*Machine Learning and Artificial Intelligence*
*A.J. Tallón-Ballesteros and C. Chen (Eds.)*

*doi:10.3233/FAIA200811*

# The Bounded Traveling Wave Solutions of a (3+1) Dimensions mKdv-ZK Equation

Yuzhong ZHANG [1]

*School of Mathematics and Information Technology, Yuxi Normal University, Yuxi, Yunnan, China*

**Abstract.** The bounded traveling waves solutions of a (3+1) dimensions mKdv-ZK equation be investigated by using method of dynamical systems. The exact expressions of bounded periodic waves, solitary waves and kink waves are given. Under fixed parameter condition, the planar simulation graphs of the bounded periodic waves, solitary waves and kinks are obtained by using the software Mathematica 7.

**Keywords.** mKdv-VZK equation, singular point, periodic waves, solitary waves, kinks

## 1. Introduction

The well-known Kdv equation has perfect dynamic properties. Since Kdv equation was proposed, many scholars have done a lot of research and obtained a lot of results. Some researchers have extended the Kdv equation, and proposed Kdv-B, mKdv, mKdv-ZK equation, etc. In reference [1], the mKdv-ZK equation is studied with homogeneous balance method, and some traveling solution is given. In this paper, the Vries – Zakharov – Kuznetsov (mKdV-ZK) equation is investigated by using the bifurcation method [2, 3, 4] of dynamical systems:

$$u_t + a_1 u^2 u_x + a_2 u_{xxx} + a_3(u_{yy} + u_{zz})_x = 0, \tag{1}$$

where $a_1$, $a_2$ and $a_3$ are real constants, the properties of singular point of mKdv-ZK plane are obtained, and the bifurcation of phase portraits are given. The exact solutions of bounded periodic waves, solitary waves and kinks of mKdv-ZK are obtained by using phase portraits. Under fixed parameter conditions, the plane simulation diagrams of the bounded periodic waves, solitary waves and kinks were obtained by using the software Mathematica 7.

[1] Corresponding Author: Yuzhong Zhang, associate professor, mainly engaged in study of differential equations, School of Mathematics and Information Technology, Yuxi Normal University, Yunnan, China; E-mail: zh1011@yxnu.net.

## 2. The Bifurcation of Phase Portraits

Letting $\xi = x + y + z - ct$, $u(x, y, z, t) = \varphi(\xi)$, where c is wave velocity, then Eq. (1) can be transformed into:

$$(a_1\varphi^2 - c)\varphi' + (a_2 + 2a_3)\varphi''' = 0. \tag{2}$$

Integrating straightforwardly Eq. (2), and taking integral constant as 0, we get

$$(a_2 + 2a_3)\varphi'' + \frac{1}{3}a_1\varphi^3 - c\varphi = 0. \tag{3}$$

Letting $\varphi' = y$, then Eq. (3) can be transformed into a plane system:

$$\begin{cases} \frac{d\varphi}{d\xi} = y \\ \frac{dy}{d\xi} = \frac{1}{a_2+2a_3}(-\frac{1}{3}a_1\varphi^3 + c\varphi) \end{cases}. \tag{4}$$

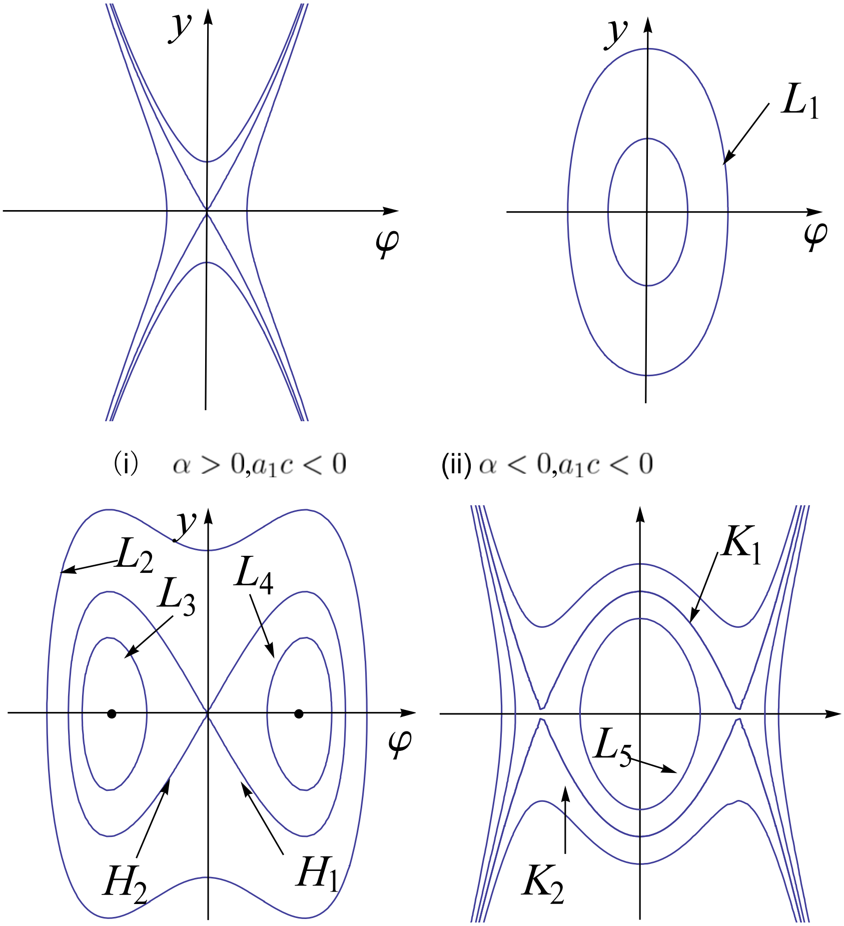
Obviously, plane system (4) is a Hamiltonian system, where Hamitonian function as follow Eq. (5):

$$H(\varphi, y) = \frac{1}{2}y^2 + \frac{1}{a_2 + 2a_3}(\frac{1}{12}a_1\varphi^4 - \frac{1}{2}c\varphi^2) = h. \tag{5}$$

Letting $\alpha = \frac{-a_1}{a_2+2a_3}$, obviously, the plane system (4) has the properties of singular points as follow:

- when $a_1c > 0$, the plane system(4) has three singular points $(-\sqrt{\frac{3c}{a_1}}, 0)$, $(0, 0)$, $(\sqrt{\frac{3c}{a_1}}, 0)$ and $H(\sqrt{\frac{3c}{a_1}}, 0) = H(-\sqrt{\frac{3c}{a_1}}, 0)$
  - If $\alpha < 0$, then $(0, 0)$ is a saddle point, $(\sqrt{\frac{3c}{a_1}}, 0)$ and $(-\sqrt{\frac{3c}{a_1}}, 0)$ are two center points.
  - If $\alpha > 0$, then $(0, 0)$ is a center point, $(\sqrt{\frac{3c}{a_1}}, 0)$ and $(-\sqrt{\frac{3c}{a_1}}, 0)$ are two saddle points.
- When $a_1c < 0$, then plane system (4) has only one singular point$(0, 0)$,
  - If $\alpha > 0$, then $(0, 0)$ is a saddle point.
  - If $\alpha < 0$, then $(0, 0)$ is a center point.

From the above analysis, the phase portraits of plane system (4) can be drawn by Eq.(5).




(i) $\alpha > 0, a_1 c < 0$ (ii) $\alpha < 0, a_1 c < 0$

(iii) $\alpha < 0, a_1 c > 0$ (iv) $\alpha > 0, a_1 c > 0$

**Figure 1** The bifurcation of phase portraits of the system (4)

## 3. The Traveling Wave Solutions

Setting $(\varphi_0, 0)$ as a initial point, substituting it into Eq.(5), then we get $H(\varphi_0, 0) = \frac{1}{a_2+2a_3}(\frac{1}{12}a_1\varphi_0^4 - \frac{1}{2}c\varphi_0^2) = h_0$.

Letting $H(\varphi, y) = h_0$, $H(\varphi, 0) = h_0$, then we get

$$\frac{1}{2}y^2 = \frac{1}{a_2+2a_2}(\frac{1}{12}a_1\varphi_0^4 - \frac{1}{2}c\varphi_0^2) - \frac{1}{a_2+2a_3}(\frac{1}{12}a_1\varphi^4 - \frac{1}{2}c\varphi^2), \tag{6}$$

$$y = \pm\sqrt{\frac{-a_1}{6(a_2+2a_3)}(\varphi-\varphi_0)(\varphi+\varphi_0)(\varphi^2+\varphi_0^2-\frac{6c}{a_1})}, \tag{7}$$

$$\frac{d\varphi}{d\xi} = \pm\sqrt{\frac{\alpha}{6}(\varphi-\varphi_0)(\varphi+\varphi_0)(\varphi^2+\varphi_0^2-\frac{6c}{a_1})}, \tag{8}$$

where $\alpha = \frac{-a_1}{a_2+2a_3}$. The traveling wave solutions can be obtained by integrating Eq. (1).

### 3.1. The Periodic Wave Solutions

Since the traveling wave corresponding to the smooth closed orbit of the phase portrait is a bounded periodic wave, the solution of the bounded smooth periodic wave can be obtained by integrating Eq.(8) along the periodic orbit.

Property 1. Under the following conditions, Eq. (1) has a bounded smooth periodic wave.

(i). If $\alpha < 0, a_1c < 0$ , $\varphi_0 < 0$, then the periodic orbit $L_1$ (see Figure 1(ii)) determines a periodic wave and its solution is:

$$\varphi = \varphi_0 \mathrm{cn}((\varphi_0^2 - \frac{3c}{a_1})\sqrt{\frac{-2\alpha}{3}}\xi, \sqrt{\frac{\varphi_0^2}{2(\varphi_0^2-\frac{3c}{a_1})}}). \tag{9}$$

(ii). If $\alpha > 0, a_1c > 0$ ,$-\sqrt{\frac{3c}{a1}} < \varphi_0 < 0$, then the periodic orbit $L_5$ (See Figure 1(vi)) determines a periodic wave and its solution is:

$$\varphi = \varphi_0^* - \frac{\varphi_0^{*2}-\varphi_0^2}{\varphi_0^*-\varphi_0+2\varphi_0\mathrm{sn}^2((\varphi_0^*-\varphi_0)\sqrt{\frac{\alpha}{12}}\xi, \frac{2\sqrt{-\varphi_0^*\varphi_0}}{\varphi_0^*-\varphi_0})}, \tag{10}$$

where $\varphi_0^* = \sqrt{\frac{6c}{a_1}-\varphi_0^2}$.

(iii). If $\alpha < 0, a_1c > 0, \varphi_0 < -\sqrt{\frac{6c}{a1}}$, then the periodic orbit $L_2$ (See Figure 1(iii)) determines a periodic wave and its solution is Eq.(9)

(vi). If $\alpha < 0, a_1c > 0, -\sqrt{\frac{6c}{a1}} < \varphi_0 < -\sqrt{\frac{3c}{a1}}$, then the periodic orbit $L_3$(See Figure 1(iii)) determines a periodic wave and its solution is:

$$\varphi = \varphi_0^* - \frac{2\varphi_0^*(\varphi_0^*-\varphi_0)}{\varphi_0^*-\varphi_0+(\varphi_0^*+\varphi_0)\mathrm{sn}^2((\varphi_0^*-\varphi_0)\sqrt{\frac{-\alpha}{12}}\xi, \frac{\varphi_0^*+\varphi_0}{\varphi_0^*-\varphi_0})}, \tag{11}$$

where $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0^2}$.

(v). If $\alpha < 0$,$a_1 c > 0$,$0 < \varphi_0 < \sqrt{\frac{3c}{a1}}$, then the periodic orbit $L_4$ (See Figure 1(iii)) determines a periodic wave and its solution is:

$$\varphi = -\varphi_0 + \frac{2\varphi_0(\varphi_0^* + \varphi_0)}{\varphi_0^* + \varphi_0 + (\varphi_0 - \varphi_0^*)\mathrm{sn}^2((\varphi_0^* + \varphi_0)\sqrt{\frac{-\alpha}{12}}\xi, \frac{\varphi_0^* - \varphi_0}{\varphi_0^* + \varphi_0})}, \tag{12}$$

where $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0^2}$.

Proof: (i). when the conditions $\alpha < 0$, $a_1 c < 0$ are satisfied, taking $\varphi_0 < 0$, then the orbit $L_1$(See Figure 1(ii)) passing through initial point $(\varphi_0, 0)$ is a smooth closed orbit, and on $\xi - \varphi$ plane, the traveling wave corresponding to it is bounded periodic wave.

Eq.(8) can be transformed into:

$$\frac{d\varphi}{\sqrt{(\varphi - \varphi_0)(-\varphi - \varphi_0)(\varphi^2 + \varphi_0^2 - \frac{6c}{a_1})}} = \pm\sqrt{\frac{-\alpha}{6}}d\xi. \tag{13}$$

Integral Eq.(13) along $L_1$

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{(-\varphi_0 - \varphi)(\varphi - \varphi_0)(\varphi^2 + \varphi_0^2 - \frac{6c}{a_1})}} = \pm\sqrt{\frac{-\alpha}{6}}\int_0^{\xi} d\xi, \tag{14}$$

Using the elliptic integral formula 259. 00 in reference [5] to calculate (20), the bounded smooth periodic wave solution (9) can be obtained.

(ii). when the conditions $\alpha > 0$, $a_1 c > 0$ are satisfied, taking $\varphi_0$ to satisfy $-\sqrt{\frac{3c}{a1}} < \varphi_0 < 0$, then the orbit $L_5$ (See Figure 1(vi)) passing through initial point $(\varphi_0, 0)$ is a smooth closed orbit, and on $\xi - \varphi$ plane, the traveling wave corresponding to it is bounded periodic wave.

Eq.(8) can be transformed into:

$$\frac{d\varphi}{\sqrt{(\varphi - \varphi_0)(\varphi + \varphi_0)(\varphi - \sqrt{\frac{6c}{a_1} - \varphi_0^2})(\varphi + \sqrt{\frac{6c}{a_1} + \varphi_0^2})}} = \pm\sqrt{\frac{\alpha}{6}}d\xi. \tag{15}$$

letting $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0^2}$, Integral Eq.(15) along $L_1$

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{(\varphi_0^* - \varphi)(-\varphi_0 - \varphi)(\varphi - \varphi_0)(\varphi + \varphi_0^*)}} = \pm\sqrt{\frac{\alpha}{6}} \int_0^{\xi} d\xi, \tag{16}$$

Using the elliptic integral formula 255. 00 in reference [5] to calculate Eq.(18), the bounded periodic wave solution Eq.(10) can be obtained.

(iii) when the conditions $\alpha < 0, a_1 c > 0$are satisfied, taking$\varphi_0 < -\sqrt{\frac{6c}{a1}}$, then the orbit$L_2$(See Figure 1(iii)) passing through the initial point$(\varphi_0, 0)$ is a smooth closed orbit, and on $\xi - \varphi$ plane, the traveling wave corresponding to it is a bounded periodic wave. In the same way as (i), the periodic wave solution is Eq.(9).

(vi) when $\alpha < 0$, $a_1 c > 0$, taking $\varphi_0$ to satisfy $-\sqrt{\frac{6c}{a1}} < \varphi_0 < -\sqrt{\frac{3c}{a1}}$, then the orbit$L_3$ (See Figure 1(iii)) passing through the initial point$(\varphi_0, 0)$ is a smooth closed orbit, and on $\xi - \varphi$ plane, the traveling wave corresponding to it is a bounded periodic wave.

(8) can be transformed into:

$$\frac{d\varphi}{\sqrt{(\varphi - \varphi_0)(-\varphi - \varphi_0)(\varphi - \sqrt{\frac{6c}{a_1} - \varphi_0^2})(\varphi + \sqrt{\frac{6c}{a_1} + \varphi_0^2})}} = \pm\sqrt{\frac{-\alpha}{6}} d\xi. \tag{17}$$

letting $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0^2}$, integral Eq.(17) along $L_3$

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{(-\varphi_0 - \varphi)(\varphi_0^* - \varphi)(-\varphi_0^* - \varphi)(\varphi - \varphi_0^*)}} = \pm\sqrt{\frac{-\alpha}{6}} \int_0^{\xi} d\xi, \tag{18}$$

Using the elliptic integral formula 253. 00 in reference [5] to calculate Eq.(18), the bounded periodic wave solution Eq.(11) can be obtained.

(v). when $\alpha < 0$, $a_1 c > 0$, takint $\varphi_0$ to satisfy $0 < \varphi_0 < \sqrt{\frac{3c}{a1}}$, then the orbit $L_4$ (See Figure 1(iii)) passing through the initial point$(\varphi_0, 0)$ is a smooth closed orbit, and on $\xi - \varphi$ plane, the traveling wave corresponding to it is a bounded periodic wave.

Eq.(8) can be transformed into:

$$\frac{d\varphi}{\sqrt{(\varphi-\varphi_0)(\varphi+\varphi_0)(-\varphi+\sqrt{\frac{6c}{a_1}-\varphi_0^2})(\varphi+\sqrt{\frac{6c}{a_1}+\varphi_0^2})}}=\pm\sqrt{\frac{-\alpha}{6}}d\xi. \tag{19}$$

Letting $\varphi_0^*=\sqrt{\frac{6c}{a_1}-\varphi_0^2}$, integral Eq.(19) along $L_4$

$$\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{(\varphi_0^*-\varphi)(\varphi-\varphi_0)(\varphi+\varphi_0)(\varphi+\varphi_0^*)}}=\pm\sqrt{\frac{-\alpha}{6}}\int_0^{\xi}d\xi, \tag{20}$$

Using the elliptic integral formula 256. 00 in reference [5] to calculate Eq.(20), the bounded periodic wave solution Eq.(12) can be obtained.

*3.2. Solitary Wave Solutions*

Since the traveling wave corresponding to the homoclinic orbit of the phase portraits is a bounded smooth solitary wave, the solution of the bounded smooth solitary wave can be obtained by integrating Eq.(8) along the homoclinic orbit.

Property 3 when $\alpha<0$, $a_1c>0$, there is a solitary wave in Eq. (1).

(i). The homoclinic orbit $H_2$ (See Figure 1(iii)) determines a downward bounded smooth solitary wave, and the solitary wave solution is:

$$\varphi=-\sqrt{\frac{6c}{a_1}}(\cosh\sqrt{\frac{-\alpha c}{a_1}}\xi)^{-1}. \tag{21}$$

(ii). The homoclinic orbit $H_1$ (See Figure1(iii)) determines a upward bounded smooth solitary wave, and the solitary wave solution is:

$$\varphi=\sqrt{\frac{6c}{a_1}}(\cosh\sqrt{\frac{-\alpha c}{a_1}}\xi)^{-1}. \tag{22}$$

Proof: (i). when the conditions $\alpha<0$, $a_1c>0$ are satisfied, then the orbit $H_2$ passing through point$(-\sqrt{\frac{6c}{a1}},0)$ is a homoclinic orbit, and on the $\xi-\varphi$ plane, the traveling wave corresponding to it is a bounded smooth solitary wave.

Eq.(8) can be transformed to:

$$\frac{d\varphi}{\sqrt{\varphi^2(\frac{6c}{a_1}-\varphi^2)}}=\pm\sqrt{\frac{-\alpha}{6}}d\xi. \tag{23}$$

Integrate Eq. (23) along $H_2$

$$\int_{-\sqrt{\frac{6c}{a_1}}}^{\varphi} \frac{d\varphi}{\sqrt{\varphi^2(\frac{6c}{a_1} - \varphi^2)}} = \pm\sqrt{\frac{-\alpha}{6}} \int_0^{\xi} d\xi. \tag{24}$$

A bounded smooth solitary wave Eq.(21) can be obtained by calculating Eq.(24).

(ii). the orbit $H_2$ passing through point$(\sqrt{\frac{6c}{a1}}, 0)$ is a homoclinic orbit, then on the $\xi - \varphi$ plane, the traveling wave corresponding to it is a bounded smooth solitary wave.

Eq.(8) can be transformed to:

$$\frac{d\varphi}{\sqrt{\varphi^2(\frac{6c}{a_1} - \varphi^2)}} = \pm\sqrt{\frac{-\alpha}{6}} d\xi. \tag{25}$$

Integrate Eq.(25) along $H_1$

$$\int_{\varphi}^{\sqrt{\frac{6c}{a_1}}} \frac{d\varphi}{\sqrt{\varphi^2(\frac{6c}{a_1} - \varphi^2)}} = \pm\sqrt{\frac{-\alpha}{6}} \int_{\xi}^{0} d\xi. \tag{26}$$

The bounded smooth solitary wave Eq.(24) can be obtained by calculating Eq.(26).

### 3.3. Kink Solutions

Since the traveling wave corresponding to the heteroclinic orbit of the phase portraits is a bounded kink, the solution of the bounded kink can be obtained by integrating Eq.(8) along the heteroclinic orbit.

Property 4 when $\alpha > 0$, $a_1 c > 0$, there are two kink in the Eq. (1).

(i). the heteroclinic orbit $K_1$(See Figure 1(iv)) determines a kink, and the solution of the kink is:

$$\varphi = -\sqrt{\frac{3c}{a_1}} + \frac{2\sqrt{\frac{3c}{a_1}}(\sqrt{\frac{3c}{a_1}} + \varphi_0)}{\sqrt{\frac{3c}{a_1}} + \varphi_0 + (\sqrt{\frac{3c}{a_1}} - \varphi_0)e^{-\sqrt{\frac{2c}{a_1}}\xi}}, \tag{27}$$

where $-\sqrt{\frac{3c}{a_1}} < \varphi_0 < \sqrt{\frac{3c}{a_1}}$.

(ii). The heteroclinic orbit $K_2$(See Figure 1(iv)) determines a kink, and the solution of the kink is:

$$\varphi = -\sqrt{\frac{3c}{a_1}} + \frac{2\sqrt{\frac{3c}{a_1}}(\sqrt{\frac{3c}{a_1}} + \varphi_0)}{\sqrt{\frac{3c}{a_1}} + \varphi_0 + (\sqrt{\frac{3c}{a_1}} - \varphi_0)e^{\sqrt{\frac{2c}{a_1}}\xi}}, \tag{28}$$

where $-\sqrt{\frac{3c}{a_1}} < \varphi_0 < \sqrt{\frac{3c}{a_1}}$.

Proof: (i)when the conditions $\alpha > 0$, $a_1 c > 0$ are satisfied, taking $\varphi_0$ to satisfy $-\sqrt{\frac{3c}{a_1}} < \varphi_0 < \sqrt{\frac{3c}{a_1}}$, the orbit $K_1, K_2$ passing through saddle point$(\frac{3c}{a_1}, 0)$ are two heteroclinic trajectories, on the $\xi - \varphi$ plane, the traveling waves corresponding to it are two bounded kinks.

Eq.(8) can be transformed to:

$$\frac{d\varphi}{\sqrt{(\sqrt{\frac{3c}{a_1}} + \varphi)^2(\sqrt{\frac{3c}{a_1}} - \varphi)^2}} = \pm\sqrt{\frac{\alpha}{6}}d\xi. \tag{29}$$

Integrate Eq.(29) along $K_1, K_2$

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{(\sqrt{\frac{3c}{a_1}} + \varphi)(\sqrt{\frac{3c}{a_1}} - \varphi)} = \pm\sqrt{\frac{\alpha}{6}}\int_0^{\varphi} d\xi. \tag{30}$$

The two bounded kinks Eq.(27) and Eq.(28) can be obtained by calculating Eq.(30).

## 4. The Plane Simulation Graphs of Traveling Wave

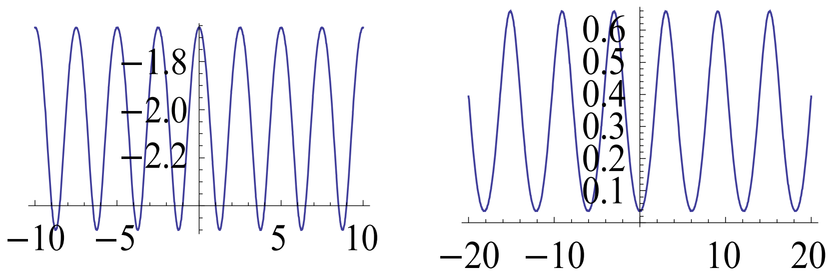
Taking fixed parameters, according to the solution of the traveling wave, then the plane simulation graphs of bounded periodic wave, solitary wave and kink can be obtained by using Mathematica7.

Example 1. Letting $\alpha = -1; a_1 = 2, c = 3$, then $-\sqrt{\frac{3c}{a_1}} \approx -2.12132$, $-\sqrt{\frac{6c}{a_1}} = -3$. Letting $\varphi_0 = -2.5$, then $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0^2} \approx 1.65831$. Substituting these data into Eq.(11), we draw a plane simulation graph of a bounded periodic wave, as shown in Figure 2(i).

Example 2. Letting $\alpha = -1; a_1 = 2, c = 3$, then $\sqrt{\frac{3c}{a_1}} \approx 2.12132$. Letting $\varphi_0 = 0.5$, then $\varphi_0^* = \sqrt{\frac{6c}{a_1} - \varphi_0} \approx 2.95804$. Substituting these data into Eq.(12), we draw a plane simulation graph of a bounded periodic wave, as shown in Figure 2(ii).

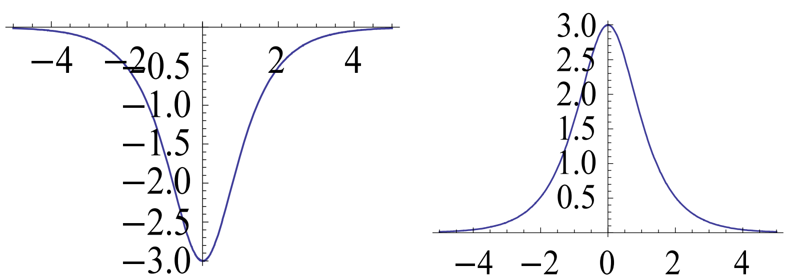
Example 3. Letting $\alpha = -1; a_1 = 2, c = 3$, Substituting these data into Eq.(21) and Eq.(22), we draw respectively two plane simulation graphs of a bounded solitary wave, as shown in Figure 3(iii) and (iv).

Example 4. Letting $\alpha = 1; a_1 = 2, c = 3$, then $\sqrt{\frac{3c}{a_1}} \approx 2.12132$. Letting $\varphi_0 = 0$, substituting these data into Eq.(27) and Eq.(28), we draw respectively two plane simulation graphs of a bounded solitary wave, as shown in Fig. 4(iii) and (vi).



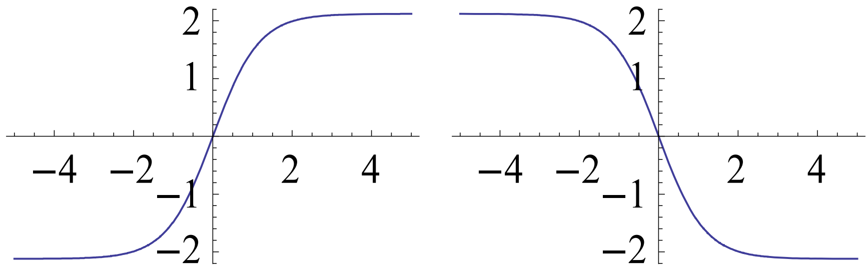

(i) $\varphi_0 = -2.5$ (ii) $\varphi_0 = 0.5$

**Figure 2**. The periodic waves of Eq (1) when $\alpha = -1; a_1 = 2, c = 3$




(iii) $\varphi_0 = -3$ (iv) $\varphi_0 = 3$

**Figure 3.** The solitary waves of Eq (1) when $\alpha = -1; a_1 = 2, c = 3$




v) $0 < y$ (vi) $y < 0$

**Figure 4.** The kinks of Eq (1) when $\alpha = 1; a_1 = 2, c = 3, \varphi_0 = 0$

## 5. Conclusion

In the study of traveling wave solutions of wave equations, the bifurcation theory of differential dynamic systems has been widely applicated. In a Hamiltonian system, the phase graphs bifurcation of a plane system can be drawn by using the bifurcation theory of differential dynamic systems. The orbit on the phase graphs correspond to different traveling wave bifurcation. The smooth closed orbit determines the periodic wave solution of the wave equations. The homoclinic orbit determines the solitary wave solutions of the wave equation. The heteroclinic orbit determines the kink wave solutions of the wave equation. In this paper, the periodic wave solutions, the solitary wave solutions, and kink solutions of mKdv-ZK equation are derived by using the bifurcation theory of differential dynamic systems, and their plane simulation graphs are given.

## References

[1] Tariq KU, Seadawy AR. Soliton solutions of (3+1)-dimensional Korteweg-de Vries Benjamin–Bona–Mahony, Kadomtsev–Petviashvili Benjamin–Bona–Mahony and odified Korteweg de Vries–Zakharov–Kuznetsov equations and their applications in water waves. Journal of King Saud University – Science http://dx. doi. org/10. 1016/j. jksus. 2017. 02. 004

[2] Xie S, Hong X, Lu J. The bifurcation traveling waves of a generalized Broer-Kaup equation. Appl. Anal. Comput., 2020, Online First.

[3] Xie S, Hong X, Jiang T. Planar bifurcation method of dynamical system for investigating different kinks of bounded traveling wave solutions of a generalized Camassa-Holm equation. J. Appl. Anal. Comput., 2017, 7(1), 78--290.

[4] Xie S, Zhang Y, He J. Two types of bounded traveling-wave solutions of a two-component Camassa-Holm equation. Appl. Math. Comput., 2013, 219(20), 10271--10282.

[5] Byrd PF, Friedman MD. Handbook of elliptic integrals for engineers and scientists. Berlin: Springer, 1971.