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The Prime Ideals of QMV*-algebras

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Abstract. QMV*-algebras were introduced in [1] as the extension of MV*-algebras and quasi-MV algebras. In the present paper, the concepts of prime ideals are introduced into QMV*-algebras. First some related properties of QMV*-algebras are listed. Second the properties of prime ideals of a QMV*-algebra are investigated and the quotient algebra by a prime ideal is characterized. Finally, maximal ideals of a QMV*-algebra are discussed.

Keywords. MV*-algebras, Quasi-MV algebras, QMV*-algebras, Ideals, Prime ideals

1. Introduction

Chang had introduced MV*-algebras in [2] for the purpose of providing a convenient abstraction of the algebra defined on the real interval $[-1,1]$, endowed with the truncated addition $\zeta \oplus \upsilon = \max\{-1, \min\{1, \zeta+\upsilon\}\}$ and the negation $-\zeta$, paralleling similar work done for MV-algebras in [3]. In [4], the algebraic study of MV*-algebras had been made by Lewin et al., and the logic L^* as a natural extension of Łukasiewicz logic was also investigated in [5]. On the other hand, quasi-MV algebras deriving from quantum computation were introduced in [6] and they were another generalization of MV-algebras. Since they were proposed, lots of properties of quasi-MV algebras were investigated in [7–10] and their corresponding logics were discussed in [11]. In [6], a standard completeness theorem for a quasi-MV algebra was shown: an equation holds in any quasi-MV algebra if and only if it holds in the standard quasi-MV algebra **D**. The standard quasi-MV algebra **D** whose universe is the set $\mathbb{C}_1 = \{\langle \zeta, \upsilon\rangle \in \mathbf{R}\times\mathbf{R} \mid (1-2\zeta)^2 + (1-2\upsilon)^2 \leq 1\}$ is a subalgebra of the standard quasi-MV algebra **S** and **S** is defined as follows: $\mathbf{S} = \langle [0,1]\times[0,1]; \oplus, ', 0, 1\rangle$ where $\langle \zeta,\upsilon\rangle \oplus \langle \kappa,\lambda\rangle = \langle \min\{(1,\zeta+\kappa), \frac{1}{2}\rangle, \langle \zeta,\upsilon\rangle' = \langle 1-\zeta, 1-\upsilon\rangle, 0 = \langle 0, \frac{1}{2}\rangle$ and $1 = \langle 1, \frac{1}{2}\rangle$.

Notice that the universe of **S** is $[0,1]\times[0,1]$, it is natural to ask whether we can generalize it to $[-1,1]\times[-1,1]$. What is the relationship between the new algebraic structure and **S**? More general, whether we can generalize quasi-MV algebras similarly as MV*-algebras extended MV-algebras. If we can, whether new algebraic structures can be obtained by quasi-MV algebras? In order to solve these questions, we introduced QMV*-algebras in [1] as an extension of quasi-MV algebras. Meanwhile, QMV*-algebras can also be viewed as a generalization of MV*-algebras.

It is well-known that ideals, especially prime ideals, play an important part in studying the algebraic structures. To take a closer look of QMV*-algebras, we introduce the

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notions of prime ideals into QMV*-algebras in the present paper. The properties of prime ideals of a QMV*-algebra are investigated and the quotient algebra using a prime ideal is characterized. The maximal ideals of a QMV*-algebra are also discussed. All results obtained in this paper will generalize the known results in MV*-algebras and expand the contents in quasi-MV algebras.

2. Preliminary

This section recalls some results of QMV*-algebras which will be used in what follows.

Definition 2.1. [1] Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be an algebra of type $\langle 2,1,0,0\rangle$. If for any $\zeta, \upsilon \in \Gamma$, we define

$\zeta^+ \in \Gamma$ with $\zeta^+ \oplus 0 = (\zeta \oplus 0)^+ = 1 \oplus ((-1) \oplus \zeta)$,
$\zeta^- \in \Gamma$ with $\zeta^- \oplus 0 = (\zeta \oplus 0)^- = (-1) \oplus (1 \oplus \zeta)$,
$\zeta \sqcup \upsilon = (\zeta^+ \oplus (-(\zeta^+) \oplus \upsilon^+)^+) \oplus (\zeta^- \oplus (-(\zeta^-) \oplus \upsilon^-)^+)$,

and the following equations hold for any $\zeta, \upsilon, \kappa \in \Gamma$,

(QMV*1) $\zeta \oplus \upsilon = \upsilon \oplus \zeta$,
(QMV*2) $(1 \oplus \zeta) \oplus (\upsilon \oplus (1 \oplus \kappa)) = ((1 \oplus \zeta) \oplus \upsilon) \oplus (1 \oplus \kappa)$,
(QMV*3) $(\zeta \oplus 1) \oplus 1 = 1$,
(QMV*4) $(\zeta \oplus \upsilon) \oplus 0 = \zeta \oplus \upsilon$,
(QMV*5) $\zeta \oplus \upsilon = (\zeta^+ \oplus \upsilon^+) \oplus (\zeta^- \oplus \upsilon^-)$,
(QMV*6) $0 = -0$,
(QMV*7) $\zeta \oplus (-\zeta) = 0$,
(QMV*8) $-(\zeta \oplus \upsilon) = (-\zeta) \oplus (-\upsilon)$,
(QMV*9) $-(-\zeta) = \zeta$,
(QMV*10) $(-\zeta \oplus (\zeta \oplus \upsilon))^+ = -(\zeta^+) \oplus (\zeta^+ \oplus \upsilon^+)$,
(QMV*11) $\zeta \sqcup \upsilon = \upsilon \sqcup \zeta$,
(QMV*12) $\zeta \sqcup (\upsilon \sqcup \kappa) = (\zeta \sqcup \upsilon) \sqcup \kappa$,
(QMV*13) $\zeta \oplus (\upsilon \sqcup \kappa) = (\zeta \oplus \upsilon) \sqcup (\zeta \oplus \kappa)$,

then $\mathbf{\Gamma}$ is called a *quasi-MV* algebra* (*QMV*-algebra* for short).

Obviously, any MV*-algebra $\mathbf{\Lambda} = \langle \Lambda; \oplus, -, 0, 1\rangle$ is a QMV*-algebra. Conversely, if $\zeta \oplus 0 = \zeta$ holds in a QMV*-algebra $\mathbf{\Gamma}$, then it is immediate to see that $\mathbf{\Gamma}$ is an MV*-algebra.

On a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$, we can define some operations on Γ by $\zeta \sqcap \upsilon = -((-\zeta) \sqcup (-\upsilon))$, $\zeta \ominus \upsilon = \zeta \oplus (-\upsilon)$ and $|\zeta| = \zeta \sqcup (-\zeta)$ for any $\zeta, \upsilon \in \Gamma$. We also define a relation $\zeta \leq \upsilon$ by $\zeta \sqcup \upsilon = \upsilon \oplus 0$, or equivalently, $\zeta \sqcap \upsilon = \zeta \oplus 0$. It is obvious to see that the relation $\leq$ is reflexivity and transitivity. For any $\zeta \in \Gamma$, if $0 \leq \zeta$, then the element ζ is called *non-negative* and if $\zeta \leq 0$, then the element ζ is called *non-positive*. We know that a QMV*-algebra does not satisfy the associativity of $\oplus$ in general. However, if ζ and κ are either non-negative or non-positive, then the equality $(\zeta \oplus \upsilon) \oplus \kappa = \zeta \oplus (\upsilon \oplus \kappa)$ always holds, we call it *restricted associativity* in this case.

Proposition 2.1. [1] *Let* $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ *be a QMV*-algebra. Then for any* $\zeta, \upsilon, \kappa, \lambda \in \Gamma$, *we have*

(1) $\zeta \oplus \upsilon = (\zeta \oplus 0) \oplus \upsilon = \zeta \oplus (\upsilon \oplus 0) = (\zeta \oplus 0) \oplus (\upsilon \oplus 0)$,
(2) $\zeta \sqcap \upsilon = (\zeta \sqcap \upsilon) \oplus 0 = (\zeta \oplus 0) \sqcap \upsilon = \zeta \sqcap (\upsilon \oplus 0) = (\zeta \oplus 0) \sqcap (\upsilon \oplus 0)$,

(3) $\kappa \ominus (\zeta \sqcup \upsilon) = (\kappa \ominus \zeta) \sqcap (\kappa \ominus \upsilon)$ *and* $\kappa \ominus (\zeta \sqcap \upsilon) = (\kappa \ominus \zeta) \sqcup (\kappa \ominus \upsilon)$,
(4) $(\zeta \sqcup \upsilon) \ominus \kappa = (\zeta \ominus \kappa) \sqcup (\upsilon \ominus \kappa)$ *and* $(\zeta \sqcap \upsilon) \ominus \kappa = (\zeta \ominus \kappa) \sqcap (\upsilon \ominus \kappa)$,
(5) $\zeta \sqcup \zeta = \zeta \oplus 0 = \zeta \sqcap \zeta$,
(6) $1 \oplus 0 = 1$ *and* $1 \oplus 1 = 1$,
(7) $0^+ = 0 = 0^-$,
(8) $(-\zeta)^+ \oplus 0 = -(\zeta^-) \oplus 0$,
(9) $\zeta \oplus 0 = (\zeta \oplus 0)^+ \oplus (\zeta \oplus 0)^-$,
(10) $\zeta \sqcup 0 = \zeta^+ \oplus 0$ *and* $\zeta \sqcap 0 = \zeta^- \oplus 0$,
(11) $\zeta^- \leq 0 \leq \zeta^+$,
(12) $-1 \leq \zeta \leq 1$ *and* $0 \leq |\zeta| \leq 1$,
(13) $\zeta \oplus 0 \leq \zeta \leq \zeta \oplus 0$,
(14) $\zeta \sqcap \upsilon \leq \zeta \leq \zeta \sqcup \upsilon$,
(15) *If* $\zeta \leq \upsilon$, *then* $\zeta^+ \leq \upsilon^+$, $\zeta^- \leq \upsilon^-$ *and* $-\upsilon \leq -\zeta$,
(16) *If* $\zeta \leq \upsilon$ *and* $\kappa \leq \lambda$, *then* $\zeta \oplus \kappa \leq \upsilon \oplus \lambda$, $\zeta \sqcup \kappa \leq \upsilon \sqcup \lambda$ *and* $\zeta \sqcap \kappa \leq \upsilon \sqcap \lambda$,
(17) *If* $\zeta \leq 0$, *then* $\zeta \oplus 0 = \zeta^- \oplus 0$ *and* $\zeta^+ \oplus 0 = 0$,
if $0 \leq \zeta$, *then* $\zeta \oplus 0 = \zeta^+ \oplus 0$ *and* $\zeta^- \oplus 0 = 0$,
(18) $\zeta \leq \upsilon$ *iff* $0 \leq \upsilon \ominus \zeta$,
(19) *If* $\zeta = \upsilon$, *then* $\upsilon \ominus \zeta = 0$, *if* $\upsilon \ominus \zeta = 0$, *then* $\zeta \oplus 0 = \upsilon \oplus 0$,
(20) $(\zeta \oplus \upsilon^+) \ominus \upsilon^+ \leq \zeta \leq (\zeta \ominus \upsilon^+) \oplus \upsilon^+$,
(21) $|\zeta| \leq \kappa$ *iff* $-\kappa \leq \zeta \leq \kappa$.

Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1 \rangle$ be a QMV*-algebra and $\emptyset \neq \Lambda \subseteq \Gamma$. We denote the set $|\Lambda| = \{|\lambda| \mid \lambda \in \Lambda\}$ and define an operation $\neg$ on $|\Lambda|$ by $\neg|\lambda| = 1 \ominus |\lambda|$ for any $|\lambda| \in |\Lambda|$. Below we will discuss the structure of $|\Gamma|$.

Lemma 2.1. *Let* $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1 \rangle$ *be a QMV*-algebra. Then* $|\Gamma|$ *is closed under operations* $\oplus$ *and* $\neg$.

Proof. For any $|\zeta|, |\upsilon| \in |\Gamma|$, then $|\zeta|, |\upsilon| \in \Gamma$ and we have $|\zeta| \oplus |\upsilon| \in \Gamma$, so $||\zeta| \oplus |\upsilon|| \in |\Gamma|$. Since $0 \leq |\zeta| \oplus |\upsilon|$ by Proposition 2.1(12),(16), we have $-(|\zeta| \oplus |\upsilon|) \leq 0$, it turns out that $||\zeta| \oplus |\upsilon|| = (|\zeta| \oplus |\upsilon|) \sqcup (-(|\zeta| \oplus |\upsilon|)) = (|\zeta| \oplus |\upsilon|) \oplus 0 = |\zeta| \oplus |\upsilon|$, so $|\zeta| \oplus |\upsilon| \in |\Gamma|$. For any $|\zeta| \in |\Gamma|$, then $|\zeta| \in \Gamma$, it follows that $\neg|\zeta| = 1 \ominus |\zeta| \in \Gamma$, so $|\neg|\zeta|| \in |\Gamma|$. Since $|\zeta| \leq 1$, we have $-1 \leq -|\zeta|$, it turns out that $0 \leq 1 \ominus |\zeta| = \neg|\zeta|$, so $|\neg|\zeta|| = (\neg|\zeta|) \sqcup (-(\neg|\zeta|)) = (\neg|\zeta|) \oplus 0 = \neg|\zeta|$. Hence $\neg|\zeta| \in |\Gamma|$.

Proposition 2.2. *Let* $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1 \rangle$ *be a QMV*-algebra. Then* $|\mathbf{\Gamma}| = \langle |\Gamma|; \oplus, \neg, 0 \rangle$ *is an MV-algebra. Moreover, the relation* $\leq$ *restricted on* $|\Gamma|$ *is partial-ordering.*

Proof. We only need to check the condition: $\neg(\neg|\zeta| \oplus |\upsilon|) \oplus |\upsilon| = \neg(\neg|\upsilon| \oplus |\zeta|) \oplus |\zeta|$ for any $|\zeta|, |\upsilon| \in |\Gamma|$. Since $\neg(\neg|\upsilon| \oplus |\zeta|) \oplus |\zeta| = (1 \ominus ((1 \ominus |\upsilon|) \oplus |\zeta|)) \oplus |\zeta| = (1 \oplus ((-1 \oplus |\upsilon|) \oplus (-|\zeta|))) \oplus |\zeta| = (1 \oplus (-1 \oplus (-|\zeta| \oplus |\upsilon|))) \oplus |\zeta| = (-|\zeta| \oplus |\upsilon|)^+ \oplus |\zeta|$ by restricted associativity and $|\zeta| \sqcup |\upsilon| = (|\zeta|^+ \oplus (-(|\zeta|^+) \oplus |\upsilon|^+)^+) \oplus (|\zeta|^- \oplus (-(|\zeta|^-) \oplus |\upsilon|^-)^+) = |\zeta| \oplus (-|\zeta| \oplus |\upsilon|)^+$ by Proposition 2.1(12),(17), we have $\neg(\neg|\upsilon| \oplus |\zeta|) \oplus |\zeta| = |\zeta| \sqcup |\upsilon|$. Similarly, we can show $\neg(\neg|\zeta| \oplus |\upsilon|) \oplus |\upsilon| = |\upsilon| \sqcup |\zeta|$. Since $|\zeta| \sqcup |\upsilon| = |\upsilon| \sqcup |\zeta|$, we get $\neg(\neg|\zeta| \oplus |\upsilon|) \oplus |\upsilon| = \neg(\neg|\upsilon| \oplus |\zeta|) \oplus |\zeta|$. Moreover, if $|\zeta| \leq |\upsilon|$ and $|\upsilon| \leq |\zeta|$, then $|\zeta| \sqcup |\upsilon| = |\upsilon| \oplus 0 = |\upsilon|$ and $|\zeta| \sqcup |\upsilon| = |\zeta| \oplus 0 = |\zeta|$, it turns out that $|\zeta| = |\upsilon|$, so the relation $\leq$ restricted on $|\Gamma|$ is antisymmetry. Hence the relation $\leq$ restricted on $|\Gamma|$ is partial-ordering.

Definition 2.2. [1] Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. The set $\emptyset \neq L \subseteq \Gamma$ is called an *ideal* of $\mathbf{\Gamma}$, if L satisfies: (I1) If $\zeta, \upsilon \in L$, then $\zeta \ominus \upsilon \in L$; (I2) If $\zeta \in L$, then $\zeta^+ \in L$; (I3) If $\zeta, \kappa \in L$ and $\upsilon \in \Gamma$ with $\zeta \leq \upsilon \leq \kappa$, then $\upsilon \in L$.

Proposition 2.3. [1] *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra and L be an ideal of $\mathbf{\Gamma}$. Then we have*

(1) $0 \in L$,
(2) *If $\zeta \in L$, then $-\zeta \in L$,*
(3) *If $\zeta, \upsilon \in L$, then $\zeta \oplus \upsilon \in L$,*
(4) *If $\zeta, \upsilon \in L$, then $\zeta \sqcup \upsilon \in L$,*
(5) *If $\zeta \in L$, then $|\zeta| \in L$,*
(6) *If $\zeta \ominus \upsilon \in L$ and $\upsilon \in L$, then $\zeta \in L$,*
(7) *If $\zeta \ominus \upsilon \in L$ and $\upsilon \ominus \kappa \in L$, then $\zeta \ominus \kappa \in L$.*

Proposition 2.4. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. If L is an ideal of $\mathbf{\Gamma}$, then* (I3) *is equivalent to the following:* (I3$'$) *If $\zeta \in L$ and $\upsilon \in \Gamma$ with $|\upsilon| \leq \zeta$, then $\upsilon \in L$.*

Proof. (I3)$\Rightarrow$(I3$'$) If $\zeta \in L$ and $\upsilon \in \Gamma$ with $|\upsilon| \leq \zeta$, then $-\zeta \in L$ using Proposition 2.3(2) and $-\zeta \leq \upsilon \leq \zeta$ by Proposition 2.1(21). Thus we have $\upsilon \in L$ by (I3).

(I3$'$)$\Rightarrow$(I3) If $\zeta, \kappa \in L$ and $\upsilon \in \Gamma$ with $\zeta \leq \upsilon \leq \kappa$, then $|\zeta|, |\kappa| \in L$ using Proposition 2.3(5). Since $\zeta \leq \upsilon \leq \kappa$, we have $-\kappa \leq -\upsilon \leq -\zeta$ by Proposition 2.1(15) and then $|\upsilon| = \upsilon \sqcup (-\upsilon) \leq \kappa \sqcup (-\zeta) \leq |\kappa| \sqcup |\zeta|$ by Proposition 2.1(16),(14). Because $|\zeta|, |\kappa| \in L$, we have $|\kappa| \sqcup |\zeta| \in L$ by Proposition 2.3(4). Thus $\upsilon \in L$ by (I3$'$).

Recall that an ideal L of an MV-algebra $\mathbf{\Lambda} = \langle \Lambda; \oplus, \neg, 0\rangle$ is a non-empty subset of Λ satisfying: (1) $0 \in L$; (2) If $\zeta, \upsilon \in L$, then $\zeta \oplus \upsilon \in L$; (3) If $\zeta \in L$ and $\upsilon \in \Lambda$ with $\upsilon \leq \zeta$, then $\upsilon \in L$.

Proposition 2.5. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. If L is an ideal of $\mathbf{\Gamma}$, then $|L|$ is an ideal of $|\mathbf{\Gamma}|$.*

Proof. Since $0 \in L$, we have $0 = |0| \in |L|$. If $|\zeta|, |\upsilon| \in |L|$, then $\zeta, \upsilon \in L$ and then $|\zeta|, |\upsilon| \in L$ by Proposition 2.3(5), so $|\zeta| \oplus |\upsilon| \in L$ by Proposition 2.3(3). Since $0 \leq |\zeta| \oplus |\upsilon|$ by Proposition 2.1(12),(16), we obtain $|\zeta| \oplus |\upsilon| = ||\zeta| \oplus |\upsilon|| \in |L|$. If $|\zeta| \in |L|$ and $|\upsilon| \in |\Gamma|$ with $|\upsilon| \leq |\zeta|$, then $|\zeta| \in L$ and then $\upsilon \in L$ by Proposition 2.4, so $|\upsilon| \in |L|$. Hence $|L|$ is an ideal of $|\mathbf{\Gamma}|$.

Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. For any $\emptyset \neq \Lambda \subseteq \Gamma$, we define $\langle \Lambda \rangle = \bigcap\{L \mid \Lambda \subseteq L$ and L is any ideal of $\mathbf{\Gamma}\}$. Then $\langle \Lambda \rangle$ is the least ideal of $\mathbf{\Gamma}$ which contains the set Λ and is called *the ideal generated* by Λ. For any $\zeta \in \Gamma$, denote $0 \cdot \zeta = 0$, $1 \cdot \zeta = \zeta$ and $n \cdot \zeta = (n-1) \cdot \zeta \oplus \zeta$ for some integer $n \geq 2$.

Proposition 2.6. [1] *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra and $\emptyset \neq \Lambda \subseteq \Gamma$. Then $\langle \Lambda \rangle = \{\zeta \in \Gamma \mid |\zeta| \leq |\lambda_1| \oplus |\lambda_2| \oplus \cdots \oplus |\lambda_n|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n \in \Lambda\}$.*

Proposition 2.7. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. IfL is an ideal of $\mathbf{\Gamma}$ and $\lambda \in \Gamma \setminus L$, then we have $\langle L \cup \{\lambda\}\rangle = \{\zeta \in \Gamma \mid |\zeta| \leq |\upsilon| \oplus n \cdot |\lambda|$, where $\upsilon \in L$ and for some integer $n \geq 1\}$.*

Given that L is an ideal of $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. For any $\zeta \in \Gamma$, we denote the equivalence class of ζ with respect to L by $\zeta/L = \{\upsilon \in \Gamma | \upsilon \ominus \zeta \in L\}$ and $\Gamma/L = \{\zeta/L | \zeta \in \Gamma\}$. For any $\zeta/L, \upsilon/L \in \Gamma/L$, we define $(\zeta/L) \oplus_L (\upsilon/L) = (\zeta \oplus \upsilon)/L$, $-_L(\zeta/L) = (-\zeta)/L$ and $(\zeta/L) \sqcup_L (\upsilon/L) = (\zeta \sqcup \upsilon)/L$, then $\mathbf{\Gamma}/L = \langle \Gamma/L; \oplus_L, -_L, 0/L, 1/L\rangle$ is a QMV*-algebra.

Proposition 2.8. *Let L be an ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. Then the quotient algebra $\mathbf{\Gamma}/L$ is an MV*-algebra.*
Proof. We only check the condition: $(\zeta/L) \oplus_L (0/L) = \zeta/L$ for any $\zeta/L \in \Gamma/L$. Since $\zeta/L \in \Gamma/L$, we have $(\zeta/L) \oplus_L (0/L) = (\zeta \oplus 0)/L = \zeta/L$. Indeed, $\upsilon \in (\zeta \oplus 0)/L$ iff $\upsilon \ominus (\zeta \oplus 0) \in L$ iff $\upsilon \ominus \zeta \in L$ iff $\upsilon \in \zeta/L$ by Proposition 2.1(1). Hence $\mathbf{\Gamma}/L$ is an MV*-algebra.

Definition 2.3. [1] Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ and $\mathbf{\Lambda} = \langle \Lambda; \oplus, -, 0, 1\rangle$ be QMV*-algebras. A function $\phi : \Gamma \to \Lambda$ is called a *QMV*-homomorphism*, if for any $\zeta, \upsilon \in \Gamma$, we have: (1) $\phi(0) = 0$; (2) $\phi(1) = 1$; (3) $\phi(\zeta \oplus \upsilon) = \phi(\zeta) \oplus \phi(\upsilon)$; (4) $\phi(-\zeta) = -\phi(\zeta)$.

Proposition 2.9. [1] *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ and $\mathbf{\Lambda} = \langle \Lambda; \oplus, -, 0, 1\rangle$ be QMV*-algebras and $\phi : \Gamma \to \Lambda$ be a homomorphism. For any $\zeta, \upsilon \in \Gamma$, then*

(1) $\phi(\zeta \ominus \upsilon) = \phi(\zeta) \ominus \phi(\upsilon)$, (5) $\phi(\zeta \sqcap \upsilon) = \phi(\zeta) \sqcap \phi(\upsilon)$,
(2) $\phi(\zeta^+ \oplus 0) = (\phi(\zeta))^+ \oplus 0$, (6) $\phi(|\zeta|) = |\phi(\zeta)|$,
(3) $\phi(\zeta^- \oplus 0) = (\phi(\zeta))^- \oplus 0$, (7) *If* $\zeta \le \upsilon$, *then* $\phi(\zeta) \le \phi(\upsilon)$.
(4) $\phi(\zeta \sqcup \upsilon) = \phi(\zeta) \sqcup \phi(\upsilon)$,

Lemma 2.2. [1] *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ and $\mathbf{\Lambda} = \langle \Lambda; \oplus, -, 0, 1\rangle$ be QMV*-algebras and $\phi : \Gamma \to \Lambda$ be a homomorphism. If $\phi(\zeta) = \phi(\upsilon)$, then $\zeta \ominus \upsilon \in ker(\phi) = \{\kappa \in \Gamma | \phi(\kappa) = 0\}$. Conversely, if $\zeta \ominus \upsilon \in ker(\phi)$, then $\phi(\zeta) \oplus 0 = \phi(\upsilon) \oplus 0$.*

Proposition 2.10. [1] *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ and $\mathbf{\Lambda} = \langle \Lambda; \oplus, -, 0, 1\rangle$ be QMV*-algebras and $\phi : \Gamma \to \Lambda$ be a homomorphism. If L is an ideal of $\mathbf{\Lambda}$, then $\phi^{-1}(L)$ is an ideal of $\mathbf{\Gamma}$.*

Suppose that L is an ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. We define a function $\phi_L : \Gamma \to \Gamma/L$ by $\phi_L(\zeta) = \zeta/L$ for any $\zeta \in \Gamma$. Then ϕ_L is the epimorphism, we call it natural homomorphism. Moreover, we have the following results.

Lemma 2.3. *Let L be an ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. If $\phi_L : \Gamma \to \Gamma/L$ is the natural homomorphism and $\upsilon \in \zeta/L$ for any $\zeta \in \Gamma$, then $\upsilon/L = \zeta/L$.*
Proof. For any $\kappa \in \upsilon/L$, then $\kappa \ominus \upsilon \in L$. Since $\upsilon \in \zeta/L$, we have $\upsilon \ominus \zeta \in L$, it turns out that $\kappa \ominus \zeta \in L$ by Proposition 2.3(7), so $\kappa \in \zeta/L$ which means that $\upsilon/L \subseteq \zeta/L$. For any $\kappa \in \zeta/L$, then $\kappa \ominus \zeta \in L$. Since $\upsilon \in \zeta/L$, we have $\upsilon \ominus \zeta \in L$ and then $\zeta \ominus \upsilon \in L$ by Proposition 2.3(2), it turns out $\kappa \ominus \upsilon \in L$, so $\kappa \in \upsilon/L$ which means that $\zeta/L \subseteq \upsilon/L$. Hence $\upsilon/L = \zeta/L$.

Proposition 2.11. *Let L be an ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ and $\phi_L : \Gamma \to \Gamma/L$ be the natural homomorphism. Then $\phi_L(L) = \{0/L\}$ and $L = ker(\phi_L)$.*
Proof. It is evident that $\{0/L\}$ is an ideal of $\mathbf{\Gamma}/L$. Now we verify that $\phi_L(L) = \{0/L\}$. For any $\upsilon/L \in \phi_L(L)$, there exists $\zeta \in L$ with $\upsilon/L = \phi_L(\zeta) = \zeta/L$. Since $\upsilon \in \upsilon/L = \zeta/L$, we have $\upsilon \ominus \zeta \in L$ and then $\upsilon \in L$ by Proposition 2.3(6), it turns out that $\upsilon \ominus 0 \in L$, so $\upsilon \in 0/L$ and then $\upsilon/L = 0/L$ by Lemma 2.3. Hence $\phi_L(L) \subseteq \{0/L\}$. Conversely, since ϕ_L is a natural homomorphism and $0 \in L$, we have $0/L = \phi_L(0) \in \phi_L(L)$, so $\{0/L\} \subseteq \phi_L(L)$. Hence $\phi_L(L) = \{0/L\}$. For any $\upsilon \in ker(\phi_L)$, then $\phi_L(\upsilon) = 0/L$, we have $\upsilon \ominus 0 \in L$ and then $\upsilon \in L$, so $ker(\phi_L) \subseteq L$. Note that $L \subseteq ker(\phi_L)$, we have $L = ker(\phi_L)$.

Proposition 2.12. *Let L be an ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. Then the mapping $T \mapsto \phi_L(T)$ is a bijection correspondence between the set of ideals of $\mathbf{\Gamma}$ containing L and the set of ideals of the quotient algebra $\mathbf{\Gamma}/L$.*

Proof. Suppose that T is an ideal of $\mathbf{\Gamma}$ with $L \subseteq T$. For any $\zeta/L, \upsilon/L \in \phi_L(T)$, then there are $\kappa, \tau \in T$ such that $\zeta/L = \phi_L(\kappa)$ and $\upsilon/L = \phi_L(\tau)$, we have $(\zeta/L) \ominus (\upsilon/L) = \phi_L(\kappa) \ominus \phi_L(\tau) = \phi_L(\kappa \ominus \tau) \in \phi_L(T)$. Moreover, because $(\zeta/L)^+ = (\zeta/L)^+ \oplus 0/L$ and $\kappa^+ \in T$, we get $(\zeta/L)^+ = (\phi_L(\kappa))^+ \oplus 0/L = \phi_L(\kappa^+ \oplus 0) \in \phi_L(T)$. For any $\zeta/L, \lambda/L \in \phi_L(T)$ and $\upsilon/L \in \Gamma/L$ with $\zeta/L \leq \upsilon/L \leq \lambda/L$, then there exist $\kappa, \omega \in T$ and $\tau \in \Gamma$ such that $\zeta/L = \phi_L(\kappa)$, $\upsilon/L = \phi_L(\tau)$ and $\lambda/L = \phi_L(\omega)$. Since $\phi_L(\kappa) \leq \phi_L(\tau) \leq \phi_L(\omega)$, we have $\phi_L(\kappa) = \phi_L(\kappa) \sqcap \phi_L(\tau) = \phi_L(\kappa \sqcap \tau)$ and $\phi_L(\omega) = \phi_L(\tau) \sqcup \phi_L(\omega) = \phi_L(\tau \sqcup \omega)$ by Proposition 2.9(4),(5), it turns out $(\kappa \sqcap \tau) \ominus \kappa \in \ker(\phi_L) = L$ and $(\tau \sqcup \omega) \ominus \omega \in \ker(\phi_L) = L$ using Proposition 2.11. Notice that $\kappa, \omega \in T$ and $L \subseteq T$, we have $\kappa \sqcap \tau \in T$ and $\tau \sqcup \omega \in T$ by Proposition 2.3(6). Since $\kappa \sqcap \tau \leq \tau \leq \tau \sqcup \omega$ by Proposition 2.1(14) and T is an ideal of $\mathbf{\Gamma}$, we have $\tau \in T$ and then $\upsilon/L = \phi_L(\tau) \in \phi_L(T)$. Hence $\phi_L(T)$ is an ideal of $\mathbf{\Gamma}/L$. For any $\zeta \in T$, we have $\zeta \in {\phi_L}^{-1}(\phi_L(\zeta)) \subseteq {\phi_L}^{-1}(\phi_L(T))$. Then $T \subseteq {\phi_L}^{-1}(\phi_L(T))$. Conversely, for any $\zeta \in {\phi_L}^{-1}(\phi_L(T))$, we have $\phi_L(\zeta) \in \phi_L(T)$, then there is $\upsilon \in T$ such that $\zeta/L = \phi_L(\upsilon) = \upsilon/L$, it follows that $\zeta \ominus \upsilon \in L \subseteq T$. Note that $\upsilon \in T$, we have $\zeta \in T$ by Proposition 2.3(6), so ${\phi_L}^{-1}(\phi_L(T)) \subseteq T$. Hence $T = {\phi_L}^{-1}(\phi_L(T))$. Now, for any ideal T' of $\mathbf{\Gamma}/L$, we have ${\phi_L}^{-1}(T')$ is an ideal of $\mathbf{\Gamma}$ by Proposition 2.10. Meanwhile, since $L = ker(\phi_L) = {\phi_L}^{-1}(0/L) \subseteq {\phi_L}^{-1}(T')$, we have $L \subseteq {\phi_L}^{-1}(T')$. For any $\upsilon \in \phi_L({\phi_L}^{-1}(T'))$, then there exists $\zeta \in {\phi_L}^{-1}(T')$ such that $\upsilon = \phi_L(\zeta) \in T'$, so $\phi_L({\phi_L}^{-1}(T')) \subseteq T'$. Conversely, for any $\upsilon \in T' \subseteq \Gamma/L$, since ϕ_L is surjective, there exists $\zeta \in \Gamma$ such that $\upsilon = \phi_L(\zeta)$, we have $\zeta \in {\phi_L}^{-1}(T')$ and then $\upsilon = \phi_L(\zeta) \in \phi_L({\phi_L}^{-1}(T'))$, so $T' \subseteq \phi_L({\phi_L}^{-1}(T'))$. Hence $\phi_L({\phi_L}^{-1}(T')) = T'$.

3. Prime ideals and maximal ideals of QMV*-algebras

In this section, we introduce prime ideals and maximal ideals of a QMV*-algebra and investigate their related properties.

In any QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1 \rangle$, an ideal L of $\mathbf{\Gamma}$ is *prime* if L is proper (i.e., $L \neq \Gamma$) and for any $\zeta \in \Gamma$, either $\zeta^+ \in L$ or $\zeta^- \in L$.

Proposition 3.1. *Let L be a proper ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1 \rangle$. Then L is prime iff the quotient algebra $\mathbf{\Gamma}/L$ is totally ordered.*

Proof. Suppose that L is a prime ideal of $\mathbf{\Gamma}$. For any $\zeta, \upsilon \in \Gamma$, then we have $(\zeta \ominus \upsilon)^+ \in L$ or $(\zeta \ominus \upsilon)^- \in L$. If $(\zeta \ominus \upsilon)^- \in L$, then $(\upsilon \ominus \zeta)^+ = -(\zeta \ominus \upsilon)^- \in L$ by Proposition 2.1(8) and Proposition 2.3(2). Since $(\zeta \sqcup \upsilon) \ominus \upsilon = (\zeta \ominus \upsilon) \sqcup (\upsilon \ominus \upsilon) = (\zeta \ominus \upsilon) \sqcup 0 = (\zeta \ominus \upsilon)^+$ and $(\zeta \sqcup \upsilon) \ominus \zeta = (\zeta \ominus \zeta) \sqcup (\upsilon \ominus \zeta) = 0 \sqcup (\upsilon \ominus \zeta) = (\upsilon \ominus \zeta)^+$ by Proposition 2.1(4),(10), we have $(\zeta \sqcup \upsilon) \ominus \upsilon \in L$ or $(\zeta \sqcup \upsilon) \ominus \zeta \in L$, it follows that $((\zeta/L) \sqcup (\upsilon/L)) \ominus (\upsilon/L) = ((\zeta \sqcup \upsilon) \ominus \upsilon)/L = 0/L$ or $((\zeta/L) \sqcup (\upsilon/L)) \ominus (\zeta/L) = ((\zeta \sqcup \upsilon) \ominus \zeta)/L = 0/L$. Note that $\mathbf{\Gamma}/L$ is an MV*-algebra, we have $(\zeta/L) \sqcup (\upsilon/L) = \upsilon/L$ or $(\zeta/L) \sqcup (\upsilon/L) = \zeta/L$ by Proposition 2.1(19), so $\zeta/L \leq \upsilon/L$ or $\upsilon/L \leq \zeta/L$. Hence $\mathbf{\Gamma}/L$ is totally ordered. Conversely, if the algebra $\mathbf{\Gamma}/L$ is totally ordered, then we have $\zeta/L \leq \upsilon/L$ or $\upsilon/L \leq \zeta/L$ for any $\zeta/L, \upsilon/L \in \Gamma/L$, it follows that $(\zeta \sqcup \upsilon)/L = \upsilon/L$ or $(\zeta \sqcup \upsilon)/L = \zeta/L$, so $((\zeta \sqcup \upsilon) \ominus \upsilon)/L = 0/L$ or $((\zeta \sqcup \upsilon) \ominus \zeta)/L = 0/L$ by Proposition 2.1(19) and then $(\zeta \sqcup \upsilon) \ominus \upsilon \in L$ or $(\zeta \sqcup \upsilon) \ominus \zeta \in L$ by Proposition 2.11. Since $(\zeta \ominus \upsilon)^+ = (\zeta \sqcup \upsilon) \ominus \upsilon$ and $(\upsilon \ominus \zeta)^+ = (\zeta \sqcup \upsilon) \ominus \zeta$, we have $(\zeta \ominus \upsilon)^+ \in L$ or $(\upsilon \ominus \zeta)^+ \in L$. Hence for any $\zeta \in \Gamma$, we have $\zeta^+ \oplus 0 = (\zeta \ominus 0)^+ \in L$ or $\zeta^- \oplus 0 = -(0 \ominus \zeta)^+ \in L$. Since $\zeta^+ \oplus 0 \leq \zeta^+ \leq \zeta^+ \oplus 0$ and $\zeta^- \oplus 0 \leq \zeta^- \leq \zeta^- \oplus 0$ by Proposition 2.1(13), we have $\zeta^+ \in L$ or $\zeta^- \in L$. So the ideal L of $\mathbf{\Gamma}$ is prime.

Proposition 3.2. *Let V be a prime ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. Then the set of all ideals of $\mathbf{\Gamma}/V$ with respect to the inclusion is totally ordered.*

Proof. It is easy to get that the set of all ideals of $\mathbf{\Gamma}/V$ with respect to the inclusion is partial order. If it is not totally ordered, we may suppose that L, T are ideals of $\mathbf{\Gamma}/V$ such that $L \nsubseteq T$ and $T \nsubseteq L$. So there exist $\zeta/V, \upsilon/V \in \Gamma/V$ such that $\zeta/V \in T\backslash L$ and $\upsilon/V \in L\backslash T$. Since $\mathbf{\Gamma}/V$ is totally ordered by Proposition 3.1, we have $\zeta/V \leq \upsilon/V$ or $\upsilon/V \leq \zeta/V$, it turns out that $|\zeta/V| \leq |\upsilon/V|$ or $|\upsilon/V| \leq |\zeta/V|$. Indeed, if $0/V \leq \zeta/V \leq \upsilon/V$, then $-(\zeta/V) \leq 0/V \leq \zeta/V$ and $-(\upsilon/V) \leq 0/V \leq \upsilon/V$, we have $|\zeta/V| = (\zeta/V) \sqcup (-(\zeta/V)) = (\zeta/V) \oplus (0/V) = \zeta/V$ and $|\upsilon/V| = (\upsilon/V) \sqcup (-(\upsilon/V)) = \upsilon/V$, so $|\zeta/V| \leq |\upsilon/V|$. If $\zeta/V \leq \upsilon/V \leq 0/V$, then $\zeta/V \leq 0/V \leq -(\zeta/V)$, $\upsilon/V \leq 0/V \leq -(\upsilon/V)$ and $0/V \leq -(\upsilon/V) \leq -(\zeta/V)$, we have $|\zeta/V| = (\zeta/V) \sqcup (-(\zeta/V)) = (-(\zeta/V)) \oplus (0/V) = -(\zeta/V)$ and $|\upsilon/V| = (\upsilon/V) \sqcup (-(\upsilon/V)) = -(\upsilon/V)$, so $|\upsilon/V| \leq |\zeta/V|$. If $\zeta/V \leq 0/V \leq \upsilon/V$, then we have $-(\upsilon/V) \leq 0/V \leq -(\zeta/V)$, so $|\zeta/V| = (\zeta/V) \sqcup (-(\zeta/V)) = -(\zeta/V)$ and $|\upsilon/V| = (\upsilon/V) \sqcup (-(\upsilon/V)) = \upsilon/V$. Since $\mathbf{\Gamma}/V$ is totally ordered and $-(\zeta/V), \upsilon/V \in \Gamma/V$, we have $0/V \leq -(\zeta/V) \leq \upsilon/V$ or $0/V \leq \upsilon/V \leq -(\zeta/V)$, so $|\zeta/V| \leq |\upsilon/V|$ or $|\upsilon/V| \leq |\zeta/V|$. The case of $\upsilon/V \leq \zeta/V$ can be proved similarly. If $|\zeta/V| \leq |\upsilon/V|$, since $\upsilon/V \in L$, we have $|\upsilon/V| \in L$ by Proposition 2.3(5) and then $\zeta/V \in L$ by Proposition 2.4. Likewise, if $|\upsilon/V| \leq |\zeta/V|$, since $\zeta/V \in T$, we have $|\zeta/V| \in T$ and then $\upsilon/V \in T$. This is a contradiction with $\zeta/V \notin L$ and $\upsilon/V \notin T$. Hence the set of all ideals of $\mathbf{\Gamma}/V$ with respect to the inclusion is totally ordered.

Proposition 3.3. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. Then we have:*

(1) *Any proper ideal of $\mathbf{\Gamma}$ containing a prime ideal is prime.*

(2) *The set of all ideals of $\mathbf{\Gamma}$ containing a prime ideal is totally ordered by the inclusion.*

Proof. (1) Suppose that L is a proper ideal of $\mathbf{\Gamma}$ and a prime ideal V of $\mathbf{\Gamma}$ with $V \subseteq L$. For any $\zeta \in \Gamma$, we have $\zeta^+ \in V$ or $\zeta^- \in V$, it follows that $\zeta^+ \in L$ or $\zeta^- \in L$. So L is prime.

(2) Suppose that V is a prime ideal of $\mathbf{\Gamma}$. Then the set of all ideals of $\mathbf{\Gamma}/V$ with respect to the inclusion is totally ordered from Proposition 3.2. Hence we get that the set of all ideals containing V is totally ordered by Proposition 2.12.

Lemma 3.1. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. Then for any $\zeta \in \Gamma$ and some integers $n, m \geq 1$, we have $(n \cdot \zeta^+) \sqcap (m \cdot (-\zeta)^+) = 0$.*

Proof. First, we prove $\zeta^+ \sqcap (-\zeta)^+ = 0$ for any $\zeta \in \Gamma$. Since $0 \leq \zeta^+ \sqcap (-\zeta)^+$, we have $0 \sqcap (\zeta^+ \sqcap (-\zeta)^+) = 0 \oplus 0 = 0$. Based on Proposition 2.1(20),(4),(8),(9),(10), we have $\zeta^+ \sqcap (-\zeta)^+ \leq ((\zeta^+ \sqcap (-\zeta)^+) \ominus (-\zeta)^+) \oplus (-\zeta)^+ = ((\zeta^+ \ominus (-\zeta)^+) \sqcap 0) \oplus (-\zeta)^+ = ((\zeta^+ \oplus \zeta^-) \sqcap 0) \oplus (-\zeta)^+ = (\zeta \sqcap 0) \oplus (-\zeta)^+ = \zeta^- \oplus (-(\zeta^-)) = 0$, so $(\zeta^+ \sqcap (-\zeta)^+) \sqcap 0 = (\zeta^+ \sqcap (-\zeta)^+) \oplus 0 = \zeta^+ \sqcap (-\zeta)^+$, then $\zeta^+ \sqcap (-\zeta)^+ = 0$. Assume $((n-1) \cdot \zeta^+) \sqcap (-\zeta)^+ = 0$ for any $\zeta \in \Gamma$ and some integer $n \geq 2$. Then $\zeta^+ \oplus 0 = (((n-1) \cdot \zeta^+) \sqcap (-\zeta)^+) \oplus \zeta^+ = (n \cdot \zeta^+) \sqcap ((-\zeta)^+ \oplus \zeta^+)$ by Proposition 2.1(4), so $\zeta^+ \sqcap (-\zeta)^+ = ((n \cdot \zeta^+) \sqcap ((-\zeta)^+ \oplus \zeta^+)) \sqcap (-\zeta)^+ = (n \cdot \zeta^+) \sqcap (((-\zeta)^+ \oplus \zeta^+) \sqcap (-\zeta)^+) = (n \cdot \zeta^+) \sqcap ((-\zeta)^+ \oplus 0) = (n \cdot \zeta^+) \sqcap (-\zeta)^+$, it turns out that $(n \cdot \zeta^+) \sqcap (-\zeta)^+ = 0$. Moreover, just like the previous, we verify that $(n \cdot \zeta^+) \sqcap (m \cdot (-\zeta)^+) = 0$ for any $\zeta \in \Gamma$ and some integers $n, m \geq 1$. Since $(n \cdot \zeta^+) \sqcap (-\zeta)^+ = 0$, we suppose that $(n \cdot \zeta^+) \sqcap ((m-1) \cdot (-\zeta)^+) = 0$ for any $\zeta \in \Gamma$ and some integer $m \geq 2$, then we can conclude that $(n \cdot \zeta^+) \sqcap (m \cdot (-\zeta)^+) = 0$ by induction.

Proposition 3.4. *Let L be a proper ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. If $\lambda \notin L$, then there is a prime ideal V satisfying $L \subseteq V$ and $\lambda \notin V$.*

Proof. By Zorn's Lemma we know that there is an ideal V satisfying $L \subseteq V$ and is maximal with the property $\lambda \notin V$. If V is not a prime ideal of $\mathbf{\Gamma}$, then assume $\zeta^+ \notin V$ and $\zeta^- \notin V$ for any $\zeta \in \Gamma$. Define $V_1 = \langle V \cup \{\zeta^+\}\rangle$ and $V_2 = \langle V \cup \{\zeta^-\}\rangle$. Since $V \subsetneqq V_1$, $V \subsetneqq V_2$ and V is maximal with the property $\lambda \notin V$, we have $\lambda \in V_1 \cap V_2$, then there exist $\upsilon, \kappa \in V$ and some integers $n, m \geq 1$ with $|\lambda| \leq |\upsilon| \oplus (n \cdot |\zeta^+|)$ and $|\lambda| \leq |\kappa| \oplus (m \cdot |\zeta^-|)$, so $|\lambda| \leq |\upsilon| \oplus (n \cdot \zeta^+)$ and $|\lambda| \leq |\kappa| \oplus (m \cdot (-\zeta)^+)$ by Proposition 2.1(8). Denote $\tau = |\upsilon| \sqcup |\kappa|$. Then $\tau \in V$ and we have $|\lambda| \leq \tau \oplus (n \cdot \zeta^+)$ and $|\lambda| \leq \tau \oplus (m \cdot (-\zeta)^+)$, it follows that $|\lambda| \leq (\tau \oplus (n \cdot \zeta^+)) \sqcap (\tau \oplus (m \cdot (-\zeta)^+))$ by Proposition 2.1(16), thus we have $|\lambda| \leq \tau \oplus ((n \cdot \zeta^+) \sqcap (m \cdot (-\zeta)^+)) = \tau \oplus 0$ by Lemma 3.1, so $|\lambda| \leq \tau$. Because V is an ideal of $\mathbf{\Gamma}$ and $\tau \in V$, we have $\lambda \in V$ by Proposition 2.4, this is a contradiction with $\lambda \notin V$. Hence $\zeta^+ \in V$ or $\zeta^- \in V$ and then V is prime.

Corollary 3.1. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. Then any proper ideal of $\mathbf{\Gamma}$ is an intersection of prime ideals.*

Proof. Suppose that T is a proper ideal of $\mathbf{\Gamma}$. Denote $\Phi = \bigcap_{i \in I}\{V_i \mid V_i$ is any prime ideal of $\mathbf{\Gamma}$ and $T \subseteq V_i\}$. Then we have $T \subseteq \Phi$. Below we verify that $\Phi \subseteq T$. If not, then there exists $\zeta \in \Phi \setminus T$. Since $\zeta \notin T$, there exists a prime ideal V with $T \subseteq V$ and $\zeta \notin V$ by Proposition 3.4, so $\zeta \notin \Phi$, this is a contradiction with $\zeta \in \Phi$.

In any QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$, an ideal T of $\mathbf{\Gamma}$ is *maximal* if it is proper and for any ideal L of $\mathbf{\Gamma}$ with $T \subsetneqq L$, then $L = \Gamma$.

Proposition 3.5. *Let T be a proper ideal of a QMV*-algebra $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$. Then the following conditions are equivalent:*

(1) *T is a maximal ideal of $\mathbf{\Gamma}$,*

(2) *For any $\zeta \in \Gamma$, $\zeta \notin T$ iff $1 \ominus (n \cdot |\zeta|) \in T$ for some integer $n \geq 1$.*

Proof. (1) $\Rightarrow$ (2) Suppose that T is a maximal ideal of $\mathbf{\Gamma}$. Then for any $\zeta \notin T$, we have $T \subsetneqq \langle T \cup \{\zeta\}\rangle$. Because T is maximal, we get $\langle T \cup \{\zeta\}\rangle = \Gamma$ and then $|\upsilon| \oplus n \cdot |\zeta| = 1$ for some $\upsilon \in T$ and $n \geq 1$. Since $0 \leq n \cdot |\zeta| \leq 1$, we have $n \cdot |\zeta| = (n \cdot |\zeta|)^+$ and then $|1 \ominus (n \cdot |\zeta|)| = 1 \ominus (n \cdot |\zeta|) = (|\upsilon| \oplus n \cdot |\zeta|) \ominus (n \cdot |\zeta|) \leq |\upsilon|$ by Proposition 2.1(17),(20). Because $\upsilon \in T$, we have $|\upsilon| \in T$ and then $1 \ominus (n \cdot |\zeta|) \in T$ by Proposition 2.4. Conversely, let $1 \ominus (n \cdot |\zeta|) \in T$ for some integer $n \geq 1$. If $\zeta \in T$, then $|\zeta| \in T$ and then $n \cdot |\zeta| \in T$ by Proposition 2.3(5),(3), so $1 = (1 \ominus (n \cdot |\zeta|)) \oplus (n \cdot |\zeta|) \in T$ which is a contradiction with T is proper.

(2) $\Rightarrow$ (1) Suppose that L is an ideal of $\mathbf{\Gamma}$ and $T \subsetneqq L$. Then for any $\zeta \in L \setminus T$, we have $1 \ominus (n \cdot |\zeta|) \in T$ for some integer $n \geq 1$. Since $\zeta \in L$, we have $|\zeta| \in L$ and then $n \cdot |\zeta| \in L$, so $1 = (1 \ominus (n \cdot |\zeta|)) \oplus (n \cdot |\zeta|) \in L$ which means that $L = \Gamma$. Hence T is maximal.

Proposition 3.6. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra. Then any proper ideal of $\mathbf{\Gamma}$ is contained in a maximal ideal.*

Proof. Denote $\mathscr{I}(\Gamma)$ the ordered set of all proper ideals in $\mathbf{\Gamma}$. Since the union of any chain of proper ideals is a proper ideal, we conclude that any chain of elements in $\mathscr{I}(\Gamma)$ has an upper bound in $\mathscr{I}(\Gamma)$. Hence for any proper ideal $T \in \mathscr{I}(\Gamma)$, there exists a maximal element $L \in \mathscr{I}(\Gamma)$ by Zorn's Lemma, i.e., L is a maximal ideal of $\mathbf{\Gamma}$ such that $T \subseteq L$.

Proposition 3.7. *Let $\mathbf{\Gamma} = \langle \Gamma; \oplus, -, 0, 1\rangle$ be a QMV*-algebra and T be any maximal ideal of $\mathbf{\Gamma}$. Then T is also a prime ideal.*

Proof. Assume that T is a maximal ideal of $\mathbf{\Gamma}$ which is not a prime ideal of $\mathbf{\Gamma}$. Then there is $\zeta \in \Gamma$ such that $\zeta^+ \notin T$ and $\zeta^- \notin T$, so we have a prime ideal V such that $T \subseteq V$ and $\zeta^+ \notin V$ by Proposition 3.4. However, because V is a prime ideal of $\mathbf{\Gamma}$ and $\zeta^+ \notin V$, we obtain $\zeta^- \in V$, which means that $T \subsetneqq V$, this is a contradiction with the maximality of T. Thus T is a prime ideal of $\mathbf{\Gamma}$.

4. Conclusion

In the present paper, the ideals of QMV*-algebras are investigated. We mainly study the properties of prime ideals and maximal ideals. It is known that filters are dual notions of ideals in MV-algebras. However, the correspondence between filters and ideals in QMV*-algebras is different from the case in MV-algebras. Hence we will focus on the filters of QMV*-algebras and characterize the prime filters in the future.

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