functions g_1, g_2 on \mathcal{N} with Π_2^0 -domains which realize f_1 and f_2 , respectively. Then $dom(\gamma) = \{p \in dom(\delta) \mid (g_1(p), g_2(p)) \in EQ(\phi)\}$. Since g_1, g_2 are continuous and have a Π_2^0 -domain, $dom(\gamma) \in \Gamma(\mathcal{N})$ by Proposition 3.3. The equation $EQ(\gamma) = (dom(\gamma) \times dom(\gamma)) \cap EQ(\delta)$ yields us $EQ(\gamma) \in \Gamma(\mathcal{N}^2)$ and $Y \in \mathsf{QCB}_0(\Gamma)$.

Now we turn our attention to co-equalizers. Co-equalizers in QCB_0 are constructed by first forming a co-equalizer in the category QCB and then, if the resulting space is non- T_0 , identifying points with the same neighbourhoods. Non- T_0 qcb-spaces do not have an admissible representation, but some of them have a quotient representation. This motivates the following definition generalizing the one from above. For a given family Γ of pointclasses, we say that a topological space X is Γ -quotient-representable, if X has a quotient representation δ such that $EQ(\delta) \in \Gamma(\mathcal{N}^2)$. We denote the class of Γ -quotient-representable spaces by $\mathsf{QTE}(\Gamma)$ and the class of Γ -quotient-representable T_0 -spaces by $\mathsf{QTE}_0(\Gamma)$. Since any admissible representation of a sequential space is a quotient representation, we have $\mathsf{QCB}_0(\Gamma) \subseteq \mathsf{QTE}_0(\Gamma)$.

We study the (non-uniform) descriptive complexity of the Kolmogorov operator \mathcal{T}_0 that maps any T_0 -space to itself and sends a non- T_0 -space X to the quotient space induced by the equivalence relation \equiv_X given by the specification order of X, i.e., $x \equiv_X x'$ iff x and x' have the same open neighbourhoods.

Proposition 4.6 Let α be a non-zero countable ordinal. Then $X \in \mathsf{QTE}(\Sigma^1_\alpha)$ implies $\mathcal{T}_0(X) \in \mathsf{QCB}_0(\Sigma^1_{\alpha+2})$. Moreover, $\mathsf{QCB}_0(\Sigma^1_\alpha) \subseteq \mathsf{QTE}_0(\Sigma^1_\alpha) \subseteq \mathsf{QCB}_0(\Sigma^1_{\alpha+2})$.

The proof is based on the following slight improvement of Proposition 4.3.

Proposition 4.7 Let $1 \leq \alpha < \omega_1$, $X \in \mathsf{QTE}(\Sigma^1_\alpha)$ and $Y \in \mathsf{QCB}_0(\Pi^1_\alpha)$. Then $Y^X \in \mathsf{QCB}_0(\Pi^1_\alpha)$.

Proof. The proof of Proposition 4.3 has been formulated to show this.

Proof. (Proposition 4.6) The Sierpinski space \mathbb{S} has $\{\bot, \top\}$ as its underlying set and $\{\emptyset, \{\top\}, \{\bot, \top\}\}$ as its topology. The space $\mathcal{T}_0(X)$ is homeomorphic to the sequential subspace of $\mathbb{S}^{\mathbb{S}^X}$ which has $\{e(y) \mid y \in \mathcal{T}_0(X)\}$ as its underlying set, cf. [Sch03]. Here the continuous injection $e \colon \mathcal{T}_0(X) \to \mathbb{S}^{\mathbb{S}^X}$ is defined by $e([x]_{\equiv X})(h) := h(x)$.

An admissible representation $\varrho_{\mathbb{S}}$ for \mathbb{S} is defined by $\varrho_{\mathbb{S}}(p) = \bot \iff \forall i \in \omega.p(i) = 0$. Clearly, $EQ(\varrho_{\mathbb{S}})$ is a Boolean combination of open sets and thus in $\Pi_0^1(\mathcal{N}^2) \subseteq \Pi_\alpha^1(\mathcal{N}^2)$. Two applications of Proposition 4.7 yield that $\gamma := \varrho_{\mathbb{S}}^{[\varrho_{\mathbb{S}}^S]}$ is an admissible representation for $\mathbb{S}^{\mathbb{S}^X}$ with $EQ(\gamma) \in \Pi_{\alpha+1}^1(\mathcal{N}^2)$ where δ is a quotient representation of X such that $EQ(\delta) \in \Sigma_\alpha^1(\mathcal{N}^2)$. We define a representation δ^* of $T_0(X)$ by $\delta^*(p) = y :\iff \gamma(p) = e(y)$. By Proposition 4.3.2 in [Sch03], δ^* is an admissible representation of $T_0(X)$. Proposition 2.2 yields a total continuous function $g \colon \mathcal{N} \to \mathcal{N}$ with u(g(r),q) = u(q,r) for all $(q,r) \in dom(u)$. One easily verifies $\gamma(g(r)) = e([\delta(r)]_{\equiv_X})$ for all $r \in dom(\delta)$. Hence for all $r \in \mathcal{N}$ we have

$$p \in dom(\delta^*) \iff \exists r \in \mathcal{N}. (r \in dom(\delta) \& (p, q(r)) \in EQ(\gamma)).$$

By Proposition 3.3, $dom(\delta^*) \in \Sigma^1_{\alpha+2}(\mathcal{N}^2)$. Since $EQ(\delta^*) = (dom(\delta^*) \times dom(\delta^*)) \cap EQ(\gamma)$, we obtain $EQ(\delta^*) \in \Sigma^1_{\alpha+2}(\mathcal{N}^2)$ and $\mathcal{T}_0(X) \in \mathsf{QCB}_0(\Sigma^1_{\alpha+2})$. If X is additionally a T_0 -space, then $\mathcal{T}_0(X) = X$ and thus $X \in \mathsf{QCB}_0(\Sigma^1_{\alpha+2})$.

Now we can formulate our result about forming co-equalizers.

Proposition 4.8 Let λ be a countable limit ordinal. Then $QCB_0(\Sigma^1_{\leq \lambda})$ is closed under forming co-equalizers in QCB_0 .

Proof. Let $\alpha < \lambda$. Let $A, X \in \mathsf{QCB}_0(\Sigma^1_\alpha)$ and let $f, g: A \to X$ be continuous. A co-equalizer $q: X \to Y$ for f, g in QCB is constructed as follows (cf. [Sch03]): Let \equiv be the equivalence relation obtained by the transitive closure of the relation

$$R := \{ (f(a), g(a)), (g(a), f(a)), (x, x) \mid a \in A, x \in X \}.$$

Let Y be the space that has the equivalence classes of \equiv as its underlying set. The function q is the surjection mapping $x \in X$ to its equivalence class $[x]_{\equiv}$. The topology of Y is the quotient topology induced by q. If Y is T_0 , then q is a co-equalizer for f, g in QCB₀ as well, otherwise the map $[.]_{\equiv_Y} \circ q \colon X \to \mathcal{T}_0(Y)$ yields a co-equalizer for f, g in QCB₀. It remains to show that $\mathcal{T}_0(Y) \in \mathsf{QCB}_0(\Sigma^1_{\leq \lambda})$.

There are admissible representations ϕ , δ for A, X such that $dom(\phi)$, $EQ(\delta)$ are Σ^1_{α} -sets. Then the partial function $\gamma := q \circ \delta$ is a quotient representation of Y. There are partial continuous F, G on \mathcal{N} with Π^0_2 -domains which realize f and g, respectively. The sets

$$\begin{split} B &:= \{(p,s) \in dom(\delta) \times dom(\phi) \,|\, (p,F(s)) \in EQ(\delta)\}, \\ C &:= \{(s,t) \in dom(\phi) \times dom(\phi) \,|\, (G(s),F(t)) \in EQ(\delta)\} \quad \text{and} \\ D &:= \{(t,p) \in dom(\phi) \times dom(\delta) \,|\, (G(t),p) \in EQ(\delta)\} \end{split}$$

are in $\Sigma^1_{\alpha}(\mathcal{N}^2)$ by Proposition 3.3. Using the projections $\pi_i \colon \mathcal{N} \to \mathcal{N}$ of the inverse of the homeomorphism $\langle . \rangle \colon \mathcal{N}^{\omega} \to \mathcal{N}$ from Section 2.2, we define for $k \geq 1$ sets $E_k, M \subseteq \mathcal{N}^3$ by

$$E_k := \{ (p, r, p') \mid (p, \pi_1(r)) \in B \& \forall 1 \le i < k. (\pi_i(r), \pi_{i+1}(r)) \in C \& (\pi_k(r), p') \in D \},$$

$$M := \{ (p, p', r) \mid (p, p') \in EQ(\delta) \text{ or } (p, r, p') \in \bigcup_{k \in \omega} E_k \text{ or } (p', r, p) \in \bigcup_{k \in \omega} E_k \}.$$

Then we have

$$EQ(\gamma) = \left\{ (p, p') \in dom(\delta)^2 \, \middle| \, \delta(p) \equiv \delta(p') \right\} = \left\{ (p, p') \in \mathcal{N}^2 \, \middle| \, \exists r \in \mathcal{N}. (p, p', r) \in M \right\},$$

because $(\delta(p), \delta(p')) \in R^{(l)}$ iff $\delta(p) = \delta(p')$ or there are $k \in \{1, \dots, l\}$ and $r \in \mathcal{N}$ such that $(p, r, p') \in E_k$ or $(p', r, p) \in E_k$. By Proposition 3.3, E_k , M and $EQ(\gamma)$ are Σ^1_{α} -sets. We conclude $Y \in \mathsf{QTE}(\Sigma^1_{\alpha})$. By Proposition 4.6, $\mathcal{T}_0(Y) \in \mathsf{QCB}_0(\Sigma^1_{\alpha+2})$. Since $\alpha + 2 < \lambda$, $\mathsf{QCB}_0(\Sigma^1_{\alpha+2}) \subseteq \mathsf{QCB}_0(\Sigma^1_{<\lambda})$. Therefore, $\mathcal{T}_0(Y) \in \mathsf{QCB}_0(\Sigma^1_{<\lambda})$.

5 Kleene-Kreisel continuous functionals of countable types

Here we extend all results in [ScS13] about the continuous functionals of finite types to the continuous functionals of countable types defined as follows:

Definition 5.1 Using the function space construction of QCB₀, we define the sequence of qcb₀-spaces $\{\mathbb{N}\langle\alpha\rangle\}_{\alpha<\omega_1}$ by induction on countable ordinals α as follows:

$$\mathbb{N}\langle 0 \rangle := \omega, \ \mathbb{N}\langle \alpha + 1 \rangle := \omega^{\mathbb{N}\langle \alpha \rangle} \text{ and } \mathbb{N}\langle \lambda \rangle := \prod_{\alpha \leq \lambda} \mathbb{N}\langle \alpha \rangle,$$

where ω denotes the space of natural numbers endowed with the discrete topology, $\alpha, \lambda < \omega_1$ and λ is a limit ordinal. We call $\mathbb{N}\langle \alpha \rangle$ the space of continuous functionals of type α over ω .

Obviously, for $k \in \omega$ the space $\mathbb{N}\langle k \rangle$ coincides with the space of Kleene-Kreisel continuous functionals of type k extensively studied in the literature [No80, No81, No99], and $\mathbb{N}\langle 1 \rangle$ coincides with the Baire space \mathcal{N} . For any finite $k \geq 2$, the sequential topology on $\mathbb{N}\langle k \rangle$ is strictly finer than the corresponding compact-open topology [Hy79]. Furthermore it is neither zero-dimensional nor regular [Sch09].

Any of the introduced spaces has a natural canonical admissible representation $\delta_{\alpha} \colon D_{\alpha} \to \mathbb{N}\langle \alpha \rangle$ induced by the constructions described in Proposition 2.3 and the proof of Proposition 4.4. They may be defined (using the notation from Sections 2.3 and 4) as follows:

$$D_0 = \{n0^\omega \mid n \in \omega\}, \ \delta_0(n0^\omega) := n, \ \delta_1 := id_{\mathcal{N}}, \ \delta_{\alpha+1} := \delta_0^{\delta_\alpha} \text{ and } \delta_\lambda := \prod_{\alpha < \lambda} \delta_\alpha,$$

where $\alpha > 0$ and λ is a limit ordinal.

The spaces of continuous functionals enjoy the following product property, which, in the case of the finite types, belongs to the folklore of sequential spaces ([EL08]).

Proposition 5.2 For all countable ordinals $\alpha \leq \beta$, $\mathbb{N}\langle \alpha \rangle \times \mathbb{N}\langle \beta \rangle$ is homeomorphic to $\mathbb{N}\langle \beta \rangle$.

Proof. We proceed by showing the following instances of our claim. For countable ordinals α, β and a countable limit ordinal $\lambda \geq \alpha$,

- (1) $\omega \times \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \alpha \rangle$
- (2) $(\mathbb{N}\langle\alpha\rangle)^{\omega} \cong \mathbb{N}\langle\max\{1,\alpha\}\rangle$
- (3) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha\rangle \cong \mathbb{N}\langle\alpha\rangle$
- (4) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\beta\rangle \cong \mathbb{N}\langle\beta\rangle$ implies $\mathbb{N}\langle\alpha+1\rangle \times \mathbb{N}\langle\beta+1\rangle \cong \mathbb{N}\langle\beta+1\rangle$
- (5) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\lambda\rangle \cong \mathbb{N}\langle\lambda\rangle$
- (6) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\lambda+1\rangle \cong \mathbb{N}\langle\lambda+1\rangle$
- (7) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha+1\rangle \cong \mathbb{N}\langle\alpha+1\rangle$
- (8) $\mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha+k\rangle \cong \mathbb{N}\langle\alpha+k\rangle$ for all $k \in \omega$.

Now we show the above claims.

(1) For $\alpha = 0$, the claim is just the statement $\omega \times \omega \cong \omega$. For a limit ordinal $\lambda > 0$ we calculate:

$$\mathbb{N}\langle\lambda\rangle = \prod_{\beta<\lambda}\mathbb{N}\langle\beta\rangle \cong \omega \times \prod_{1\leq\beta<\lambda}\mathbb{N}\langle\beta\rangle \cong \omega \times \omega \times \prod_{1\leq\beta<\lambda}\mathbb{N}\langle\beta\rangle \cong \omega \times \prod_{\beta<\lambda}\mathbb{N}\langle\beta\rangle = \omega \times \mathbb{N}\langle\lambda\rangle.$$

Now let α be a successor cardinal $\alpha = \alpha' + 1$. We fix an element $x_0 \in \mathbb{N}\langle \alpha' \rangle$ and define functions $S \colon \mathbb{N}\langle \alpha \rangle \times \omega^2 \to \mathbb{N}\langle \alpha \rangle$, $\varphi \colon \omega \times \mathbb{N}\langle \alpha \rangle \to \mathbb{N}\langle \alpha \rangle$ and $\psi \colon \mathbb{N}\langle \alpha \rangle \to \omega \times \mathbb{N}\langle \alpha \rangle$ by

$$S(f, a, b)(x) := \begin{cases} b & \text{if } f(x) = a \\ a & \text{if } f(x) = b \\ f(x) & \text{otherwise,} \end{cases}$$
$$\varphi(a, f) := S(f, \langle a, f(x_0) \rangle, f(x_0)) \text{ and }$$
$$\psi(f) := (\pi_1(f(x_0)), S(f, \pi_2(f(x_0)), f(x_0)))$$

for $f \in \mathbb{N}\langle \alpha \rangle$, $x \in \mathbb{N}\langle \alpha' \rangle$ and $a, b \in \omega$, where $\pi_1, \pi_2 : \omega \to \omega$ are the projections of the inverse of the pairing function $\langle ., . \rangle$. By cartesian closedness of QCB₀, these functions are continuous. One easily checks that φ and ψ are inverses of each other. Hence $\omega \times \mathbb{N}\langle \alpha \rangle$ is homeomorphic to $\mathbb{N}\langle \alpha \rangle$.

(2) Clearly $\mathbb{N}\langle 0 \rangle^{\omega} = \mathbb{N}\langle 1 \rangle$. For a successor ordinal $\alpha = \alpha' + 1$, the cartesian closedness of QCB_0 and Claim (1) yield us

$$\mathbb{N}\langle\alpha\rangle^{\omega} \cong \left(\omega^{\mathbb{N}\langle\alpha'\rangle}\right)^{\omega} \cong \omega^{\omega \times \mathbb{N}\langle\alpha'\rangle} \cong \omega^{\mathbb{N}\langle\alpha'\rangle} \cong \mathbb{N}\langle\alpha\rangle.$$

For limit ordinals $\lambda > 0$ we proceed by induction and get by the induction hypothesis

$$\begin{split} \mathbb{N}\langle\lambda\rangle &\cong \mathbb{N}\langle0\rangle \times \mathbb{N}\langle1\rangle \times \prod_{2\leq \beta<\lambda} \mathbb{N}\langle\beta\rangle \cong \prod_{1\leq \beta<\lambda} \mathbb{N}\langle\beta\rangle \\ &\cong \prod_{1\leq \beta<\lambda} (\mathbb{N}\langle\beta\rangle)^\omega \cong \left(\prod_{1\leq \beta<\lambda} \mathbb{N}\langle\beta\rangle\right)^\omega \cong \mathbb{N}\langle\lambda\rangle^\omega \,. \end{split}$$

(3) We proceed by induction. For $\alpha = 0$, the statement is simply $\omega \times \omega \cong \omega$. For any limit ordinal $\lambda \neq 0$, the induction hypothesis yields

$$\mathbb{N}\langle\lambda\rangle\times\mathbb{N}\langle\lambda\rangle = \left(\prod_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle\right)\times\left(\prod_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle\right) \cong \prod_{\alpha<\lambda}\left(\mathbb{N}\langle\alpha\rangle\times\mathbb{N}\langle\alpha\rangle\right) \cong \prod_{\alpha<\lambda}\mathbb{N}\langle\alpha\rangle = \mathbb{N}\langle\lambda\rangle.$$

For any successor ordinal $\alpha = \alpha' + 1$, the cartesian closedness of QCB₀ and the induction hypothesis yield

$$\mathbb{N}\langle\alpha\rangle\times\mathbb{N}\langle\alpha\rangle = \omega^{\mathbb{N}\langle\alpha'\rangle}\times\omega^{\mathbb{N}\langle\alpha'\rangle} \cong (\omega\times\omega)^{\mathbb{N}\langle\alpha'\rangle} \cong \omega^{\mathbb{N}\langle\alpha'\rangle} = \mathbb{N}\langle\alpha\rangle.$$

(4) We calculate using the cartesian closedness of QCB₀ and Claim (1):

$$\mathbb{N}\langle \alpha+1\rangle \times \mathbb{N}\langle \beta+1\rangle = \omega^{\mathbb{N}\langle \alpha\rangle} \times \omega^{\mathbb{N}\langle \beta\rangle} \cong \omega^{\mathbb{N}\langle \alpha\rangle} \times \omega^{\mathbb{N}\langle \alpha\rangle} \times \omega^{\mathbb{N}\langle \alpha\rangle} \cong \omega^{\mathbb{N}\langle \alpha\rangle} \times \left(\omega^{\mathbb{N}\langle \beta\rangle}\right)^{\mathbb{N}\langle \alpha\rangle} \\ \cong \left(\omega \times \omega^{\mathbb{N}\langle \beta\rangle}\right)^{\mathbb{N}\langle \alpha\rangle} \cong \left(\omega^{\mathbb{N}\langle \beta\rangle}\right)^{\mathbb{N}\langle \alpha\rangle} \cong \omega^{\mathbb{N}\langle \alpha\rangle} \times \mathbb{N}\langle \beta\rangle \cong \omega^{\mathbb{N}\langle \beta\rangle} = \mathbb{N}\langle \beta+1\rangle.$$

(5) We calculate using Claim (3):

$$\begin{split} \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\lambda\rangle &\cong \mathbb{N}\langle\alpha\rangle \times \prod_{\beta<\lambda} \mathbb{N}\langle\beta\rangle \cong \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha\rangle \times \prod_{\beta<\lambda,\beta\neq\alpha} \mathbb{N}\langle\beta\rangle \\ &\cong \mathbb{N}\langle\alpha\rangle \times \prod_{\beta<\lambda,\beta\neq\alpha} \mathbb{N}\langle\beta\rangle \cong \prod_{\beta<\lambda} \mathbb{N}\langle\beta\rangle = \mathbb{N}\langle\lambda\rangle \,. \end{split}$$

(6) For a successor ordinal $\alpha = \alpha' + 1 < \lambda$, we have $\mathbb{N}\langle \alpha' \rangle \times \mathbb{N}\langle \lambda \rangle \cong \mathbb{N}\langle \lambda \rangle$ by Claim (5) and thus $\mathbb{N}\langle \alpha \rangle \times \mathbb{N}\langle \lambda + 1 \rangle \cong \mathbb{N}\langle \lambda + 1 \rangle$ by Claim (4). For a limit ordinal α with $0 < \alpha \le \lambda$, we proceed by ordinal induction and get by Claim (2):

$$\begin{split} \mathbb{N}\langle\lambda+1\rangle &\cong (\mathbb{N}\langle\lambda+1\rangle)^\omega \cong \prod_{i\in\omega} \mathbb{N}\langle\lambda+1\rangle \cong \prod_{\beta<\alpha} \mathbb{N}\langle\lambda+1\rangle \cong \prod_{\beta<\alpha} (\mathbb{N}\langle\beta\rangle \times \mathbb{N}\langle\lambda+1\rangle) \\ &\cong \left(\prod_{\beta<\alpha} \mathbb{N}\langle\beta\rangle\right) \times \left(\prod_{\beta<\alpha} \mathbb{N}\langle\lambda+1\rangle\right) \cong \left(\prod_{\beta<\alpha} \mathbb{N}\langle\beta\rangle\right) \times (\mathbb{N}\langle\lambda+1\rangle)^\omega \cong \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\lambda+1\rangle \,. \end{split}$$

- (7) We proceed by ordinal induction. For $\alpha = 0$, $\mathbb{N}\langle \alpha \rangle \times \mathbb{N}\langle \alpha + 1 \rangle \cong \mathbb{N}\langle \alpha + 1 \rangle$ is an instance of Claim (1). For a successor ordinal, $\mathbb{N}\langle \alpha \rangle \times \mathbb{N}\langle \alpha + 1 \rangle \cong \mathbb{N}\langle \alpha + 1 \rangle$ follows from the induction hypothesis and Claim (4). For a limit ordinal $\lambda > 0$, we know $\mathbb{N}\langle \lambda \rangle \times \mathbb{N}\langle \lambda + 1 \rangle \cong \mathbb{N}\langle \lambda + 1 \rangle$ from Claim (6).
- (8) We proceed by induction on $k \in \omega$. For $k \in \{0,1\}$, we know the claim from (3) and (7). For $k \geq 2$ we obtain by Claim (7) and by the induction hypothesis

$$\begin{split} \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha+k\rangle &\cong \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\alpha+k-1\rangle \times \mathbb{N}\langle\alpha+k\rangle \\ &\cong \mathbb{N}\langle\alpha+k-1\rangle \times \mathbb{N}\langle\alpha+k\rangle \cong \mathbb{N}\langle\alpha+k\rangle \,. \end{split}$$

Now let $\alpha \leq \beta < \omega_1$. Then either there is some $k \in \omega$ such that $\beta = \alpha + k$ or the largest limit ordinal λ with $\lambda \leq \beta$ satisfies $\alpha < \lambda$. In the first case, $\mathbb{N}\langle \alpha \rangle \times \mathbb{N}\langle \beta \rangle \cong \mathbb{N}\langle \beta \rangle$ follows from Claim (8). Otherwise we choose $l \in \omega$ such that $\beta = \lambda + l$ and calculate using Claims (5) and (8):

$$\mathbb{N}\langle\beta\rangle = \mathbb{N}\langle\lambda+l\rangle \cong \mathbb{N}\langle\lambda\rangle \times \mathbb{N}\langle\lambda+l\rangle \cong \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\lambda\rangle \times \mathbb{N}\langle\lambda+l\rangle \cong \mathbb{N}\langle\alpha\rangle \times \mathbb{N}\langle\beta\rangle \,.$$

From this propositions, we deduce the following basic properties of continuous functionals of countable types.

Lemma 5.3 (1) For all $\alpha < \beta < \omega_1$, the spaces $\mathbb{N}\langle \alpha \rangle$, $\omega \times \mathbb{N}\langle \beta \rangle$ and $\mathcal{N} \times \mathbb{N}\langle \beta \rangle$ are continuous retracts of $\mathbb{N}\langle \beta \rangle$.

- (2) $(\mathbb{N}\langle 0 \rangle)^{\omega} \cong \mathbb{N}\langle 1 \rangle$ and $(\mathbb{N}\langle \alpha \rangle)^{\omega} \cong \mathbb{N}\langle \alpha \rangle$ for $1 \leq \alpha < \omega_1$.
- (3) For all $\beta < \omega_1$, $\prod_{\alpha \leq \beta} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle$.
- (4) For all $\alpha < \beta < \omega_1$, $\prod_{\alpha < \gamma \le \beta} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle$.
- (5) For all countable ordinals $\beta_0 < \beta_1 < \cdots$, we have $\prod_{k \in \omega} \mathbb{N} \langle \beta_k \rangle \cong \mathbb{N} \langle \sup \{ \beta_k \mid k \in \omega \} \rangle$.
- (6) For all countable ordinals $\alpha_0, \alpha_1, \ldots, we$ have $\prod_{i \in \omega} \mathbb{N} \langle \alpha_i \rangle \cong \mathbb{N} \langle \sup\{1, \alpha_i \mid i \in \omega\} \rangle$.
- (7) For all $\alpha, \beta < \omega_1$, we have $\mathbb{N}\langle \alpha \rangle^{\mathbb{N}\langle \beta \rangle} \cong \mathbb{N}\langle \max\{\alpha, \beta + 1\} \rangle$.

Proof.

- (1) This follows immediately from Proposition 5.2.
- (2) This has been shown inside the proof of Proposition 5.2.

(3) We proceed by induction, the case $\beta=0$ being trivial. If $\beta=\gamma+1$ is successor then we have

$$\prod_{\alpha \leq \beta} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle \times \prod_{\alpha \leq \gamma} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle \times \mathbb{N}\langle \gamma \rangle \cong \mathbb{N}\langle \beta \rangle.$$

If β is limit then we have

$$\prod_{\alpha \leq \beta} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle \times \prod_{\alpha < \beta} \mathbb{N}\langle \alpha \rangle \cong \mathbb{N}\langle \beta \rangle \times \mathbb{N}\langle \beta \rangle \cong \mathbb{N}\langle \beta \rangle \,.$$

(4) By induction on β , the case $\beta = \alpha + 1$ being trivial. If $\beta = \gamma + 1 > \alpha + 1$ is successor then we have

$$\prod_{\alpha<\delta\leq\beta}\mathbb{N}\langle\delta\rangle\cong\mathbb{N}\langle\beta\rangle\times\prod_{\alpha<\delta\leq\gamma}\mathbb{N}\langle\delta\rangle\cong\mathbb{N}\langle\beta\rangle\times\mathbb{N}\langle\gamma\rangle\cong\mathbb{N}\langle\beta\rangle.$$

If β is limit then we have by Claim (3)

$$\begin{split} \prod_{\alpha < \delta \leq \beta} \mathbb{N}\langle \delta \rangle &\cong \mathbb{N}\langle \beta \rangle \times \prod_{\alpha < \delta < \beta} \mathbb{N}\langle \delta \rangle \cong \mathbb{N}\langle \beta \rangle \times \mathbb{N}\langle \alpha \rangle \times \prod_{\alpha < \delta < \beta} \mathbb{N}\langle \delta \rangle \\ &\cong \mathbb{N}\langle \beta \rangle \times \prod_{\delta \leq \alpha} \mathbb{N}\langle \delta \rangle \times \prod_{\alpha < \delta < \beta} \mathbb{N}\langle \delta \rangle \cong \mathbb{N}\langle \beta \rangle \times \mathbb{N}\langle \beta \rangle \cong \mathbb{N}\langle \beta \rangle. \end{split}$$

(5) Clearly, $\beta := \sup\{\beta_k \mid k \in \omega\}$ is a limit ordinal. By Claims (3) and (4) we get

$$\mathbb{N}\langle\beta\rangle\cong\prod_{\alpha<\beta}\mathbb{N}\langle\beta\rangle\cong\prod_{\alpha\leq\beta_0}\mathbb{N}\langle\alpha\rangle\times\prod_{\beta_0<\alpha\leq\beta_1}\mathbb{N}\langle\alpha\rangle\times\dots\cong\mathbb{N}\langle\beta_0\rangle\times\mathbb{N}\langle\beta_1\rangle\times\dots=\prod_k\mathbb{N}\langle\beta_k\rangle.$$

(6) If $\alpha_i = 0$ for all i then $\prod_i \mathbb{N}\langle \alpha_i \rangle = \mathbb{N}\langle 1 \rangle$. So let $\alpha_i > 0$ for some $i \in \omega$. First we assume that there is some j with $\alpha_j = \alpha := \sup\{1, \alpha_i \mid i \in \omega\}$. From Proposition 5.2 and Claim (2) we get

$$\begin{split} \prod_{i \in \omega} \mathbb{N} \langle \alpha_i \rangle &\cong \prod_{i \neq j} \mathbb{N} \langle \alpha_i \rangle \times \mathbb{N} \langle \alpha \rangle \cong \prod_{i \in \omega} \mathbb{N} \langle \alpha_i \rangle \times (\mathbb{N} \langle \alpha \rangle)^{\omega} \\ &\cong \prod_{i \in \omega} \mathbb{N} \langle \alpha_i \rangle \times \prod_{i \in \omega} \mathbb{N} \langle \alpha \rangle \cong \prod_{i \in \omega} (\mathbb{N} \langle \alpha_i \rangle \times \mathbb{N} \langle \alpha \rangle) \cong \prod_{i \in \omega} \mathbb{N} \langle \alpha \rangle \cong \mathbb{N} \langle \alpha \rangle \,. \end{split}$$

It remains to consider the case when $\alpha_j < \alpha$ for all $j \in \omega$ (so, in particular, α is limit). Choose a sequence $i_0 < i_1 < \cdots$ of indices such that $\alpha_{i_0} < \alpha_{i_1} < \cdots$ and $\sup\{\beta_k \mid k \in \omega\} = \alpha$ where $\beta_k := \alpha_{i_k}$ for each $k \in \omega$. Claims (3) and (4) yield us

$$\prod \{ \mathbb{N} \langle \alpha_i \rangle \mid \alpha_i \leq \beta_0 \} \cong \mathbb{N} \langle \beta_0 \rangle \text{ and } \prod \{ \mathbb{N} \langle \alpha_i \rangle \mid \beta_{j-1} < \alpha_i \leq \beta_j \} \cong \mathbb{N} \langle \beta_j \rangle,$$

for all $j \geq$. From Claim (5) we deduce

$$\mathbb{N}\langle\alpha\rangle\cong\prod_{k}\mathbb{N}\langle\beta_{k}\rangle\cong\prod\{\mathbb{N}\langle\alpha_{i}\rangle\mid\alpha_{i}\leq\beta_{0}\}\times\prod\{\mathbb{N}\langle\alpha_{i}\rangle\mid\beta_{0}<\alpha_{i}\leq\beta_{1}\}\times\cdots\cong\prod_{i}\mathbb{N}\langle\alpha_{i}\rangle.$$

(7) If $\alpha = 0$, then $\mathbb{N}\langle \alpha \rangle^{\mathbb{N}\langle \beta \rangle} = \omega^{\mathbb{N}\langle \beta \rangle} = \mathbb{N}\langle \beta + 1 \rangle$. If $\alpha = \gamma + 1$ is successor then we have by Proposition 5.2

$$\mathbb{N}\langle\alpha\rangle^{\mathbb{N}\langle\beta\rangle} = (\omega^{\mathbb{N}\langle\gamma\rangle})^{\mathbb{N}\langle\beta\rangle} \cong \omega^{\mathbb{N}\langle\gamma\rangle\times\mathbb{N}\langle\beta\rangle} \cong \omega^{\mathbb{N}\langle\max\{\gamma,\beta\}\rangle} = \mathbb{N}\langle\max\{\alpha,\beta+1\}\rangle.$$

Now let $\alpha > 0$ be a limit ordinal. First consider the case $\beta < \alpha$. There are non-limit ordinals α_i such that $\beta < \alpha_0 < \alpha_1 < \cdots$ and $\sup\{\alpha_i \mid i \in \omega\} = \alpha$. From the result for non-limit ordinals and Claim (5) we deduce

$$\mathbb{N}\langle\alpha\rangle^{\mathbb{N}\langle\beta\rangle}\cong (\prod_i \mathbb{N}\langle\alpha_i\rangle)^{\mathbb{N}\langle\beta\rangle}\cong \prod_i (\mathbb{N}\langle\alpha_i\rangle^{\mathbb{N}\langle\beta\rangle})\cong \prod_i \mathbb{N}\langle\alpha_i\rangle\cong \mathbb{N}\langle\alpha\rangle.$$

Let now $\alpha \leq \beta$. Then there are non-limit ordinals α_i such that $\alpha_0 < \alpha_1 < \cdots$ and $\sup\{\alpha_i \mid i \in \omega\} = \alpha$. We get by the result for non-limit ordinals and by Claims (2) and (5)

$$\mathbb{N}\langle\alpha\rangle^{\mathbb{N}\langle\beta\rangle} \cong (\prod_i \mathbb{N}\langle\alpha_i\rangle)^{\mathbb{N}\langle\beta\rangle} \cong \prod_i (\mathbb{N}\langle\alpha_i\rangle^{\mathbb{N}\langle\beta\rangle}) \cong \prod_i \mathbb{N}\langle\beta+1\rangle \cong \mathbb{N}\langle\beta+1\rangle.$$

6 The relationship of the continuous functionals to the hyperprojective hierarchies

In this section we investigate the relationship of the continuous functionals to both the hyperprojective hierarchy of subsets of the Baire space \mathcal{N} and the hyperprojective hierarchy of qcb₀-spaces.

The first result generalizes the characterization of projective subsets of \mathcal{N} with the help of Kleene-Kreisel continuous functionals in [ScS13, Theorem 7.6] to hyperprojective subsets.

Theorem 6.1 Let α be a non-zero countable ordinal and B a non-empty subset of \mathcal{N} . Then $B \in \Sigma^1_{\alpha}(\mathcal{N})$ iff there is a continuous function $f : \mathbb{N}\langle \alpha \rangle \to \mathcal{N}$ with rng(f) = B.

The theorem is based on the following proposition about complementation in \mathcal{N} , which has been shown in [ScS13].

Proposition 6.2 Let Y be a qcb₀-space and let $f: Y \to \mathcal{N}$ be a continuous function with $rng(f) \neq \mathcal{N}$. Then there exists a continuous function $g: \mathcal{N} \times \omega^Y \to \mathcal{N}$ with $rng(g) = \mathcal{N} \setminus rng(f)$.

Proof (Theorem 6.1). We proceed by induction. The case of non-limit levels are considered precisely as for the finite ordinals in the proof of Proposition 7.5 and Theorem 7.6 in [ScS13]. Let now λ be a limit ordinal and $f \colon \mathbb{N}\langle \lambda \rangle \to \mathcal{N}$ a continuous function with rng(f) = B. Then $rng(f \circ \delta_{\lambda}) = B$, where $\delta_{\lambda} : D_{\lambda} \to \mathbb{N}\langle \lambda \rangle$ is the canonical admissible representation of $\mathbb{N}\langle \lambda \rangle$. Since D_{λ} is in $(\Sigma^{1}_{<\lambda})_{\delta}$ by Proposition 4.4(2), $B \in \Sigma^{1}_{\lambda}(\mathcal{N})$ by Proposition 3.3(3).

For the other direction, let $B \in \Sigma^1_{\lambda}(\mathcal{N})$. Then B = g(A) for some $A \in (\Sigma^1_{<\lambda})_{\delta}$ and some continuous function $g \colon \mathcal{N} \to \mathcal{N}$. Since $\Sigma^1_{\leq \lambda} = \Pi^1_{<\lambda}$ by Proposition 3.3, there are sets $A_0, A_1, \ldots \in \Pi^1_{<\lambda}$ such that $A = \bigcap_{k \in \omega} A_k$. Then $\overline{A} = \bigcup_{k \in \omega} \overline{A_k}$, where $\overline{A} = \mathcal{N} \setminus A$. Choose ordinals $\alpha(0), \alpha(1), \ldots$ below λ such that $\overline{A_k} \in \Sigma^1_{\alpha(k)}$ for each $k \in \omega$. By the induction hypothesis, there are continuous functions $f_k \colon \mathbb{N}\langle \alpha(k) \rangle \to \mathcal{N}$ such that $rng(f_k) = \overline{A_k}$ for each $k \in \omega$. Then there is a continuous function $h \colon X \to \mathcal{N}$ such that $rng(h) = \overline{A_k}$ where $X = \bigoplus_k \mathbb{N}\langle \alpha(k) \rangle$. By Proposition 6.2 there is a continuous function $u \colon \mathcal{N} \times \omega^X \to \mathcal{N}$ such that rng(u) = A. Let \mathcal{B} be the set $\{1\} \cup \{\alpha(k) + 1 \mid k \in \omega\}$. Since QCB₀ is cartesian closed and closed under countable products and countable coproducts, we have

$$\mathcal{N} \times \omega^X \cong \mathcal{N} \times \prod_{k \in \omega} \omega^{\mathbb{N}\langle \alpha(k) \rangle} \cong \mathbb{N}\langle 1 \rangle \times \prod_{k \in \omega} \mathbb{N}\langle \alpha(k) + 1 \rangle.$$

By Lemma 5.3(1) and (2), $\mathbb{N}\langle 1 \rangle \times \prod_{k \in \omega} \mathbb{N}\langle \alpha(k) + 1 \rangle$ is a continuous retract of $\prod_{\beta \in \mathcal{B}} \mathbb{N}\langle \beta \rangle$. This set is in turn a continuous retract of $\prod_{\beta < \lambda} \mathbb{N}\langle \beta \rangle = \mathbb{N}\langle \lambda \rangle$. So there is a continuous surjection r from $\mathbb{N}\langle \lambda \rangle$ onto $\mathcal{N} \times \omega^X$. Since B = g(A), the continuous function $f := g \circ u \circ r$ satisfies B = rng(f).

Below we make use of the following lemma.

Lemma 6.3 (1) Let X, Y be qcb_0 -spaces and $f: Y \to Y$ a continuous function without fixed points. Then there is no continuous surjection from X onto Y^X .

(2) For any qcb_0 -space X, there is no continuous surjection from X onto ω^X .

Proof. The second assertion follows from the first one, where $Y = \omega$ and f(y) = y + 1. So it suffices to prove the first one. Suppose for a contradiction that $h: X \to Y^X$ is a continuous surjection. Define a function $g: X \to Y$ by g(x) = f(h(x)(x)). By cartesian closedness of QCB₀, g is continuous, i.e. $g \in Y^X$. Since h is a surjection, g = h(a) for some $a \in X$. Then g(a) = f(h(a)(a)) = f(g(a)), hence $g(a) \in Y$ is a fixed point of f. A contradiction.

Finally, we relate the continuous functionals of countable types to the hyperprojective hierarchy of qcb₀-spaces (extending Theorem 7.7 of [ScS13]). The next result provides the exact estimation of the spaces of continuous functionals of countable types in the hyperprojective hierarchy of qcb₀-spaces. On the other hand, the result provides "natural" witnesses for the non-collapse property of this hierarchy.

Theorem 6.4 For any non-zero countable ordinal α , $\mathbb{N}\langle \alpha+1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha) \setminus \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$. For any countable limit ordinal λ , $\mathbb{N}\langle \lambda \rangle \in \mathsf{QCB}_0((\mathbf{\Pi}^1_{\leq \lambda})_\delta) \setminus \mathsf{QCB}_0((\mathbf{\Sigma}^1_{\leq \lambda})_\sigma)$.

Proof. We proceed by induction. Obviously, $\mathbb{N}\langle 0 \rangle$, $\mathbb{N}\langle 1 \rangle \in \mathsf{QCB}_0(\Delta_0^1)$. By Proposition 4.3, $\mathbb{N}\langle 2 \rangle \in \mathsf{QCB}_0(\Pi_1^1)$.

Let now $\alpha \geq 2$. If α is a successor ordinal, then $\mathbb{N}\langle \alpha+1\rangle \in \mathsf{QCB}_0(\Pi^1_{\alpha})$ by Proposition 4.3, because $\mathbb{N}\langle \alpha \rangle \in \mathsf{QCB}_0(\Pi^1_{\alpha}) \subseteq \mathsf{QCB}_0(\Sigma^1_{\alpha+1})$ by the induction hypothesis. If $\alpha = \lambda$ is a limit ordinal, we have

$$\mathbb{N}\langle\lambda\rangle\in\mathsf{QCB}_0((\Sigma^1_{<\lambda})_\delta)=\mathsf{QCB}_0((\Pi^1_{<\lambda})_\delta)\subseteq\mathsf{QCB}_0(\Sigma^1_\lambda)$$

by the induction hypothesis and Propositions 3.3, 4.4. Again by Proposition 4.3 we obtain $\mathbb{N}(\lambda+1) \in \mathsf{QCB}_0(\Pi^1_\lambda)$. This completes the proof of the "upper bounds".

Now we turn to the "lower bounds" and first show that $\mathbb{N}\langle \alpha+1\rangle \notin \mathsf{QCB}_0(\Sigma_\alpha^1)$ for any non-zero countable ordinal α . We prove the stronger assertion that $\mathbb{N}\langle \alpha+1\rangle$ has no continuous representation δ with $D=dom(\delta)\in\Sigma_\alpha^1(\mathcal{N})$. Suppose for a contradiction that δ is such a continuous representation of $\mathbb{N}\langle \alpha+1\rangle$. Then there is a continuous surjection f from $\mathbb{N}\langle \alpha\rangle$ onto D by Theorem 6.1. Then δf is a continuous surjection from $\mathbb{N}\langle \alpha\rangle$ onto $\mathbb{N}\langle \alpha+1\rangle=\omega^{\mathbb{N}\langle \alpha\rangle}$. This contradicts Lemma 6.3(2).

It remains to show that $\mathbb{N}\langle\lambda\rangle \notin \mathsf{QCB}_0((\Sigma^1_{<\lambda})_\sigma)$. Again we prove a stronger assertion that there is no continuous surjection δ from a set $D \in (\Sigma^1_{<\lambda})_\sigma$ onto $\mathbb{N}\langle\lambda\rangle$. Suppose for a contradiction that $\delta: D \to \mathbb{N}\langle\lambda\rangle$ is such a surjection. Choose non-zero ordinals α_k and non-empty sets $D_k \in \Sigma^1_{\alpha_k}$ such that $\alpha_0 < \alpha_1 < \cdots$, $\sup\{\alpha_k \mid k \in \omega\} = \lambda$, and $D = \bigcup_k D_k$. By Theorem 6.1, for any $k \in \omega$ there is a continuous surjection from $\mathbb{N}\langle\alpha_k\rangle$ onto D_k . Then there is a continuous surjection from $X := \bigoplus_k \mathbb{N}\langle\alpha_k\rangle$ onto D. Then δf is a continuous surjection from X onto $\mathbb{N}\langle\lambda\rangle$. Since by Lemma 5.3 $\mathbb{N}\langle\lambda\rangle\cong\prod_k \mathbb{N}\langle\alpha_{k+1}\rangle\cong\omega^X$, there is a continuous surjection from X onto ω^X . This contradicts Lemma 6.3(2).

7 Categories of hyperprojective qcb₀-spaces

In this section we show that the hyperprojective hierarchy of qcb₀-spaces gives rise to a nice cartesian closed category. This is the full subcategory of the category QCB₀ consisting of the spaces in $\bigcup_{\alpha<\omega_1}$ QCB₀(Σ^1_{α}) as objects and all continuous function between them as morphisms. We denote this category by QCB₀(**HP**) and call its objects hyperprojective qcb₀-spaces.

Recall that a category is countably complete (resp. countably co-complete), if it is closed under countable limits (resp. co-limits). We do not recall here the rather technical notions of limits and co-limits, but remind the reader that a category is countably complete (resp. co-complete) iff it is closed under countable products and equalizers (resp. countable co-products and co-equalizers). A category is cartesian closed, if it has finite products and admits for any two objects X, Y an exponential Y^X and an evaluation morphism $ev: Y^X \times X \to Y$ allowing curry and uncurry.

According to [Sch03], the category QCB_0 is cartesian closed, countably complete and countably co-complete. Here we discuss closure properties of some natural subcategories of QCB_0 including $QCB_0(\mathbf{HP})$ and the category $QCB_0(\mathbf{P})$ of projective qcb₀-spaces from [ScS13].

Theorem 7.1 The category $QCB_0(\mathbf{HP})$ of hyperprojective qcb_0 -spaces is cartesian closed, countably complete and countably co-complete. It inherits its exponentials, countable products, countable co-products, equalizers and co-equalizers from QCB_0 .

Proof. By Propositions 4.3 and 4.4, $QCB_0(\mathbf{HP})$ is closed under QCB_0 -exponentials and countable products. Since $QCB_0(\mathbf{HP})$ is a full subcategory of the cartesian closed category QCB_0 , it is also cartesian closed. The remaining properties follow from the corresponding results about countable products, countable co-products, equalizers and co-equalizers in Section 4.

The next result provides a characterization of $QCB_0(\mathbf{HP})$ that avoids explicit mention of the hyperprojective hierarchy. We thank Matthew de Brecht for pointing out this fact to us and for allowing us to include it into the paper.

Theorem 7.2 The category $QCB_0(\mathbf{HP})$ is (up to homeomorphic equivalence) the smallest full subcategory of QCB_0 that has the closure properties from the previous theorem and contains the Sierpinski space as an object. So any full subcategory C of QCB_0 which contains the Sierpinski space as an object and inherits exponentials, countable limits and countable co-limits from QCB_0 contains a homeomorphic copy of any space in $QCB_0(\mathbf{HP})$.

Proof. Let C be a full subcategory of QCB_0 which contains the Sierpinski space as an object and inherits exponentials, countable limits and countable co-limits from QCB_0 . So C is cartesian closed, countably complete and countably co-complete. Without loss of generality we assume that C is closed under homeomorphic equivalence.

First we show that ω is an object of C. Since $\mathbb S$ is a C-object and C is closed under countable product, $\mathbb S^{\omega}$ is a C-object. By being homeomorphic to $\mathbb S^{\omega}$, $P\omega$ is in C as well. Clearly, ω is homeomorphic to the subspace M of $P\omega$ consisting of all singleton subsets of ω . We define $f,g\colon P\omega\to P\omega$ by

$$f(p) := \left\{ \begin{array}{ll} \emptyset & \text{if } p = \emptyset \\ \{0\} & \text{otherwise} \end{array} \right. \quad \text{and} \quad g(p) := \left\{ \begin{array}{ll} \{0,1\} & \text{if } p \text{ contains at least two numbers} \\ \{0\} & \text{otherwise} \end{array} \right.$$

Clearly, f and g are continuous and satisfy $f(p) = g(p) \iff p \in X$. Hence M (together with its inclusion into $P\omega$) is an equalizer to f,g in QCB₀ and thus in C. We conclude $\omega \in C$.

Next we show that any proper hyperprojective subset D of the Baire space \mathcal{N} , endowed with the subspace topology, is a C-object. To see this, choose a non-zero countable ordinal α such that $\mathcal{N} \setminus D \in \Sigma^1_{\alpha}(\mathcal{N})$. By Theorem 6.1 there is a continuous surjection f from $\mathbb{N}\langle \alpha \rangle$ onto $\mathcal{N} \setminus D$. We define $F, G: \mathcal{N} \to \mathbb{S}^{\mathbb{N}\langle \alpha \rangle}$ by

$$F(x)(y) := \top$$
 and $G(x)(y) := \begin{cases} \bot & f(y) = x \\ \top & \text{otherwise} \end{cases}$

for all $x \in \mathcal{N}$ and $y \in \mathbb{N}\langle \alpha \rangle$. Since \mathcal{N} is a Hausdorff space and QCB₀ is cartesian closed, F and G are continuous. Moreover every $x \in \mathcal{N}$ satisfies F(x) = G(x) iff $x \notin rng(f)$ iff $x \in D$. Hence D is an equalizer to F, G in QCB₀. Since $\mathbb{S}^{\mathbb{N}\langle \alpha \rangle} \in \mathbb{C}$, we obtain $D \in \mathbb{C}$. As \mathcal{N}^2 is homeomorphic to \mathcal{N} , every subspace of \mathcal{N}^2 with a hyperprojective carrier set is in \mathbb{C} as well.

Finally, we employ co-equalizers to get all of the hyperprojective qcb₀-spaces. Let X be a qcb₀-space having an admissible representation $\delta \colon D \to X$ such that $EQ(\delta)$ is hyperprojective. Thus D is hyperprojective and δ is a quotient map. Above we have seen that D and $EQ(\delta)$ with the respective subspace topologies are spaces in C. We let $p_1, p_2 \colon EQ(\delta) \to D$ be the respective projections. One easily checks that X and δ form a co-equalizer to p_1, p_2 in QCB₀ and thus in C. Hence X is a C-object.

Next we identify a natural "small" cartesian closed subcategory of $QCB_0(\mathbf{HP})$ closed under countable product. By 1 we denote a fixed one-point space.

Theorem 7.3 The full subcategory $F := \{1, \mathbb{N}\langle \alpha \rangle \mid \alpha < \omega_1 \}$ of QCB₀ is cartesian closed and closed under countable QCB₀-product. Moreover, F is the smallest such subcategory in the following sense: if C is a full cartesian closed subcategory of QCB₀ which is closed under countable product inherited from QCB₀ and contains the space ω as an object, then any F-object is homeomorphic to a C-object.

Proof. By Lemma 5.3(6), F is closed under countable QCB₀-product. By Lemma 5.3(7), F is closed under QCB₀-exponentiation, hence F is also cartesian closed.

Let now C be a subcategory with the specified properties. From the proof of Proposition 8.2 in [ScS13] it follows that exponentials formed in C are homeomorphic to the corresponding QCB₀-exponentials. Since ω is a C-object, C contains homeomorphic copies of all the spaces of continuous functionals of countable types.

Corollary 7.4 There is no full cartesian closed subcategory C of QCB₀ such that C inherits countable products from QCB₀, contains the discrete space ω of natural numbers and is contained itself in QCB₀(Σ^1_{α}) for some $\alpha < \omega_1$.

Proof. Suppose for a contradiction that C were a cartesian closed subcategory of QCB_0 with the specified properties. By the previous theorem, there is a space $E \in C$ homeomorphic to $\mathbb{N}\langle \alpha+1\rangle$. Hence $\mathbb{N}\langle \alpha+1\rangle$ has an admissible representation δ such that $EQ(\delta) \in \Sigma^1_{\alpha}(\mathcal{N}^2)$. This contradicts Theorem 6.4.

We conclude this section by formulating "finite" versions of the results above. They are proved in the same way and provide a bit of new information to some results in [ScS13]. A category is finitely complete (resp. co-complete) if it is closed under finite limits (resp. co-limits).

Theorem 7.5 The category $QCB_0(\mathbf{P})$ is the smallest full subcategory of QCB_0 that inherits exponentials, finite limits and finite co-limits from QCB_0 and contains the Sierpinski space and the space ω as objects.

Theorem 7.6 The full subcategory $F := \{1, \mathbb{N}\langle k \rangle \mid k \in \omega\}$ of QCB₀ is cartesian closed and closed under finite QCB₀-product. Moreover, F is the smallest such subcategory in the following sense: if C is a full cartesian closed subcategory of QCB₀ which is closed under finite product inherited from QCB₀ and contains the space ω as an object, then any F-object is homeomorphic to a C-object.

Remark 7.7 The above results characterize some subcategories of QCB₀ in terms of minimality in a class of subcategories of QCB₀ that enjoy certain structural properties. In most cases we have demanded that the corresponding constructions of new spaces are inherited from QCB₀. However, for some of these constructions this requirement is not necessary. We omit the details.

8 Final Remarks

We hope that the established closure properties of $QCB_0(\mathbf{HP})$ motivates to study many other spaces of interest for Computable Analysis. We give two examples of sequences of function spaces which are interesting objects of investigation.

The first example is the sequence of hyperprojective qcb₀-spaces $\{\mathbb{R}\langle\alpha\rangle\}_{\alpha<\omega_1}$, defined by induction on α as follows:

$$\mathbb{R}\langle 0\rangle := \mathbb{R}, \ \mathbb{R}\langle \alpha+1\rangle := \mathbb{R}^{\mathbb{R}\langle \alpha\rangle}, \ \text{and} \ \mathbb{R}\langle \lambda\rangle := \prod_{\alpha<\lambda} \mathbb{R}\langle \alpha\rangle,$$

where \mathbb{R} denotes the space of real numbers endowed with the standard Euclidean topology, $\alpha, \lambda < \omega_1$ and λ is a non-zero limit ordinal. We call $\mathbb{R}\langle \alpha \rangle$ the space of continuous functionals of type α over \mathbb{R} . Again, for finite ordinals we obtain the functionals over \mathbb{R} of finite types which are rather popular. Propositions 4.3 and 4.4 yield $\mathbb{R}\langle \alpha+1 \rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_\alpha)$ with an analogous proof as for Theorem 6.4. With more effort one can establish $\mathbb{R}\langle \alpha+1 \rangle \notin \mathsf{QCB}_0(\mathbf{\Sigma}^1_\alpha)$.

Recall that for any qcb₀-space X by $\mathcal{O}(X)$ we denote the hyperspace of open sets in X endowed with the Scott topology. The space $\mathcal{O}(X)$ is homeomorphic to the function space \mathbb{S}^X , where \mathbb{S} is the Sierpinski space. We define the sequence of hyperprojective qcb₀-spaces $\{\mathcal{O}^{\alpha}(X)\}_{\alpha<\omega_1}$ by induction on α as follows: $\mathcal{O}^0(X) := X$, $\mathcal{O}^{\alpha+1}(X) := \mathcal{O}(\mathcal{O}^{\alpha}(X))$, and $\mathcal{O}^{\lambda}(X) := \prod_{\alpha<\lambda} \mathcal{O}^{\alpha}(X)$, where again $\alpha, \lambda < \omega_1$ and $\lambda \neq 0$ is a limit ordinal. It seems to be worth investigating the behavior of $\{\mathbb{R}\langle\alpha\rangle\}_{\alpha<\omega_1}$ and $\{\mathcal{O}^{\alpha}(X)\}_{\alpha<\omega_1}$ with respect to the hyperprojective hierarchy of qcb₀-spaces.

References

- [Br13] M. de Brecht. Quasi-Polish spaces. Annals of pure and applied logic, 164 (2013), 356–381.
- [En89] R. Engelking. General Topology. Heldermann, Berlin, 1989.
- [EL08] M. Escardó, J. Lawson. Personal communication.
- [ELS04] M. Escardó, J. Lawson, A. Simpson. Comparing Cartesian closed Categories of Core Compactly Generated Spaces. *Topology and its Applications* 143 (2004), 105–145.
- [GH80] G. Giertz, K.H. Hoffmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott. A compendium of Continuous Lattices, Berlin, Springer, 1980.

- [Hy79] J.M.L. Hyland. Filter spaces and continuous functionals. *Annals of Mathematical Logic*, 16 (1979), 101–143.
- [Ke83] A.S. Kechris. Suslin cardinals, k-Suslin sets and the scale property in the hyperprojective hierarchy. The Cabal Seminar, v. 1: Games, Scales and Suslin Cardinals, Eds. A.S. Kechris, B. Löwe, J.R. Steel, Lecture Notes in Logic, 31, 2008, p. 314–332. (Reprinted from Lecture Notes in mathematica, No 1019, Berlin, Springer, 1983).
- [Ke95] A.S. Kechris. Classical Descriptive Set Theory. Springer, New York, 1995.
- [Kl59] S.C. Kleene. Countable functionals. *Constructivity in Mathematics* (A. Heyting, Ed.), North Holland, Amsterdam, 1959, 87–100.
- [Kr59] G. Kreisel. Interpretation of analysis by means of constructive functionals of finite types. *Constructivity in Mathematics* (A. Heyting, Ed.), North Holland, Amsterdam, 1959, 101–128.
- [KW85] C. Kreitz and K. Weihrauch. Theory of representations. *Theoretical Computer Science*, 38 (1985), 35–53.
- [No80] D. Normann. Recursion on the Countable Functionals. Lecture Notes in Mathematics, 811 (1980), Springer, Heidelberg.
- [No81] D. Normann. Countable functionals and the projective hierarchy. *Journal of Symbolic Logic*, 46.2 (1981), 209–215.
- [No99] D. Normann. The continuous functionals. *Handbook of Computability Theory* (E.R. Griffor Ed.), 1999, 251–275, Elsevier, Amsterdam.
- [Sch02] M. Schröder. Extended admissibility. Theoretical Computer Science, 284 (2002), 519–538.
- [Sch03] M. Schröder. Admissible representations for continuous computations. PhD thesis, Fachbereich Informatik, FernUniversität Hagen, 2003.
- [Sch09] M. Schröder. The sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular. Mathematical Structures in Computer Science, 19 (2009), 943–957.
- [ScS13] M. Schröder and V. Selivanov. Some Hierarchies of qcb₀-Spaces. *Mathematical Structures in Computer Science*, to appear. arXiv:1304.1647
- [ScS14] M. Schröder and V. Selivanov. Hyperprojective Hierarchy of qcb₀-space. In: Computability in Europe: Language, Life, Limits. *Lecture Notes in Computer Science*, to appear.
- [Se06] V.L. Selivanov. Towards a descriptive set theory for domain-like structures. *Theoretical Computer Science*, 365 (2006), 258–282.
- [Se13] V.L. Selivanov. Total representations. Logical Methods in Computer Science, 9(2) (2013), p. 1–30. DOI: 10.2168/LMCS-9(2:5)2013.
- [We00] K. Weihrauch. Computable Analysis. Berlin, Springer, 2000.