Computational Models of Argument
F. Toni et al. (Eds.)

doi:10.3233/FAIA220150

Composite Argumentation Systems with ML Components

Nguyen Duy HUNG [a,1], Nam-Van HUYNH [b] and Thanaruk THEERAMUNKONG [a]
Tho-Quy NHU [c]

[a] *Sirindhorn International Institute of Technology, Thailand*
[b] *Japan Advanced Institute of Science and Technology, Japan*
[c] *Hanoi University of Industry, Vietnam*

Abstract. Today AI systems are rarely made without Machine Learning (ML) and this inspires us to explore what aptly called composite argumentation systems with ML components. Concretely, against two theoretical backdrops of PABA (Probabilistic Assumption-based Argumentation) and DST (Dempster-Shafer Theory), we present a framework for such systems called c-PABA. It is argued that c-PABA lends itself to a development tool as well and to demonstrate we show that DST-based ML classifier combination and multi-source data fusion can be implemented as simple c-PABA frameworks.

1. Introduction

Today AI systems are rarely made without Machine Learning (ML) though one may criticize the overuse of ML especially in tasks demanding explainability. On the other hand, Argumentation is widely viewed as an inherently explainable AI formalism, however practical argumentation systems are still hard to develop. This inspires us to explore a synthesis of ML and Argumentation that fosters AI systems with the elements of both formalisms. Concretely, against two theoretical backdrops of PABA (Probabilistic Assumption-based Argumentation [5]) and DST (Dempster-Shafer Theory), we present a framework for composite argumentation systems with ML components called c-PABA. It is argued that c-PABA lends itself to a development tool for these systems as well and to demonstrate we show that DST-based ML classifier combination and multi-source data fusion can be implemented as simple c-PABA frameworks. The rationale behind our selection of two theoretical backdrops (DST and PABA) is as follows. DST, which alone is often described as a generalisation of the probability theory, has long been proven very suitable for representing knowledge under uncertainty and ignorance. Predictions of ML models belong to knowledge of this kind, and hence ML models can be viewed as sources generating DST data. Since a composite argumentation system in our view can contain many components some of which may be ML models, DST lends itself to an appropriate model for the information exchanged between these components. To model the workings of these components as well as the whole composite argumentation system,

[1] Corresponding author; E-mail: hung.nd.siit@gmail.com

we choose PABA rather than an abstract PA model such as [10] because of two reasons: a) PABA deals the material out of which arguments are constructed, and hence allows us to go down to the level of DST data exchanged between components; b) PABA reasoning engines for precise results [7] as well as approximate any-time results [9] are recently available. Such engines can run c-PABA frameworks (since as will be seen, c-PABA can be translated back to PABA), and hence one can use c-PABA as a design as well as development tool for the above described kind of composite argumentation systems. The remaining of this paper is structured as follows. We recall DST and PABA in Section 2. Then we develop three complementary techniques respectively for: translating DST data to PABA (Section 3), generating DST data from PABA, and fusing DST data with existing PABA frameworks (Section 4). We then accumulate these techniques to present the c-PABA framework (Section 5). Due to the lack of space, the proofs of theorems and lemmas are moved to an on-line appendix[2].

2. Background

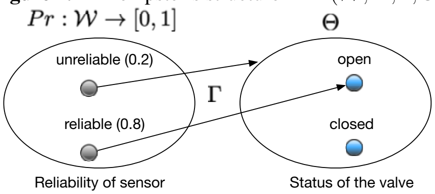
2.1. Dempster Shafer Theory (DST [3,13])

Definition 1. *A* ***Demspter's structure*** *is a tuple $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$ where Θ is an exhaustive set of mutually exclusive answers for some question (Frame of Discernment or FoD for short); $\mathcal{W}$ is a finite set of possible worlds; $Pr : \mathcal{W} \rightarrow [0,1]$ is a probability distribution; $\Gamma : \mathcal{W} \rightarrow 2^{\Theta}$ is a multi-valued mapping from $\mathcal{W}$ into Θ. For $X \subseteq \Theta$,* ***the degree of belief*** *and* ***the degree of plausibility in*** *X are defined as follows.*

$$Bel_{\mathcal{D}}(X) \triangleq \sum_{\omega \in \mathcal{W}: \emptyset \neq \Gamma(\omega) \subseteq X} Pr(\omega) \quad \text{and} \quad Pl_{\mathcal{D}}(X) \triangleq \sum_{\omega \in \mathcal{W}: \emptyset \neq \Gamma(\omega) \cap X} Pr(\omega)$$

Intuitively Γ says that if ω is the actual world, then the answer is in $\Gamma(\omega)$. The interval $[Bel_{\mathcal{D}}(X), Pl_{\mathcal{D}}(X)]$ delineates the probability that the answer is in X. For example, suppose that a sensor which is unreliable in 20% of the times, is installed to check a valve's status. If the sensor indicates "valve open", the best conclusion one can make is $0.8 \leq Prob(open) \leq 1$ because if the sensor is unreliable, one has no information about the valve's status. Hence one does not want to represent the observation by a standard probability distribution over space $\Theta = \{open, closed\}$ but by a Dempster's structure depicted in Fig. 1 with: $\Theta = \{open, closed\}$, $\Gamma = \{reliable \mapsto \{open\}, unreliable \mapsto \Theta\}$, and $\mathcal{W} = \{reliable, unreliable\}$ with two possible worlds having probabilities 0.8 and 0.2. Clearly $[Bel_{\mathcal{D}}(\{open\}), Pl_{\mathcal{D}}(\{open\})] = [0.8, 1]$.

Figure 1. A Dempster's structure $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$



[2] http://ict.siit.tu.ac.th/~hung/comma22-proofs.pdf

Definition 2. *A* ***mass function over FoD*** Θ *is a function* $m : 2^{\Theta} \rightarrow [0,1]$ *such that* $\sum_{X \subseteq \Theta} m(X) = 1$. *For* $X \subseteq \Theta$, $Bel_m(X) \triangleq \sum_{Y \subseteq \Theta : \emptyset \neq Y \subseteq X} m(Y)$ *and* $Pl_m(X) \triangleq \sum_{Y \subseteq \Theta : \emptyset \neq X \cap Y} m(Y)$.

And here are some more definitions. $X \subseteq \Theta$ is said to be a **focal** element of $m : 2^{\Theta} \rightarrow [0,1]$ iff $m(X) \neq 0$. The set of all focal elements of m is denoted by $focal(m)$. The set $\{(X_i, \mu_i)\}_{i=1}^{|focal(m)|}$ where $X_i \in focal(m)$ and $\mu_i = m(X_i)$ is called the **focal specification** of m. For a Demspter's structure $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$, $m_{\mathcal{D}}$ denotes the mass function: $m_{\mathcal{D}}(X) = \sum_{\omega \in \Theta : \Gamma(\omega) = X} Pr(\omega)$. Clearly $Bel_{\mathcal{D}}(X) = Bel_{m_{\mathcal{D}}}(X)$ and $Pl_{\mathcal{D}}(X) = Pl_{m_{\mathcal{D}}}(X)$.

Example 1. *Consider* $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$ *where* $\Theta = \{\theta_1, \theta_2, \theta_3\}$*;* $(\mathcal{W}, Pr)$ *is generated from two independent events* α_1, α_2 *with* $Pr(\alpha_1) = 0.4$ *and* $Pr(\alpha_2) = 0.7$*; and* Γ *is shown in the table below. The focal specification of* $m_{\mathcal{D}}$ *is* $\{(\emptyset, 0.28), (\{\theta_2\}, 0.12), (\{\theta_2, \theta_3\}, 0.6)\}$.

$\mathcal{W}$	$Pr(\omega_i)$	$\Gamma(\omega_i)$
$\omega_1 = \{\alpha_1, \alpha_2\}$	$0.4 \times 0.7 = 0.28$	$\{\}$
$\omega_2 = \{\alpha_1, \neg\alpha_2\}$	$0.4 \times 0.3 = 0.12$	$\{\theta_2\}$
$\omega_3 = \{\neg\alpha_1, \alpha_2\}$	$0.6 \times 0.7 = 0.42$	$\{\theta_2, \theta_3\}$
$\omega_4 = \{\neg\alpha_1, \neg\alpha_2\}$	$0.6 \times 0.3 = 0.18$	$\{\theta_2, \theta_3\}$

In general mass functions can represent real-world data directly. Moreover those over the same FoD can be combined using various combination rules. Due to the lack of space, we focus on only the Smet's rule.

Definition 3. *Given two mass functions* m_1, m_2 *over the same FoD* Θ*,* ***Smet's rule*** *returns a combined mass function* $m_1 \otimes m_2(X) \triangleq \sum_{B \cap C = X} m_1(B) m_2(C)$, $\forall X \subseteq \Theta$.

For a set $\mathcal{M}$ of mass functions over the same FoD (aka *a DST evidence base*), any order of applying $\otimes$ yields the same result, denoted $\bigotimes \mathcal{M}$. Hence the Smet's rule derives two functions $Bel_{\mathcal{M}}(X) \triangleq Bel_{\bigotimes \mathcal{M}}(X)$ and $Pl_{\mathcal{M}}(X) \triangleq Pl_{\bigotimes \mathcal{M}}(X)$, which together define the Smet's semantics for $\mathcal{M}$.

2.2. Argumentation frameworks

An **Abstract Argumentation** (AA [4]) framework a pair $(\mathcal{A}rg, \mathcal{A}tt)$ of a set $\mathcal{A}rg$ of arguments and an attack relation $\mathcal{A}tt \subseteq \mathcal{A}rg \times \mathcal{A}rg$. An argument $A \in \mathcal{A}rg$ is *acceptable wrt to* $S \subseteq \mathcal{A}rg$ iff S attacks every argument attacking A. S is *admissible* iff S does not attack itself (aka conflict-free) and each argument in S is acceptable wrt S; a *preferred* extension iff S is a maximal (wrt $\subseteq$) admissible set. $A \in \mathcal{A}rg$ is *credulously* (resp. *skeptically*) acceptable if it is acceptable wrt a preferred extension (resp. any preferred extension).

Assuming a logical language $\mathcal{L}$, an **Assumption-based Argumentation** (ABA [1]) framework is a tuple $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \overline{})$ where: $\mathcal{R}$ is a set of inference rules of the form $l_0 \leftarrow l_1, \ldots, l_n$ ($n \geq 0$, $l_i \in \mathcal{L}$); $\mathcal{A} \subseteq \mathcal{L}$ is a set of assumptions; $\overline{} : \mathcal{A} \rightarrow \mathcal{L}$ maps each assumption to its contrary. In this paper we restrict ourselves to *flat* ABA frameworks where assumptions do not appear in the heads of inference rules. An argument (Q, π) for $\pi \in \mathcal{L}$ supported by a set of assumptions $Q \subseteq \mathcal{A}$ is a backward deduction from π to Q. An argument (Q, π) attacks an argument (Q', π') if $\pi = \overline{a}$ for some $a \in Q'$. A proposition π is said to be credulously/skeptically acceptable, denoted $\mathcal{F} \vdash_{cr} \pi$ (resp. $\mathcal{F} \vdash_{sk} \pi$) if in the

AA framework consisting of above defined arguments and attacks, there is a credulously (skeptically) acceptable argument (Q,π). For short we may specify an ABA framework $(\mathcal{R},\mathcal{A},\overline{})$ by just a pair $(\mathcal{R},\mathcal{A})$ and for each assumption $a \in \mathcal{A}$, we write $\overline{a}$ in inference rules of $\mathcal{R}$ as if $\overline{a}$ were a "legal" sentence (which of course refers to the one returned by the omitted contrary function $\overline{}$). Inference rules of $\mathcal{R}$ with the same head are grouped together by connecting their bodies with symbol | as demonstrated by the example below.

Example 2. *A flat ABA framework describing $\Theta = \{\theta_1,\theta_2,\theta_3\}$ as a set of exhaustive and mutually exclusive propositions is $\mathcal{F} = (\mathcal{A},\mathcal{R})$ with $\mathcal{A} = \{\theta_1,\theta_2,\theta_3,\{\theta_1,\theta_2\},\{\theta_1,\theta_3\},\{\theta_2,\theta_3\},$ $\{\theta_1,\theta_2,\theta_3\}\}$ saying that one can assume any proposition $\theta_i \in \Theta$ to be true. Consequently one can also assume any disjunction of these propositions. Note that these disjunctions are written in the set-based clausal form (e.g. $\{\theta_1,\theta_2\}$ means $\theta_1 \vee \theta_2$) so that in the general case (as in Def. 6) we can simply write $\mathcal{A} = 2^{\Theta} \setminus \{\emptyset\}$. $\mathcal{R}$ consists of two groups of rules:*

- $\overline{\theta_1} \leftarrow \neg\theta_1 \mid \theta_2 \mid \theta_3.$ $\quad \overline{\theta_2} \leftarrow \theta_1 \mid \neg\theta_2 \mid \theta_3.$ $\quad \overline{\theta_3} \leftarrow \theta_1 \mid \theta_2 \mid \neg\theta_3.$ *saying that each assumption $\theta_i \in \Theta$ can be disproved by either proving its classical negation $\neg\theta_i$ or proving any mutually exclusive assumption θ_j, $j \neq i$.*
- $\overline{\{\theta_1,\theta_2\}} \leftarrow \overline{\theta_1},\overline{\theta_2}.$ $\quad \overline{\{\theta_1,\theta_3\}} \leftarrow \overline{\theta_1},\overline{\theta_3}.$ $\quad \overline{\{\theta_2,\theta_3\}} \leftarrow \overline{\theta_2},\overline{\theta_3}.$ $\quad \overline{\{\theta_1,\theta_2,\theta_3\}} \leftarrow \overline{\theta_1},\overline{\theta_2},\overline{\theta_3}.$ *saying that a disjunction of several assumptions is disproved by proving each and every contraries of the assumptions.*

Some arguments of $\mathcal{F}$ are $(\{\theta_i\},\{\theta_1,\theta_2,\theta_3\})$ and $(\{\theta_i\},\theta_i)$ with $i \in \{1,2,3\}$. Clearly $\mathcal{F} \vdash_{sk} \{\theta_1,\theta_2,\theta_3\}$ (saying that some element of Θ holds certainly) and $\mathcal{F} \vdash_{cr} \theta_i$ but $\mathcal{F} \not\vdash_{sk} \theta_i$ (saying that it is probable but not certain that θ_i holds).

2.3. PABA

A PABA [5] framework can be seen as a probability distribution of ABA frameworks. In this paper, we focus on a class of PABA frameworks, called Bayesian.

Definition 4. *A (Bayesian) PABA framework is a triple $\mathcal{P} = (\mathcal{V},Pr,\mathcal{F})$ where $\mathcal{F} = (\mathcal{R},\mathcal{A})$ is an ABA framework, and*

1. *$\mathcal{V}$ is a finite set of so-called **probabilistic assumptions** such that no elements of $\mathcal{V} \cup \neg\mathcal{V}$[3] occurs in $\mathcal{A}$ or in the head of a rule in $\mathcal{R}$.*
2. *Pr is a probability distribution over the set of all possible worlds, where a* possible world *is a maximal (wrt set inclusion)* consistent[4] *subset of $\mathcal{V} \cup \neg\mathcal{V}$.*

Definition 5. *Given $\mathcal{P} = (\mathcal{V},Pr,\mathcal{F})$, the **acceptability probability** of a proposition π under semantics s is $Prob(\mathcal{P} \vdash_s \pi) \triangleq \sum_{\omega \in \mathcal{W}: \mathcal{F}_\omega \vdash_s \pi} Pr(\omega)$, where $\mathcal{W}$ is the set of all possible worlds and $\mathcal{F}_\omega$ is the ABA framework obtained from $\mathcal{F}$ by adding facts $\{\alpha \leftarrow \mid \alpha \in \omega\}$.*

Note that the above definitions leave it open the representation of *Pr* (hence they do not demand any probabilistic relationships between probabilistic assumptions). However for convenience we shall specify *Pr* by a Problog [6] program, using especially *probabilistic facts* and *annotated disjunctions* as done in [9]. Note that a probabilistic

[3] $\neg\mathcal{V} = \{\neg\alpha \mid \alpha \in \mathcal{V}\}$

[4] No α and $\neg\alpha$ co-exist in the set.

fact of Prolog is a sentence of the form "$p :: x$." where $p \in [0,1]$ and x is a proposition saying that x holds with probability p. An annotated disjunction is of the form "$p_1 :: x_1; \ldots; p_i :: x_i; \ldots; p_n :: x_n$." saying that propositions $x_1, \ldots, x_i, \ldots x_n$ are mutually exclusive and hold with respective probabilities $p_1, \ldots, p_i, \ldots p_n$ whose sum must equal 1. Let's have an example for illustration.

Example 3. *Consider PABA framework $\mathcal{P} = (\mathcal{V}, Pr, \mathcal{F})$ where $\mathcal{V} = \{\alpha_1, \alpha_2\}$; Pr is Problog program $\{0.4 :: \alpha_1.\ \ 0.7 :: \alpha_2.\}$ with two probabilistic facts (saying that α_1, α_2 hold with probabilities 0.4 and 0.7 respectively); $\mathcal{F}$ is obtained from the ABA framework in Example 2 by adding the following rules:*

$$\neg\theta_1 \leftarrow \alpha_1, \alpha_2. \quad \neg\theta_1 \leftarrow \alpha_1, \neg\alpha_2. \quad \neg\theta_1 \leftarrow \neg\alpha_1, \alpha_2.$$
$$\neg\theta_2 \leftarrow \alpha_1, \alpha_2. \quad \neg\theta_3 \leftarrow \alpha_1, \neg\alpha_2. \quad \neg\theta_1 \leftarrow \neg\alpha_1, \neg\alpha_2$$
$$\neg\theta_3 \leftarrow \alpha_1, \alpha_2.$$

From the acceptabilities of θ_2 in different possible worlds shown in the table below, we have $Prob(\mathcal{P} \vdash_{sk} \theta_2) = 0.12$ but $Prob(\mathcal{P} \vdash_{cr} \theta_2) = 0.12 + 0.42 + 0.18$.

ω	$\mathcal{F}_\omega \vdash_{cr} \theta_2$?	$\mathcal{F}_\omega \vdash_{sk} \theta_2$?	$Pr(\omega)$
$\omega_1 = \{\alpha_1, \alpha_2\}$	*No*	*No*	0.28
$\omega_2 = \{\alpha_1, \neg\alpha_2\}$	*Yes*	*Yes*	0.12
$\omega_3 = \{\neg\alpha_1, \alpha_2\}$	*Yes*	*No*	0.42
$\omega_4 = \{\neg\alpha_1, \neg\alpha_2\}$	*Yes*	*No*	0.18

3. Translating DST data into PABA

In this section we show that any DST data, be it a Dempster's structure, a DST mass function or a DST evidence base, can always be translated into a PABA framework. Let's start by translating FoDs - the basic component of all forms of DST data, to ABA frameworks. As suggested by Example 2, each possible answer θ of the given FoD Θ is represented by an assumption the contrary of which can be proven by either proving either the classical negation $\neg\theta$, or by assuming an alternative answer θ' from Θ.

Definition 6. *For a FoD $\Theta = \{\theta_1, \ldots, \theta_i, \ldots, \theta_k\}$, **the canonical ABA translation of** Θ, denoted $\mathcal{FD}_\Theta$, is the ABA framework $(\mathcal{A}_\Theta, \mathcal{R}_\Theta)$ where*

1. *$\mathcal{A}_\Theta = 2^\Theta - \{\emptyset\}$ saying that any non-empty subset of Θ may contain the actual answer (For simplicity, a singleton set $\{\theta\} \in \mathcal{A}_\Theta$ is written as θ[5]).*
2. *$\mathcal{R}_\Theta$ is the minimal set such that*
   - (a) *For each $\theta_i \in \Theta$, $\mathcal{R}_\Theta$ contains $\overline{\theta_i} \leftarrow \theta_1 \mid \cdots \mid \theta_{i-1} \mid \neg\theta_i \mid \theta_{i+1} \mid \cdots \mid \theta_k$. saying that θ_i can be disproved by proving its classical negation $\neg\theta_i$ or taking an alternative assumption θ_j, $j \neq i$.*
   - (b) *For each subset $X \in \mathcal{A}_\Theta$ where $|X| \geq 2$, $\mathcal{R}_\Theta$ contains a rule with head $\overline{X}$ and body $\{\overline{\theta} \mid \theta \in X\}$ saying that the answer is not in X if every $\theta \in X$ is disproved.*

Example 4. *For FoD $\Theta = \{\theta_1, \theta_2, \theta_3\}$, $\mathcal{FD}_\Theta$ coincides with the ABA framework given in Example 2.*

[5]So Θ is a subset as well as an element of $\mathcal{A}_\Theta$.

The following lemma asserts that $\mathcal{FD}_\Theta$ and FoD Θ are "semantically equivalent"

Lemma 1. *Let $\Theta = \{\theta_1, \ldots, \theta_k\}$ be a FoD. For any $X \in 2^\Theta$,*

1. *If $X \neq \Theta, \emptyset$, then $\mathcal{FD}_\Theta \vdash_{cr} X$ but $\mathcal{FD}_\Theta \not\vdash_{sk} X$ (representing that X possibly but not surely contains the answer).*
2. *If $X = \Theta$, then $\mathcal{FD}_\Theta \vdash_{sk} X$ (representing that Θ surely contains the answer); If $X = \emptyset$, then $\mathcal{FD}_\Theta \not\vdash_s X$ for any semantics s.*

Note that there are different ABA frameworks with the same semantics as $\mathcal{FD}_\Theta$, for example the one obtained from $\mathcal{FD}_\Theta$ by adding a rule $\neg\theta_i \leftarrow false$. However $\mathcal{FD}_\Theta$ is clearly the most obvious. Likewise, a Dempster's structure $\mathcal{D}$ can be translated to PABA in many ways but the PABA framework $PABA_\mathcal{D}$ defined below is the most obvious.

Definition 7. *Let $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$ be a Demspter's structure.* ***The canonical PABA translation of*** *$\mathcal{D}$ is the PABA framework $PABA_\mathcal{D} = (\mathcal{V}, Pr, \mathcal{F})$ where*

1. *$PABA_\mathcal{D}$ and $\mathcal{D}$ have the same set of possible worlds $\mathcal{W}$ and probability distribution Pr, and*
2. *$\mathcal{F} = (\mathcal{A}_\Theta, \mathcal{R}_\Theta \cup \mathcal{R}_\Gamma)$ is the ABA framework obtained from ABA framework $\mathcal{FD}_\Theta = (\mathcal{A}_\Theta, \mathcal{R}_\Theta)$ by adding a set of rules $\mathcal{R}_\Gamma = \bigcup_{\omega \in \mathcal{W}} \{\neg\theta \leftarrow \omega \mid \theta \in \Theta - \Gamma(\omega)\}$.*

Here $\mathcal{R}_\Gamma$ represents the multi-valued function $\Gamma : \mathcal{W} \rightarrow 2^\Theta$. Recall that for a possible world ω, Γ says that the answer must be in $\Gamma(\omega)$ or equivalently must not be some $\theta \in \Theta - \Gamma(\omega)$. Hence $\neg\theta \leftarrow \omega$ occurs as an inference rule in $\mathcal{R}_\Gamma$.

Example 5. *For the Demspter's structure $\mathcal{D}$ in Example 1, $PABA_\mathcal{D}$ coincides with the PABA framework in Example 3.*

The theorem below asserts that Demspter's structure $\mathcal{D}$ is semantically equivalent to the PABA framework $PABA_\mathcal{D}$.

Theorem 1. *Let $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$ be a Demspter's structure. Then for any $X \in 2^\Theta$, $Pl_\mathcal{D}(X) = Prob(PABA_\mathcal{D} \vdash_{cr} X)$ and $Bel_\mathcal{D}(X) = Prob(PABA_\mathcal{D} \vdash_{sk} X)$.*

Now let's switch our attention to the remaining forms of DST data - mass functions and evidence bases. Recall that any mass function, say m, can be specified by a set of ordered pairs $\{(X_i, \mu_i)\}_{i=1}^{|focal(m)|}$ with X_i being a focal element and $\mu_i \in [0,1]$ being the mass of X_i. Obviously m can be translated to PABA in many ways, and the so-called *canonical PABA translation $PABA_m$* defined below uses a set of probabilistic assumptions $\{\phi_i^m \mid i \in \{1, \ldots, |focal(m)|\}\}$ where the probability of ϕ_i^m is set to μ_i. To represent the mutually exclusiveness of focal elements, $PABA_m$ uses a Problog annotated disjunction "$\mu_1 :: \phi_1^m; \mu_2 :: \phi_2^m; \ldots; \mu_{|focal(m)|} :: \phi_{|focal(m)|}^m$.". Finally the content of each focal element X_i is represented in $PABA_m$ by a set of rules $\{\neg\theta \leftarrow \phi_i^m \mid \theta \in \Theta - X_i\}$, where $\neg\theta \leftarrow \phi_i^m$ says that if the i^{th} focal element of m occurs then any $\theta \in \Theta - X_i$ cannot be the answer.

Definition 8. *Let $m = \{(X_i, \mu_i)\}_{i=1}^{|focal(m)|}$ be a mass function with FoD Θ.* ***The canonical PABA translation of*** *m, denoted $PABA_m$, is the PABA framework $(\mathcal{V}_m, Pr_m, \mathcal{F}_m)$ where*

1. *$\mathcal{V}_m = \{\phi_1^m, \ldots, \phi_{|focal(m)|}^m\}$ and Pr_m is a Problog program consisting of only one annotated disjunction: $\mu_1 :: \phi_1^m; \mu_2 :: \phi_2^m; \ldots; \mu_{|focal(m)|} :: \phi_{|focal(m)|}^m$.*

2. *$\mathcal{F}_m$ is the ABA framework obtained from the canonical ABA translation $\mathcal{FD}_\Theta$ of Θ by adding a set of rules* $\bigcup_{1 \le i \le |focal(m)|} \{\neg\theta \leftarrow \phi_i^m \mid \theta \in \Theta - X_i\}$.

The canonical PABA translation of a DST evidence base $\mathcal{M}$ (a set of mass functions over the same FoD Θ) defined below is simply the set union of the canonical PABA translations of individual mass functions.

Definition 9. ***The canonical PABA translation of a DST evidence base*** *$\mathcal{M}$ is the PABA framework $PABA_\mathcal{M} = (\mathcal{V}_\mathcal{M}, Pr_\mathcal{M}, \mathcal{F}_\mathcal{M})$ where $\mathcal{V}_\mathcal{M} = \bigcup_{m \in \mathcal{M}} \mathcal{V}_m$, $Pr_\mathcal{M} = \bigcup_{m \in \mathcal{M}} Pr_m$ and $\mathcal{F}_\mathcal{M} = \bigcup_{m \in \mathcal{M}} \mathcal{F}_m$ (where $\mathcal{V}_m$, Pr_m and $\mathcal{F}_m$ are defined in Def. 8).*

Theorem 2 asserts the semantic equivalence between m and the above defined canonical PABA translation of m.

Theorem 2. *Let m be a mass function over FoD Θ. Then for any $X \subseteq \Theta$, $Bel_m(X) = Prob(PABA_m \vdash_{sk} X)$ and $Pl_m(X) = Prob(PABA_m \vdash_{cr} X)$.*

More generally, for a DST evidence base $\mathcal{M}$, the Smet's semantics of $\mathcal{M}$ and the semantics of $PABA_\mathcal{M}$ coincide.

Theorem 3. *Let $\mathcal{M}$ be a DST evidence base over FoD Θ. Then for any $X \subseteq \Theta$, $Bel_\mathcal{M}(X) = Prob(PABA_\mathcal{M} \vdash_{sk} X)$ and $Pl_\mathcal{M}(X) = Prob(PABA_\mathcal{M} \vdash_{cr} X)$.*

4. Fusion and generation of DST data

In this section we present two techniques for: 1) fusing DST data with; and 2) generating DST data from existing PABA frameworks.

4.1. DST data fusion with PABA frameworks

Given a DST evidence base $\mathcal{M}$ and a PABA framework $\mathcal{P}$, the so-called *the s-union of $\mathcal{P}$ and $\mathcal{M}$* defined below is a structure obtained by taking set unions of the corresponding components of $\mathcal{P}$ and $PABA_\mathcal{M}$.

Definition 10. *Let $\mathcal{M}$ be a DST evidence base and $\mathcal{P} = (\mathcal{V}, Pr, \mathcal{F})$ be a PABA framework.* ***The s-union of $\mathcal{P}$ and $\mathcal{M}$****, denoted $\mathcal{P} \uplus \mathcal{M}$, is simply the triple $(\mathcal{V} \cup \mathcal{V}_\mathcal{M}, Pr \cup Pr_\mathcal{M}, \mathcal{F} \cup \mathcal{F}_\mathcal{M})$ obtained by taking set-unions of the corresponding components of $\mathcal{P}$ and the canonical PABA translation $PABA_\mathcal{M} = (\mathcal{V}_\mathcal{M}, Pr_\mathcal{M}, \mathcal{F}_\mathcal{M})$ of $\mathcal{M}$.*

Let's introduce a condition to ensure that $\mathcal{P} \uplus \mathcal{M}$ is a well-formed PABA framework.

Definition 11. *We say that a PABA framework $\mathcal{P} = (\mathcal{V}, Pr, \mathcal{F})$* ***syntactically complies*** *with a FoD Θ iff: 1) all assumptions and rules occurring $\mathcal{FD}_\Theta$ also occur in $\mathcal{F}$; and 2) for any $\theta \in \Theta$, $\neg\theta$ is not an assumption of $\mathcal{F}$.*

Lemma 2. *Let $\mathcal{M}$ be a DST evidence base over FoD Θ and $\mathcal{P}$ be a PABA framework that is syntactically complies with Θ. Then $\mathcal{P} \uplus \mathcal{M}$ is a well-formed PABA framework syntactically complying with Θ.*

Hence $\uplus$ can be viewed as a knowledge fusion operator. Though its applicability is limited (e.g. $\uplus$ cannot fuse two arbitrary PABA frameworks), $\uplus$ suffices for our purpose which is to fuse a DST evidence base $\mathcal{M}$ with an existing PABA framework $\mathcal{P}$. Now let's examine several properties of $\uplus$. Lemma 3 below says that $\uplus$ in fact encapsulates the translation technique from DST data to PABA presented in the previous section.

Lemma 3. *Let $\mathcal{M}$ be a DST evidence base over FoD Θ and $\mathcal{P} = (\emptyset, \emptyset, \mathcal{FD}_\Theta)$. Then $\mathcal{M} \uplus \mathcal{P}$ is exactly the canonical PABA translation of $\mathcal{M}$.*

As the underlying operation in $\uplus$ is set union, $\uplus$ inherits many desirable properties from $\cup$. For example, $\mathcal{P} \uplus \emptyset = \mathcal{P}$; and $\mathcal{P} \uplus \mathcal{M} = (\mathcal{P} \uplus \mathcal{M}_1) \uplus \mathcal{M}_2$ if $\mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}$.

4.2. DST data generation from PABA frameworks

Obviously a Dempster's structure generated from a PABA framework should share the same probability space with the framework.

Definition 12. *We say that a PABA framework $\mathcal{P} = (\mathcal{V}, Pr, \mathcal{F})$ **generates** a Dempster's structure $\mathcal{D} = (\mathcal{W}, Pr, \Gamma, \Theta)$, written $\mathcal{P} \xrightarrow{\Theta} \mathcal{D}$, if*

1. *$\mathcal{D}$ and $\mathcal{P}$ have the same set of possible worlds $\mathcal{W}$ and the same probability distribution $Pr : \mathcal{W} \to [0,1]$, and*
2. *for each $\omega \in \mathcal{W}$, $\Gamma(\omega) = \{\theta \in \Theta \mid \mathcal{F}_\omega \vdash_{cr} \theta\}$.*

That is, we add θ into $\Gamma(\omega)$ just in case there is a credulously accepted argument for θ in $\mathcal{F}_\omega$. Generated Dempster's structures can be converted into mass functions, and so:

Definition 13. *A PABA framework $\mathcal{P}$ **generates** a mass function m, written $\mathcal{P} \xrightarrow{\Theta} m$, if $\mathcal{P} \xrightarrow{\Theta} \mathcal{D}$ and m coincides $m_{\mathcal{D}}$.*

It is easy to see that:

Lemma 4. *Let $PABA_{\mathcal{M}}$ be the canonical PABA translation of DST evidence base $\mathcal{M}$ over FoD Θ. Then $PABA_{\mathcal{M}} \xrightarrow{\Theta} \bigotimes_S \mathcal{M}$ (hence for any $m \in \mathcal{M}$, $PABA_m \xrightarrow{\Theta} m$).*

It is worth noting that the generating PABA framework $\mathcal{P}$ does not have to satisfy any constraint with respect to the FoD Θ of the generated DST data. Concretely:

Lemma 5. *Let $\mathcal{P}$ be a PABA framework. For any FoD Θ, there is an unique Dempster's structure $\mathcal{D}$ (resp. mass function m) such that $\mathcal{P} \xrightarrow{\Theta} \mathcal{D}$ (resp. $\mathcal{P} \xrightarrow{\Theta} m$).*

Hence a PABA framework can generate DST data over any FoD. This flexibility, however, may lead to possible semantic differences between the generated DST data and the generating PABA framework. Let's study conditions for preventing such differences.

4.3. Relationships between generated DST data and generating PABA framework

Theorem 4 below says that the degree of plausibility wrt generated DST data and the credulous semantics of the generating PABA framework always coincide.

Theorem 4. *Suppose $\mathcal{P} \xrightarrow{\Theta} m$. Then for any $\theta \in \Theta$, $Pl_m(\theta) = Prob(\mathcal{P} \vdash_{cr} \theta)$.*

However in general $Bel_m(\theta)$ and $Prob(\mathcal{P} \vdash_{sk} \theta)$ may be different as illustrated by the following example.

Example 6. *Consider $\mathcal{P} = (\emptyset, \emptyset, \mathcal{F})$ with $\mathcal{F} = (\emptyset, \{a \leftarrow, b \leftarrow\})$. For $\Theta = \{a, b\}$, we have $\mathcal{P} \xrightarrow{\Theta} m = \{\{a, b\} \mapsto 1\}$. Hence $Bel_m(a) = Bel_m(b) = 0$. However $Prob(\mathcal{P} \vdash_{sk} a) = Prob(\mathcal{P} \vdash_{sk} b) = 1$.*

Now let's introduce a condition that ensures that $Bel_m(\theta) = Prob(\mathcal{P} \vdash_{sk} \theta)$.

Definition 14. *We say that a PABA framework $\mathcal{P}$ **semantically complies** with a FoD Θ if for each possible world ω and $\theta \in \Theta$, $\mathcal{F}_\omega \vdash_{sk} \theta$ iff $\{x \in \Theta \mid \mathcal{F}_\omega \vdash_{cr} x\} = \{\theta\}$.*

For example, it is easy to see that for a DST evidence base $\mathcal{M}$ over FoD Θ, the PABA canonical translation $PABA_{\mathcal{M}}$ of $\mathcal{M}$ always semantically complies with Θ.

Theorem 5 given below and the previous Theorem 4 say that semantic compliance is a sufficient condition for ensuring the semantic coincidence between generated DST data and its generating PABA framework.

Theorem 5. *Suppose $\mathcal{P} \xrightarrow{\Theta} m$. If $\mathcal{P}$ semantically complies with Θ then for any $\theta \in \Theta$, $Bel_m(\theta) = Prob(\mathcal{P} \vdash_{sk} \theta)$.*

One might ask whether syntactic compliance (defined in Def. 11) ensures semantic compliance. The following example shows that it does not.

Example 7. *Consider FoD $\Theta = \{\theta_1, \theta_2\}$ and $\mathcal{P} = (\emptyset, \emptyset, \mathcal{F})$ where $\mathcal{F}$ is the ABA framework obtained from $\mathcal{FD}_\Theta$ by adding rules $\{\neg\theta_1 \leftarrow .\ \neg\theta_2 \leftarrow a.\ \overline{a} \leftarrow b.\ \overline{b} \leftarrow a\}$ and assumptions a, b. Clearly $\mathcal{P}$ syntactically complies with Θ. However $\mathcal{P}$ does not semantically comply with Θ. To see this consider the possible world $\omega = \{\}$ (the only possible world of $\mathcal{P}$), clearly $\{x \in \Theta \mid \mathcal{F}_\omega \vdash_{cr} x\} = \{\theta_2\}$ but $\mathcal{F}_\omega \not\vdash_{sk} \theta_2$.*

However, syntactic compliance ensures a weaken version of semantic compliance defined as follows.

Definition 15. *We say that a PABA framework $\mathcal{P}$ **semantically semi-complies** with a FoD Θ if for each possible world ω and $\theta \in \Theta$, if $\mathcal{F}_\omega \vdash_{sk} \theta$ then $\{x \in \Theta \mid \mathcal{F}_\omega \vdash_{cr} x\} = \{\theta\}$.*

Basically $\mathcal{P}$ semantically semi-complies but not semantically complies with Θ if for some possible world ω and answer $\theta \in \Theta$, $\{x \in \Theta \mid \mathcal{F}_\omega \vdash_{cr} x\} = \{\theta\}$ but $\mathcal{F}_\omega \not\vdash_{sk} \theta$. The PABA framework in Example 7 falls into this case.

Lemma 6. *If a PABA framework $\mathcal{P}$ syntactically complies with Θ, then $\mathcal{P}$ semantically semi-complies with Θ.*

Obviously semantic semi-compliance could not ensure that $Bel_m(\theta) = Prob(\mathcal{P} \vdash_{sk} \theta)$. However it ensures a half of this equality as follows.

Lemma 7. *Suppose $\mathcal{P} \xrightarrow{\Theta} m$. If $\mathcal{P}$ semantically semi-complies with Θ then $\forall \theta \in \Theta$, $Bel_m(\theta) \geq Prob(\mathcal{P} \vdash_{sk} \theta)$.*

So a corollary of the above lemmas is that if $\mathcal{P} \xrightarrow{\Theta} m$ and $\mathcal{P}$ syntactically complies with Θ, then for any $\theta \in \Theta$, $Bel_m(\theta) \geq Prob(\mathcal{P} \vdash_{sk} \theta)$ and $Pl_m(\theta) = Prob(\mathcal{P} \vdash_{cr} \theta)$.

5. Composite PABA frameworks

In this section, we accumulate three presented techniques to propose so-called c-PABA which lends itself to a development tool for composite argumentation systems.

5.1. Structure and Semantics

A c-PABA framework contains components of two kinds: data-consuming and data-generating. The latter provides DST data which is consumed by the former.

Definition 16. *A **composite PABA (c-PABA) framework** is a structure of the form $\mathcal{S} = (\{(\Theta_i, S_i)\}_{i=1}^{k}, \mathcal{P})$ where $\mathcal{P}$, which is referred to as the **data-consuming component** of $\mathcal{S}$, is a PABA framework; $\mathcal{P}_i$, which is referred to as a **data-generating component** of $\mathcal{S}$, is a c-PABA framework; and Θ_i is a FoD.*

*The set of all mass functions $\{m_i \mid i \in \{1,\ldots,k\}, S_i \xrightarrow{\Theta_i} m_i\}$ generated by all data-generating components is referred to as **the internal information flow** in $\mathcal{S}$.*

For example, a c-PABA framework $(\{\}, \mathcal{P})$, which shall be written as $\mathcal{P}$ for short, is just a PABA framework. In general, we want to see a c-PABA framework as the combination of its main module and its internal information flow. To ensure that this combination can be computed by the s-fusion operator and always results in a well-formed PABA framework, let's introduce a class of *well-formed* c-PABA frameworks.

Definition 17. *A c-PABA framework $\mathcal{S} = (\{(\Theta_i, S_i)\}_{i=1}^{k}, \mathcal{P})$ is said to be **well-formed** if for each $i, j \in \{1,\ldots,k\}$, $\mathcal{P}$ syntactically complies with FoD Θ_i and S_i semantically complies with Θ_i; further either $\Theta_i = \Theta_j$ or $\Theta_i \cap \Theta_j = \emptyset$ for any $j \in \{1,\ldots,k\}$.*

For example, c-PABA framework $\mathcal{S} = (\{(\Theta, PABA_{m_i})\}_{i=1}^{k}, (\emptyset, \emptyset, \mathcal{FD}_{\Theta}))$ representing a DST evidence base $\mathcal{M} = \{m_i\}_{i=1}^{k}$ over FoD Θ, is well-formed[6]. The following lemma follows directly from Lemma 2.

Lemma 8. *Let $\mathcal{S} = (\{(\Theta_i, S_i)\}_{i=1}^{k}, \mathcal{P})$ be a well-formed c-PABA framework with internal information flow $\mathcal{M}$, and $\{M_1, M_2, \ldots, M_n\}$ be the partition on $\mathcal{M}$ such that M_i is a DST evidence base[7]. Then $\mathcal{P} \uplus M_1 \uplus \cdots \uplus M_n$ is a well-formed PABA framework.*

The above PABA framework $\mathcal{P} \uplus M_1 \uplus \cdots \uplus M_n$ will be referred to as the PABA representation of the given c-PABA framework $\mathcal{S}$ and denoted by $PABA_{\mathcal{S}}$. The semantics of $\mathcal{S}$ is then defined by that of $PABA_{\mathcal{S}}$, concretely:

Definition 18. *Let $\mathcal{S} = (\{(\Theta_i, S_i)\}_{i=1}^{k}, \mathcal{P})$ be a well-formed c-PABA framework and π is a proposition. Define $Prob(\mathcal{S} \vdash_s \pi) \triangleq Prob(PABA_{\mathcal{S}} \vdash_s \pi)$.*

Of course we will say that $\mathcal{S}$ generates a mass function m over FoD Θ if $PABA_{\mathcal{S}} \xrightarrow{\Theta} m$; $\mathcal{S}$ semantically/syntactically complies with Θ if so does $PABA_{\mathcal{S}}$; and so on.

[6] It is easy to see that the internal information flow of $\mathcal{S}$ coincides with $\mathcal{M}$ because $PABA_{m_i} \xrightarrow{\Theta} m_i$

[7] That is, the mass functions in M_i share the same FoD.

5.2. Two sample applications: DST-based data fusion and ML classifier combination

DST-based data fusion can be implemented by a simple c-PABA framework as follows.

Lemma 9. *Let $\mathcal{M} = \{m_i\}_{i=1}^{k}$ be a DST evidence base over FoD Θ. For any $X \subseteq \Theta$, $Bel_{\mathcal{M}}(X) = Prob(\mathcal{S} \vdash_{sk} X)$ and $Pl_{\mathcal{M}}(X) = Prob(\mathcal{S} \vdash_{cr} X)$ where $\mathcal{S}$ is the c-PABA framework $(\{(\Theta, PABA_{m_i})\}_{i=1}^{k}, (\emptyset, \emptyset, \mathcal{FD}_\Theta))$.*

ML classifier combination can be implemented by c-PABA as well. Note that a classifier is an algorithm that assigns to each input pattern x a single class from a set of classes $\Theta = \{\theta_1, \ldots, \theta_{|\Theta|}\}$ - which can be viewed as a FoD. In practice, however a classifier built by ML often returns a vector $[s_1, \ldots, s_{|\Theta|}]$ where s_i indicates some kind of confidence degree that x belongs to class θ_i. It is a common practice to test a ML classifier against test datasets, computing various performance indexes such as recognition rate r, substitution rate s and rejection rate $q = 1 - r - s$. Such indexes are then used to interpret what the classifier actually says. For example, in [16] Xu et al argue that $[s_1, \ldots, s_{|\Theta|}]$ should be interpreted as such a mass function m that: if $[s_1, \ldots, s_{|\Theta|}] = [0, \ldots, 0]$, then $m = \{(\Theta, 1)\}$; if $[s_1, \ldots, s_{|\Theta|}] = [0, \ldots, s_i = 1, \ldots, 0]$, then $m = \{(\{\theta_i\}, r), (\Theta - \{\theta_i\}, s), (\Theta, q)\}$. Now as different classifiers potentially offer complementary information about patterns to be classified, one wants to combine the outputs of multiple classifiers for the classification problem at hand. This combination problem is formalized as follows: given classifiers, $f_1, \ldots, f_k$, return a combined classifier f^* that for a given input x, assigns a class $\theta^* \in \Theta$ to x, by: 1) for each output vector $f_i(x)$, construct a mass function m_i; and 2) combine $m_1, \ldots, m_k$ to obtain one mass function m^* which then derives θ^*. Clearly both steps allow choices. Suppose that step (1) uses Xu et al's rule to compute m_i; and step (2) uses Smet's rule to compute m^* then returns $\theta^* = argmax_{\theta \in \Theta} Pl_{m^*}(\theta)$. The lemma below says that f^* can be implemented in c-PABA.

Lemma 10. *Suppose f^* combines classifiers $f_1, \ldots, f_k$ (by using Xu et al's rule, Smet's rule to arrive at a combined mass function m^*) where f_i has recognition/substitution/rejection rates $f_i.r, f_i.s, f_i.q$. Then*

$$f^*(x) \triangleq argmax_{\theta \in \Theta} Pl_{m^*}(\theta) = argmax_{\theta \in \Theta} Prob(\mathcal{S} \vdash_{cr} \theta)$$

where $\mathcal{S} = (\{(\Theta, \mathcal{P}_i)\}_{i=1}^{k}, (\emptyset, \emptyset, \mathcal{FD}_\Theta))$, with $\mathcal{P}_i$ being any PABA framework that generates m_i - the mass function that Xu et al's rule derives from the output vector $f_i(x) = [s_{i1}, \ldots, s_{i|\Theta|}]$ of f_i.

For example, $\mathcal{P}_i$ may be $(\mathcal{V}, Pr, \mathcal{OI}_\Theta \cup \{op([s_{i1}, \ldots, s_{i|\Theta|}]) \leftarrow)\})$ with $\mathcal{V} = \{reg, sub, rej\}$, $Pr = \{f_i.r : reg; f_i.s :: sub; f_i.q :: rej.\}$ (saying that the probabilities of random variables reg, sub, rej coincide with the recognition/substitution/rejection rates of f_i), and $\mathcal{OI}_\Theta$ is the ABA framework obtained from $\mathcal{FD}_\Theta$ by adding the following rules for each vector $[0, \ldots, s_i = 1, \ldots, 0]$:

- $\neg\theta_j \leftarrow reg, op([0, \ldots, s_i = 1, \ldots, 0])$ where $j \neq i$ representing that θ_i is the right class with probability equal the recognition rate.
- $\neg\theta_i \leftarrow sub, op([0, \ldots, s_i = 1, \ldots, 0])$ representing that $\Theta - \{\theta_i\}$ contains the right class with probability equal the substitution rate.

Note that $op([s_{i1}, \ldots, s_{i|\Theta|}]) \leftarrow$ is a just a fact encoding the output vector of f_i.

6. Conclusion and related work

Against two theoretical backdrops: PABA and DST, we present a development tool called c-PABA for composite argumentation systems with ML components. Demonstratively we use c-PABA to implement two key applications of DST: multi source data fusion and multi-classifier combination. To the best of our knowledge, the only work in the current literature that involves both DST and PABA is [8] which uses PABA to re-construct DST, but does not deal with composite argumentation systems as in the current paper. However there is a rich line of work combining DST and logic-based reasoning (but not necessarily argumentative). For example, in [15,2,14] the authors combine DST with deductive reasoning. [11] associate probability mass with formula and compute measures-like belief degrees of the reasoning with these formula. The notion of arguments of this work, however, is limited to conjunctions of literals. In [12] the authors define argumentation semantics for subjective logic, a logic that incorporates measures from DST. It is argued that reasoning systems in these above reasoning formalisms can be viewed as components in composite argumentation systems that our c-PABA proposal captures.

References

[1] A. Bondarenko, P.M. Dung, R.A. Kowalski, and F. Toni. An abstract, argumentation-theoretic approach to default reasoning. *Artificial Intelligence*, 93(1):63 – 101, 1997.

[2] P. Chatalic, D. Dubois, and H. Prade. An approach to approximate reasoning based on the dempster rule of combination. *Int. J. Expert Syst.*, 1(1):67–85, September 1987.

[3] A. P. Dempster. Upper and Lower Probabilities Induced by a Multivalued Mapping. *The Annals of Mathematical Statistics*, 38(2):325 – 339, 1967.

[4] Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321 – 357, 1995.

[5] Phan Minh Dung and Phan Minh Thang. Towards (probabilistic) argumentation for jury-based dispute resolution. In *COMMA 2010*, pages 171–182, 2010.

[6] Daan Fierens, Guy Van Den Broeck, Joris Renkens, Dimitar Shterionov, Bernd Gutmann, Ingo Thon, Gerda Janssens, and Luc De Raedt. Inference and learning in probabilistic logic programs using weighted boolean formulas. *Theory and Practice of Logic Programming*, 15(3):358–401, 2015.

[7] Nguyen Duy Hung. Inference procedures and engine for probabilistic argumentation. *International Journal of Approximate Reasoning*, 90:163–191, 2017.

[8] Nguyen Duy Hung. Probabilistic assumption-based argumentation with dst evidence. In *IFSA-SCIS*, pages 1–6, 2017.

[9] Nguyen Duy Hung. Progressive inference algorithms for probabilistic argumentation. In *PRIMA 2018*, volume 11224, pages 371–386, 2018.

[10] Anthony Hunter. A probabilistic approach to modelling uncertain logical arguments. *International Journal of Approximate Reasoning*, 54(1):47 – 81, 2013.

[11] J. Kohlas, D. Berzati, and R. Haenni. Probabilistic argumentation systems and abduction. *Annals of Mathematics and Artificial Intelligence*, 34(1):177–195, 2002.

[12] Nir Oren, Timothy J. Norman, and Alun Preece. Subjective logic and arguing with evidence. *Artificial Intelligence*, 171(10):838 – 854, 2007.

[13] Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, 1976.

[14] Philippe Smets. Probability of deductibility and belief functions. In *ECSQARU*, pages 332–340, 1993.

[15] Yuqing Tang, Chung-Wei Hang, Simon Parsons, and Munindar P. Singh. Towards argumentation with symbolic dempster-shafer evidence. In *COMMA 2012*, pages 462–469.

[16] L. Xu, A. Krzyzak, and C.Y. Suen. Methods of combining multiple classifiers and their applications to handwriting recognition. *IEEE Transactions on Systems, Man, and Cybernetics*, 22(3):418–435, 1992.