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A Theorem About the Existence of Minimax Rules for Statistical Decision Problems with Trapezoidal Fuzzy Losses

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Abstract. This paper considers the problem of choice of an optimal decision under unknown state of nature when losses are not known exactly for some or all decisions and some or all states of nature. However, it is possible to determine probabilities for all states of nature on the basis of statistical data. So, there are two kinds of uncertainty, randomness and vagueness. That is why both methods of the probability theory and methods of the theory of fuzzy sets are used. There is no a unified approach to choice of an optimal decision even in the classical statistical decision theory. One of existing approaches is to use minimax decision rules when losses are crisp. In this paper, the approach is generalized for problems with trapezoidal fuzzy losses. A theorem about the existence of a minimax decision rule is proved when the set of all possible states of nature is finite.

Keywords. fuzzy set, decision rule, risk function, loss function

1. Introduction

A decision rule, which allocates a decision to statistical data, is the main object of study in the statistical decision theory. It is an important theory since many well-known problems of mathematical statistics are particular cases of problems from this theory. Many of the basic methods of statistical decision theory were developed by Wald [1]. Also, some results connected with minimax decisions were obtained by Von Neumann and Morgenstern [2]. The statistical decision theory proved to be useful in many fields, see, e.g., Berger [3] and Robert [4]. In the classical statistical decision theory, losses are crisp and depend on the decision and the state of nature. But when a decision should be made, the state of nature is not known. However, on the basis of statistical data, which describe the state of nature, it may be possible to make a reasonable decision. The decision rule should satisfy some conditions of optimality. For example, the decision rule may be Bayesian or minimax. In the statistical decision theory, both problems with an infinite set of all possible states of nature Θ and problems with a finite set of all possible states of nature $\Theta = \{\theta_1, ..., \theta_m\}$ are considered. When Θ is

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finite, there is a theorem about the existence of minimax decision rules, see, e.g., Berger [3] and Schervish [5].

However, the assumption that losses are known exactly for all decisions and states of nature is not realistic on many occasions. The theory of fuzzy sets can be used to overcome this lack of certainty. The fuzzy statistical decision theory is a relatively new field of research. Some results of the theory were obtained by Gil and Jain [6], Gil and López-Díaz [7], Gil et al. [8], Hryniewicz [9], Viertl and Sunanta [10]. However, these papers do not consider minimax decision rules. To the best of our knowledge, no theorems about the existence of minimax rules for statistical decision problems with fuzzy losses were published before.

This paper develops the fuzzy statistical decision theory. A theorem about the existence of minimax decision rules, which generalizes the result presented by Berger [3] and Schervish [5] for the case of trapezoidal fuzzy losses, is proved. Section 2 contains some definitions. In Section 3, the main theorem is given.

2. Definitions

Let D be a set of all possible decisions. Denote by $\tilde{L}_i(d)$ a loss under the state of nature θ_i and the decision $d \in D$. The membership function of the fuzzy set $\tilde{L}_i(d)$ has the following form. There are real numbers $a_i(d)$ and $b_i(d)$ such that $a_i(d) \leq b_i(d)$, $\mu_{\tilde{L}_i(d)}(\xi) = 1$ for $\xi \leq a_i(d)$, and $\mu_{\tilde{L}_i(d)}(\xi) = 0$ for $\xi > b_i(d)$. When $a_i(d) < b_i(d)$, the membership function is linear on the closed interval $\left[a_i(d), b_i(d)\right]$ and continuous on the set of all real numbers. So, the choice of a fuzzy set $\tilde{L}_i(d)$ is equivalent to the choice of a two-dimensional vector $c_i(d) = \left(a_i(d), b_i(d)\right)$, where $a_i(d) \leq b_i(d)$. It is reasonable to consider semi-infinite trapeziums because we are interested in minimization of maximum losses only.

Suppose $\tilde{M}_1$ and $\tilde{M}_2$ are semi-infinite trapezoidal fuzzy sets; $\tilde{M}_1$ is defined by a two-dimensional vector (α_1, β_1), where $\alpha_1 \leq \beta_1$; $\tilde{M}_2$ is defined by a two-dimensional vector (α_2, β_2), where $\alpha_2 \leq \beta_2$. Let k be a non-negative real number. Then $\tilde{M}_1 + \tilde{M}_2$ is a semi-infinite trapezoidal fuzzy set that is defined by a two-dimensional vector $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ and $k\tilde{M}_1$ is a semi-infinite trapezoidal fuzzy set that is defined by a two-dimensional vector $(k\alpha_1, k\beta_1)$.

We say that $\tilde{M}_1 \leq \tilde{M}_2$ if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. We say that $\tilde{M}_1 = \tilde{M}_2$ if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. We say that $\tilde{M}_1 < \tilde{M}_2$ if $\tilde{M}_1 \leq \tilde{M}_2$ and $\tilde{M}_1 \neq \tilde{M}_2$. So, the set of semi-infinite trapezoidal fuzzy sets is a partially ordered set. Similar ranking of fuzzy sets is considered by Ramík and Římánek [11]. If a semi-infinite trapezoidal fuzzy set $\tilde{M}_1$ is defined by a two-dimensional vector $c_1 = (\alpha_1, \beta_1)$ and a semi-infinite trapezoidal fuzzy set $\tilde{M}_2$ is defined by a two-dimensional vector $c_2 = (\alpha_2, \beta_2)$ we say that $c_1 \leq c_2$, $c_1 = c_2$, and $c_1 < c_2$ if and only if $\tilde{M}_1 \leq \tilde{M}_2$, $\tilde{M}_1 = \tilde{M}_2$, and $\tilde{M}_1 < \tilde{M}_2$, respectively.

Denote by $p_i(x)$ a probability of an outcome x under the state of nature θ_i. Assume that x takes values from a set that is at most countable. Obviously,

$$\sum_x p_i(x) = 1$$

for all i. Consider a risk function

$$\tilde{R}_i(\delta) = \sum_x p_i(x)\tilde{L}_i(\delta(x)),$$

where δ is a decision rule.

Denote by $c_i(\delta) = \left(a_i(\delta), b_i(\delta)\right)$ a two-dimensional vector that corresponds to the semi-infinite trapezoidal fuzzy set $\tilde{R}_i(\delta)$. Suppose that the losses are bounded below. This means that there is a real number K such that $a_i(\delta) > K$ for all i and δ. Without loss of generality, we can assume that $K > 0$. Consider two-dimensional vectors

$$\lambda(\delta) = \left(\max_{1\le i\le m} a_i(\delta), \max_{1\le i\le m} b_i(\delta)\right)$$

for all decision rules δ.

A decision rule δ_0 is said to be *minimax* if there is no decision rule δ such that $\lambda(\delta) < \lambda(\delta_0)$.

Denote by S a set of two-dimensional vectors $c = \left(a, b\right)$, where $a \le b$. Denote by U a set whose elements are $\left(u_1,...,u_m\right)$, where $u_i \in S$ for all i. Consider a set

$$V_c = \left\{\left(u_1,...,u_m\right) \in U : u_i \le c \textit{ for all } i\right\},$$

where $c \in S$. Denote by W a set of elements $\left(c_1(\delta),...,c_m(\delta)\right)$ for all δ. Consider a set

$$T = \left\{c \in S : W \cap V_c \ne \varnothing\right\}.$$

The elements of the set U can be considered as elements of $2m$-dimensional Euclidean space. By the convergence in the set U we understand the convergence in $2m$-dimensional Euclidean space. Any subset $Q \subseteq U$ can be considered as a subset of $2m$-dimensional Euclidean space. When we define the closure and the interior of the subset Q, we take the closure and the interior of the corresponding subset of $2m$-dimensional Euclidean space, respectively. We denote by $\bar{Q}$ the closure of Q.

We say that an element $\left(u_1',...,u_m'\right) \in U$ lies *directly underneath* an element $\left(u_1,...,u_m\right) \in U$ if $u_i' \le u_i$ for all i and the inequality is strict at least for one i.

The *lower boundary* of a set $Q \subseteq U$ is the set of elements $\left(u_1,...,u_m\right) \in \bar{Q}$ such that there are no elements $\left(u_1',...,u_m'\right)$ belonging to the interior of Q and lying directly underneath $\left(u_1,...,u_m\right)$.

We say that a set Q is *closed from below* if its lower boundary belongs to Q.

3. The main theorem

Theorem. If W is closed from below then a minimax decision rule δ_0 exists.

Proof. By $\|c\|$ denote the Euclidean norm for $c \in S$. Let

$$\rho = \inf_{c \in T} \|c\|$$

and

$$\|e_n\| \downarrow \rho \quad \text{as} \quad n \to \infty$$

for elements $e_n \in T$. Since the sequence e_n is bounded, it is possible to choose a convergent subsequence e_q from the sequence. Thus,

$$e_q \to e_0 \quad \text{as} \quad q \to \infty .$$

It is clear that $\|e_0\| = \rho$.

Taking a point from each set $W \cap V_{e_q}$, we obtain a bounded sequence $\left(c_1\left(\delta_q\right), \ldots, c_m\left(\delta_q\right)\right)$. It is possible to choose a convergent subsequence from this sequence. Let

$$\left(c_1\left(\delta_r\right), \ldots, c_m\left(\delta_r\right)\right) \to \left(c_1^*, \ldots, c_m^*\right) \text{ as } r \to \infty ,$$

where $c_i^* \in S$ for all i. From the definition of V_{e_r} we get

$$c_1\left(\delta_r\right) \le e_r, \ldots, c_m\left(\delta_r\right) \le e_r$$

for all r. Hence, $\lambda\left(\delta_r\right) \le e_r$ for all r. Obviously, $\left(c_1^*, \ldots, c_m^*\right) \in \overline{W} \cap V_{e_0}$.

It follows from the condition $\|e_0\| = \rho$ that a point from the interior of W can not belong to V_{e_0}. But a point that does not belong to V_{e_0} can not lie directly underneath a point from V_{e_0}. Consequently, the point $\left(c_1^*, \ldots, c_m^*\right)$ belongs to the lower boundary of the set W. Since W is closed from below, there is a decision rule δ_0 such that

$$\left(c_1^*, \ldots, c_m^*\right) = \left(c_1\left(\delta_0\right), \ldots, c_m\left(\delta_0\right)\right).$$

Clearly, $\lambda\left(\delta_0\right) \le e_0$. It follows from the definition of ρ that there is no decision rule δ for which $\lambda(\delta) < e_0$.

The theorem is proved.

4. Conclusion

When losses are not known exactly, it is reasonable to consider fuzzy-valued losses in statistical decision problems. In this paper, the theorem about the existence of minimax decision rules for the problems with trapezoidal fuzzy losses is proved. Remaining questions concern the existence of minimax decision rules for the problems with non-trapezoidal fuzzy losses.

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