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Category of CFR $(Y)^{\mathbf{C}}$ and Weak Topos[1]

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Abstract. Topos theory plays an important role in modern mathematics. It can be viewed as a generalization of set from the category aspect. However, the category of all fuzzy sets do not form topos. In order to investigate the category properties about fuzzy sets, the notion of weak topos is introduced. Factor space is an effective approach in knowledge representation and factor rattan is a crucial concept of a factor space. Rattans over Y could be viewed as an abstraction of factor rattan. In addition, the category of rattans over Y is not a topos. In this paper, two comments about the paper titled "Factor rattans, category **FR**(Y), and factor space" (J MATH ANAL APPL, 1994) are presented, and the notion of rattan over Y is revised to complete rattan over Y. The corresponding category is denoted **CFR**(Y). Topoi properties of the functor category **CFR** $(Y)^{\mathbf{C}}$ are investigated and it is proved that **CFR** $(Y)^{\mathbf{C}}$ is a weak topos, but not a topos.

Keywords. Factor space, Rattan over Y, category of fuzzy sets, Weak topos

1. Introduction

Set theory is an important field in morden mathematics and provides a general fundamental framework of mathematics. The notion of fuzzy sets, which can be considered as a generalization of classical sets, was first introduced in 1965 by Zadeh [1].

The sets with the functions between two sets could construct a category **Set**. Topoi, which are cartesian closed categories with subobject classifiers, can be regarded as categories which are "essentially the same" as **Set** [2, 3]. That is, topoi can be viewed as another generalization of sets. Topoi theory plays an important role in mathematics, especially in logic. By the notion of topos, the logical operators (such as implication, negation) of classical sets can be obtained. More generally, each topos carries its own logical calculus [2].

Some correlations between fuzzy sets and topoi are inspected by various researchers [4–11]. However, the logical operators can not be obtained reasonably by the theory of topoi since **Fuz** which is the category of fuzzy sets is not a topos. To address this issue, the concepts of middle object and weak topos are introduce [9, 11, 12]. It is proved that **Fuz** is a weak topos and middle object can serve a similar function as the

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subobject classifier in a topos. Besides, the category **FuzFuz** of morphisms in category **Fuz** and the functor category $\mathbf{Fuz}^{\mathbf{C}}$ from a small category **C** to **Fuz** are also weak topoi [10, 12].

Another important instance of weak topos is **FR** (Y), which is the category of rattans over Y [13]. With the background of factor spaces [14], the notion of rattans over Y is introduced in [13]. Factor space is an important concept and It can be widely applied to various fields, such as knowledge representation, information science and big data [15–18]. Factor rattan is a crucial concept of a factor space, and rattans over Y [13] could be viewed as an abstraction of factor rattan. It is proved that the category of rattans over Y is a weak topos [13].

Definition 1.1 ([13])**.** Let Y be a fixed non-empty set and let X be any set. (X, ξ) is called a rattan over Y if $\xi : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ is a mapping satisfying the following conditions.

(1) $\xi(\emptyset) = Y$ and there exists a fixed element $y_0 \in Y$ such that $y_0 \in \xi(A)$ for any $A \in \mathcal{P}(X)$.

(2) $\xi(A \cup B) = \xi(A) \cap \xi(B)$ for any $A, B \in \mathcal{P}(X)$.

A rattan over Y is called a Y-rattan for short.

The purpose of this paper is to inspect the topos properties of the category $\mathbf{FR}(Y)^{\mathbf{C}}$, which is the functors category from a small category **C** to **FR** (Y). Two comments about [13] are proposed. Then the notion of complete rattans over Y is given, and the corresponding category is denoted by **CFR** (Y). It is proved that $\mathbf{CFR}(Y)^{\mathbf{C}}$ is not a topos since it has no subobject classifier. However there exists a middle object in $\mathbf{CFR}(Y)^{\mathbf{C}}$. Therefore, $\mathbf{CFR}(Y)^{\mathbf{C}}$ forms a weak topos.

2. Topos and weak topos

A topos **C** is a category satisfying the following properties.

1. **C** has all finite products.
2. There is a terminal object t in **C**.
3. An equalizer exists in **C** for any $f, g \in \mathcal{M}(a, b)$ with $a, b \in \mathcal{O}(\mathbf{C})$.
4. Exponentials exist in **C**.
5. There is a subobject classifier in **C**. Specifically, there is an object s and a morphism ν from t to s such that for each monomorphism $f \in \mathcal{M}(a, b)$ with $a, b \in \mathcal{O}(\mathbf{C})$, there exists a unique morphism $\chi_f \in \mathcal{M}(b, s)$ so that the following diagram is a pullback.

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & t \arrow[d, "\nu"] \\ b \arrow[r, "\chi_f"] & s \end{tikzcd}$$

Topoi have a great influence on logic. However, some significant categories, such as **Fuz**, **FuzFuz**, are not topoi since there is no subobject classifier in those categories. The notion of middle object defined in [12] can be viewed as a generalization of subobject classifier.

Definition 2.1 ([12]). A middle object in a category $\mathbf{C}$ is a monomorphism $m \in \mathcal{M}(c_1, c_2)$ where $c_1, c_2 \in \mathcal{O}(\mathbf{C})$ satisfy the following conditions.

1. $\mathcal{M}(c, c_2)$ is partially ordered for all object $c \in \mathcal{O}(\mathbf{C})$.
2. For any $c \in \mathcal{O}(\mathbf{C})$, there is an unique smallest morphism $\alpha \in \mathcal{M}(c, c_2)$ such that the square is a pullback.

$$\begin{tikzcd} c \arrow[r] \arrow[d, "1_c"'] & c_1 \arrow[d, "m"] \\ c \arrow[r, "\alpha"] & c_2 \end{tikzcd}$$

3. For any monomorphism $f \in \mathcal{M}(c, d)$ with any $c, d \in \mathcal{O}(\mathbf{C})$, there is an unique characteristic mapping $\chi_f \in \mathcal{M}(d, c_2)$ such that $\chi_f \leq \alpha$ and the following square is a pullback.

$$\begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & c_1 \arrow[d, "m"] \\ d \arrow[r, "\chi_f"] & c_2 \end{tikzcd}$$

With the definition of middle object, the concept of weak topos was also introduced.

Definition 2.2 ([12]). A weak topos is a category which satisfies the first four properties in the definition of topos and has a middle object.

In fact, a weak topos is a cartesian closed category with middle object.

3. Comments on “Factor rattans, category FR(Y), and factor space" (J MATH ANAL APPL, 1994)

The category of rattans over Y is denoted by $\mathbf{FR}(Y)$ where the morphisms from (X_1, ξ_1) to (X_2, ξ_2) are mappings $f : X_1 \longrightarrow X_2$ satisfying $\xi_2(f^{\rightarrow}(A)) \supseteq \xi_1(A)$ for any $A \in \mathcal{P}(X_1)$. In addition, it is proved that $\mathbf{FR}(Y)$ is a weak topos.

The topic of [13] is meaningful and the results are beautiful. However, we think that there are two minor errors neglected by the authors.

Comment 1. It is proved in [13] that $\mathbf{FR}(Y)$ has finite product (Theorem 3.1 in [13]). Specifically, the product of any two rattans (X_1, ξ_1) and (X_2, ξ_2) can be denoted by (X, ξ) such that $X = X_1 \times X_2$ and $\xi(A) = \xi_1(A_1) \cap \xi_2(A_1)$ for any $A \in \mathcal{P}(X)$, where $A_1 = \{x_1 \in X_1 : \exists x_2 \in X_2 \text{ with } (x_1, x_2) \in A\}$ and $A_2 = \{x_2 \in X_2 : \exists x_1 \in X_1 \text{ with } (x_1, x_2) \in A\}$. However, the products do not always satisfy condition (1) in Definition 1.1 (see the following example).

Example 3.1. Let $Y = \{\bot, \top\}$, $X_1 = X_2 = \{x\}$ and let $\xi_i : \mathcal{P}(X_i) \longrightarrow \mathcal{P}(Y)$ $(i = 1, 2)$ such that $\xi_1(\emptyset) = \xi_2(\emptyset) = Y$, $\xi_1(X_1) = \{\top\}$ and $\xi_2(X_2) = \{\bot\}$. It is trivial to verify that (X_1, ξ_1) and (X_2, ξ_2) are two rattans over Y. However, we have $\xi(X) = \xi_1(X_1) \cap \xi_2(X_2) = \emptyset$ where (X, ξ) is the product of (X_1, ξ_1) and (X_2, ξ_2).

Indeed, according to the definitions of factor spaces and factor rattans, we can see "there exists a fixed element $y_0 \in Y$ such that $y_0 \in \xi(A)$ for any $A \in \mathcal{P}(X)$" is not a pivotal property. Thus we can drop this condition in Definition 1.1.

Comment 2. It is proved that a middle object exists in $\mathbf{FR}(Y)$ (Theorem 3.2 in [13]). More specifically, let $M = \{\{y_0\} \cup A : A \in \mathcal{P}(Y)\}$, $N = \mathcal{P}(Y)$ and let $\xi_M : \mathcal{P}(M) \longrightarrow \mathcal{P}(Y)$ be a mapping such that $\xi_M(\mathcal{A}) = \bigcap \mathcal{A}$ if $\emptyset \neq \mathcal{A} \in \mathcal{P}(M)$ and $\xi_M(\mathcal{A}) = Y$ if $\mathcal{A} = \emptyset$. Then (M, ξ_M) and (N, ξ_N) with $\xi_N(A) = Y$ for any $A \subseteq Y$, are two rattans. Besides, let $l_M : M \longrightarrow N$ be a mapping such that $l_M(A) = A$ for any $A \in M$. Thus l_M is a morphism from (M, ξ_M) to (N, ξ_N)

Let (X_1, ξ_1) and (X, ξ) be two rattans and let $m : X_1 \longrightarrow X$ be a monomorphism. Then two morphisms, $\phi_m : X \longrightarrow N$ and $f_1 : X_1 \longrightarrow M$ are defined in [13], where

$$\phi_m(x) = \begin{cases} \xi_1(\{x_1\}) & \exists x_1 \in X_1 \text{ s.t. } m(x_1) = x; \\ \emptyset & \text{otherwise,} \end{cases}$$

and $f_1(x_1) = \xi_1(\{x_1\})$ for any $x \in X$ and $x_1 \in X_1$, respectively. Thus we deduce that $\phi_m \cdot m = l_M \cdot f_1$. For any rattan (X', ξ') and any two morphisms $n : X' \longrightarrow X$ and $f' : X' \longrightarrow M$ with $\phi_m \cdot n = l_M \cdot f'$, let $\overline{n} : X' \longrightarrow X_1$ be the mapping such that $n = m \cdot \overline{n}$. Then we can deduce that $\overline{n}(x') = x_1$ where x_1 satisfies $m(x_1) = n(x')$ for any $x' \in X'$. According to the definitions, $\overline{n}$ is well-defined. Besides, it is said in [13] that "$\overline{n}$ is a morphism". However, $\overline{n}$ is not always a morphism (see the following example).

Example 3.2. Let $Y = [0, +\infty] = [0, +\infty) \cup \{+\infty\}$ with $a < +\infty$ for any $a \in [0, +\infty)$ and let $y_0 = +\infty$. Let $X_1 = X = [0, +\infty)$ with $\xi_1(A) = (\bigvee A, +\infty) \cup \{+\infty\}$ and $\xi(A) = [\bigvee A, +\infty]$ for any $A \subseteq [0, +\infty)$. Thus (X_1, ξ_1) and (X, ξ) form two rattans, and the mapping $m : X_1 \longrightarrow X$ with $m(x) = x$ for any $x \in [0, +\infty)$ is a monomorphism. Besides, according to the definitions in [13], we have $M = \{\{+\infty\} \cup A : A \subseteq [0, +\infty]\}$, $N = \mathcal{P}([0, +\infty])$, $f_1(x) = (x, +\infty]$ and $\phi_m(x) = (x, +\infty]$, $\forall x \in [0, +\infty)$. Thus we have $\phi_m \cdot m = l_M \cdot f_1$.

Let $X' = [0, +\infty)$ and $\xi' : \mathcal{P}(X') \longrightarrow \mathcal{P}(Y)$ such that:

$$\xi'(A) = \begin{cases} [\bigvee A, +\infty], \text{ if } \bigvee A \notin A, \\ (\bigvee A, +\infty) \cup \{+\infty\}, \text{ if } \bigvee A \in A, \end{cases}$$

for any $A \in \mathcal{P}(X')$. It is easy to verify that (X', ξ') is a rattan over Y. Then we get $\phi_m \cdot n = m \cdot f'$ where $n : X' \longrightarrow X$ and $f' : X' \longrightarrow M$ such that $n(x) = x$, $f'(x) = \xi'(\{x\})$ are two morphisms.

Then according to the definition of $\overline{n}$ in [13], we get $\overline{n}(x) = x$ for any $x \in [0, +\infty)$. However, $\overline{n}$ is not a morphism since $\xi_1(\overline{n}^{\rightarrow}([0, 1))) = \xi_1([0, 1)) = (1, +\infty] \subsetneq [1, +\infty] = \xi'([0, 1))$.

The main reason is that ξ satisfies $\xi\left(A\cup B\right)=\xi\left(A\right)\cap\xi\left(B\right)$ but not $\xi\left(\bigcup_{i\in I}A_i\right)=\bigcap_{i\in I}\xi\left(A_i\right)$ for any index set I. However, according to the definition of factor rattan, ξ and η satisfy $\xi\left(\bigcup_{i\in I}A_i\right)=\bigcap_{i\in I}\xi\left(A_i\right)$ and $\eta\left(\bigcup_{i\in I}A_i\right)=\bigcap_{i\in I}\eta\left(A_i\right)$ for any index set I, respectively.

According to the Comment 1 and Comment 2, we can revise the definition of rattan over Y as follows.

Definition 3.3. Let Y be a fixed non-empty set and let X be any set. (X,ξ) is called an complete rattan over Y if $\xi:\mathcal{P}\left(X\right)\longrightarrow\mathcal{P}\left(Y\right)$ is a mapping satisfying the following conditions.

1. $\xi\left(\emptyset\right)=Y$.
2. $\xi\left(\bigcup_{i\in I}A_i\right)=\bigcap_{i\in I}\xi\left(A_i\right)$ for any $A_i\in\mathcal{P}\left(X\right)\quad(i\in I)$ where I is an index set.

Under this revised definition, all complete rattans over Y also form a category, and we denote it by $\mathbf{CFR}\left(Y\right)$.

Remark 3.4. This concept is equivalent to a relation R from X to Y: $\xi_R(A)=\{y\in Y|\ A\times\{y\}\subseteq R\}$ and $(x,y)\in R_\xi$ iff $y\in\xi(\{x\})$ [19].

4. Topoi properties of category $\mathbf{CFR}\left(Y\right)^{\mathbf{C}}$

Let $\mathbf{C}$ be a small category and $\mathbf{CFR}\left(Y\right)^{\mathbf{C}}$ be the functor category from $\mathbf{C}$ to $\mathbf{CFR}\left(Y\right)$.

Theorem 4.1. *Category $\mathbf{CFR}\left(Y\right)^{\mathbf{C}}$ has all topoi properties except subobject classifiers.*

Proof. We only need to prove that category $\mathbf{FR}\left(Y\right)^{\mathbf{C}}$ has exponentials since others are trivial.

Let $F,G\in\mathcal{O}\left(\mathbf{CFR}\left(Y\right)^{\mathbf{C}}\right)$ be two functors from $\mathbf{C}$ to $\mathbf{CFR}\left(Y\right)$ with $F\left(c\right)=\left(X_c^F,\xi_c^F\right)$ and $G\left(c\right)=\left(X_c^G,\xi_c^G\right)$ for any $c\in\mathcal{O}\left(\mathbf{C}\right)$. $c\uparrow\mathbf{C}$ denotes the comma category.

Let F_c,G_c be two “forgetful" functors from $c\uparrow\mathbf{C}$ to **Set**. More specifically, for any $f:c\longrightarrow d\in\mathcal{O}\left(c\uparrow\mathbf{C}\right)$ and $h\in\mathcal{M}\left(f:c\longrightarrow d,g:c\longrightarrow e\right)$, we have $F_c\left(f\right)=X_d^F$, $F_c\left(h\right)=F\left(h\right)$, $G_c\left(f\right)=X_d^G$ and $G_c\left(h\right)=G\left(h\right)$. In addition, let $Nat[F_c,G_c]$ be the family of natural transformations from F_c to G_c. Then for any $A\subseteq Nat[F_c,G_c]$, we define:

$$\Sigma_c^A=\left\{C\in\mathcal{P}\left(Y\right):\begin{array}{c}\xi_d^F\left(B\right)\cap C\subseteq\xi_d^G\left(A_f^{\rightarrow}\left(B\right)\right)\\ \text{for any } f:c\longrightarrow d,B\in\mathcal{P}\left(X_d^F\right)\end{array}\right\}$$

where $A_f^{\rightarrow}\left(B\right)=\bigcup_{\tau\in A}\tau_f^{\rightarrow}\left(B\right)$. In addition, let $\xi_c\left(A\right)=\bigcup\Sigma_c^A$.

According to the definition of ξ_c, it is trivial that $\xi_c\left(\emptyset\right)=Y$. Moreover, for any index set I, $A_i\subseteq Nat[F_c,G_c]$, $C_i\in\Sigma_c^{A_i}\ (i\in I)$, $f:c\longrightarrow d$ and any $B\in\mathcal{P}\left(X_d^F\right)$, we can get $\xi_d^F\left(B\right)\cap\bigcap_{i\in I}C_i=\bigcap_{i\in I}\xi_d^F\left(B\right)\cap C_i\subseteq\bigcap_{i\in I}\xi_d^G\left(\left(A_i\right)_f^{\rightarrow}\left(B\right)\right)=\xi_d^G\left(\left(\bigcup_{i\in I}A_i\right)_f^{\rightarrow}\left(B\right)\right)$. Thus $\bigcap_{i\in I}C_i\in\Sigma_c^{\bigcup_{i\in I}A_i}$, which implies $\bigcap_{i\in I}\xi_c\left(A_i\right)=\bigcap_{i\in I}\bigcup\Sigma_c^{A_i}\subseteq\bigcup\Sigma_c^{\bigcup_{i\in I}A_i}=\xi_c\left(\bigcup_{i\in I}A_i\right)$. Besides,

$\xi_c(A_1)\cap\xi_c(A_2)\supseteq\ \xi_c(A_1\cup A_2)$ is trivial. Thus $(Nat[F_c,G_c],\xi_c)\in\mathcal{O}(\mathbf{CFR}(Y))$ is a Y-rattan.

For any $c\in\mathcal{O}(\mathbf{C})$ and $f:c\longrightarrow d\in\mathcal{M}(c,d)$, let $G^F(c)=(Nat[F_c,G_c],\xi_c)$ and $G^F(f):Nat[F_c,G_c]\longrightarrow Nat[F_d,G_d]$, where $\left(G^F(f)(\tau)\right)_g=\tau_{g\cdot f}$ for any $\tau\in Nat[F_c,G_c]$ and any $g:d\longrightarrow e\in\mathcal{O}(d\uparrow\mathbf{C})$. Then for any $A\subseteq Nat[F_C,G_C]$, we have

$$
\begin{aligned}
\Sigma_d^{\left(G^F(f)\right)^{\rightarrow}(A)} &=\left\{C\in\mathcal{P}(Y):\begin{array}{l}\xi_e^F(B)\cap C\subseteq\xi_e^G\left(\left(\left(G^F(f)\right)^{\rightarrow}(A)\right)_g^{\rightarrow}(B)\right),\\ \forall\, g:d\longrightarrow e,B\in\mathcal{P}\left(X_e^F\right)\end{array}\right\}\\
&\supseteq\left\{C\in\mathcal{P}(Y):\begin{array}{l}\xi_e^F(B)\cap C\subseteq\xi_e^G\left(A_h^{\rightarrow}(B)\right),\\ \forall\, h:c\longrightarrow e,B\in\mathcal{P}\left(X_e^F\right)\end{array}\right\}\\
&=\Sigma_c^A.
\end{aligned}
$$

Hence, for any $A\subseteq Nat[F_C,G_C]$, we get: $\xi_d\left(\left(G^F(f)\right)^{\rightarrow}(A)\right)=\bigcup\Sigma_d^{\left(G^F(f)\right)^{\rightarrow}(A)}\supseteq\bigcup\Sigma_c^A=\xi_c(A)$. Thus we know that $G^F(f)$ is a morphism from $G^F(c)$ to $G^F(d)$. In addition, it is trivial that G^F preserves the identity and composition of morphisms. So G^F is a functor from category $\mathbf{C}$ to category $\mathbf{CFR}(Y)$.

For any $c\in Ob(\mathbf{C})$, let $ev_c:Nat[F_c,G_c]\times X_c^F\longrightarrow X_c^G,\ (\tau,x)\mapsto\tau_{1_c}(x)$, where $1_c:c\longrightarrow c$ is the identity of c. For any $A\subseteq Nat[F_c,G_c]\times X_c^F$, we denote $A_1=\left\{\tau\in Nat[F_c,G_c]:\text{ there exsits }x\in X_c^F\text{ s.t. }(\tau,x)\in A\right\}$ and $A_2=\left\{x\in X_c^F:\text{ there exists }\tau\in Nat[F_c,G_c]\text{ s.t. }(\tau,x)\in A\right\}$. Then we get that $\xi_c^G\left(ev_c^{\rightarrow}(A)\right)=\xi_c^G\left(\{\tau_{1_c}(x):(\tau,x)\in A\}\right)\supseteq\xi_c^G\left(\bigcup_{\tau\in A_1}\tau_{1_c}^{\rightarrow}(A_2)\right)\supseteq\bigcup_{B\in\Sigma_c^{A_1}}\xi_c^F(A_2)\cap B=\xi_c\times\xi_c^F(A)$. Thus $ev_c\in\mathcal{M}\left(G^F(c)\times F(c),G(c)\right)$.

For any $f:c_1\longrightarrow c_2\in\mathcal{M}(c_1,c_2)$ and any $(\tau,x)\in G^F(c_1)\times F(c_1)$, according to the definition of τ, we can reduce that $G(f)\cdot\tau_{\left(1_{c_1}\right)}(x)=\tau_f\cdot F(f)(x)$ for any $x\in X_{c_1}^F$, which implies that $G(f)\cdot ev_{c_1}(\tau,x)=G(f)\cdot\tau_{1_{c_1}}(x)=\tau_{\left(1_{c_2}\cdot f\right)}\cdot F(f)(x)=ev_{c_2}\left(G^F(f)(\tau),F(f)(x)\right)=ev_{c_2}\cdot\left(G^F\times F\right)(f)(\tau,x)$. So $ev:G^F\times F\longrightarrow G$ is a natural transformation from $G^F\times F$ to G.

Now let $H:\mathbf{C}\longrightarrow\mathbf{CFR}(Y)$ be a functor from $\mathbf{C}$ to $\mathbf{CFR}(Y)$, where $H(c)=\left(X_c^H,\xi_c^H\right)$ for any $c\in\mathcal{O}(\mathbf{C})$, and let $\sigma:H\times F\longrightarrow G$ be a natural transformation. Then we define $\bar{\sigma}_c:X_c^H\longrightarrow Nat[F_c,G_c]$ such that for any $y\in X_c^H$ and any $f:c\longrightarrow d\in\mathcal{M}(c,d)$, $(\bar{\sigma}_c(y))_f:X_d^F\longrightarrow X_d^G$ is a mapping where $(\bar{\sigma}_c(y))_f(x)=\sigma_d(H(f)(y),x)$ for any $x\in X_d^F$.

For any $f:c\longrightarrow d\in\mathcal{M}(c,d)$, $g:c\longrightarrow e\in\mathcal{M}(c,e)$ and any $h:d\longrightarrow e\in\mathcal{M}(d,e)$ with $g=h\cdot f$, we have $(\bar{\sigma}_c(y))_g\cdot F(h)(x)=\sigma_e(H(g)(y),F(h)(x))=\sigma_e(H(h)\cdot H(f)(y),F(h)(x))=G(h)\cdot(\bar{\sigma}_c(y))_f(x)$ for any $x\in X_d^F$. Hence, we have $\bar{\sigma}_c(y):F_c\longrightarrow G_c$ is a natural transformation. So $\bar{\sigma}_c:X_c^H\longrightarrow Nat[F_c,G_c]$ is well-defined.

For any $A\in\mathcal{P}\left(X_c^H\right)$, $f:c\longrightarrow d\in\mathcal{M}(c,d)$ and any $B\in\mathcal{P}\left(X_d^F\right)$, we get $\xi_d^G\left((\bar{\sigma}_c^{\rightarrow}(A))_f^{\rightarrow}(B)\right)=\xi_d^G\left(\{\sigma_d(H(f)(y),x):x\in B,y\in A\}\right)=\xi_d^G\left(\sigma_d^{\rightarrow}(\{H(f)^{\rightarrow}(A)\}\times B)\right)\supseteq\xi_d^H\left(\{H(f)^{\rightarrow}(A)\}\right)\cap\xi_d^F(B)$, which implies $\xi_c\left(\bar{\sigma}_c^{\rightarrow}(A)\right)=\bigcup\Sigma_c^{\bar{\sigma}_c^{\rightarrow}(A)}\supseteq\xi_d^H\left(H(f)^{\rightarrow}(A)\right)\supseteq\xi_c^H(A)$. Hence, we get $\bar{\sigma}_c\in\mathcal{M}\left(H(c),G^F(c)\right)$.

For any $c \in Ob(\mathbf{C})$, $f : c \longrightarrow d \in \mathcal{M}(c,d)$, $y \in X_c^H$, $g : d \longrightarrow e \in \mathcal{M}(d,e)$ and any $x \in X_e^F$, we have $\left(G^F(f) \cdot \bar{\sigma}_c(y)\right)_g(x) = (\bar{\sigma}_c(y))_{g \cdot f}(x) = \sigma_e(H(g \cdot f)(y), x) = \sigma_e(H(g) \cdot H(f)(y), x) = (\bar{\sigma}_d \cdot H(f)(y))_g(x)$. It follows that $\bar{\sigma} : H \longrightarrow G^F$ is a natural transformation, and according to the definition of $\bar{\sigma}$, the following diagram is commutative.

$$\begin{tikzcd}
G^F \times F \arrow[r, "ev"] & G \\
H \times F \arrow[u, "\bar{\sigma} \times 1_F"] \arrow[ur, "\sigma"] &
\end{tikzcd}$$

In addition, it is easy to verify such $\bar{\sigma}$ is unique. So, $\left\{G^F, ev\right\}$ is an exponential of F and G. It follows the theorem. $\square$

From [13], we know that **CFR** (Y) has no subobject classifier. Similarly, we also have the following proposition.

Proposition 4.2. *Let Y be a non-empty setting. Category* **CFR** $(Y)^{\mathbf{C}}$ *has no subobject classifier.*

Proof. Let $T : \mathbf{C} \longrightarrow$ **CFR** (Y) be the terminal object in **CFR** $(Y)^{\mathbf{C}}$. That is, for any $c \in \mathcal{O}(\mathbf{C})$, $T(c) = (\{\top\}, \xi_0)$ where $\xi_0(\emptyset) = \xi_0(\{\top\}) = Y$. If **CFR** $(Y)^{\mathbf{C}}$ has a subobject classifier, then there exists a functor $F : \mathbf{C} \longrightarrow$ **CFR** (Y) and a natural transformation $\omega : T \longrightarrow F$ such that F is a subobject classifier.

Let $F_i : \mathbf{C} \longrightarrow$ **CFR** (Y) $(i = 1,2,3)$ be three functors such that $F_i(c) = (X_i, \xi_i)$ for any $c \in \mathcal{O}(\mathbf{C})$, where $X_1 = X_2 = X_3 = \{\top\}$ and $\xi_1(\{\top\}) = \emptyset$, $\xi_2 = \xi_3 = \xi_0$. It is trivial that $\iota : F_1 \longrightarrow F_2$ such that $\iota_c : X_1 \longrightarrow X_2$ with $\iota_c(\top) = \top$ for any $c \in \mathcal{O}(\mathbf{C})$ is a natural transformation. In addition, ι is a monomorphism in **CFR** $(Y)^{\mathbf{C}}$. Thus there exists a natural transformation $\chi_\iota : F_2 \longrightarrow F$ such that $\chi_\iota \cdot \iota = \omega \cdot !$. Similarly, we can define a natural transformation $\kappa : F_3 \longrightarrow F_2$ such that $\chi_\iota \cdot \kappa = \omega \cdot !$. Thus there exists a unique natural transformation $\overline{\kappa} : F_3 \longrightarrow F_1$ with $\kappa = \iota \cdot \overline{\kappa}$. So for any $c \in \mathcal{O}(\mathbf{C})$, $\xi_1(\overline{\kappa}_c^{\to}(\{\top\})) \supseteq \xi_3(\{\top\})$. However, according to the definitions, we have $\xi_1(\overline{\kappa}_c^{\to}(\{\top\})) = \xi_1(\{\top\}) = \emptyset \subsetneq Y = \xi_3(\{\top\})$, which is a contradiction. Thus there is no subobject classifier in **CFR** $(Y)^{\mathbf{C}}$. $\square$

5. Category **CFR** $(Y)^{\mathbf{C}}$ is a weak topos

In this section, we prove **CFR** $(Y)^{\mathbf{C}}$ is a weak topos.

Theorem 5.1. *Category* **CFR** $(Y)^{\mathbf{C}}$ *has a middle object.*

Proof. Let $*$ be an universal element and $\mathcal{P}(Y)^* = \mathcal{P}(Y) \cup \{*\}$. For convenience, we stipulate that $* \subseteq \emptyset$.

For any $c \in \mathcal{O}(\mathbf{C})$, let $M(c) = \left(X_c^M, \xi_c^M\right)$ and $N(c) = \left(X_c^N, \xi_c^N\right)$, where

$$X_c^M = \left\{ f : \mathcal{O}(c \uparrow \mathbf{C}) \longrightarrow \mathcal{P}(Y) : \begin{matrix} f(k \cdot l) \supseteq f(l), \forall l \in \mathcal{M}(c, a), \\ \forall k \in \mathcal{M}(a, b) \end{matrix} \right\},$$

$$X_c^N = \left\{ f : \mathcal{O}(c \uparrow \mathbf{C}) \longrightarrow \mathcal{P}(Y)^* : \begin{matrix} f(k \cdot l) \supseteq f(l), \forall l \in \mathcal{M}(c, a), \\ \forall k \in \mathcal{M}(a, b) \end{matrix} \right\},$$

$$\xi_c^M(A) = \bigcap_{f \in A} f(1_c), \ \forall A \in \mathcal{P}\left(X_c^M\right),$$

$$\xi_c^N(B) = Y, \ \forall B \in \mathcal{P}\left(X_c^N\right).$$

It is trivial that $\left(X_c^M, \xi_c^M\right)$ and $\left(X_c^N, \xi_c^N\right)$ are two Y-rattans.

In addition, for any $c, a \in \mathcal{O}(C)$ and $k \in \mathcal{M}(c, a)$, let $M(k) : X_c^M \longrightarrow X_a^M$ and $N(k) : X_c^N \longrightarrow X_a^N$ be two mappings such that $M(k)(f)(g) = f(g \cdot k)$ and $N(k)(f)(g) = f(g \cdot k)$. Thus for any $A \in \mathcal{P}\left(X_c^M\right)$, we have $\xi_a^M\left(M(k)^{\rightarrow}(A)\right) = \bigcap_{f \in A} M(k)(f)(1_a) = \bigcap_{f \in A} f(1_a \cdot k) = \bigcap_{f \in A} f(k) = \bigcap_{f \in A} f(k \cdot 1_c) \geq \bigcap_{f \in A} f(1_c) = \xi_c^M(A)$. So $M : \mathbf{C} \longrightarrow \mathbf{CFR}(Y)$ is a functor, and analogously, $N : \mathbf{C} \longrightarrow \mathbf{CFR}(Y)$ is also a functor.

For any $c \in \mathcal{O}(C)$, we define $m_c : X_c^M \longrightarrow X_c^N$ such that $m_c(f) = f$ for any $f \in X_c^M$. Then we get $N(k) \cdot m_c = m_a \cdot M(k)$ for any $k \in \mathcal{M}(\mathbf{C})$, which implies that $m : M \longrightarrow N$ is a natural transformation. Besides, it is clear that m is a monomorphism.

Now we want to prove that $m : M \longrightarrow N$ is a middle object.

(1) Let $F : \mathbf{C} \longrightarrow \mathbf{CFR}(Y)$ be a functor and let $\iota, \kappa \in \mathcal{M}(F, N)$ be two natural transformations. Besides, we let $\iota \leq \kappa \iff \iota_c \leq \kappa_c, \ \forall c \in \mathcal{O}(C) \iff \iota_c(x) \leq \kappa_c(x), \ \forall x \in X_c^F, \ c \in \mathcal{O}(C) \iff \iota_c(x)(k) \subseteq \kappa_c(x)(k), \ \forall k \in \mathcal{O}(c \uparrow \mathbf{C}), \ x \in \left(X_c^F\right), \ c \in \mathcal{O}(C)$. Then the pair $(\mathcal{M}(F, N), \leq)$ forms a partial order set.

(2) Let $F : \mathbf{C} \longrightarrow \mathbf{CFR}(Y)$ be a functor and let $a, c \in \mathcal{O}(\mathbf{C})$. For any $x \in X_c^F$ and $k \in \mathcal{M}(c, a)$, we define $\alpha_c(x)(k) = \xi_a^F(\{F(k)(x)\})$. Thus for any $l \in \mathcal{M}(a, b)$, we reduce that $\alpha_c(x)(l \cdot k) = \xi_b^F(\{F(l \cdot k)(x)\}) = \xi_b^F\left(F(l)^{\rightarrow}\{F(k)(x)\}\right) \geq \xi_a^F(\{F(k)(x)\}) = \alpha_c(x)(k)$, which implies $\alpha_c(x) \in X_c^N$. Besides, we have $(\alpha_a \cdot F(k)(x))(l) = \xi_b^F(\{F(l) \cdot F(k)(x)\}) = \xi_b^F(\{F(l \cdot k)(x)\}) = \alpha_c(x)(l \cdot k) = (N(k) \cdot \alpha_c(x))(l)$. Thus $\alpha : F \longrightarrow N$ is a natural transformation since $\alpha_c \in \mathcal{M}(F(c), N(c))$ is trivial.

Let $\tau_c(x) = \alpha_c(x) \in X_c^M$ for any $c \in \mathcal{O}(\mathbf{C})$ and $x \in X_c^F$. Hence, for any $A \in \mathcal{P}\left(X_c^F\right)$, we know that $\xi_c^M\left(\tau_c^{\rightarrow}(A)\right) = \bigcap_{x \in A} \tau_c(x)(1_c) = \bigcap_{x \in A} \xi_c^F(\{F(1_c)(x)\}) = \bigcap_{x \in A} \xi_c^F(\{x\}) = \xi_c^F(A)$, which implies $\tau_c \in \mathcal{M}(F(c), M(c))$. Similarly, we can prove $\tau : F \longrightarrow M$ is a natural transformation.

It is clear that $1_F : F \longrightarrow F$ with $(1_F)_c = F(1_c) : X_c^F \longrightarrow X_c^F$ for any $c \in \mathcal{O}(\mathbf{C})$ is the identical morphism in $\mathbf{CFR}(Y)^{\mathbf{C}}$. According to the definition of m, α and τ, we can get $m_c \cdot \tau_c = \alpha_c \cdot F(1_c)$. In addition, it is trivial to verify that this is a pullback.

If there exist α' and τ' such that the following diagram is a pullback.

$$\begin{tikzcd} F \arrow[r, "\tau'"] \arrow[d, "1_F"] & M \arrow[d, "m"] \\ F \arrow[r, "\alpha'"] & N \end{tikzcd}$$

Thus for any $c \in \mathcal{O}(\mathbf{C})$ and $x \in X_c^F$, we have $\alpha'_c(x) = (\alpha'_c \cdot F(1_c))(x) = (m_c \cdot \tau'_c)(x) = \tau'_c(x)$. Thus for any $k \in \mathcal{M}(c,a)$, we have $\alpha'_c(x)(k) = \tau'_c(x)(k) = \tau'_c(x)(1_a \cdot k) = (M(k) \cdot \tau'_c(x))(1_a) = (\tau'_a \cdot F(k)(x))(1_a) = \xi_a^M(\{(\tau'_a \cdot F(k))(x)\}) \supseteq \xi_a^F(\{F(k)(x)\}) = \alpha_c(x)(k)$, which means $\alpha' \geq \alpha$.

(3) Let $\sigma : F \longrightarrow G$ be a monomorphism in $\mathbf{CFR}(Y)^{\mathbf{C}}$ with $F, G \in \mathcal{O}\left(\mathbf{CFR}(Y)^{\mathbf{C}}\right)$. For any $c \in \mathcal{O}(\mathbf{C})$, $y \in X_c^G$ and any $k : c \longrightarrow a \in \mathcal{O}(c \uparrow \mathbf{C})$, we define

$$(\chi_\sigma)_c(y)(k) = \begin{cases} \xi_a^F(\{z\}) & \exists z \in X_a^F \text{ s.t. } G(k)(y) = \sigma_a(z), \\ * & \text{else.} \end{cases}$$

Since $\sigma : F \longrightarrow G$ is a monomorphism, then $(\chi_\sigma)_c(y)(k)$ is well-defined. Thus if $(\chi_\sigma)_c(y)(k) \neq *$, we have $(\chi_\sigma)_c(y)(l \cdot k) = \xi_b^F(\{F(l)(z)\}) \geq \xi_a^F(\{z\}) = (\chi_\sigma)_c(y)(k)$ for any $l : a \longrightarrow b \in \mathcal{M}(a,b)$. This implies $(\chi_\sigma)_c(y) \in X_c^N$. Besides, it is trivial to verify $\chi_\sigma : G \longrightarrow N$ is a natural transformation.

For any $c \in \mathcal{O}(\mathbf{C})$ and $x \in X_c^F$, let $\varsigma_c(x) = (\chi_\sigma)_c \cdot \sigma_c(x) \in X_c^M$. Then for any $A \in \mathcal{P}\left(X_c^F\right)$, we have $\xi_c^M(\varsigma_c^{\rightarrow}(A)) = \bigcap_{x \in A} \varsigma_c(x)(1_c) = \bigcap_{x \in A}((\chi_\sigma)_c \cdot \sigma_c(x))(1_c) = \bigcap_{x \in A} \xi_c^F(\{x\}) = \xi_c^F(A)$. Thus we get $\varsigma_c : X_c^F \longrightarrow X_c^M \in \mathcal{M}(F(c), M(c))$. Therefor, $\varsigma : F \longrightarrow M$ is a natural transformation. Besides, we have $\chi_\sigma \cdot \sigma = m \cdot \varsigma$.

Let $\eta : H \longrightarrow G$ and $\theta : H \longrightarrow M$ be two natural transformations with $\chi_\sigma \cdot \eta = m \cdot \theta$. Then we have $\theta_c(x)(k) = ((\chi_\sigma)_c \cdot \eta_c(x))(k)$ for any $x \in X_c^H$ and any $k \in \mathcal{O}(c \uparrow \mathbf{C})$ with any $c \in \mathcal{O}(\mathbf{C})$. Thus $((\chi_\sigma)_c \cdot \eta_c(x))(1_c) \neq *$. So there exists a unique $y \in X_c^F$ such that $\eta_c(x) = \sigma_c(y)$. Let $\overline{\eta}_c(x) = y$. Then $\overline{\eta}_c$ is well-defined and $\eta_c(x) = \sigma_c \cdot \overline{\eta}_c(x)$. So $\theta_c(x) = m_c \cdot \theta_c(x) = (\chi_\sigma)_c \cdot \eta_c(x) = (\chi_\sigma)_c \cdot \sigma_c \cdot \overline{\eta}_c(x) = m_c \cdot \varsigma_c \cdot \overline{\eta}_c(x) = \varsigma_c \cdot \overline{\eta}_c(x)$. In addition, for any $A \in \mathcal{P}\left(X_c^H\right)$, we have $\xi_c^F(\overline{\eta}_c^{\rightarrow}(A)) = \bigcap_{x \in A}((\chi_\sigma)_c \cdot \eta_c(x))(1_c) = \bigcap_{x \in A}(\theta_c(x))(1_c) = \xi_c^M(\theta_c^{\rightarrow}(A)) \supseteq \xi_c^H(A)$, which implies $\overline{\eta}_c \in \mathcal{M}(H(c), F(c))$.

For any $k \in \mathcal{M}(c,a)$ and $x \in X_c^H$, we have $(\overline{\eta}_a \cdot H(k))(x) = y \iff \sigma_a(y) = (\eta_a \cdot H(k))(x) \iff \sigma_a(y) = G(k) \cdot \eta_c(x) \iff \sigma_a(y) = G(k) \cdot \sigma_c \cdot \overline{\eta}_c(x) \iff \sigma_a(y) = \sigma_a \cdot F(k) \cdot \overline{\eta}_c(x) \iff F(k) \cdot \overline{\eta}_c(x) = y$. Thus $\overline{\eta} : H \longrightarrow F$ is a natural transformation with $\eta = \sigma \cdot \overline{\eta}$ and $\theta = \varsigma \cdot \overline{\eta}$. The proof of uniqueness of $\overline{\eta}$ is trivial. □

Theorem 5.2. $\mathbf{CFR}(Y)^{\mathbf{C}}$ *is a weak topos.*

Proof. The proof of this theorem follows from Theorem 4.1 and Theorem 5.1. □

6. Conclusion

In this paper, we present two comments about category of rattan over Y in [13] by two counter-examples. According to those comments and the background of rattan over Y, we revise the definition of rattan over Y. Then we show that $\mathbf{CFR}(Y)^{\mathbf{C}}$, which is the functors category from a small category to the category of rattan over Y (revised), is a weak topos, but it is not a topos since there is no subobject classifier in it. More specifically, it is a cartesian closed category with a middle object.

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