COMPUTATIONAL COMPLEXITY AND THE CONJUGACY PROBLEM

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ABSTRACT. The conjugacy problem for a finitely generated group G is the two-variable problem of deciding for an arbitrary pair (u, v) of elements of G, whether or not u is conjugate to v in G. We construct examples of finitely generated, computably presented groups such that for every element u_0 of G, the problem of deciding if an arbitrary element is conjugate to u_0 is decidable in quadratic time but the worst-case complexity of the global conjugacy problem is arbitrary: it can be any c.e. Turing degree , can exactly mirror the Time Hierarchy Theorem, or can be \mathcal{NP} -complete. Our groups also have the property that the conjugacy problem is generically linear time: that is, there is a linear time partial algorithm for the conjugacy problem whose domain has density 1, so hard instances are very rare. We also consider the complexity relationship of the "half-conjugacy" problem to the conjugacy problem. In the last section we discuss the extreme opposite situation: groups with algorithmically finite conjugation.

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1. INTRODUCTION

The word problem is a special case of the conjugacy problem since an element in a group G is equal to the identity if and only if it is conjugate

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to the identity. The *individual conjugacy problem*, $ICP(u_0)$, for a fixed element u_0 of a group G is the problem of deciding whether or not arbitrary elements v of G are conjugate to u_0 . The complexity of an individual conjugacy problem depends on the fixed element u_0 , and its complexity may be much less than that of the conjugacy problem in G. The word problem for semigroups resembles the conjugacy problem for groups in that it is a two variable problem since an equation u = v cannot generally be reduced to an equation with one fixed side. In 1965, J.C. Shepherdson [20] constructed finitely presented semigroups in which each individual problem $u_0 = v$? with u_0 fixed and v arbitrary, is decidable but the global word problem is undecidable of arbitrary c.e. degree. Shortly thereafter Donald Collins [7] constructed finitely presented groups in which each individual conjugacy problem was decidable but the groups had undeciable conjuacy problem of arbitrary c.e. degree. Indeed, both papers show that if one has a uniformly computable set d_i of c.e. degrees then the complexity of each individual problem problem is bounded by the join of a finite number of the d_i while the global two-variable problem can be of any c.e. degree d greater than or equal to any of the d_i .

The constructions of Collins used HNN extensions. Although HNN extensions may generally have complicated conjugacy problems, the time complexity of many individual conjugacy problems may be quite low. Miller [15] constructed a group K which is an HNN extension of a free group of finite rank with finitely generated associated subgroups in which the conjugacy problem is undecidable. However, it was shown in [5] that the individual conjugacy problem for a generic element $w_0 \in K$ is decidable in at most cubic time. See [3, 4] for other examples of HNN extensions and free products with amalgamation where the the complexity of the individual conjugacy problems of many elements is low.

In this article we construct finitely generated, computably presented groups where *all* individual conjugacy problems are decidable in quadratic time but the global conjugacy problem can have arbitrary worst-case complexity. complexity. (Note in particular that the word problem is decidable in quadratic time.) We discuss this from the viewpoints of the general theory of computability, the Time Hierarchy Theorem and \mathcal{NP} -completeness. There is also now a general awareness that many decision problems are generically easy and this phenomenon for decision problems in group theory was investigated in detail in [10]. Although having arbitrary worst-case complexity, the conjugacy problem in the groups we construct will have linear time generic-case complexity. This means that there is a linear time partial algorithm for the conjugacy problem whose domain has density 1.

The paper [6] raised the fascinating question of the "half-conjugacy problem". Suppose that we have a finitely generated group G with an algorithm which decides, given an arbitrary pair (u, v) of elements of G, whether or not u is conjugate to one of v or v^{-1} : Must G have solvable conjugacy problem? One supposes that the answer is "No", but the question seems very subtle. We do not answer the basic question but we show that for every computable function $f : \mathbb{N} \to \{0, 1\}$, there is a group P with solvable conjugacy problem in which the half-conjugacy problem is decidable in quadratic time while the conjugacy problem has time complexity greater than f. The group P also satisfies the above constraints on individual conjugacy problems and generic-case complexity.

Finally, in the last section we discuss the extreme opposite situation where complexity is the worst possible. A finitely generated group G with a computably enumerable set of defining relators is *algorithmically finite* if every infinite computably enumerable subset has two distinct words which define elements equal in G. Miasnikov and Osin [16] showed how to use the Golod-Shafarevich inequality to construct such groups. We say that a finitely generated group G has *algorithmically finite conjugation* if G has infinitely many conjugacy classes and every infinite computablely enumerable set of elements of G must contain two elements which are conjugate. We show that algorithmically finite groups have algorithmically finite conjugation.

We obtain very precise control over complexity by using non-metric small cancellation theory so we first review the condition which which use. This condition ensures that the structure of conjugacy diagrams in the groups we construct is very simple, thus proving the desired results. We note that our groups require using an infinite number of relators since any group with a finite presentation satisfying our condition has global conjugacy problem decidable in quadratic time. We then review each desired complexity condition and discuss the corresponding groups in separate sections. Howevery, there is really only one basic construction and the different cases require only small adjustments in the defining relators.

We thank the referee for very helpful suggestions.

2. Non-metic small cancellation theory

We construct the desired groups using small cancellation theory. For this we need results developed in Chapter V of Lyndon and Schupp [13] but we recall some essential definitions and details here.

Definition 1. Let $F = \langle X \rangle$ be a finitely generated free group. A subset R of F is symmetrized if all elements of R are cyclically reduced and, for each $r \in R$, all cyclic permutations of both r and r^{-1} are also in R. We write $r \equiv bc$ if b is an initial segment of r and c consists of the remaining letters of r, so bc is reduced without cancellation. If $r_1 \neq r_2$ are distinct elements of R with $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$ where b is nonempty, then b is called a piece relative to R.

The basic non-metric small cancellation condition is

Condition C(p): No element of R is a product of fewer than p pieces.

The sets of defining relators which we construct will satisfy the condition C(20). Even though we will not have a metric condition on the lengths of pieces, we need a good notion of *reduction*.

Definition 2. Fix a symmetrized subset R of the free group $F = \langle X \rangle$. We assume that all generators are pieces. If w is any cyclically reduced word, consider the factorization $w \equiv b_1b_2...b_l$ of w into maximal pieces. That is, each b_i is a piece and, if b_i is not a suffix of w then b_iy is not a piece where y is the letter following b_i in w. The integer l is the piece length of w with respect to R. We denote the piece length of w by $||w||_R$.

The following lemma is immediate.

Lemma 2.1. Let R be a symmetrized subset of F. If w is a cyclically reduced word and w' is a cyclic permutation of w then $||w||_R$ and $||w'||_R$ differ by at most 1.

We need a notion of reduction tailored to the sets of relators which we will use.

Definition 2.2. A word w contains an element of R with at most k pieces missing if $w \equiv usv$ and there is an element $r \in R$ with $r \equiv st$ where $||t||_R \leq k$. A word w is weakly cyclically R-reduced if w is cyclically reduced in the free group F, w does not begin and end with powers of the same generator unless w is simply a power of a generator, and no cyclic permutation of w contains a relator from R with at most 7 pieces missing.

The basic result of small cancellation theory is that if $G = \langle X; R \rangle$ where R satisfies the condition C(p) with $p \ge 6$ then every nontrivial word w which is equal to the identity in G contains a subword which is an element of R with at most 3 pieces missing. (If R satisfies the metric condition $C'(\frac{1}{6})$, so that the length of any piece is less than $\frac{1}{6}$ of the length of any element of R in which it occurs, we then have Dehn's Algorithm.)

3. Groups with global conjugacy problem of arbitrary c.e. Degree

A set A of positive integers is *computably enumerable*, written c.e., if there is a Turing machine M which enumerates all the elements of A. The basic relation between computably enumerable sets is that of Turing reducibility. A set A is *Turing reducible* to a set B, denoted $A \leq_T B$, if there is an oracle Turing machine M^B with an oracle for B which computes A. Two sets Aand B are *Turing equivalent* if $A \leq_T B$ and $B \leq_T A$. Turing equivalence is indeed an equivalence relation and equivalence classes are called *Turing degrees*. A Turing degree d is *c.e.* if it contains a c.e. set. (Not all sets in a nonzero c.e. degree are themselves c.e. since a set is always Turing equivalent to its complement.) The c.e. Turing degrees are partially ordered by Turing reducibility and it is an important fact that there infinitely many distinct c.e. Turing degrees. Traditionally, a recursive presentation of a group is a group presentation $\langle X; R \rangle$ where the set X of generators is finite and the set R of defining relators is computably enumerable. A computable presentation of a group is a group presentation $\langle X; R \rangle$ where the set X of generators is finite and the set R of defining relators is computable. We use these terms as distinguishing two different classes of presentations. If a group G has a recursive presentation it also has a computable presentation, but this requires changing the presentation.

It was shown in the early 1970's that the word and conjugacy problems for recursively presented, indeed, finitely presented, groups mirror all possible relations of Turing reducibility between c.e. Turing degrees: The c.e. Turing degrees d_1 and d_2 satisfy $d_1 \leq_T d_2$ if and only if there is a recursively presented (finitely presented) group G with word problem of Turing degree d_1 and conjugacy problem of Turing degree d_2 . The result for recursive presentations is in Miller [14] and the stronger version for finite presentations is due to Collins [8].

We will show

Theorem 3.1. For every c.e. Turing degree **d** there is computably presented group G such that for every fixed element u_0 of G, the problem of deciding if an arbitrary element is conjugate to u_0 is decidable in quadratic time but the conjugacy problem of G has degree **d** but linear time generic-case complexity.

Alexander Ol'shanskii and Mark Sapir [17] proved the deep theorem that any finitely generated group with solvable conjugacy problem can be embedded in a finitely presented group with solvable conjugacy problem, establishing a direct conjugacy analog of the Higman Embedding Theorem. Such a result depends on Ol'shanskii and Sapir's development of the theory of Smachines. It seems plausible that a very detailed analysis of the definitions and lemmas in their monograph applied to our groups would yield finitely presented groups where the complexity of all the individual conjugacy problems is bounded by a fixed tower of exponentials while the global conjugacy problem is undecidable, but this would need to be carefully verified.

Since we want to discuss computational complexity, we need to be precise about how the lengths of inputs are measured. The standard length function for free groups is essentially a unary notation since it requires reduced words to be completely written out. Thus the length of a^i is *i*. We therefore use a unary notation for positive integers, representing *n* by a repetition of *n* identical symbols. It is a basic fact that if *A* is any infinite c.e. set, there is a *computable bijection* $f : \mathbb{N}^+ \to A$. Given an infinite c.e. set *A* not containing 0, fix such an *f* and let *M* be a Turing machine which, on input *i* written in unary, computes j = f(i) in unary. Let t_i be the number of steps used by *M* in computing f(i). Let $F = \langle a_1, \ldots, a_{20}, b_1, \ldots, b_{20}, c_1, c_2, c_3, d_1, d_2, d_3 \rangle$. The defining relators for the group *G* for *A* will be the symmetrized closure *R* of the the set $\{r_i : i \in \mathbb{N}^+\}$ of relators where $r_i = a_1{}^j \dots a_{20}{}^j c_1{}^i d_1{}^{t_i} c_2{}^i d_2{}^{t_i} c_3{}^i d_3{}^{t_i} b_{20}{}^{-j} \dots b_1{}^{-j} d_3{}^{-t_i} c_3{}^{-i} d_2{}^{-t_i} c_2{}^{-i} d_1{}^{-t_i} c_1{}^{-i}, j = f(i)$

Given the defining relators R, it will be immediate from small cancellation theory that

$$a_1{}^j \dots a_{20}{}^j \sim b_1{}^j \dots b_{20}{}^j \iff j \in A.$$

and thus the conjugacy problem in G has the same Turing degree as A. After discussing the geometry of conjugacy diagrams it will be clear that this is essentially the only difficult case, establishing the desired results.

We first verify that R satisfies the piece condition C(20). The conjugating subwords of the relator r_i are the subword

$$c_1{}^i d_1{}^{t_i} c_2{}^i d_2{}^{t_i} c_3{}^i d_3{}^{t_i}$$

and its inverse. A conjugating part of a relator is of course a piece. Note that for any full power of a generator a_k or b_k or c_l occurring in an r_i , say $a_k{}^j$, that power flanked on *both* sides by occurrences of the neighboring generators, say $a_{k-1}a_k{}^ja_{k+1}$, is *not* a piece. Since the function f is one-to-one, $a_k{}^j$ and $c_l{}^i$ occur only in the relator r_i where f(i) = j. On the other hand, the subword consisting of two successive powers, say $a_k{}^ja_{k+1}{}^j$, is a piece because the set A is infinite and there are i' > i, j' > j with $f(i') = j' \in A$ so $a_k{}^{j'}a_{k+1}{}^{j'}$ occurs in the relator $r_{i'}$. Since the time t_i required to compute f(i) and the time $t_{i'}$ required to compute f(i') may be the same for different values of i and i', the three syllables at either end of the conjugating part, say $c_3{}^{i}d_3{}^{t_i}b_{20}{}^{-j}$, may occur in different relators.

The relators have been chosen so that the following lemma holds. Indeed, it will hold for the set of relators of all the groups which we construct.

Lemma 3.2 (Reduction Lemma). There is an algorithm which, given an arbitrary word w, calculates a weakly cyclically R-reduced conjugate of w in quadratic time.

Proof. First, in linear time we calculate a cyclically reduced conjugate w' in the free group which does not begin and end with powers of the same generator, unless it is simply a power of a generator. In the latter case, it weakly cyclically *R*-reduced. The point of the form of the relators is that a relator with at most 7 pieces missing must contain a *critical subword* of either the form

$$a_{20}{}^{j}c_{1}{}^{i}d_{1}{}^{t_{i}}c_{2}{}^{i}d_{2}{}^{t_{i}}c_{3}{}^{i}d_{3}{}^{t_{i}}b_{20}{}^{-j}$$

or the form

$$b_1^{-j}d_3^{-t_i}c_3^{-i}d_2^{-t_i}c_2^{-i}d_1^{-t_i}c_1^{-i}a_1^{-j}d_1^{-t_i}c_1^{-i}a_1^{-j}d_1^$$

or their inverses. This is because there are 9 pieces between the occurrences of critical subwords in the defining relators. On seeing such a subword in w', run the Turing machine M which calculates f for t_i steps on input i

and see if M calculates j in exactly that number of steps. If so, we indeed have part of a conjugate of the relator r_i or its inverse. Next, check that the part of w' containing the above subword has the correct form, that is, generators are in the correct order and have the appropriate powers j or -j. If we expand the occurrence to a subword s which is part of a relator $r \equiv st$ and t is a product of at most 7 pieces, replace s by t^{-1} and freely cyclically reduce the resulting word to obtain w''. Note that the piece length of w'' is at least 12 less than the piece length of w. Repeat until we do not find any critical subwords. The proof for the other sets of relators we construct will be the same.

We review the idea of generic-case complexity from the paper [10]. Let Σ be a nonempty finite alphabet and let $S \subseteq \Sigma^*$. The density of S at n, written $\rho_n(S)$, is the number of words in S of length less than or equal to n divided by the number of all words of length less than or equal to n. If limit $\rho(S) = \lim_{n\to\infty} \rho_n(S) = 1$ we say that S is generic in Σ^* . A particular decision problem D on words over Σ^* is said to generically computable in time T(n) if there is a partial algorithm Φ for D which answers correctly on an input w in time T(|w|) or else does not give an answer and such that the domain of Φ is generic in Σ^* .

We point out that while worst-case complexity of the word or conjugacy problems is independent of a given presentation for a finitely generated group, this is not the case for generic-case complexity. To show that a decision problem having a certain generic-case complexity is a property of the group G one needs to show that for every finitely generated presentation of G there is a partial algorithm working in the given time bound. For the groups which we construct, the conjugacy problem is generically linear time by Theorem C of [10] since our groups have infinite abelianizations containing \mathbb{Z}^6 . From the form of the relators, after abelianization the c and d generators disappear, so they generate a free abelian subgroup in the abelianization, and the abelianization is independent of presentation.

4. The groups for the Time Hierarchy Theorem

We now want to mirror the Time Hierarchy Theorem. We discuss a few details of the proof following the presentation in Arora and Barak [2]. They consider Turing machines with a special input tape, some number $k \geq 1$ of work tapes and an output tape. Such a machine is completely determined by a complete listing of its transition function, which can easily be encoded by a string of 0's and 1's beginning with 1, and thus can be regarded as a binary number. Many strings are not valid codes of Turing machines. We then express strings coding Turing machines as unary numbers and M_x is the Turing machine coded by a unary x. Of course, there are infinitely many numbers coding machines with exactly the same behavior. (Just add an arbitrary number of non-reachable states.) If $f(n) \ge n$ is a fully time constructable function then there is a language $L_f \subseteq \{1\}^*$ with $L \in TIME(f^2(n))$ but such that $L_f \notin TIME(f(n))$. It is a fact that there is a universal Turing machine \mathcal{U} which, on input x, simulates M_x on input x in time |x|log(|x|). Now consider the Turing machine $\widehat{\mathcal{M}}$ which, on unary input x, halts and outputs 1 if x does not code a Turing machine and otherwise uses the universal machine \mathcal{U} to simulate M_x on input x for $f(x)^{1.5}$ steps. If M_x halts and outputs 1 then $\widehat{\mathcal{M}}$ halts and outputs 0. Otherwise $\widehat{\mathcal{M}}$ halts and outputs 1. Let L_f be the set of unary inputs on which $\widehat{\mathcal{M}}$ halts and outputs 1. Then $L_f \subseteq DTIME(f^2(n))$ but $L_f \notin DTIME(f(n))$. If L_f were in DTIME(f(n)), there would a Turing machine \mathcal{M} which obtains the same output as $\widehat{\mathcal{M}}$ in time f(|n|). For any constant c, there is an n_0 such that $n^2 > cnlog(n)$ for all $n > n_0$. Since there are infinitely many Turing machines with the same behavior as \mathcal{M} , let x be the code of such a machine with $x > n_0$. Then M_x would obtain the same result as $\widehat{\mathcal{M}}$ on input x in time f(|x|), a contradiction.

We again use the free group $F = \langle a_1, \ldots, a_{20}, b_1, \ldots, b_{20}, c_1, c_2, c_3, d_1, d_2, d_3 \rangle$. Now let t_i be the time used by \widehat{M} in deciding if $i \in L_f$. We will use almost the same set of relators as before. The defining relators for the group H for f will be the symmetrized closure R of the following set $\{r_j : j \in L_f\}$ of relators where we now have

$$r_j = a_1{}^j \dots a_{20}{}^j c_1{}^j d_1{}^{t_j} c_2{}^j d_2{}^{t_j} c_3{}^j d_3{}^{t_j} b_{20}{}^{-j} \dots b_1{}^{-j} d_3{}^{-t_j} c_3{}^{-j} d_2{}^{-t_j} c_2{}^{-j} d_1{}^{-t_j} c_1{}^{-j}, j \in L_f$$

The same considerations as before shows that this R satisfies the piece condition C(20) and that there is a quadratic time algorithm which, given w, calculates a weakly cyclically reduced conjugate of w. Given the defining relators R, it will again be immediate from small cancellation theory that

$$a_1{}^j \dots a_{20}{}^j \sim b_1{}^j \dots b_{20}{}^j \iff j \in L_f.$$

and that the conjugacy problem for H is not in DTIME(f(n)) but is in $DTIME(f^2(n))$.

We have

Theorem 4.1. For every fully time constructable function f with $f(n) \ge n^2$, there is a computably presented group H such that for every fixed element u_0 of H, the problem of deciding if an arbitrary element is conjugate to u_0 is decidable in quadratic time while the conjugacy problem of H is decidable in time $f^2(n)$ but is not decidable in time f(n). The conjugacy problem of H has linear time generic-case complexity.

5. The half-conjugacy problem

Our result on the half-conjugacy problem is the following.

Theorem 5.1. For every computable function $f : \mathbb{N} \to \{0, 1\}$ there is a computably presented group P with solvable conjugacy problem for which the

half-conjugacy problem is solvable in quadratic time but the time complexity function of the conjugacy problem satisfies T(n) > f(n) for all $n \ge 1$. The group P retains the property that all individual conjugacy problems are decidable in quadratic time and the conjugacy problem has linear time genericcase complexity.

For the half-conjugacy problem, now let f be any computable function f: $\mathbb{N}^+ \to \{0, 1\}$. One can construct a Turing machine \widehat{M} which, on unary input i, halts and outputs 1 if i does not code a Turing machine and otherwise uses the universal machine \mathcal{U} to simulate M_i on input i for f(40i) steps. If M_i halts and outputs 1 then \widehat{M} halts and outputs 0. Otherwise \widehat{M} halts and outputs 1. The machine \widehat{M} computes a total function g. Let t_i be the time used by \widehat{M} in computing g(i). Let L_g be the set of unary inputs on which outputs 1. Since we have diagonalized over f, L_g is computable in time g but not in time f.

Again let $F = \langle a_1, \ldots, a_{20}, b_1, \ldots, b_{20}, c_1, c_2, c_3, d_1, d_2, d_3 \rangle$. The set R of defining relators will be $\{r_i : i \ge 1\}$ where for $i \in L_g$,

$$r_i = a_1{}^i \dots a_{20}{}^i c_1{}^i d_1{}^{t_i} c_2{}^i d_2{}^{t_i} c_3{}^i d_3{}^{t_i} b_{20}{}^{-i} \dots b_1{}^{-i} d_3{}^{-t_i} c_3{}^{-i} d_2{}^{-t_i} c_2{}^{-i} d_1{}^{-t_i} c_1{}^{-i},$$

while for $i \notin L_a$,

$$r_i = a_1^{i} \dots a_{20}^{i} c_1^{i} d_1^{t_i} c_2^{i} d_2^{t_i} c_3^{i} d_3^{t_i} b_1^{i} \dots b_{20}^{i} d_3^{-t_i} c_3^{-i} d_2^{-t_i} c_2^{-i} d_1^{-t_i} c_1^{-i}.$$

The set R again satisfies the piece condition C(20). Note that the relators show that for all $i \ge 1$ the element $a_1^i \ldots a_{20}^i$ is conjugate to either $b_1^i \ldots b_{20}^i$ or to its inverse, but which possibility holds depends on whether or not $i \in L_g$. Furthermore, any algorithm for the conjugacy problem decides membership in L_g and so takes as much time as g.

6. The geometry of conjugacy diagrams

Although various results of small cancellation theory are often stated for a metric small cancellation condition, results about the geometry of the relevant diagrams depend only on the appropriate piece condition. We have seen that given an arbitrary element w we can effectively find a weakly cyclically *R*-reduced conjugate u of w in quadratic time. What we now need is that given two weakly cyclically *R*-reduced words u and v which are not equal to the identity in G and which are conjugate in G, the conjugacy diagram Δ for u and v satisfies the conclusion of Theorem 5.5 of Chapter V of Lyndon-Schupp. ([13], page 257.)

Theorem 6.1. Fix any group G among the groups we have constructed. Let u and v be two nontrivial weakly cyclically R-reduced words which are conjugate in G and let Δ be a reduced conjugacy diagram for u and v with outer boundary σ and inner boundary τ . Then every region of Δ has edges on both σ and τ , has at most two interior edges and has no interior vertices.

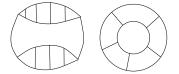


FIGURE 1. Possible conjugacy diagrams

In short, the theorem says that Δ "looks like" one of the diagrams in Figure 1. The essential difference between the two pictures is whether or not the inner and outer boundaries have any vertices in common.

That u and v are weakly cyclically R-reduced means that no cyclic permutation of either contains an element of R with at most 7 pieces missing. Since R satisfies the condition C(20), this means that any region D which intersected only σ or τ and such that the intersection is a consecutive part of the boundary would have interior degree at least 12, which will be impossible by the counting formulas. These formulas depend only on the piece condition and thus the conclusion of Theorem 5.5 follows just as in the metric case.

We give the detailed argument for the groups G with conjugacy problem of desired c.e. degree. Given an arbitrary nonidentity element u_0 of G, fix a weakly cyclically reduced conjugate u of u_0 . For powers a_k^l or b_p^q which occur in u, we need to know if k or q are in A and, if so, what arguments of the function f give those values. We claim that this finite amount of information suffices to decide conjugacy to u and thus conjugacy to u_0 .

Given an element v', we can calculate a weakly cyclically *R*-reduced conjugate v of v' in quadratic time by the Reduction Lemma, Lemma 3.2. If $v \sim u$ in the free group F, they are conjugate in G. If not, but $v \sim u$ in G, there is a minimal conjugacy diagram Δ for u and v containing at least one region. Let σ , labelled by u, be the outer boundary of Δ , and let τ , labelled by v, be the inner boundary of Δ . The structure theorem shows that, since the piece length of the intersection of the boundary of any region D with either the outer or inner boundary of Δ is a most 12, the intersection of the boundary of D with both σ and τ must contain several occurrences of generators to the same power j. Since we are just considering conjugacy to u, the free group length C = |u| of u is a constant in this algorithm.

In a conjugacy diagram Δ for u and v, say that an edge $e \in \sigma$ and an edge $f \in \tau$ are "opposite each other" if one of the following conditions holds:

- (1) the two edges are on both boundaries and coincide;
- (2) the edges form the beginning of an island in that the edge preceding them is on both boundaries and $f^{-1}e$ are successive edges on the boundary of a region of Δ ;
- (3) $f^{-1}he$ is a successive part of the boundary of a region D and h is the label on an interior edge separating D from another region.

The main point is that if we choose a letter y in u and a letter z in v and suppose that they are the labels on edges which are opposite each other in Δ , then is only one way to fill in the rest of the conjugacy diagram and we can calculate whether or not a valid conjugacy diagram with this initial condition exists in linear time. If we succeed in constructing a valid conjugacy diagram for one of the 3C|v| possible initial conditions, then u and v are conjugate, and if we cannot construct a conjugacy diagram then they are not conjugate.

It is also clear from the structure of conjugacy diagrams that

$$a_1{}^j \dots a_{20}{}^j \sim b_1{}^j \dots b_{20}{}^j \iff j \in A.$$

so a solution to the conjugacy problem for G decides membership in A. On the other hand, given an oracle for the set A we can calculate weakly cyclically R-reduced conjugates for any pair of elements and then apply the method above to decide conjugacy and the conjugacy problem for G is Turing equivalent to deciding membership in A.

For the groups H for the Time Hierarchy Theorem it is clear that

$$a_1^i \dots a_{20}^i \sim b_1^i \dots b_{20}^i \iff i \in L_f$$

so the conjugacy problem cannot be calculated in time f.

The conjugating part of the defining relators has been chosen so that the only way that $a_{20}c_1{}^i d_1$ or $d_1c_2{}^i d_2$ or $d_2c_2{}^i d_3$ or their inverses can be subwords of the label on an interior edge in a reduced diagram is if they are matched against the corresponding generators of the other conjugating part of the same relator. And since the boundary labels of a conjugacy diagram are freely reduced, the entire conjugating parts must then be exactly matched and appear as the label on the interior edge. If all interior edges are so labelled, the diagram shows that a power of some $a_1{}^i \dots a_{20}{}^i$ is conjugacy to the same power of $b_1{}^i \dots b_{20}{}^i$.

If it is the case that for some region of the conjugacy diagram a conjugating part of the relator is not completely matched against its inverse, then some $c_l{}^i d_l{}^{t_i}$ occurs on one of the boundaries and the padding allows us to calculate in linear time if this is indeed a correct part of a relator. In this case we can again see if one can construct a valid conjugacy diagram in quadratic time. So the algorithm in time $f^2(n)$ for membership in L_f solves the conjugacy problem for H.

The remarks for the groups H apply exactly to the groups P for the halfconjugacy problem. The only hard case is when a power of $a_1{}^i \dots a_{20}{}^i$ is conjugate to the corresponding power of $b_1{}^i \dots b_{20}{}^i$ or to its inverse. The relators force one of the two possibilities to hold but deciding which one requires deciding membership in L_g . So the conjugacy problem for these groups is solvable in time g but not in time f.

7. A group with \mathcal{NP} -complete conjugacy problem

We now want to construct a computably presented group G with \mathcal{NP} complete conjugacy problem while keeping the constraint that all individual conjugacy problems are decidable in quadratic time and that the conjugacy problem for the given presentation is strongly generically quadratic time. The previous results on imitating the Time-Hierarchy Theorem and the half-conjugacy problem depended on using the free group unary notation since this notation gives enough padding in the conjugating parts of the relators to check the correctness of relators. Of course, elements of a free group also have a unique normal form with exponents, where we write powers of generators as the name of the generator with a decimal exponent. A sullable is such a power of a generator and we require that adjacent sullables are powers of distinct generators. For example, one element of the free group $\langle a, b, c \rangle$ is $w = a^{25}b^{-17}a^{-33}c^3$. The *length* of a normal form with exponents is the total number of symbols in the normal form. Thus |w| = 13 for the example just given. Basic decision problems in free groups mainly retain their polynomial-time decidability in this notation. For example, Gurevich and Schupp [9] show that the uniform membership problem for finitely generated subgroups of a free group remains in polynomial time when elements are written in exponent normal form. We need to use exponent normal form in order to have a coding of the satisfiability problem for Boolean expressions where the coded length is proportional to the length of the standard coding of such expressions.

The problem 3-SAT is the satisfiability problem for Boolean expressions which are conjunctions of clauses, each of which contains exactly three literals. A literal is the symbol x with positive decimal subscript, representing a variable, or its negation. For example, the expression

$$(x_1 \lor x_3 \lor \neg x_7) \land (\neg x_4 \lor x_7 \lor x_{11}) \land (x_1 \lor x_7 \lor \neg x_9) \land (\neg x_3 \lor x_4 \lor x_9)$$

is an instance of 3-SAT. A basic result of complexity theory is that 3-SAT is \mathcal{NP} -complete. Variables may be repeated in a clause but we assume that a clause does not both a variable and its negation.

We will represent the clause $(x_i \vee x_j \vee x_k)$ as $a^i b^j c^k$ with the exponent negative if the variable is negated. We represent a conjunction of clauses by the concatenation of the representatives of the clauses. Thus we represent the example of 3-SAT given above by

$$a^{1}b^{3}c^{-7}a^{-4}b^{7}c^{11}a^{1}b^{7}c^{-9}a^{-3}b^{4}c^{9}$$

So the length of our basic coding is even shorter than the standard coding of instances of 3-SAT.

In order to represent 3-SAT we need to code all its instances. To do this we put a "short-lex" well-ordering on \mathbb{Z}^3 as follows. The *index* of a triple (z_1, z_2, z_3) is $|z_1| + |z_2| + |z_3|$, the sum of the absolute values of the z_i . We order triples first by index, and within the same index lexicographically but with positive values preceding negative ones. Note that we consider only triples which do not contain 0 since subscripts of variables are positive.

We define an enumeration $\mathcal{E} = \{\eta_i\}$ of all instances of 3-SAT as follows. We enumerate all pairs (m, n) of positive integers in the usual way. When a pair (m, n) is enumerated, we then enumerate all instances of 3-SAT where there are at most m clauses and the maximum index of any clause is at most n and the instance has not previously been enumerated. The clauses in an instance are concatenated in the short-lex order defined above and we order instances lexicograpically.

A language $L \in \mathcal{NP}$ is characterized by the fact that for every instance which is in L, there is a short *certificate*, given which one can verify in polynomial time that the instance is indeed in L. For 3-SAT this certificate is an assignment of truth values showing that the instance is actually satisfiable. For an instance of 3-SAT which is satisfiable we choose the the first satisfying assignment in the usual truth-table order.

We code this satisfying assignment using generators x, y, z in the following way. For our running example given above, the first line of the truth-table assigning all variables the value false satisfies the instance. So we code this assignment as

$$x^{-1}y^{-3}z^{-7}x^{-4}y^{-7}z^{-11}x^{-1}y^{-7}z^{-9}x^{-3}y^{-4}z^{-9}$$

The absolute values of the exponents again represent the variables and the sign of the exponent is negative if the variable is assigned the value false and positive if the variable is assigned the value.

The defining relators we use will be words in exponential normal form in the free group F on generators

$a_1, b_1, c_1, \dots, a_{20}, b_{20}, c_{20}, d_1, e_1, f_1, \dots, d_{20}, e_{20}, f_{20}, u_1, v_1, w_1, \dots, u_3, v_3, w_3, x_1, y_1, z_1, x_2, y_2, z_2, \dots, u_{10}, u_{10}$

If η_j is a satisfiable instance in the enumeration \mathcal{E} defined above, let $\alpha_{\eta_j,l}$ denote the coding of this instance on the generators a_l, b_l, c_l for $1 \leq l \leq 20$ and let $\beta_{\eta_j,l}$ denote the coding of this instance on the generators d_l, e_l, f_j for $1 \leq l \leq 21$. Let $\gamma_{\eta_j,l}$ represent the coding of this instance on the generators u_l, v_l, w_l for $1 \leq l \leq 3$. Finally, let $\delta_{\eta_j,l}$ represent the coding of the satisfying truth assignment for this instance on the generators $x_l, y_l, z_l, l = 1, 2$.

The basic defining relators for our group G is the set $R = \{r_{\eta_j}\}$ where r_{η_j} is

$$\alpha_{\eta_j,1}...\alpha_{\eta_j,20}(\gamma_{\eta_j,1}\delta_{\eta_j,1}\gamma_{\eta_j,2}\delta_{\eta_j,2}\gamma_{\eta_j,3})\beta_{\eta_j,20}^{-1}...\beta_{\eta_j,1}^{-1}(\gamma_{\eta_j,3}^{-1}\delta_{\eta_j,2}^{-1}\gamma_{\eta_j,2}^{-1}\delta_{\eta_j,2}^{-1}\gamma_{\eta_j,1}^{-1})$$

and where η_j ranges over all *satisfiable* instances in the enumeration \mathcal{E} . Note that although the Greek letters now represent long words on the given generators, taking them to correspond to the Roman letters of the previous groups shows that the general form of the defining relators is essentially the same as before.

We now need to use small cancellation theory over free products, which is essentially like small cancellation theory over free groups. For technical details we again refer to Lyndon-Schupp[13], but we review some basic definitions. A free group F with a specified free basis can be viewed as the free product of the infinite cyclic groups generated by the specified generators. The free product normal form is given by the normal form with exponents but the *free product length* |u| of a normal form is just the number of syllables in u. We now view F as this free product.

If $u = ys_1$ and $v = s_2z$ are free product normal forms with the last syllable of u and the first syllable of v in the same factor, then there is *cancellation* in the product uv if $s_2 = s_1^{-1}$ and *consolidation* in the product uv if $s_2 \neq s_1^{-1}$. An element $u = y_1...y_n$ is *weakly cyclically reduced* if $|u| \leq 1$ or $y_n \neq y_1^{-1}$. A set $R \subset F$ is symmetrized if every element of R is weakly cyclically reduced and if $r \in R$ then every weakly cyclically reduced conjugate of r and r^{-1} is also in R.

An element w has semi-reduced form uv if there is no cancellation in the product uv. Note that consolidation is allowed. An element p is a piece relative to R if R contains distinct elements r_1 and r_2 with semi-reduced forms $r_1 = py_1$ and $r_2 = p^{-1}y_2$. The metric small cancellation condition is now

Condition $C'(\lambda)$: If $r \in R$ has semi-reduced form r = py where p is a piece, then $|p| < \lambda |r|$. Also, every element of R has length greater than $1/\lambda$.

The set R of defining relators we now use is the symmetrized closure of the set R defined above, that is, all weakly cyclically reduced conjugates of elements of R and their inverses. Since there can be only one basic relator corresponding to a particular instance of 3-SAT, it is easy to see that the maximum length of a piece relative to R is at most the length of the conjugating part of the relator. Thus our set R of relators satisfies C'(1/9).

Given this small cancellation condition the geometry of conjugacy diagrams is the same as in the case of quotients of free groups (not viewed as free products). Thus in the quotient group F/N where N is the normal closure of R we have

$\alpha_{\eta_j,1}...\alpha_{\eta_j,20} \sim \beta_{\eta_j,1}...\beta_{\eta_j,20}$

if and only if the instance η_j is satisfiable. It follows exactly as in our previous discussion that powers of variants of the above are the only hard instances. Thus each individual conjugacy problem is decidable in quadratic time and the conjugacy problem as a whole is generically linear time. Thus the conjugacy problem of G is \mathcal{NP} -complete and the other requirements are met.

8. GROUPS WITH ALGORITHMICALLY FINITE CONJUGATION

We have shown that one can bound the complexity of all individual conjugacy problems while making the global conjugacy problem arbitrarily complex. In Miller's famous examples of residually finite, finitely presented groups G with undecidable conjugacy problem, there is always some element q for which ICP(q) is undecidable (Lemma 4 on page 27 in [?]). However, [5] shows that the individual conjugacy problems ICP(w) in Miller's group are solvable in polynomial time for for all w from an exponentially generic subset of G. This leads one to ask about the opposite phenomenon.

Question 1. Are there recursively presented groups G with solvable word problem such that if the individual conjugacy problems are decidable on a computably enumerable subset $Y \subseteq G$ then Y is negligible, or indeed exponentially negligible?

Although Theorem 3.1 shows that there is no general effective way to build a uniform decision algorithm for the conjugacy problem from solutions of the individual conjugacy problems, the following general lemma holds.

Lemma 8.1. Let $G = \langle X; R \rangle$ be a recursively presented group. If $W = \{w_1, ..., w_n\}$ is a finite set of pairwise nonconjugate elements of G then there is a partial algorithm Φ which decides the conjugacy problem on the union $Z = \bigcup_{i=1,...,n} w_i^G$ of the conjugacy classes of the $w_i \in W$.

Proof. The partial algorithm Φ works as follows. Since G is recursively presented, when given elements $u, v \in G$, we can begin enumerating in parallel all words equal in G to conjugates of the $w_i \in W$. If u and v are in Z, they will both eventually be enumerated in this process. If they are enumerated as conjugates of the same w_i they represent conjugate elements of G. If they are enumerated as conjugates of distinct w_i and w_j they are not conjugate in G.

Corollary 1. A recursively presented group with only finitely many conjugacy classes has solvable conjugacy problem.

We say that a recursively presented group G has algorithmically finite conjugation if G has infinitely many conjugacy classes and every infinite c.e. set of elements of G must contain two elements which are conjugate. We note that this condition implies that if Y is a c.e. set of elements of G for which there is a partial algorithm Φ solving the conjugacy problem for elements of Y then Y must have elements from only finitely many conjugacy classes and we are in the situation of the above lemma. Given a partial algorithm Φ deciding conjugacy for a set Y containing infinitely many pairwise nonconjugate elements, we could computably enumerate an infinite set S of pairwise non-conjugate elements of G as follows. Let s_0 be the first element in the enumeration of Y. Now use Φ and the enumeration Y until we find an element s_1 not conjugate to s_0 . Continue this process, finding at the *n*-th stage, an element s_n not conjugate to any of $s_0, ..., s_{n-1}$.

Thus the conjugacy problem is as bad as possible in a group with algorithmically finite conjugation.

Recall that a group G generated by a finite set X is termed algorithmically finite [16, 11] if every infinite computably enumerable subset of F(X) has two distinct words which define equal elements in G. In other words one can computably enumerate only a finite set of words in F(X) which define pairwise distinct elements of G. Infinite, recursively presented, algorithmically finite groups are also called Dehn Monsters and have been shown to exist [16]. Indeed, there are even residually finite Dehn Monsters[11, 12]. We next observe that any Dehn Monster has algorithmically finite conjugacy.

Theorem 8.1. Let G be an infinite, recursively presented, algorithmically finite group generated by a finite set X. Then:

- 1) G has infinitely many conjugacy classes;
- 2) G has algorithmically finite conjugation;

Proof. Since $G = \langle X; R \rangle$ has unsolvable word problem it has unsolvable conjugacy problem and hence must have infinitely many conjugacy classes by the above corollary. That G has algorithmically finite conjugacy is immediate since any infinite c.e. set must have two distinct words which are equal in G and thus certainly conjugate.

In Dehn Monsters one can solve the conjugacy problem only on finite unions of conjugacy classes, but there is still a question about the asymptotic density of single conjugacy classes. There are non-amenable finitely generated groups with finitely many conjugacy classes [18], so not all conjugacy classes in such groups are negligible. Also, the question of whether or not there are finitely generated, residually finite, non-amenable groups with only finitely many conjugacy classes seems to be open.

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