Intersection points of planar curves can be computed

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Abstract

Consider two paths $\phi, \psi : [0; 1] \to [0; 1]^2$ in the unit square such that $\phi(0) = (0, 0)$, $\phi(1) = (1, 1)$, $\psi(0) = (0, 1)$ and $\psi(1) = (1, 0)$. By continuity of ϕ and ψ there is a point of intersection. We prove that from ϕ and ψ we can compute closed intervals $S_{\phi}, S_{\psi} \subseteq [0; 1]$ such that $\phi(S_{\phi}) = \psi(S_{\psi})$.

1 Introduction

A path in the Euclidean plane is a continuous function $f:[0;1] \to \mathbb{R}^2$, a curve is the range of a path. The following is known about planar curves.

Theorem 1.1 Let $\phi, \psi : [0;1] \to [0;1]^2$ be two paths in the unit square such that

$$\phi(0) = (0,0), \ \phi(1) = (1,1), \ \psi(0) = (0,1) \ \text{and} \ \psi(1) = (1,0).$$
 (1)

Then the two curves range(ϕ) and range(ψ) intersect.

Figure 1 visualizes the theorem. In Markov-style computable analysis [4] Manukyan [7] has proved a related theorem (in Russian), cited in [5, Page 279] as follows:

Theorem 1.2 (Manukyan) There are two constructive (and therefore continuous) planar curves φ_1 and φ_2 such that

$$\varphi_1(0) = (0,0), \ \varphi_1(1) = (1,1), \ \varphi_2(0) = (0,1), \ \varphi_1(1) = (1,0),$$
 (2)

for every
$$0 < t < 1$$
 both $\varphi_1(t)$ and $\varphi_2(t)$ belong to the open unit square, (3)

the paths of
$$\varphi_1$$
 and φ_2 do not intersect. (4)

While in (Grzegorczyk-Lacombe- [2, 6]) computable analysis the following has been proved [10]:

Theorem 1.3 (Weihrauch) If ϕ and ψ in Theorem 1.1 are computable then there is a computable point $x \in \text{range}(\phi) \cap \text{range}(\psi)$.

This is not contradictory. In Markov's approach only functions on the computable real numbers which are encoded by Gödel numbers are considered and computations transform Gödel numbers to Gödel numbers. While in the Grzegorczyk-Lacombe-approach all real numbers are considered, where real numbers are encoded by (fast converging) Cauchy sequences of rational numbers and computations transform infinite Cauchy sequences to infinite Cauchy sequences.

In this article we prove Claim 6.2 from [10]:

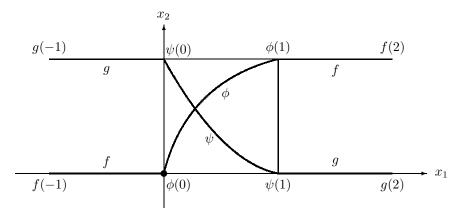


Figure 1: Intersecting curves ϕ and ψ with extensions f and g.

Theorem 1.4 Let \mathcal{T} be the multi-valued operator mapping every pair $\phi, \psi : [0;1] \to [0;1]^2$ of paths in the unit square such that

$$\phi(0) = (0,0), \ \phi(1) = (1,1), \ \psi(0) = (0,1) \ and \ \psi(1) = (1,0)$$
 (5)

to some pair (S_{ϕ}, S_{ψ}) of closed intervals such that $\phi(S_{\phi}) = \psi(S_{\psi})$. Then the operator \mathcal{T} is computable.

Theorem 1.3 follows straightforwardly from Theorem 1.4. In the proof from ϕ and ψ we compute sequences $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ and $J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots$ of closed intervals with rational endpoints such that $\phi(\bigcap_i I_i) = \psi(\bigcap_i J_i)$.

Curves (even computable ones) can be much more complicated than the examples shown in Figure 1. Consider, for example, space-filling curves or curves with infinitely many spirals, each of which containing infinitely many sub-spirals etc. infinitely often or curves with "completely" chaotic behavior.

This article is a contribution to computable analysis. There are various non equivalent definitions of computability in analysis. One of these is Markov's constructive analysis [4, 5]. Theorem 1.2 is a result in this theory. We use "TTE", an approach which is based on ideas from [2, 3, 6]. In TTE computability on $\{0,1\}^*$ and Cantor space $\{0,1\}^\omega$ (the finite and infinite 0-1-sequences) is defined explicitly (e.g. by Turing machines with finite or infinite one-way input and output tapes) and computability on other sets X is induced via representations $\delta : \subseteq \{0,1\}^* \to X$ or $\delta : \subseteq \{0,1\}^\omega \to X$ (partial surjective) where finite or infinite 0-1-sequences are interpreted as names and computations are performed on names. We consider canonical representations of the real numbers, open subsets, closed subsets, compact subsets and real functions. Equivalently any finite alphabet Σ (with at least two elements) can be used instead of $\{0,1\}$. We assume that the reader is familiar with the basic concepts of TTE. Details can be found in [9,1,11].

For technical reasons we extend ϕ and ψ trivially to continuous functions $f, g : [-1; 2] \to \mathbb{R}^2$ which intersect in the same way as ϕ and ψ , that is, $\phi(s) = \psi(t) \iff f(s) = g(t)$ (see Figure 1):

$$f(t) := \begin{cases} (t,0) & \text{if } -1 \le t \le 0\\ \phi(t) & \text{if } 0 \le t \le 1\\ (t,1) & \text{if } 1 \le t \le 2, \end{cases}$$
 (6)

$$g(t) := \begin{cases} (t.1) & \text{if } -1 \le 0 \\ \psi(t) & \text{if } 0 \le t \le 1 \\ (t,0) & \text{if } 1 \le t \le 2, \end{cases}$$
 (7)

We will consider closed rational sub-intervals $I = [a_I; b_I]$ and $J = [a_J; b_J]$ of the real interval [-1; 2] such that for the restrictions $f|_I$ of f to I and $g|_J$ of g to J,

the end-points
$$f|_{I}(a_{I})$$
 and $f|_{I}(b_{I})$ are not in $g(J)$ and the end-points $g|_{J}(a_{J})$ and $g|_{J}(b_{J})$ are not in $f(I)$.

Since f(I) and g(J) are compact this means

$$\alpha_{IJ} := \min(d_s(\{f(a_I), f(b_I)\}, g(J)), d_s(\{g(a_J), g(b_J)\}, f(I))) > 0.$$
(8)

where $d_s(A_1, A_2) := \inf\{||z_1 - z_2|| \mid z_1 \in A_1, z_2 \in A_2\}.$

We will approximate $f|_I$ and $g|_J$ by rational polygon paths h and h', respectively, and consider the intersections (s,t), that is, pairs such that h(s) = h'(t). In order to keep this number finite we consider only pairs (h,h') such that range $(h) \cap \text{range}(h')$ contains no straight line segment. For such pairs every intersection (s,t) is either a crossing or tangent. As a central lemma we will prove that the parity (even or odd) of the number of crossings does not depend on h and h' (it is an invariant of $(f|_I,g|_J)$). We call it the crossing parity of the pair $(f|_I,g|_J)$. In the proof we will apply transformations of polygon paths which may change the number of crossings but do not change the parity (even or odd) of the number of crossings.

2 The crossing parity

For points $x, y \in \mathbb{R}^2$ such that $x \neq y$ let $\overline{xy} \subseteq \mathbb{R}^2$ be the straight line segment from x to y. In this article \overline{xx} is not a straight line segment.

Definition 2.1

- 1. A track is a sequence $p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$ such that $s_i < s_{i+1}$ and $x_i \neq x_{i+1}$ for $0 \leq i < k$. The points x_0, x_1, \dots, x_k are the vertices of p.
- 2. The track p spans a (polygon) path $h_p: [s_0; s_k] \to \mathbb{R}^2$ by

$$h_p(s) = x_i + \frac{s - s_i}{s_{i+1} - s_i} (x_{i+1} - x_i) \text{ if } s_i \le s \le s_{i+1}.$$
 (9)

By (9), $h_p(s_i) = x_i$ and $h_p[s_i; s_{i+1}] = \overline{x_i x_{i+1}}$.

For tracks p, q we want to count the number of crossings of the paths h_p and h_q . In order to keep this number finite we consider only pairs (p, q) such that range (h_p) and range (h_q) have no common straight line segment. Furthermore, we will not count all intersections of h_p and h_q but only "proper" crossings.

Definition 2.2 Let (p,q) where $p = ((s_0,x_0),s_1,x_1),\ldots,(s_k,x_k))$ and $q = ((t_0,y_0),t_1,y_1),\ldots,(t_l,y_l)$ be a pair of tracks such that range (h_p) and range (h_q) have no common straight line segment.

1. An intersection of p and q is a pair (s,t) such that $s_0 < s < s_k$, $t_0 < t < t_l$ and $h_p(s) = h_q(t)$. We call $x := h_p(s) = h_q(t)$ the corresponding intersection point.

2. For an intersection (s,t) of p and q with intersection point $h_p(s) = h_q(t) = x$ let $\delta_{st} > 0$ be a number such that $B(x, \delta_{st}) \setminus \{x\}$ contains no vertex of p and no vertex of q. Let

$$\begin{split} s_{<} &:= \inf\{s' < s \mid h_p[s'; s] \subseteq B(x, \delta_{st})\}, \quad x_{<} := h_p(s_{<})\,, \\ s_{>} &:= \sup\{s' > s \mid h_p[s; s'] \subseteq B(x, \delta_{st})\}, \quad x_{>} := h_p(s_{>})\,, \\ t_{<} &:= \inf\{t' < t \mid h_q[t'; t] \subseteq B(x, \delta_{st})\}, \quad y_{<} := h_q(t_{<})\,, \\ t_{>} &:= \sup\{t' > t \mid h_q[t; t'] \subseteq B(x, \delta_{st})\}, \quad y_{>} := h_q(t_{>})\,. \end{split}$$

If on the boundary of $B(x, \delta_{st})$ the four points $x_{<}$, $x_{>}$, $y_{<}$ and $y_{>}$ occur in the order $(x_{<}, y_{<}, x_{>}, y_{>})$ or in the order $(x_{<}, y_{>}, x_{>}, y_{<})^{-1}$, we call (s,t) a crossing and x the corresponding crossing point, else x is a touch point.

3. Let CN(p,q) the number of crossings of p and q and let $\pi(p,q) := CN(p,q) \mod 2$ be its parity $(0 = even \ and \ 1 = odd)$.

Obviously, $s_{<} < s_i < s$ for no number i, $x_{<} = h_p(s_{<}) \in \partial B(x, \delta_{st})$ and $h_p[s_{<}; s] = \overline{x_{<}x}$ where ∂A denotes the boundary of $A \subseteq \mathbb{R}^2$. This is true correspondingly for $s_{>}, t_{<}$ and $t_{>}$.

Figure 2 shows several kinds of intersection of p and q (thin lines for h_q and thick lines for h_p) the first two of which are crossings. In (a) possibly the center x is no vertex of p or no vertex of q.

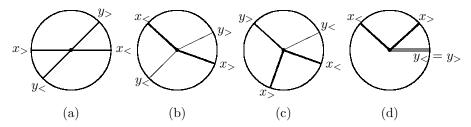


Figure 2: h_p and h_q in the ball $B(x, \delta_{st})$ for $x = h_p(s) = h_q(t)$.

The definition of a crossing (s,t) depends on the number δ_{st} only formally.

Lemma 2.3 The definition of a crossing does not depend on the choice of δ_{st} .

Proof Let (s,t) be a crossing of p and q defined via some δ_{st} . Let $0 < \overline{\delta} < \delta_{st}$.

Let $\overline{s}_{<} := \inf\{s' < s \mid h_p[s'; s] \subseteq B(x, \overline{\delta})\}$ and $\overline{x}_{<} := h_p(\overline{s}_{<})$. Then $\overline{x}_{<} = \overline{x}_{<} \overline{x} \cap \delta B(x, \overline{\delta})$. This is true correspondingly for the other three cases. Obviously the four points on $B(x, \overline{\delta})$ alternate in the same way as the four corresponding points on $B(x, \delta_{st})$.

Notice that $\{x_{<}, x_{>}\} \cap \{y_{<}, y_{>}\} = \emptyset$ since $\operatorname{range}(h_p)$ and $\operatorname{range}(h_q)$ have no common straight line segment. In the applications below the endpoints of h_p are not in $\operatorname{range}(h_q)$ and the endpoints of h_q are not in $\operatorname{range}(h_p)$. Therefore it suffices to consider $s_0 < s < s_k$ and $t_0 < t < t_l$ in the definition of intersections.

We introduce a separation concept for tracks p and q which induces that range (h_p) and range (h_q) have no common straight line segment.

¹that is, on the boundary of $B(x, \delta_{st})$ the four points alternate in x and y

Definition 2.4 Let $p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$ be a track.

- 1. Define $V(p) := \{x_0, x_1, \dots, x_k\}$.
- 2. For $x, y \in \mathbb{R}^2$ with $x \neq y$ let l(x, y) be the straight line through x and y.
- 3. Define

$$\mathcal{L}(p) := \left\{ \left| \{ l(x_{i-1}, x_i) \mid 1 \le i \le k \} \right. \right\}, \tag{10}$$

$$\overline{\mathcal{L}}(p) := \bigcup \{ l(x_i, x_j) \mid 0 \le i < j \le k, \ x_i \ne x_j \}.$$

$$\tag{11}$$

4. We call tracks p and q weakly separated, $p \bowtie q$, iff

$$\mathcal{V}(p) \cap \mathcal{L}(q) = \emptyset \quad and$$
 (12)

$$\mathcal{V}(q) \cap \mathcal{L}(p) = \emptyset. \tag{13}$$

Figure 3 shows on the left the set $\mathcal{L}(p)$ of a track p and a straight line through a point $x \notin \mathcal{L}(p)$ and on the right the set $\overline{\mathcal{L}}(q)$ of a track q and a straight line through a point $y \notin \overline{\mathcal{L}}(q)$.

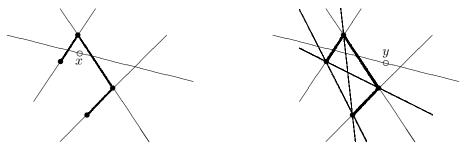


Figure 3: $\mathcal{L}(p)$ and $\overline{\mathcal{L}}(q)$

Lemma 2.5

- 1. If $y \notin \mathcal{L}(p)$ then every straight line through the point y intersects every straight line from $\mathcal{L}(p)$ at most once.
- 2. If $V(p) \cap \mathcal{L}(q) = \emptyset$ or $V(q) \cap \mathcal{L}(p) = \emptyset$ then then $\operatorname{range}(h_p)$ and $\operatorname{range}(h_q)$ have no common straight line segment.

Proof If a straight line through y intersects a straight line from $\mathcal{L}(p)$ twice then $y \in \mathcal{L}(p)$. Contradiction. Let $q = ((t_0, y_0), \dots, (t_m, y_m))$. Suppose for some i and j, $\overline{x_i x_{i+1}}$ and $\overline{y_j y_{j+1}}$ have a common straight line segment. Then $x_i \in l(y_j, y_{j+1}) \subseteq \mathcal{L}(q)$, but $\mathcal{V}(p) \cap \mathcal{L}(q) = \emptyset$. Correspondingly, $\mathcal{V}(q) \cap \mathcal{L}(p) \neq \emptyset$. \square

As an essential tool we will use local transformations of tracks which leave the crossing parity invariant. The following lemma justifies these transformations.

Lemma 2.6 Let

$$p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k)),$$

$$q_1 = ((r, y), (t_1, z_1), (r', y')) \text{ and }$$

$$q_2 = ((r, y), (t_2, z_2), (r', y'))$$

be tracks and let B be a ball such that

$$\{y, y'\} \cap \mathcal{L}(p) = \emptyset, \tag{14}$$

$$\{y, y', z_1, z_2\} \subseteq B,$$
 (15)

$$\{x_0, x_k\} \cap B = \emptyset. \tag{16}$$

Then $\pi(p, q_1) = \pi(p, q_2)$.

Proof Remember that by Definition 2.1, $r < t_1 < r'$, $r < t_2 < r'$, $z_1 \notin \{y, y'\}$ and $z_2 \notin \{y, y'\}$. By (14), $y, y' \notin \text{range}(h_p)$.

If range (h_p) and range (h_{q_1}) have a common straight line segment then $y \in \mathcal{L}(p)$ or $y' \in \mathcal{L}(p)$, but $\{y, y'\} \notin \mathcal{L}(p)$. Therefore, $CN(p, q_1)$ is well-defined. Correspondingly, $CN(p, q_2)$ is well-defined. In Figure 4 the "open ended" line segments are parts of range (h_p) . We distinguish several cases, see Figure 4)

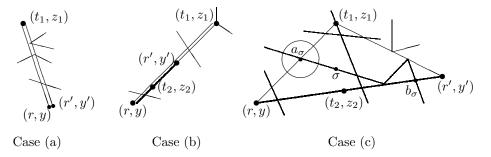


Figure 4: Illustration for Lemma 2.6.

Case (a) y = y': Obviously, $CN(p, q_1)$ and $CN(p, q_2)$ are even, hence $\pi(p, q_1) = 0 = \pi(p, q_2)$.

Case (b) $y \neq y'$, $z_2 \in \overline{yy'}$ and $z_1 \in l(y, y')$: In Figure 4(b) the track q_2 is drawn in thick lines. If $z_1 \in \overline{y, y'}$ then $h_{q_1} = h_{q_2}$, hence $CN(p, q_1) = CN(p, q_2)$. If $z_1 \notin \overline{y, y'}$ then $CN(p, q_1)$ and $CN(p, q_2)$ differ by an even number. In both cases $\pi(p, q_1) = 0 = \pi(p, q_2)$.

Case (c) $y \neq y'$, $z_2 \in \overline{yy'}$ and $z_1 \notin l(y, y')$: In Figure 4(c) the track q_2 is drawn in thick lines. Let Δ be the closed triangle with boundary $\partial \Delta := \text{range}(h_{q_1}) \cup \text{range}(h_{q_2}) \ (= \overline{yz_1} \cup \overline{z_1y'} \cup \overline{yy'})$ and let Δ° be its interior.

If $\overline{yz_1}$ and range (h_p) have a common straight line segment then $y \in \mathcal{L}(p)$, a contradiction by (14). Therefore $\overline{yz_1}$ and range (h_p) have no common straight line segment. This is true correspondingly for z_1y' and $\overline{yy'}$. Therefore

$$\partial \Delta$$
 and range(h_p) have no common straight line segment. (17)

Let $s_0 < \sigma < s_k$ such that $h_p(\sigma) \in \Delta^{\circ}$. Let $[a_{\sigma}; b_{\sigma}]$ be the longest interval such that $s_0 \leq a_{\sigma} < \sigma < b_{\sigma} \leq s_k$ and $h_p[a_{\sigma}; b_{\sigma}] \subseteq \Delta$. Figure 4(c) shows an example for σ with a_{σ} and b_{σ} positioned at the images under h_p

and $h_p(a_{\sigma}) \in \text{range}(h_{q_1})$. Obviously, $h_p(a_{\sigma}), h_p(b_{\sigma}) \in \Delta$. We show that $h_p(a_{\sigma})$ and $h_p(b_{\sigma})$ are crossing points of p and q_1 or of p and q_2 .

Suppose $h_p(a_\sigma) \in \text{range}(h_{q_1})$

Since range (h_p) is a chain of straight line segments there is some $a_{\sigma} < s' < \sigma$ such that $\lambda := h_p([a_{\sigma}; s'])$ is a straight line segment with $h_p(a_{\sigma}) \in \lambda \cap \partial \Delta$ and by (17) $\lambda \setminus \{h_p(a_{\sigma})\} \subseteq \Delta^{\circ}$.

Since range (h_p) is a chain of straight line segments and a_{σ} is the smallest number a with $h_p[a;\sigma] \in \Delta$, there is some $s'' < a_{\sigma}$ such that $\lambda'' := h_p[s''; a_{\sigma}]$ is straight line segment with $h_p(a_{\sigma}) \in \lambda'' \cap \partial \Delta$ and $(\lambda'' \setminus \{h_p(a_{\sigma})\}) \cap \Delta = \emptyset$ by (17).

If we draw a sufficiently small circle around $h_p(a_{\sigma})$ then (with the terminology from Definition 2.2)) the intersections $x_{<}$ and $x_{>}$ of it with range (h_p) and the intersections $y_{<}$ and $y_{>}$ of it with range (h_{q_1}) alternate on this circle in x and y. Therefore, for every σ such that $h_p(\sigma) \in \Delta^{\circ}$,

 $h_p(a_{\sigma})$ is a crossing point of p and q_1 if $h_p(a_{\sigma}) \in \text{range}(h_{q_1})$ and correspondingly,

 $h_p(a_\sigma)$ is a crossing point of p and q_2 if $h_p(a_\sigma) \in \text{range}(h_{q_2})$,

 $h_p(b_\sigma)$ is a crossing point of p and q_1 if $h_p(b_\sigma) \in \text{range}(h_{q_1})$ and

 $h_p(b_\sigma)$ is a crossing point of p and q_2 if $h_p(b_\sigma) \in \text{range}(h_{q_2})$.

On the other hand, every crossing point of p and q_1 or q_2 is equal to $h_p(a_\sigma)$ or $h_p(b_\sigma)$ for some σ with $h_p(\sigma) \in \Delta^{\circ}$. Therefore, the number N of crossings of p with q_1 or q_2 is even. Since by (14) $\{y, y'\} \cap \operatorname{range}(h_p) = \emptyset$, $N = \operatorname{CN}(p, q_1) + \operatorname{CN}(p, q_2)$ is an even number, hence $\pi(p, q_1) = \pi(p, q_2)$.

Case (d) $y \neq y'$, $z_2 \in l(y, y') \setminus \overline{yy'}$ and $z_1 \notin l(y, y')$: Let $q_3 := (r, y)(t_1, z_3)(r', y')$ for $z_3 := (z_1 + z_2)/2$. By Case (b), $\pi(p, q_2) = \pi(p, q_3)$ and by Case (c), $\pi(p, q_1) = \pi(p, q_3)$.

Case (e) $y \neq y'$, $z_1, z_2 \notin l(y, y')$: Let $q_3 := (r, y)(t_1, z_3)(r', y')$ for $z_3 := (z_1 + z_2)/2$. By Case (c), $\pi(p, q_2) = \pi(p, q_3)$ and $\pi(p, q_1) = \pi(p, q_3)$.

Case (f)
$$y \neq y'$$
 and $z_1, z_2 \in l(y, y') \setminus \overline{yy'}$: Proof via q_3 as in (d) and (e).

For tracks p, q and q', by tiny shifts of the vertices of p we can obtain a track $\overline{p} \bowtie q$ and $\overline{p} \bowtie q'$.

Lemma 2.7

- 1. Let $p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$ and let q and q' be tracks. Then for every $\delta > 0$ there is some track $\overline{p} = ((s_0, y_0), (s_1, y_1), \dots, (s_k, y_k))$ such that $\overline{p} \bowtie q$, $\overline{p} \bowtie q'$ and $||x_i y_i|| < \delta$ for $0 \le i \le k$.
- 2. If $p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$ and $\overline{p} = ((s_0, y_0), (s_1, y_1), \dots, (s_k, y_k))$ are tracks such that $||x_i y_i|| < \delta$ for $0 \le i \le k$ then $||h_p(s) h_{\overline{p}}(s)|| < \delta$ for all $s_0 \le s \le s_k$.

Proof

1. We must find (prove the existence of) points y_0, y_1, \ldots, y_k such that $||x_i - y_i|| < \delta$ and for $\overline{p} := (s_0, y_0), (s_1, y_1), \ldots, (s_k, y_k)$,

$$\mathcal{V}(\overline{p}) \cap (\mathcal{L}(q) \cup \mathcal{L}(q')) = \emptyset \text{ and }$$
 (18)

$$\mathcal{L}(\overline{p}) \cap (\mathcal{V}(q) \cup \mathcal{V}(q')) = \emptyset$$
 (19)

The following statements are true since $V(q) \cup V(q')$ is a finite set of points and $\mathcal{L}(q) \cup \mathcal{L}(q')$ is a finite set of straight lines.

- There is some $y_0 \in B(x_0, \delta)$ such that $y_0 \notin \mathcal{L}(q) \cup \mathcal{L}(q')$.
- Suppose y_i has been determined for some $0 \le i < k$. There is some $y_{i+1} \in B(x_{i+1}, \delta)$ such that

$$y_{i+1} \neq y_i$$
, $y_{i+1} \notin (\mathcal{L}(q) \cup \mathcal{L}(q'))$ and $l(y_i, y_{i+1}) \cap (\mathcal{V}(q) \cup \mathcal{V}(q')) = \emptyset$,

Then $\overline{p} := ((s_0, y_0), (s_1, y_1), \dots, (s_k, y_k))$ is a track such that $\overline{p} \bowtie q$, $\overline{p} \bowtie q'$ and $||x_i - y_i|| < \delta$ for $0 \le i \le k$.

2. For $s_i \le s \le s_{i+1}$ by (9),

$$||h_{p}(s) - h_{\overline{p}}(s)|| = ||h_{p}(s_{i}) + \frac{s - s_{i}}{s_{i+1} - s_{i}}(h_{p}(s_{i+1}) - h_{p}(s_{i}))$$

$$-h_{\overline{p}}(s_{i}) - \frac{s - s_{i}}{s_{i+1} - s_{i}}(h_{\overline{p}}(s_{i+1}) - h_{\overline{p}}(s_{i}))||$$

$$= ||x_{i} + \frac{s - s_{i}}{s_{i+1} - s_{i}}(x_{i+1} - x_{i})$$

$$-y_{i} - \frac{s - s_{i}}{s_{i+1} - s_{i}}(y_{i+1} - y_{i})||$$

$$= ||(1 - \frac{s - s_{i}}{s_{i+1} - s_{i}}) \cdot (x_{i} - y_{i})$$

$$+ \frac{s - s_{i}}{s_{i+1} - s_{i}}) \cdot (x_{i+1} - y_{i+1})||$$

$$< \delta$$

In the following let $f, g: [-1; 2] \to \mathbb{R}^2$, let $I = [a_I; b_I]$ and $J = [a_J; b_J]$ be intervals with rational endpoints such that $I, J \subseteq [-1; 2]$ and let α_{IJ} be the number from (8).

Since f and g are continuous, there is a modulus of uniform continuity $\mathrm{md}: \mathbb{N} \to \mathbb{N}$ for f and g, that is, md is increasing and for all $t, t' \in [-1; 2]$,

$$||f(t) - f(t')|| < 2^{-n} \text{ and } ||g(t) - g(t')|| < 2^{-n} \text{ if } |t - t'| < 2^{-\text{md}(n)}.$$
 (20)

We approximate $f|_I$ and $g|_J$ by tracks.

Definition 2.8 A track $p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$ is an approximation with precision 2^{-n} (shortly an n-approximation) of $f|_I$ if

$$s_0 = a_I \quad and \quad s_k = b_I \,, \tag{21}$$

$$s_{i+1} - s_i < 2^{-\text{md}(n)}$$
 and (22)

$$||f(s_i) - x_i|| < 2^{-n}. (23)$$

Remember that by Definition 2.1, $x_i = h_p(s_i)$. Approximations of $g|_J$ are defined accordingly.

Lemma 2.9 In Definition 2.8, for all meaningful i and s,

$$||x_i - x_{i+1}|| < 3 \cdot 2^{-n},$$
 (24)

$$||f(s) - h_p(s)|| < 5 \cdot 2^{-n}.$$
 (25)

Proof
$$||x_i - x_{i+1}|| \le ||x_i - f(s_i)|| + ||f(s_i) - f(s_{i+1})|| + ||f(s_{i+1}) - x_{i+1})|| < 3 \cdot 2^{-n}$$
.
For $s_i \le s \le s_{i+1}$, $||f(s) - h_p(s)|| \le ||f(s) - f(s_i)|| + ||f(s_i) - h_p(s_i)|| + ||h_p(s_i) - h_p(s)|| < 2^{-n} + 2^{-n} + ||x_i - x_{i+1}|| < 5 \cdot 2^{-n}$ by (9) and (24).

After these technical preparations prove the following central lemma.

Lemma 2.10 Let $2^{-n} < \alpha_{IJ}/16$. Let p and p' be n-approximations of $f|_I$ and let q and q' be napproximations of $g|_J$ such that $p \bowtie q$ and $p' \bowtie q'$. Then $\pi(p,q) = \pi(p',q')$.

Proof Let

$$p = ((s_0, x_0), (s_1, x_1), \dots, (s_k, x_k))$$

where $s_0 = a_I$ and $s_k = b_I$. Notice that possibly p and q' as well as p' and q have common straight line segments such that $\pi(p,q')$ and $\pi(p',q)$ are not defined. By Lemma 2.7 there is some track

$$\overline{p} = ((s_0, y_0), (s_1, y_1), \dots, (s_k, y_k))$$

such that

$$\overline{p} \bowtie q, \quad \overline{p} \bowtie q' \quad \text{and} \quad ||x_i - y_i|| < 2^{-n} \text{ for } 0 \le i \le k.$$
 (26)

Figure 5 shows the \bowtie -relation (indicated by lines) between the tracks p, q, \overline{p}, p' and q'. It suffices to prove

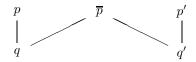


Figure 5: The \bowtie -relation between the tracks p, q, \overline{p}, p' and q'.

$$\pi(p,q) = \pi(\overline{p},q) = \pi(\overline{p},q') = \pi(p',q'). \tag{27}$$

We use the facts that by the definitions the points f(t), $h_p(t)$, $h_{\overline{p}}(t)$ and $h_{p'}(t)$ are close together and that the points g(t), $h_q(t)$ and $h_{q'}(t)$ are close together.

In the following we prove the second equation $\pi(\overline{p},q) = \pi(\overline{p},q')$ of (27) in detail. Let

$$q = ((r_0, u_0), (r_1, u_1), \dots, (r_{l-1}, u_{l-1}), (r_l, u_l)),$$
(28)

$$q' = ((r'_0, u'_0), (r'_1, u'_1), \dots, (r'_{m-1}, u'_{m-1}), (r'_m, u'_m))$$
where $r_0 = r'_0 = a_J$ and $r_l = r'_m = b_J$. (30)

where
$$r_0 = r'_0 = a_J$$
 and $r_l = r'_m = b_J$. (30)

be n-approximations of $g|_J$. As an example, Figure 6 shows the arguments r_0, \ldots, r_6 of a track q on the real line and and the arguments r'_0, \ldots, r'_7 of a track q' on the real line (thick black dots).

From (26) we know that $\mathcal{V}(q) \cap \mathcal{L}(\overline{p}) = \emptyset$ and $\mathcal{V}(q') \cap \mathcal{L}(\overline{p}) = \emptyset$. In a first step we add "redundant" vertices to q and q' such that the resulting tracks q_1 and q'_1 have the same number of vertices and such that $h_q = h_{q_1}$, $h_{q'} = h_{q'_1}$, $\mathcal{V}(q_1) \cap \mathcal{L}(\overline{p}) = \emptyset$ and $\mathcal{V}(q'_1) \cap \mathcal{L}(\overline{p}) = \emptyset$.

Let i be a number such that $r_i \notin \{r'_0, \dots, r'_m\}$. There is a unique j such that $r'_j < r_i < r'_{j+1}$ (e.g. i = 3 and j = 1 in Figure 6).

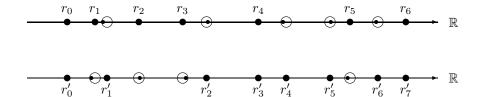


Figure 6: Insertion of redundant vertices into q and q'.

The point $h_{q'}(r_i)$ is an element of the straight line segment $\overline{u'_j u'_{j+1}}$ and different from u'_j and u'_{j+1} . We would like to add the "redundant" pair $(r_i, h_{q'}(r_i))$ to q' with result

$$((r'_0, u'_0), \dots, (r'_i, u'_i), (r_i, h_{q'}(r_i), (r'_{i+1}, u'_{i+1}), \dots, (r'_m, u'_m))$$

But possibly $h_{q'}(r_i) \in \mathcal{L}(\overline{p})$. Instead, we add a pair (t, v) with $v = h_{q'}(t) \notin \mathcal{L}(\overline{p})$ to q' which is very close to $(r_i, h_{q'}(r_i))$. Let

$$\gamma := \min\{|c - d| \mid c, d \in \{r_0, \dots, r_l, r'_0, \dots, r'_m\} \text{ and } c \neq d\}.$$
(31)

Since $u'_j \notin \mathcal{L}(\overline{p})$, by Lemma 2.5 $l(u'_j u'_{j+1})$ intersects every straight line from $\mathcal{L}(\overline{p})$ at most once. Therefore, the straight line segment $\overline{u'_j u'_{j+1}}$ contains only finitely many points of $\mathcal{L}(\overline{p})$. Thus there is some t such that $|t - r_i| < \gamma/8$ and $v := h_{q'}(t) \notin \mathcal{L}(\overline{p})$. Then $v \in \overline{u'_j u'_{j+1}}$ (see (9)). We add the pair (t, v) to q' with result

$$\overline{q} := ((r'_0, u'_0), \dots, (r'_j, u'_j), (t, h_{q'}(t)(r'_{j+1}, u'_{j+1}), \dots, (r'_m, u'_m)).$$

Then $h_{q'} = h_{\overline{q}}$. Let q'_1 be the track obtained from q' by adding a pair (t, v) in this way for every i such that $r_i \notin \{r'_0, \dots, r'_m\}$ in turn. Then $h_{q'} = h_{q'_1}$ and $\mathcal{V}(q'_1) \cap \mathcal{L}(\overline{p}) = \emptyset$.

Correspondingly let q_1 be the track obtained from q in the same way. Then $h_q = h_{q_1}$ and $\mathcal{V}(q_1) \cap \mathcal{L}(\overline{p}) = \emptyset$. In summary,

$$h_q = h_{q_1}, \ h_{q'} = h_{q'}, \ \mathcal{V}(q_1) \cap \mathcal{L}(\overline{p}) = \emptyset \text{ and } \mathcal{V}(q'_1) \cap \mathcal{L}(\overline{p}) = \emptyset.$$
 (32)

In Figure 6 the inserted arguments t are within the the circles. The new tracks can be written as

$$q_1 = ((t_0, v_0), (t_1, v_1), \dots, (t_{\mu}, v_{\mu})), \tag{33}$$

$$q_1' = ((t_0', v_0'), (t_1', v_1'), \dots, (t_{\mu}', v_{\mu}')) \tag{34}$$

where $a_J = r_0 = t_0 = t_0'$ and $b_J = r_l = t_\mu = t_\mu'$. By the condition $|t - r_i| < \gamma/8$ in the definition of (t, v) above,

$$|t_{\nu} - t_{\nu}'| < \gamma/8 \text{ and } \{t_{\nu}, t_{\nu}'\} < \{t_{\nu+1}, t_{\nu+1}'\}.$$
 (35)

The next Proposition prepares the proof or Proposition 2.12.

Proposition 2.11 For the tracks q_1 and q'_1 , for all $0 < \nu < \mu$,

$$\{v_{\nu-1}, v_{\nu}, v_{\nu+1}, v'_{\nu-1}, v'_{\nu}, v'_{\nu+1}\} \subseteq B(g(t_{\nu}), 5 \cdot 2^{-n}) \quad and$$
 (36)

$$\{y_0, y_k\} \cap B(g(t_\nu), 14 \cdot 2^{-n}) = \emptyset,$$
 (37)

$$B(g(t_1), 9 \cdot 2^{-n}) \cap \operatorname{range}(h_{\overline{p}}) = \emptyset, \tag{38}$$

$$B(g(t_{\mu-1}), 9 \cdot 2^{-n}) \cap \operatorname{range}(h_{\overline{p}}) = \emptyset.$$
(39)

Proof (Proposition 2.11)

Proof of (36): Consider $((t_{\nu-1}, v_{\nu-1}), (t_{\nu}, v_{\nu}), (t_{\nu+1}, v_{\nu+1}))$ as a part of q_1 . Then (t_{ν}, v_{ν}) has already been in q, that is, $(t_{\nu}, v_{\nu}) = (r_i, u_i)$ for some i or it has been inserted via some j such that $r'_j \notin \{r_0, \ldots, r_l\}$. Therefore, there is some i such that (see Figure 6)

$$r_{i-1} \le t_{i-1} < r_i < t_{\nu+1} \le r_{i+1} \quad \text{or}$$
 (40)

$$r_i \le t_{\nu-1} < t_{\nu} < t_{\nu+1} \le r_{i+1} \,. \tag{41}$$

Consider (40). Then $\|g(t_{\nu}) - v_{\nu}\| = \|g(r_i) - u_i\| < 2^{-n}$ by (23). Furthermore, $\|g(t_{\nu}) - v_{\nu-1}\| \le \|g(t_{\nu}) - v_{\nu}\| + \|v_{\nu} - v_{\nu-1}\| < 2^{-n} + \|u_i - v_{\nu-1}\|$. Since $h_q = h_{q_1}, v_{\nu-1} \in \overline{u_{i-1}u_i}$, hence $\|u_i - v_{\nu-1}\| \le \|u_{i-1} - u_i\| < 3 \cdot 2^{-n}$ by (24). Therefore, $\|g(t_{\nu}) - v_{\nu-1}\| < 4 \cdot 2^{-n}$. By symmetry $\|g(t_{\nu}) - v_{\nu+1}\| < 4 \cdot 2^{-n}$. Therefore, $\{v_{\nu-1}, v_{\nu}, v_{\nu+1}\} \subseteq B(g(t_{\nu}), 4 \cdot 2^{-n})$.

Consider (41). Then

 $||g(t_{\nu}) - v_{\nu}|| = ||g(t_{\nu}) - h_{q}(t_{\nu})|| < 5 \cdot 2^{-n} \text{ by}(25). \text{ Furthermore, } ||g(t_{\nu}) - v_{\nu-1}|| \le ||g(t_{\nu}) - v_{\nu}|| + ||v_{\nu} - v_{\nu-1}|| < 2^{-n} + ||u_{i+1} - u_{i}|| < 2^{-n} + 3 \cdot 2^{-n} = 4 \cdot 2^{-n} \text{ (since } v_{\nu}, v_{\nu-1} \in \overline{u_{i}u_{i+1}} \text{ and by (24)). By symmetry } ||g(t_{\nu}) - v_{\nu+1}|| < 4 \cdot 2^{-n}. \text{ Therefore, } \{v_{\nu-1}, v_{\nu}, v_{\nu+1}\} \subseteq B(g(t_{\nu}), 4 \cdot 2^{-n}).$

As a summary, in both cases (40) and (41), $\{v_{\nu-1}, v_{\nu}, v_{\nu+1}\} \subseteq B(g(t_{\nu}), 4 \cdot 2^{-n})$. By symmetry, $\{v'_{\nu-1}, v'_{\nu}, v'_{\nu+1}\} \subseteq B(g(t'_{\nu}), 4 \cdot 2^{-n})$. Since $|t_{\nu} - t'_{\nu}| < \gamma/4 < 2^{-\operatorname{md}(n)}$, $||g(t_{\nu}) - g(t'_{\nu}|| < 2^{-n})$ by (35), (31), (22) and (20). Therefore, $\{v_{\nu-1}, v_{\nu}, v_{\nu+1}, v'_{\nu-1}, v'_{\nu}, v'_{\nu+1}\} \subseteq B(g(t_{\nu}), 5 \cdot 2^{-n})$.

Proof of (37): By (8), (23) and Lemma 2.7 for all ν

 $16 \cdot 2^{-n} < \alpha_{IJ} \le \|f(s_0) - g(t_\nu)\| \le \|f(s_0) - x_0\| + \|x_0 - y_0\| + \|y_0 - g(t_\nu)\| < 2^{-n} + 2^{-n} + \|y_0 - g(t_\nu)\|,$ hence $\|y_0 - g(t_\nu)\| > 14 \cdot 2^{-n}$. By symmetry, $\|y_k - g(t_\nu)\| > 14 \cdot 2^{-n}$.

Proof of (38): For every $s_0 \le s \le s_k$, $||f(s) - g(r_0)|| \ge \alpha_{IJ} > 16 \cdot 2^{-n}$. By (25) and Lemma 2.7, $||h_{\overline{p}}(s) - g(r_0)|| > 10 \cdot 2^{-n}$. Since $|r_0 - t_1| \le |s_0 - s_1| < 2^{\text{md}}(n)$, $||h_{\overline{p}}(s) - g(t_1)|| > 9 \cdot 2^{-n}$, hence $B(g(t_1), 9 \cdot 2^{-n}) \cap \text{range}(h_{\overline{p}}) = \emptyset$.

Proof of (39): From (38) by symmetry.

 \square (Proposition 2.11)

Proposition 2.12 $\pi(\overline{p}, q_1) = \pi(\overline{p}, q'_1)$.

Proof (Proposition 2.12): We transform the track q'_1 in two phases to the track q_1 preserving the crossing parity in each step. Remember (32) and Proposition 2.11 for

$$q_1 = ((t_0, v_0), (t_1, v_1), (t_2, v_2), \dots, (t_{\mu}, v_{\mu})) \text{ and }$$

$$q'_1 = ((t'_0, v'_0), (t'_1, v'_1), (t'_2, v'_2), \dots, (t'_{\mu}, v'_{\mu})).$$

In the first phase for every $0 \le i < \mu$ we replace every v'_i by some w_i without changing q'_i . For $0 \le i < \mu$ let Q(i) be the following property:

There are points w_0, \ldots, w_i such that for all $1 \leq j \leq i$

$$||w_j - v_j'|| < 2^{-n}/16, \quad w_j \notin \mathcal{L}(\overline{p}), \quad w_j \notin \{v_{j-1}, w_{j-1}, v_{j+1}'\}) \quad \text{and}$$
 (42)

$$\pi(\overline{p}, q_1') = \pi(\overline{p}, \overline{q}_i) \tag{43}$$

where \overline{q}_i is the track

$$\overline{q}_i := ((t'_0, w_0), \dots, (t'_i, w_i), (t'_{i+1}, v'_{i+1}), \dots, (t'_{\mu}, v'_{\mu})).$$

Let $w_0 := v_0'$. Then $q_1' = \overline{q}_0$, hence Q(0) is true. Suppose for some $0 \le i \le \mu - 2$ we have proved Q(i). There is some w_{i+1} such that

$$||w_{i+1} - v'_{i+1}|| < 2^{-n}/16, \quad w_{i+1} \notin \mathcal{L}(\overline{p}), \quad w_{i+1} \notin \{v_i, w_i, v'_{i+2}\}.$$
 (44)

By definition,

$$\overline{q}_{i} = ((t'_{0}, w_{0}), \dots, (t'_{i}, w_{i}), (t'_{i+1}, v'_{i+1}), (t'_{i+2}, v'_{i+2}), \dots, (t'_{\mu}, v'_{\mu})) \text{ and }
\overline{q}_{i+1} = ((t'_{0}, w_{0}), \dots, (t'_{i}, w_{i}), (t'_{i+1}, w_{i+1}), (t'_{i+2}, v'_{i+2}), \dots, (t'_{\mu}, v'_{\mu})).$$

Since \overline{q}_i is a track, by (44), \overline{q}_{i+1} is a track (neighboring vertices must be different). Also by (44), (42) is true also for i+1. By Proposition 2.11 Lemma 2.6 can be applied to the boldface sub-tracks $\tau_1 := ((t'_i, w_i), (t'_{i+1}, v'_{i+1}), (t'_{i+2}, v'_{i+2}))$ and $\tau_2 := ((t'_i, w_i), (t'_{i+1}, w_{i+1}), (t'_{i+2}, v'_{i+2}))$ such that $\pi(\overline{p}, \tau_1) = \pi(\overline{p}, \tau_2)$. Since the vertices of \overline{q}_i and \overline{q}_{i+1} are not in $\mathcal{L}(\overline{p})$, hence not in range $(h_{\overline{p}})$, $\pi(\overline{p}, q'_1) = \pi(\overline{p}, \overline{q}_i) = \pi(\overline{p}, \overline{q}_{i+1})$. Therefore, we have proved Q(i+1).

By induction $Q(\mu - 1)$ is true, hence

$$\pi(\overline{p}, q_1') = \pi(\overline{p}, \overline{q}_{\mu-1}). \tag{45}$$

Figure 7 shows the \neq -relation for the vertices of $q_1 = ((t_0, v_0), (t_1, v_1), (t_2, v_2), \dots, (t_{\mu}, v_{\mu}))$ (top) and $\overline{q}_{\mu-1} = ((t'_0, w_0), (t'_1, w_1), \dots, (t'_{\mu-1}, w_{\mu-1}), (t'_{\mu}, v'_{\mu}))$ (bottom). A line between x and y means $x \neq y$.

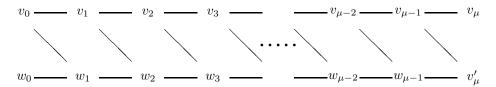


Figure 7: The \neq -relation for the vertices of q_1 (top) and $\overline{q}_{\mu-1}$ (bottom).

In the second phase two we transform $\overline{q}_{\mu-1}$ step by step to q_1 without changing the crossing parity with \overline{p} . For $0 \le i \le \mu - 1$ let

$$q_i'' := ((t_0, v_0), \dots, (t_i, v_i)(t'_{i+1}, w_{i+1}), \dots, (t'_{u-1}, w_{u-1}), (t'_u, v'_u)).$$

Since q_1 and $\overline{q}_{\mu-1}$ are tracks and $v_i \neq w_{i+1}$, q_i'' is a track.

By (36) and by $||w_1 - v_1'|| < 2^{-n}/16$ (42), $\overline{v_0w_1} \cup \overline{w_0w_1} \subseteq B(g(t_1), 6 \cdot 2^{-n})$. By (38), $\overline{v_0w_1}$ and $\overline{w_0w_1}$ do not intersect $h_{\overline{p}}$. Therefore,

$$\pi(\overline{p}, \overline{q}_{\mu-1}) = \pi(\overline{p}, q_0'') \tag{46}$$

Suppose for $0 \le i \le \mu - 3$

$$\pi(\overline{p}, \overline{q}_{n-1}) = \pi(\overline{p}, q_i'')$$
.

By definition,

$$q_{i+1}'' = ((t_0, v_0), \dots, (t_i, v_i), (t'_{i+1}, w_{i+1}), (t'_{i+1}, w_{i+2}), \dots, (t'_{\mu}, v'_{\mu}))$$
 and $q_{i+1}'' = ((t_0, v_0), \dots, (t_i, v_i), (t_{i+1}, v_{i+1}), (t'_{i+1}, w_{i+2}), \dots, (t'_{\mu}, v'_{\mu})),$

By Proposition 2.11 Lemma 2.6 can be applied to the boldface sub-tracks

$$\tau_1 := ((t_i, v_i), (t'_{i+1}, w_{i+1}), (t'_{i+1}, w_{i+2}) \text{ and }$$

$$\tau_2 := ((t_i, v_i)(t_{i+1}, v_{i+1}), (t'_{i+1}, w_{i+2}))$$

such that $\pi(\overline{p}, \tau_1) = \pi(\overline{p}, \tau_2)$.

Since the vertices of \overline{q}_i'' and \overline{q}_{i+1}'' are not in $\mathcal{L}(\overline{p})$, hence not in range $(h_{\overline{p}})$, $\pi(\overline{p}, \overline{q}_{\mu-1}) = \pi(\overline{p}, q_i'') =$ $\pi(\overline{p}, q_{i+1}^{"})$. By induction,

$$\pi(\overline{p}, \overline{q}_{\mu-1}) = \pi(\overline{p}, q''_{\mu-2})$$

Finally we compare

$$q_1 = ((t_0, v_0), \dots, (t_{\mu-2}, v_{\mu-2}), (t_{\mu-1}, v_{\mu-1}), (t_{\mu}, v_{\mu})) \text{ and } q''_{\mu-2} = ((t_0, v_0), \dots, (t_{\mu-2}, v_{\mu-2}), (t'_{\mu-1}, w_{\mu-1}), (t'_{\mu}, v'_{\mu})).$$

Since $||w_{\mu-1} - v'_{\mu-1}|| < 2^{-n}$ by Proposition 2.11 and

$$(\overline{v_{\mu-2}v_{\mu-1}}\cup\overline{v_{\mu-1}v_{\mu}}\cup\overline{v_{\mu-2}w_{\mu-1}}\cup\overline{w_{\mu-1}v_{\mu}'})\cap\mathrm{range}(h_{\overline{p}})=\emptyset\,,$$

$$\pi(\overline{p}, q_{\mu-2}'') = \pi(\overline{p}, q_1), \text{ hence } \pi(\overline{p}, q_1') = \pi(\overline{p}, q_1).$$

We continue the proof of Lemma 2.10. Since $h_q = h_{q_1}$ and $h_{q'} = h_{q'_1}$ by (32), $\pi(\overline{p}, q') = \pi(\overline{p}, q)$, which is the second equation from (27). The first and the third equation can be poved accordingly. We omit the details. □ (Lemma 2.10)

Lemma 2.10 allows to define the parity of the pair $(f|_{I}, g|_{J})$.

Definition 2.13 $\pi(f|_I,g|_J) := \pi(p,q)$ for an arbitrary n-approximation p of $f|_I$ and an arbitrary napproximation q of $g|_J$ such that $p \bowtie q$ and $2^{-n} < \alpha_{IJ}/16$.

Lemma 2.14 Let f, g, J and $I = [a_I; b_I]$ be as before such that (8). Let $I_1 := [a_I; c]$ and $I_2 := [c; b_I]$ where $a_I < c < b_I$ and $f(c) \notin g(J)$. Then

$$\pi(f|_{I}, g|_{J}) = (\pi(f|_{I_{1}}, g|_{J}) + \pi(f|_{I_{2}}, g|_{J})) \bmod 2. \tag{47}$$

Proof There are numbers $\alpha > 0$ and $n \in \mathbb{N}$ such that

 $\alpha < d_s(\{f(a_I), f(c), f(b_I)\}, g(J)), \alpha < d_s(\{g(a_J), g(b_J)\}, f(I)) \text{ and } 16 \cdot 2^{-n} < \alpha.$ There are *n*-approximations $p = ((s_0, x_0), \dots, (s_k, x_x))$ and q of $f|_I$ and $g|_q$, respectively, such that $c = s_i$ for some i and $p \bowtie q$. Let $p_1 := ((s_0, x_0), \dots, (s_i, x_i))$ and $p_2 := ((s_i, x_i), \dots, (s_k, x_k))$.

Then $p_1 \bowtie q$ and $p_2 \bowtie q$. Therefore $CN(p_1,q)$ and $CN(p_2,q)$ are well-defined such that (see Definition 2.2)

$$CN(p,q) = CN(p_1,q) + CN(p_2,q),$$

hence $\pi(p,q) = \text{CN}(p,q) \mod 2 = (\text{CN}(p_1,q) \mod 2 + \text{CN}(p_2,q) \mod 2) \mod 2 = (\pi(p_1,q) + \text{CN}(p_2,q) \mod 2)$ $\pi(p_2,q)$) mod 2. (47) follows by Definition 2.13.

3 Crossing parities can be computed.

In TTE computability is defined on represented sets (X, δ) . A representation is a function $\delta : \subseteq \Sigma^* \to X$ (if X is finite) or $\delta :\subset \Sigma^{\omega} \to X$ (if X has at most continuum cardinality). If $\delta(w) = x$ then w is considered as a "name" (or a " δ -name") of x. Every $x \in X$ must have a name (and may have many names) but not every $w \in \Sigma^*$ must be a name of some $x \in X$, hence δ is a partial surjective function.

For represented sets (X_i, δ_i) where $\delta_i :\subseteq A_i \to X_i$, $A_i \in \{\Sigma^*, \Sigma^\omega\}$, a partial function $f :\subseteq X_1 \to X_2$ is computable, if there is a computable function $h :\subseteq A_1 \to A_2$ which realizes f, that is, if w is a δ_1 -name of $x \in \text{dom}(f)$ then h(w) is a δ_2 -name of f(x) (accordingly functions on Cartesian products). For most sets in Analysis there are canonical (or standard or effective or obvious) representations which we use here. We will say "computable" without mentioning the (standard) representations.

We need the concept of computable multi-functions on represented sets. As an example, for the standard represented sets (\mathbb{R}, ρ) , $(\mathbb{Q}, \nu_{\mathbb{Q}})$ and $(\mathbb{N}, \nu_{\mathbb{N}})$ there is no computable function $f: \mathbb{R} \times \mathbb{N} \to \mathbb{Q}$ such that $|x - f(x, n)| < 2^{-n}$. But there is a computable function $h: \Sigma^{\omega} \times \Sigma^* \to \Sigma^*$ which from every name of x and every name of n computes a name of some $a \in \mathbb{Q}$ such that $|x - a| < 2^{-n}$. The computed number a depends on the names of x and n and not only on (x, n). The function h realizes the multi-function $f: \mathbb{R} \times \mathbb{N} \rightrightarrows \mathbb{Q}$ such that $f(x, n) = \{a \in \mathbb{Q} \mid |x - q| < 2^{-n}\}$. Informally we say: f maps every (x, n) to some $a \in \mathbb{Q}$ such that $|x - a| < 2^{-n}$, more formally, $f: (x, n) \bowtie a$ such that $|x - a| < 2^{-n}$. In the following we apply results about computability on sets with standard representations from [9, 1, 11, 8].

In Lemma 3.1 we consider n-approximations p and q of $f|_I$ and $g|_J$, respectively such that $p \bowtie q$. We prove that p and q can be computed from (f, g, I, J, n). We use from TTE, that $(f, s, n) \bowtie y$ $(s \in \mathbb{Q}, y \in \mathbb{Q}^2)$ such that $||f(s) - y|| < 2^{-n}$) is computable.

Lemma 3.1 Let $\mathcal{R}: (f,g,I,J,n) \mapsto (p,q)$ be the multi-function mapping continuous functions $f,g: [-1;2] \to \mathbb{R}^2$, closed rational intervals $I,J \subseteq [-1;2]$ and a number $n \in \mathbb{N}$ to a pair $(p,q), p = ((s_0,x_0),\ldots,(s_k,x_k))$ and $q = ((t_0,y_0),\ldots,(t_l,y_l))$, of rational tracks such that

$$p$$
 is an n -approximation of $f|_{I}$, (48)

$$q$$
 is an n -approximation of $g|_J$ and (49)

$$p \bowtie q$$
. (50)

Then \mathcal{R} is computable.

Since rational tracks are "discrete" objects there is no compuble funtion but only a computable multifunction (known from TTE).

Proof It is known [9, 1, 11] that from continuous functions $f, g : [-1; 2] \to \mathbb{R}^2$ we can compute a modulus of continuity md_f of f and a modulus of continuity md_g of g. We can compute $\mathrm{md} := \mathrm{max}(\mathrm{md}_f, \mathrm{md}_g)$, which is a modulus of continuity of f and of g. Let $I = [a_I; b_I]$ and $J = [a_J; b_J]$. Consider Definition 2.8. First we compute p.

Choose some k and rational numbers $s_0 := a_I$, $s_1, \ldots, s_{k-1}, s_k := b_I$ such that $0 < s_{i+1} - s_i < 2^{-\operatorname{md}(n)}$ for $0 \le i < k$. Choose some $x_0 \in \mathbb{Q}^2$ such that $\|f(s_0) - x_0\| < 2^{-n}$. Suppose, x_i has been chosen for i < k. Then choose some $x_{i+1} \in \mathbb{Q}^2$ such that $\|f(s_{i+1}) - x_{i+1}\| < 2^{-n}$ and $x_{i+1} \ne x_i$. Since $x_i \ne x_{i+1}$ for i < k, $p := ((s_0, x_0), \ldots, (s_k, x_k))$ is a rational track. It is an n-approximation of $f|_I$.

Next we compute q.

Choose some l and rational numbers $t_0 := a_J, t_1, \ldots, t_{l-1}, t_l := b_J$ such that $0 < t_{i+1} - t_i < 2^{-\operatorname{md}(n)}$ for $0 \le i < l$. Choose some $y_0 \in \mathbb{Q}^2$ such that $\|g(t_0) - y_0\| < 2^{-n}$ and $y_0 \notin \mathcal{L}(p)$. Suppose, y_0, \ldots, y_i have been chosen for some i < l. Then choose some $y_{i+1} \in \mathbb{Q}^2$ such that $y_{i+1} \ne y_i, \|g(t_{i+1}) - y_{i+1}\| < 2^{-n}, y_{i+1} \notin \mathcal{L}(p)$ and $l(y_i, y_{i+1}) \cap \mathcal{V}(p) = \emptyset$.

Since $y_i \neq y_{i+1}$ for 0i < l, $q = ((t_0, y_0), \dots, (t_l, y_l))$ is a rational track. It is an n-approximation of $g|_J$. We have $\mathcal{V}(q) \cap \mathcal{L}(p) = \emptyset$ since $y_i \notin \mathcal{L}(p)$ for all i. We have $\mathcal{L}(q) \cap \mathcal{V}(p) = \emptyset$ since $l(y_i, y_{i+1})) \cap \mathcal{V}(p) = \emptyset$ for all i < l. Therefore, $p \bowtie q$.

All of this can be computed from continuous functions $f, g : [-1; 2] \to \mathbb{R}^2$, rational intervals $I, J \subseteq [-1; 2]$ and n.

Corollary 3.2 Let $f, g : [-1; 2] \to \mathbb{R}^2$ be continuous functions and let $I, J \subseteq [-1; 2]$ be closed rational intervals such that $f(I) \cap g(J) = \emptyset$. Then $\pi(f|_{I}, g|_{J}) = 0$.

Proof Since f(I) and g(J) are compact, $d_s(f(I), g(J)) > 0$ and hence $\alpha_{IJ} > 0$. There is some number n such that $11 \cdot 2^{-n} < d_s(f(I), g(J))$ and $2^{-n} < \alpha_{IJ}/16$.

By Lemma 3.1 there are *n*-approximations p and q of $(f|_I \text{ and } g|_J, \text{ respectively, such that } p \bowtie p$. By definition 2.13, $\pi(f|_I, g|_J) = \pi(p, q)$. By (25), $\operatorname{range}(h_p) \subseteq U(\operatorname{range}(f_I), \dots \dots \cap f(f_I), \dots \cap f(f_I)) \subseteq U(\operatorname{range}(g_J), \dots \cap f(f_I)) \subseteq U(\operatorname{range}(f_I), \dots \cap f(f_I)) \subseteq U(\operatorname{range}(f_$

Lemma 3.3

- 1. From continuous functions $f, g: [-1; 2] \to \mathbb{R}^2$ and closed rational intervals $I, J \subseteq [-1; 2]$ we can compute α_{IJ} .
- 2. From continuous functions $f, g: [-1; 2] \to \mathbb{R}^2$ and closed rational intervals $I, J \subseteq [-1; 2]$ such that $\alpha_{IJ} > 0$ we can compute $\pi(f|_I, g|_J)$.

Remember that $f(a_I), f(b_I) \notin g(J)$ and $g(a_J), g(b_J) \notin f(I)$ is equivalent to $\alpha_{IJ} > 0$.

- **Proof** (1) From f, a_I and b_I we can compute the compact set $\{a_I, b_I\}$. from g and J we can compute the compact set g(J). The function $(A, B) \mapsto d_s(A, B)$ on (non-empty) compact set is computable. Therefore, from f, g, I, J we can compute $d_s(\{f(a_I), f(b_I)\}, g(J))$. Correspondingly we can compute $d_s(\{g(a_J), g(b_J)\}, f(I))$ and hence their minimum α_{IJ} .
- (2) Compute α_{IJ} . Find some n such that $2^{-n} < \alpha/16$. By Lemma 3.1 we can compute n-approximations p and q of f_I and g_J , respectively such that $p \bowtie q$. Compute the number of crossings CN(p,q) and its parity $\pi(p,q)$. By Definition 2.13, $\pi(f|_I,g|_J) = \pi(p,q)$

4 The crossing parity of f and g.

The crossing parity $\pi(f,g)$ does not depend on f and g as long as $\phi, \psi : [0;1] \to [0;1]^2$ are continuous and (1) is true. For showing $\pi(p,q) = 1$ we construct special tracks p and q. For tracks p define $\overline{\mathcal{L}}(p) := \bigcup \{l(x,y) \mid x,y \in \mathcal{V}(p), \ x \neq y\}$ ((11) and Figure 3).

Lemma 4.1 There are 5-approximations $p = ((s_0, x_0), \dots, (s_k, x_k))$ and $q = ((t_0, y_0), \dots, (t_l, y_l))$ of f and g, respectively, such that

$$\operatorname{card} \mathcal{V}(p) = k+1 \quad and \quad \operatorname{card} V(q) = l+1 \quad and \tag{51}$$

$$V(p) \cap \overline{\mathcal{L}}(q) = \emptyset \quad and \quad V(q) \cap \overline{\mathcal{L}}(p) = \emptyset.$$
 (52)

Proof There is a modulus of uniform continuity md of f and g (cf. (20)). There are some k and rational numbers $s_0 := -1, s_1, \ldots, s_{k-1}, s_k := 2$ such that $0 < s_{i+1} - s_i < 2^{-\text{md}(5)}$ for $0 \le i < k$. There are points $x_0, \ldots, x_k \in \mathbb{Q}^2$ such that $||f(s_0) - x_0|| < 2^{-5}$ and $x_i \ne x_j$ for all $i \ne j$. Then $p = ((s_0, x_0), \ldots, (s_k, x_k))$ is a 5-approximation of f such that card $\mathcal{V}(p) = k + 1$. There are some l and

²where $U(A, \delta) := \{ x \in \mathbb{R}^2 \mid (\exists y \in A) ||x - y|| < \delta \}$

rational numbers $t_0 := -1$, $t_1, \ldots, t_{k-1}, t_k := 2$ such that $0 < t_{i+1} - t_i < 2^{-\operatorname{md}(5)}$ for $0 \le i < l$. There is some $y_0 \in \mathbb{Q}^2$ such that $\|g(t_0) - y_0\| < 2^{-5}\|$ and $y_0 \notin \overline{\mathcal{L}}(p)$. Suppose y_0, \ldots, y_i (i < l) have been found. Then there is some $y_{i+1} \in \mathbb{Q}^2$ such that $\|g(s_{i+1}) - y_{i+1}\| < 2^{-5}$, $y_{i+1} \notin \{y_0, \ldots, y_i\}$, $y_{i+1} \notin \overline{\mathcal{L}}(p)$ and $(l(y_0, y_{i+1}) \cup \ldots \cup l(y_i, y_{i+1})) \cap \mathcal{V}(p) = \emptyset$. By induction, we can define y_0, \ldots, y_l . The obviously (51) and (52) are true for p and q.

Lemma 4.2 $\pi(f,g) = 1$.

Proof For $I_0 := J_0 := [-1; 2]$ we have $f|_{I_0} = f$, $g|_{J_0} = g$ and $\alpha_{I_0J_0} = 1$ ((8) and Figure 1). Since $2^{-5} < \alpha_{IJ}/16$ by Definition 2.13, $\pi(f,g) = \pi(p,q)$ for (arbitrary) 5-approximations p and q of f and g such that (51) and (52). Notice that (52) implies $p \bowtie q$. In the following we prove $\pi(p,q) = 1$. We define

$$z := (0.5, 0.5),$$
 (53)

$$R_{f0} := [-1 - 2^{-4}; -0.5 + 3 \cdot 2^{-4}] \times [-2^{-4}; 2^{-4}] \text{ and}$$
 (54)

$$R_{f1} := [1.5 - 3 \cdot 2^{-4}; 2 + 2^{-4}] \times [-2^{-4}; 2^{-4}].$$
 (55)

Figure 8 shows f and g, the ball B(z, 1.31) and the two boxes R_{f0} and R_{f1} . By the definitions of f and g in (6) and (7), f(s) = (s, 0) for $-1 \le s \le 0$. First we prove

$$h_p(s) \in R_{f0} \quad \text{for} \quad s < -0.5 + 2^{-5} \quad \text{and}$$
 (56)

$$||h_p(s) - z|| < 1.31$$
 for $-0.5 - 2^{-5} < s \le 1.5 + 2^{-5}$ and (57)

$$h_p(s) \in R_{f1} \quad \text{for} \quad s > 1.5 - 2^{-5} \,.$$
 (58)

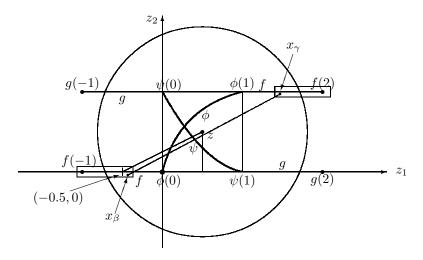


Figure 8: Intersecting curves ϕ and ψ with extensions f and g

We show (56). Let $s < -0.5 + 2^{-5}$. Then for some $i, s_i \le s < s_{i+1} < -0.5 + 2^{-4}$. Then $x_i \in R_{f0}$ since $||f(s_i) - x_i|| - 2^{-5}$. Correspondingly, $x_{i+1} \in R_{f0}$. Since $h_p(s) \in \overline{x_i x_{i+1}}$ and R_{f0} is convex, $h_p(s) = R_{f0}$. (58) is true by symmetry.

We show (57). If $-0.5-2^{-5} < s \le -0.5$ then $||h_p(s)-z|| \le ||h_p(s)-f(s)|| + ||f(s)-f(-0.5)|| + ||f(-0.5)-z|| < 5 \cdot 2^{-5} + 2^{-5} + ||f(-0.5)-z|| \le 6 \cdot 2^{-5} + \sqrt{1+1/4}) < 0.19 + 1.12 = 1.31$. If $-0.5 \le s \le 0$

then $||h_p(s) - z|| \le ||h_p(s) - f(s)|| + ||f(s) - z|| < 5 \cdot 2^{-5} + ||f(-0.5) - z|| < 1.31$ (from above). Hence $||h_p(s) - z|| < 1.31$ for $-0.5 - 2^{-5} < s \le 0$. By symmetry, $||h_p(s) - z|| < 1.31$ also for $1 \le s < 1.5 + 2^{-5}$. For $0 \le s \le 1$, $||h_p(s) - z|| \le ||h_p(s) - f(s)|| + ||f(s) - z|| \le 5 \cdot 2^{-5} + \sqrt{2}/2 < 1.31$. Therefore (57) is true. Since $|s_{i+1} - s_i| < 2^{-\text{md}(5)} \le 2^{-5}$ there are indices $\beta < \gamma$ such that

$$-0.5 - 2^{-5} < s_{\beta} < -0.5 + 2^{-5}$$
 and $1.5 - 2^{-5} < s_{\gamma} < 1.5 + 2^{-5}$. (59)

By (56) - (58),

$$x_i \in R_{f0} \quad \text{if} \quad i \le \beta \,, \tag{60}$$

$$x_i \in B((z, 1.31) \quad \text{if} \quad \beta \le i \le \gamma,$$
 (61)

$$x_i \in R_{f1} \quad \text{if} \quad \gamma \le i \,.$$
 (62)

We simplify the track p step by step by means of Lemma 2.6. For $\beta < i \le \gamma$ define

$$p_i := ((s_0, x_0), \dots, (s_\beta, x_\beta), (s_i, x_i), \dots, (s_k, x_k))$$

Since the x_i are pairwise different (by $\operatorname{card}(\mathcal{V}(p)) = k + 1$), $x_{\beta} \neq x_i$. Therefore, p_i is a track, see Definition 2.1. We prove by induction

$$\pi(p_i, q) = \pi(p, q) \quad \text{for} \quad \beta < i \le \gamma.$$
 (63)

Since $p_{\beta+1} = p$, $\pi(p_{\beta+1}, q) = \pi(p, q)$.

Suppose $\pi(p_i, q) = \pi(p, q)$ for some $\beta < i < \gamma$. We define

$$p_a := ((s_0, x_0), \dots, (s_\beta, x_\beta)),$$
 (64)

$$p_b := ((s_\beta, x_\beta), (s_i, x_i), (s_{i+1}, x_{i+1})), \tag{65}$$

$$p_c := ((s_\beta, x_\beta), (s_i, z_1), (s_{i+1}, x_{i+1}))$$
 (66)

for some $z_1 \in \overline{x_{\beta}x_{i+1}} \setminus \{x_{\beta}, x_{i+1}\},$

$$p_d := ((s_\beta, x_\beta), (s_{i+1}, x_{i+1}))$$
 (67)

$$p_e := ((s_{i+1}, x_{i+1}), \dots, (s_k, x_k))$$
 (68)

We apply Lemma 2.6 to p_b and p_c . We have (cf. (14 - 16))

- $-\{x_{\beta}, x_{i+1}\} \cap \overline{\mathcal{L}}(q) = \emptyset$, (since $\mathcal{V}(p) \cap \overline{\mathcal{L}}(q) = \emptyset$),
- $-\{x_{\beta}, x_i, x_{i+1}, z_2\} \in B(z, 1.31)$ (by (61)) and
- $-\{y_0, y_l\} \cap B(z, 1.31) = \emptyset.$

(For the last line: $1.58 < \sqrt{1.5^2 + 0.5^2} = ||g(-1) - z|| \le ||g(-1) - y_0|| + ||y_0 - z|| < ||y_0 - z|| + 2^{-5} < ||y_0 - z|| + 0.04$, hence $||y_0 - z|| > 1.31$. Accordingly $||y_l - z|| > 1.31$.) By Lemma 2.6, $\pi(p_b, q) = \pi(p_c, q)$. Since $z_1 \in \overline{x_\beta x_{i+1}} \setminus \{x_\beta, x_{i+1}\}$, $\pi(p_c, q) = \pi(p_d, q)$, hence $\pi(p_b, q) = \pi(p_d, q)$. Therefore, replacing p_b by p_d in $p_{\beta,i}$ does not change the crossing parity. We show this in detail.

Since $V(p) \cap \overline{\mathcal{L}}(q) = \emptyset$, $x_i \notin \text{range}(h_q)$ for all i, hence

$$\begin{array}{rcl} \mathrm{CN}(p_i,q) & = & \mathrm{CN}(p_a,q) + \mathrm{CN}(p_b,q) + \mathrm{CN}(p_e,q) \ \ \mathrm{and} \\ \mathrm{CN}(p_{i+1},q) & = & \mathrm{CN}(p_a,q) + \mathrm{CN}(p_d,q) + \mathrm{CN}(p_e,q) \ . \end{array}$$

Since $(m+n) \mod 2 = (m \mod 2 + n \mod 2) \mod 2$,

$$\pi(p_i, q) = (\pi(p_a, q) + \pi(p_b, q) + \pi(p_e, q)) \mod 2$$

$$= (\pi(p_a, q) + \pi(p_d, q) + \pi(p_e, q)) \mod 2$$

$$= \pi(p_{i+1}, q).$$

By induction

$$\pi(p,q) = \pi(p_{\gamma},q) \text{ for } p_{\gamma} = ((s_0, x_0), \dots, (s_{\beta}, x_{\beta}), (s_{\gamma}, x_{\gamma}), \dots, (s_k, x_k)).$$
 (69)

By (56) and (58), $h_{p_{\gamma}}$ runs in the box R_{f1} for $-1 \le s \le s_{\beta}$, then its graph is the straight line segment $\overline{x_{\beta}x_{\gamma}}$ and finally $h_{p_{\gamma}}$ remains in the box R_{f2} for $s_{\gamma} < s$, see Figure 8.

Since $V(p_{\gamma}) \subseteq V(p)$, $V(q) \cap \overline{\mathcal{L}}(p_{\gamma}) = \emptyset$. Keeping p_{γ} fixed we can simplify the track q accordingly. Therefore there are indices $\mu < \nu$ such that for $q_{\nu} := ((t_0, y_0), \dots, (t_{\mu}, y_{\mu})(t_{\nu}, y_{\nu}), \dots, (t_l, y_l)), \ \pi(p_{\gamma}, q) = \pi(p_{\gamma}, q_{\nu})$, where $h_{q_{\nu}}$ runs in the box R_{g1} for $-1 \le t_{\mu}$, then its graph is the straight line segment $\overline{y_{\mu}y_{\nu}}$ and finally $h_{p_{\gamma}}$ remains in the box R_{g2} for $t_{\nu} < t$ as shown in Figure 9.

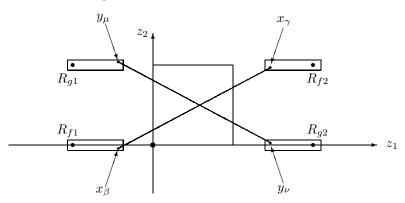


Figure 9: The tracks p_{γ} and q_{ν} .

Since $(R_{f1} \cup R_{f2}) \cap \text{range}(h_{q_{\nu}}) = \emptyset$ and $(R_{g1} \cup R_{g2}) \cap \text{range}(h_{p_{\gamma}}) = \emptyset$ (we omit the straightforward formal verifications), $CN(p_{\gamma}, q_{\nu}) = 1$. We obtain $\pi(p, q) = \pi(p_{\gamma}, q) = \pi(p_{\gamma}, q_{\nu}) = 1$.

5 The proof of the main theorem

For $A \in \mathbb{R}^2$ and a number $\delta > 0$ let $U(A, \delta) := \{x \in \mathbb{R}^2 \mid (\exists y \in A) \|x - y\| < \delta\}$ be the δ -neighborhood of A. From f and g we will compute sequences $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ and $J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots$ of rational closed intervals such that $f(\bigcap_i I_i) = g(\bigcap_i J_i)$. First we prove the following lemma.

Lemma 5.1 From f, g, rational intervals $I, J \subseteq [-1; 2]$ and $n \subseteq \mathbb{N}$ such that such that $2^{-n} < \alpha_{IJ}$ and $\pi(f|_{I}, g|_{J}) = 1$ we can compute a rational interval $K \subseteq I$ such that

$$\alpha_{KJ} > 0, \ f(K) \subseteq U(g(J), 2^{-n}) \ \text{and} \ \pi(f|_K, g|_J) = 1.$$
 (70)

Proof Perform the following computations:

- From J and g compute the compact set g(J).
- Compute some sequence $(a_I = s_0, s_1, \dots, s_{k-1}, s_k = b_I)$ of rational numbers such that $0 < s_{i+1} s_i < 2^{-md(n+4)}$.

For every index i do the following:

- From f and s_i compute $x_i := f(s_i) \in \mathbb{R}^2$.
- From x_i and g(J) compute $c_i := d_s(\{x_i\}, g(J)) \in \mathbb{R}$.

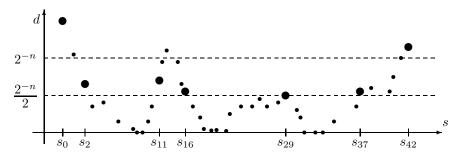


Figure 10: $d_i \approx d_s(f(s_i), g(J))$

- From $c_i \in \mathbb{R}$ and n compute some $d_i \in \mathbb{Q}$ such that $|c_i - d_i| < 2^{-n}/16$. Figure 10 shows the points $d_i \approx d_s(f(s_i), g(J))$.

Suppose $|s_i - s| < 2^{-md(n+4)}$ and $d_i \ge 2^{-n}/2$. Then for every $z \in g(J)$, $2^{-n}/2 \le d_i < c_i + 2^{-n}/16 \le ||x_i - z|| + 2^{-n}/16 \le ||f(s_i) - f(s)|| + ||f(s) - z|| + 2^{-n}/16 < 2^{-n}/16 + ||f(s) - z|| + 2^{-n}/16$, hence $||f(s) - z|| > 6/16 \cdot 2^{-n}$. Therefore,

if
$$|s_i - s| < 2^{-md(n+4)}$$
 and $d_i \ge 2^{-n}/2$ then
$$(\forall z \in f(J)) ||f(s) - z|| > 6/16 \cdot 2^{-n}. \tag{71}$$

Suppose $|s_i - s| < 2^{-md(n+4)}$ and $d_i < 2^{-n}/2$. Then $c_i < d_i + 2^{-n}/16 \le 9/16 \cdot 2^{-n}$, hence for some $z \in g(J)$, $||f(s_i) - z|| < 9/16 \cdot 2^{-n}$ and $||f(s) - z|| \le ||f(s) - f(s_i)|| + ||f(s_i) - z|| \le 10/16 \cdot 2^{-n}$. Therefore,

if
$$|s_i - s| < 2^{-md(n+4)}$$
 and $d_i < 2^{-n}/2$ then
$$(\exists z \in f(J)) ||f(s) - z|| < 10/16 \cdot 2^{-n}. \tag{72}$$

Since $2^{-n} < \alpha_{IJ}$ and $s_0 = a_I$, $2^{-n} < d_s(f(a_I), g(J)) = c_0 < d_0 + 2^{-n}/16$, hence

$$d_0 > 2^{-n}/2$$
, correspondingly, $d_k > 2^{-n}/2$. (73)

Therefore, we can find a sub-sequence $0 = i_0 < i_1 < \ldots < i_m = k$ of $0, 1, \ldots, k$ such that

$$(\forall j) d_{i_j} \ge 2^{-n}/2 \quad \text{and} \tag{74}$$

either
$$(\forall i_i < l < i_{j+1}) d_l \ge 2^{-n}/2$$
 (75)

or
$$(\forall i_i < l < i_{i+1}) d_l < 2^{-n}/2$$
. (76)

Let $K_j := [s_{i_j}; s_{i_{j+1}}]$. (In Figure 10, m = 6 and $(i_0, i_1, \dots, i_6) = (0, 2, 11, 16, 29, 37, 42)$.)

Suppose (75), that is, $(\forall i_j < l < i_{j+1}) d_l \ge 2^{-n}/2$. (In Figure 47 this is true for j = 0, 2, 5.) By (71) for all $z \in g(J)$ and $s \in K_j$, $||f(s) - z|| > 6/16 \cdot 2^{-n}$, hence $f(K_j) \cap g(J) = \emptyset$, therefore,

$$\pi(f|_{K_i}, g|_J) = 0. (77)$$

Suppose (76), that is, $(\forall i_j < l < i_{j+1}) d_l < 2^{-n}/2$. (In Figure 47 this is true for j = 1, 3, 4.) By (72) for all $s \in K_j$ there is some $z \in g(J)$ such that $||f(s) - z|| < 10/16 \cdot 2^{-n}$, hence $f(K_j) \subseteq U(g(J), 2^{-n})$. Furthermore by (74), $2^{-n}/2 - 2^{-n}/16 \le d_{i_j} - 2^{-n}/16 < c_{i_j} = d_s(\{f(s_{i_j})\}, g(J))$. The same is true for (j+1) instead of j, therefore $0 < d_s(\{f(s_{i_j}), f(s_{i_{j+1}})\}, g(J))$.

Since $K_j \subseteq I$ and $\alpha_{IJ} > 0$, $d_s(f(K_j), \{g(a_J), g(b_J)\} \ge d_s(f(I), \{g(a_J), g(b_J)\} \ge \alpha_{IJ}$. In summary,

$$0 < \alpha_{K_j J} \text{ and } f(K_j) \subseteq U(g(J), 2^{-n}). \tag{78}$$

By Lemma 2.14,

$$(\pi(f|_{K_0}, g|_J) + \dots + (\pi(f|_{K_{m-1}}, g|_J)) \mod 2 = \pi(f|_I, g|_J) = 1.$$

Therefore, there is some j such that $\pi(f|_{K_i}, g|_J) = 1$. By (77), (75) cannot be true for j, hence (76) is true for j, hence (78) is true for K_i .

For computing K from f, g, I, J and n, first compute (d_0, \ldots, d_k) , then compute the sub-sequence $0 = i_0 < i_1 < \dots, < i_m = k \text{ of } 0, 1, \dots, k \text{ such that } (74-76).$ The find some j such that $\pi(f|_{K_j}, g|_J) = 1$. Let $K := K_j$. Then (70) is true by (78).

Lemma 5.2 From functions f and g, closed rational intervals $I, J \subseteq [-1, 2]$ and $m \in \mathbb{N}$ such that $\alpha_{IJ} > 0$ and $\pi(f|_{I}, g|_{J}) = 1$ we can compute closed rational intervals I', J' such that

$$\alpha_{I'J'} > 0 \quad and \quad \pi(f|_{I'}, g|_{J'}) = 1,$$
 (79)

$$I' \subseteq I \quad and \quad J' \subseteq J \,, \tag{80}$$

$$f(I') \subseteq U(g(J), 2^{-m}) \quad and \quad g(J') \subseteq U(f(I'), 2^{-m}).$$
 (81)

Proof Compute α_{IJ} and some number n > m such that $2^{-n} < \alpha_{IJ}$.

By Lemma 5.1 we can compute a rational interval $I' \subseteq I$ ($I_m = K$ in the lemma) such that by (70),

$$\alpha_{I'J} > 0, \quad f(I') \in U(g(J), 2^{-n}) \quad \text{and} \quad \pi(f|_{I'}, g|_{J}) = 1.$$
 (82)

Again by Lemma 5.1 with (f and g exchanged) first compute $\alpha_{I'J}$ and some number n' > m such that $2^{-n'} < \alpha_{I'J}$, then compute some $J' \subseteq J$ such that

$$\alpha_{I'J'} > 0, \quad g(J') \in U(f(I'), 2^{-n'}) \quad \text{and} \quad \pi(f|_{I'}, g|_{J'}) = 1.$$
 (83)

(79) follows from (83), (80) is true by the construction and (81) follows from (82) and (83) by m < n and m < n'.

Lemma 5.3 From functions f and g (as defined in (6) and (7)) we can compute sequences $I_0 \supseteq I_1, \supseteq$ I_2, \ldots and $J_0 \supseteq J_1 \supseteq J_2, \ldots$ of rational closed intervals such that $f(\bigcap_m I_m) = g(\bigcap_m J_m)$.

Proof By Lemma 5.2 starting with $I_0 := J_0 := [-1, 2]$ we can compute sequences $I_0 \supseteq I_1, \supseteq I_2, \ldots$ and $J_0 \supseteq J_1 \supseteq J_2, \ldots$ of rational closed intervals such that for all $m \ge 1$,

$$f(I_m) \subseteq U(g(J_{m-1}), 2^{-(m-1)}) \text{ and } g(J_m) \subseteq U(f(I_m)), 2^{-m}).$$
 (84)

The following equations follow from continuity of f and g.

$$f(\bigcap_{m} I_m) = \bigcap_{m} f(I_m) = \bigcap_{m} U(f(I_m), 2^{-m}), \qquad (85)$$

$$f(\bigcap_{m} I_{m}) = \bigcap_{m} f(I_{m}) = \bigcap_{m} U(f(I_{m}), 2^{-m}),$$

$$g(\bigcap_{m} J_{m}) = \bigcap_{m} g(J_{m}) = \bigcap_{m} U(g(J_{m}), 2^{-m}).$$
(85)

We give elementary proofs. Since $f(\bigcap_k I_k) \subseteq f(I_m) \subseteq U(f(I_m), 2^{-m})$ for all $m, f(\bigcap_k I_k) \subseteq \bigcap_m f(I_m) \subseteq I$ $\bigcap_m U(f(I_m), 2^{-m}).$

Suppose $x \in \bigcap_m f(I_m)$. Then there is a sequence a_0, a_1, a_2, \ldots such that for all $m, a_m \in I_m$ and $f(a_m) = x$. Since I_0 is compact there is a subsequence $a_{m_0}, a_{m_1}, a_{m_2}, \ldots$ converging to some $a \in I_0$. Since f is continuous, $x = \lim_i f(a_{m_i}) = f(\lim_i a_{m_i}) = f(a)$. Assume $a \notin I_{m_k}$ for some k. Then for some j > k, $a_{m_j} \notin I_{m_k}$, hence $a_{m_j} \notin I_{m_j}$, a contradiction. Therefore $a \in I_{m_k}$ for all k, hence $a \in \bigcap_m I_m$ and $x = f(a) \in f(\bigcap_m I_m).$

Suppose $x \in \bigcap_m U(f(I_m), 2^{-m})$. Assume $x \notin f(I_k)$ for some k. Since $f(I_k)$ is compact $x \notin U(f(I_k), 2^{-j})$ for some j > k. Therefore, $x \notin U(f(I_j), 2^{-j})$, hence $x \notin \bigcap_m U(f(I_m), 2^{-m})$, a contradiction. Therefore $x \in f(I_k)$ for all k, hence $x \in \bigcap_m f(I_m)$.

Therefore (85) is true. By symmetry also (86) is true. By (84–86),
$$f(\bigcap_m I_m) = \bigcap_m f(I_m) = \bigcap_{m \geq 1} f(I_m) \subseteq \bigcap_{m \geq 1} U(g(J_{m-1}), 2^{-(m-1)}) = \bigcap_m U(g(J_m), 2^{-m}) = g(\bigcap_m J_m).$$
 Accordingly, $g(\bigcap_m J_m) \subseteq f(\bigcap_m I_m)$.

After these preparations Theorem 1.4 can be proved straightforwardly. For the canonical representation (most conveniently the canonical multi-representation [11]) of functions $h:\subseteq\mathbb{R}\to\mathbb{R}^2$ the function f can be computed from ϕ and the function g can be computed from ψ . By the outer representation of the set \mathcal{I} of the closed real intervals, a name of S is a sequence $I_0 \supseteq I_1 \supseteq \ldots$ of closed rational intervals such that $S = \bigcap_n I_n$. Therefore, by Lemma 5.3 the multi-function $(\phi, \psi) \Longrightarrow (S_\phi, S_\psi)$ such that $\phi(S_\phi) = \psi(S_\psi)$ is computable. Thus we have proved Theorem 1.4.

The main theorem from [10] follows straightforwardly from Theorem 1.4.

Corollary 5.4

- 1. If ϕ and ψ in Theorem 1.1 are computable then there are computable numbers a and b such that $\phi(a) \in \text{range}(\psi)$ (hence $x := \phi(a) \in \text{range}(\phi) \cap \text{range}(\psi)$) and $\psi(b) \in \text{range}(\phi)$ (hence $y := \psi(b) \in \text{range}(\phi) \cap \text{range}(\psi)$).
- 2. Restricted to the pairs (ϕ, ψ) which have a unique intersection point, the point x such that $\{x\} = \operatorname{range}(\phi) \cap \operatorname{range}(\psi)$ can be computed from ϕ and ψ .
- 3. Restricted to the pairs (ϕ, ψ) such that $\phi(a) = \psi(b)$ for a unique pair $(a, b) \in [0; 1]^2$ the function $(\phi, \psi) \mapsto (a, b)$ is computable.

Proof

1. Suppose S_{ϕ} has length > 0. Then $a \in S_{\phi}$ for some $a \in \mathbb{Q}$, which is a computable real number such that $\phi(a) \in \psi(S_{\beta}) \subseteq \text{range}(\psi)$.

Suppose S_{ϕ} has length 0, that is, $S_{\phi} = \{a\}$ for some a. Since $\{a\} = S_{\phi} = \bigcap_{n} I_{n}$ for a computable sequence $I_{0} \supseteq I_{1} \supseteq \ldots$ of rational intervals, a is a computable real number such that $\phi(a) \in \psi(S_{\beta}) \subseteq \operatorname{range}(\psi)$.

By symmetry there is a computable number b such that $\psi(b) \in \text{range}(\phi)$.

- 2. Notice that possibly S_{ϕ} and S_{ψ} have positive lengths. Suppose ϕ and ψ have a single intersection point x. Then $\{x\} = \phi(S_{\phi}) = \psi(S_{\psi})$. For every m > 0, $\emptyset = \{x\} \setminus B(x, 2^{-m}) = \bigcap_{n \in \mathbb{N}} \phi(I_n) \setminus B(x, 2^{-m}) = \bigcap_{n \in \mathbb{N}} \phi(I_n) \setminus B(x, 2^{-m})$ by (85). The countable intersection of closed subsets of the compact set $[0; 1]^2$ is empty. Therefore finitely many suffice: there is some N such that $\emptyset = \bigcap_{n \leq N} (\phi(I_n) \setminus B(x, 2^{-m})) = \bigcap_{n \leq N} \phi(I_n) \setminus B(x, 2^{-m}) = \phi(I_N) \setminus B(x, 2^{-m})$, hence $\phi(I_N) \subseteq B(x, 2^{-m})$. Therefore, for every m > 0 there are some $N \in \mathbb{N}$ and some $z \in \mathbb{Q}^2$ such that $x \in \phi(I_N) \subseteq B(z, 2 \cdot 2^{-m})$. Since $(\phi, N) \mapsto \phi(I_N)$ is computable and the set (K, z, j) such that K is compact and $K \subseteq B(z, j)$ is c.e. 3 [11], from ϕ and the list I_0, I_1, \ldots we can compute a sequence of rational balls contracting to x. Therefore we can compute the single intersection point of ϕ and ψ .
- 3. We can compute a sequence $I_0 \supseteq I_1 \supseteq \dots$ converging to a. Therefore we can compute a. Correspondingly we can compute b.

In Corollary 5.4.1 in general $\phi(a) \neq \psi(b)$. From ϕ and ψ we cannot compute some a such that $\phi(a) \in \text{range}(\psi)$ or some $x \in \text{range}(\phi) \cap \text{range}(\psi)$. We do not even know whether for the computable number a there is a computable number c such that $\phi(a) = \psi(c)$. In Corollary 5.4.2 from ϕ and ψ we cannot compute some a such that $\phi(a) = x$.

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 $^{^{3}}$ computably enumerable or recursively enumerable

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