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Preliminary Test Estimator in a Linear Regression Model

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Abstract. In this paper, we study the preliminary test method in a linear regression model. The preliminary test Liu-type estimator is introduced when it is suspected that the regression parameter may be constraint to a subspace. We also compare the preliminary test Liu-type estimator to the preliminary test estimator, preliminary test ridge estimator and preliminary test Liu estimator in the mean squared error sense.

Keywords. Linear restrictions, Preliminary test estimator, Preliminary test ridge estimator

1. Background Study

Let us discuss the linear regression model

$$y = X\beta + \varepsilon \tag{1}$$

where y defines an $n \times 1$ known vectors, X shows an $n \times p$ matrix with $rank(X) = p$, β defines a $p \times 1$ vectors of unknown parameters, $E(\varepsilon) = 0$ and $Cov(\varepsilon) = \sigma^2 I$.

We also consider the following linear restrictions for β

$$R\beta = r \tag{2}$$

where R shows a $q \times p$ known matrix and $rank(R) = q(q < p)$, r shows a $q \times 1$ known vector.

The restricted least squares estimator (RLSE) of β denotes as follows:

$$\hat{\beta}_{RLSE} = \hat{\beta}_{OLSE} - S^{-1}R'(RS^{-1}R')^{-1}(R\hat{\beta}_{OLSE} - r) \tag{3}$$

where $\hat{\beta}_{OLSE} = S^{-1}X'y$ and $S = X'X$.

When the prior information $R\beta = r$ is suspected, the statisticians have been combined the OLSE and RLSE to obtain a better performance of the estimators, which make the preliminary test least squares estimator (PTE) and defined as

$$\hat{\beta}_{PTE} = \hat{\beta}_{RLSE} I(F \leq F_\alpha) + \hat{\beta}_{OLSE} I(F > F_\alpha) \tag{4}$$

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which $I(A)$ defines the indicator of A, F shows the general test statistic for testing the null hypothesis H_0: $R\beta = r$ against H_1: $R\beta \neq r$ and

$$F = \frac{(R\hat{\beta}_{OLSE} - r)'(RS^{-1}R')^{-1}(R\hat{\beta}_{OLSE} - r)}{qS_\varepsilon^2} \tag{5}$$

where $S_\varepsilon^2 = (y - X\hat{\beta}_{OLSE})'(y - X\hat{\beta}_{OLSE})/(n-p)$ shows the unbiased estimator of σ^2. We can see that the test statistic F satisfy a central F -distribution with $(q, n-p)$ degrees of freedom under H_0 and F_α is the upper α -level critical value of ζ. If H_0 does not hold, then ζ follows a noncentral F -distribution with $(q, n-p)$ degrees of freedom noncentrality parameter $(1/2)\Delta$, where

$$\Delta = \frac{(R\beta - r)'(RS^{-1}R')^{-1}(R\beta - r)}{\sigma^2} \tag{6}$$

But if the multicollinearity exists in linear regression model, the statisticians have found that the OLSE is no longer a good estimator. To deal with multicollinearity, we can use biased estimator to do it, such as Stein estimator [1], ridge estimator [2], and Liu estimator [3], two-parameter estimator [4], almost unbiased two-parameter estimator [5] and Liu-type estimator [6].

Another method to deal with the multicollinearity is to use the linear restrictions. Sarkar [7] introduced the restricted ridge regression estimator (RRRE). Kaçıranlar et al. [8] proposed the restricted Liu estimator (RLE). Kibria [9] proposed PTRE and which is defined as

$$\hat{\beta}_{PTRE}(k) = W\hat{\beta}_{PTE} = W\left(\hat{\beta}_{RLSE}I(F \leq F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha)\right) \tag{7}$$

where $W = S_k^{-1}S$, $S_k = S + kI$ and $k > 0$. And Yuksel and Akdeniz [10]introduced the preliminary test LE (PTLE) and which is given as follows:

$$\hat{\beta}_{PTLE}(d) = F_d\hat{\beta}_{PTE} = F_d\left(\hat{\beta}_{RLSE}I(F \leq F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha)\right) \tag{8}$$

where $F_d = (S+I)^{-1}(S+dI)$ and $0 < d < 1$. Except these estimators, many authors have studied the preliminary test estimator, such as Billah and Saleh [11-12], Khan and Saleh [13], Kibria [14-15], Rao [16], Yang and Xu [17], Kibria and Saleh [18], Chang and Yang [19], Xu and Yang [20], Liu and Yang [21], Wu [22], Chang and Wu [23],Yüzbaşı et al. [24].

Our primary purpose of this paper is to consider the preliminary test Liu-type estimator (PTLTE) by combining Liu-type estimation approach and PTE in the linear regression model. The rest of this paper is organized as follows. We propose the preliminary test Liu-type estimator in Section 2 and the comparison results of these estimator are given in Section 3 and some conclusions are given in Section 4.

2. The New Estimator

The Liu-type estimator proposed by Xue and Liang [25] is given as follows:

$$\hat{\beta}(k,d) = F_{k,d}\hat{\beta} \tag{9}$$

where $F_{k,d} = (S+kI)^{-1}(S+dI)$ and $k > d, d > 0$.

Firstly, we introduced the following restricted LTE

$$\hat{\beta}_{RLTE}(k,d) = F_{k,d}\hat{\beta}_{RLSE} \tag{10}$$

Based on LTE and RLTE, we introduced a new estimator:

$$\hat{\beta}_{PTLTE}(k,d) = F_{k,d}\hat{\beta}_{PTE} = F_{k,d}\left(\hat{\beta}_{RLSE}I(F \le F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha)\right) \tag{11}$$

where $F_{k,d} = (S+kI)^{-1}(S+dI)$, $k > d > 0$. And we call this estimator as PTLTE. It is easy to know that

$$\hat{\beta}_{PTLTE}(0,0) = \hat{\beta}_{PTE} = \hat{\beta}_{RLSE}I(F \le F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha) \tag{12}$$

$$\hat{\beta}_{PTLTE}(k,0) = \hat{\beta}_{PTRE}(k) = W\left(\hat{\beta}_{RLSE}I(F \le F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha)\right) \tag{13}$$

$$\hat{\beta}_{PTLTE}(1,d) = \hat{\beta}_{PTLE}(d) = F_d\left(\hat{\beta}_{RLSE}I(F \le F_\alpha) + \hat{\beta}_{OLSE}I(F > F_\alpha)\right) \tag{14}$$

3. Performance of the New Estimator

In this section, we will compare the new estimator to the PTE, PTRE and PTLE in the MSE criterion. Firstly we compute the MSE of the new estimator.

Through (11), we can get that

$$E\left(\hat{\beta}_{PTLTE}(k,d)\right) = F_{k,d}\beta - F_{k,d}\eta G_{q+2,n-p}(l_1;\Delta) \tag{15}$$

$$Bias\left(\hat{\beta}_{PTLTE}(k,d)\right) = -B\beta - F_{k,d}\eta G_{q+2,n-p}(l_1;\Delta) \tag{16}$$

where $B = (k-d)S_k^{-1}$

$$\begin{aligned} MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) = {} & \sigma^2\left[tr(F_{k,d}S^{-1}F'_{k,d}) - tr(F_{k,d}AF'_{k,d})G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta' F'_{k,d}F_{k,d}\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\ & +2G_{q+2,n-p}(l_1;\Delta)\beta' B' F_{k,d}\eta + \beta' B' B\beta \end{aligned} \tag{17}$$

where $B = (k-d)S_k^{-1}$, $\eta = S^{-1}R'(RS^{-1}R')^{-1}(R\beta - r)$, $A = S^{-1}R'(RS^{-1}R')^{-1}RS^{-1}$, $l_1 = \frac{q}{q+2}F_{q,n-p}(\alpha)$, $l_2 = \frac{q}{q+4}F_{q,n-p}(\alpha)$, $G_{m,n}(\cdot;\Delta)$ shows the cumulative non-central F-distribution with (m,n) degrees of freedom and the non-central parameter $1/2\Delta$.

Thus we have

$$\begin{aligned} MSE\left(\hat{\beta}_{PTE}\right) = {} & \sigma^2\left[tr(S^{-1}) - tr(A)G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta'\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \end{aligned} \tag{18}$$

$$\begin{aligned} MSE\left(\hat{\beta}_{PTRE}(k)\right) = {} & \sigma^2\left[tr(WS^{-1}W') - tr(WAW')G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta' W' W\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \end{aligned}$$

$$+2G_{q+2,n-p}(l_1;\Delta)\beta' B_1' W\eta + \beta' B_1' B_1 \beta \tag{19}$$

$$\begin{aligned} MSE\left(\hat{\beta}_{PTLE}(d)\right) = \sigma^2 & \left[tr(F_d S^{-1} F_d') - tr(F_d A F_d') G_{q+2,n-p}(l_1;\Delta)\right] \\ & + \eta' F_d' F_d \eta \left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\ & + 2G_{q+2,n-p}(l_1;\Delta)\beta' B_2' F_d \eta + \beta' B_2' B_2 \beta \end{aligned} \tag{20}$$

where $B_1 = -kS_k^{-1}, B_2 = (d-1)(S+I)^{-1}$.

Now we suppose that Q is the orthogonal matrix such that $QSQ' = diag(\lambda_1, \cdots, \lambda_p)$ where $\lambda_1, \cdots, \lambda_p > 0$ define the ordered eigenvalues of S. Thus we obtain the following representations

$$tr\left(F_{k,d} S^{-1} F_{k,d}'\right) = \sum_{i=1}^{p} \frac{(\lambda_i + d)^2}{\lambda_i(\lambda_i + k)^2} \tag{21}$$

$$tr\left(F_{k,d} A F_{k,d}'\right) = \sum_{i=1}^{p} \frac{\tilde{a}_{ii}(\lambda_i + d)^2}{(\lambda_i + k)^2} \tag{22}$$

$$\eta' F_{k,d}' F_{k,d} \eta = \sum_{i=1}^{p} \frac{\tilde{\eta}_i^2(\lambda_i + d)^2}{(\lambda_i + k)^2} \tag{23}$$

$$\beta' B' F_{k,d} \eta = \sum_{i=1}^{p} \frac{(k-d)\tilde{\eta}_i \theta_i (\lambda_i + d)}{(\lambda_i + k)^2} \tag{24}$$

$$\beta' B' B \beta = \sum_{i=1}^{p} \frac{(d-k)^2 \theta_i^2}{(\lambda_i + k)^2} \tag{25}$$

where $\theta = Q'\beta = (\theta_1, \cdots, \theta_p)'$, $\tilde{\eta} = Q'\eta = (\tilde{\eta}_1, \cdots, \tilde{\eta}_p)'$, $\tilde{a}_{ii}$ denotes the ith diagonal element of the matrix $\tilde{A} = QAQ'$.

3.1. Comparison of the PTLTE and PTE

When the null hypothesis satisfy, we consider the following difference:

$$\begin{aligned} & MSE\left(\hat{\beta}_{PTE}\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) \\ = & \sum_{i=1}^{p} \frac{\sigma^2 (k-d)(2\lambda_i + k + d)}{\lambda_i(\lambda_i + k)^2} \left[1 - \tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0)\right] - \sum_{i=1}^{p} \frac{(d-k)^2 \theta_i^2}{(\lambda_i + k)^2} \end{aligned} \tag{26}$$

Since $\lambda_i > 0$ and $0 < \tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0) < 1$, thus when $d > max\left(0, \frac{(\theta_i^2 \lambda_i - \sigma^2}{\theta_i^2 \lambda_i + \sigma^2 + 2\lambda_i}\right)$, $MSE\left(\hat{\beta}_{PTE}\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$.

Under the alternative hypothesis, we consider the following difference:

$$\begin{aligned}
& MSE\left(\hat{\beta}_{PTE}\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) \\
= & \sigma^2\left[tr(S^{-1}) - tr(A)G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta'\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& -\{\sigma^2\left[tr(F_{k,d}S^{-1}F'_{k,d}) - tr(F_{k,d}AF'_{k,d})G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta'F'_{k,d}F_{k,d}\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta'B'F_{k,d}\eta + \beta'B'B\beta\} \\
= & \sigma^2\left[tr(S^{-1} - F_{k,d}S^{-1}F'_{k,d}) - tr(A - F_{k,d}AF'_{k,d})G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta'(I_p - F'_{k,d}F_{k,d})\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& -2G_{q+2,n-p}(l_1;\Delta)\beta'B'F_{k,d}\eta - \beta'B'B\beta \qquad (27)
\end{aligned}$$

Since $G_{q+2,n-p}(l_1;\Delta) > 0$ and $2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta) > 0$, then

$$MSE\left(\hat{\beta}_{PTE}\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$$

if and only

$$\eta'(I_p - F'_{k,d}F_{k,d})\eta \geq \frac{f_1(k,d,\alpha,\Delta)}{\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \qquad (28)$$

where

$$\begin{aligned}
f_1(k,d,\alpha,\Delta) = & \sigma^2\left[tr(A - F_{k,d}AF'_{k,d})G_{q+2,n-p}(l_1;\Delta) - tr(S^{-1} - F_{k,d}S^{-1}F'_{k,d})\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta'B'F_{k,d}\eta + \beta'B'B\beta \qquad (29)
\end{aligned}$$

By Anderson (1984), we have

$$\lambda_p\left((I_p - F'_{k,d}F_{k,d})S^{-1}\right) \leq \frac{\eta'(I_p - F'_{k,d}F_{k,d})\eta}{\eta'S\eta} \leq \lambda_1\left((I_p - F'_{k,d}F_{k,d})S^{-1}\right) \qquad (30)$$

So when $\Delta > \Delta_1$, then $MSE\left(\hat{\beta}_{PTE}\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$, where

$$\Delta_1 = \frac{f_1(k,d,\alpha,\Delta)}{\lambda_1\left((I_p - F'_{k,d}F_{k,d})S^{-1}\right)\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \qquad (31)$$

Summarize these, we may get the following theorems:

Theorem 3.1 When the null hypothesis satisfy, when $k > d > max\left(0, \frac{(\theta_i^2\lambda_i - \sigma^2}{\theta_i^2\lambda_i + \sigma^2 + 2\lambda_i}\right)$, the PTLTE is superior to the PTE in the MSE criterion.

Theorem 3.2 Under the alternative hypothesis, when $k > d > 0$, and $\Delta > \Delta_1$, then the PTLTE is superior the PTE in the MSE criterion.

3.2. Comparison of the PTLTE and PTRE

When the null hypothesis satisfy, we discuss the following difference:

$$MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right)$$

$$= \sum_{i=1}^{p} \frac{\sigma^2(-d)(2\lambda_i + d)}{\lambda_i(\lambda_i + k)^2} \left[1 - \tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0)\right] + \sum_{i=1}^{p} \frac{d(2k-d)\theta_i^2}{(\lambda_i + k)^2} \tag{32}$$

Though $\lambda_i > 0$ and $0 < \tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0) < 1$, then when $k > max\left(d, \frac{d(\theta_i^2\lambda_i+\sigma^2)+2\sigma^2\lambda_i}{2\theta_i^2\lambda_i}\right) > 0$, $MSE\left(\hat{\beta}_{PTRE}(k)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$.

Under the alternative hypothesis, we have

$$\begin{aligned}
& MSE\left(\hat{\beta}_{PTRE}(k)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) \\
= & \sigma^2\left[tr(WS^{-1}W') - tr(WAW')G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta' W'W\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta' B_1' W\eta + \beta' B_1' B_1\beta \\
& -\{\sigma^2\left[tr(F_{k,d}S^{-1}F_{k,d}') - tr(F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta' F_{k,d}' F_{k,d}\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta' B' F_{k,d}\eta + \beta' B' B\beta\} \\
= & \sigma^2\left[tr(WS^{-1}W' - F_{k,d}S^{-1}F_{k,d}') - tr(WAW' - F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\
& +\eta'(W'W - F_{k,d}'F_{k,d})\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta'(B_1'W - B'F_{k,d})\eta + \beta'(B_1'B_1 - B'B)\beta
\end{aligned} \tag{33}$$

Though $G_{q+2,n-p}(l_1;\Delta) > 0$ and $2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta) > 0$, then

$$MSE\left(\hat{\beta}_{PTRE}(k)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$$

if and only

$$\eta'(W'W - F_{k,d}'F_{k,d})\eta \geq \frac{f_2(k,d,\alpha,\Delta)}{\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \tag{34}$$

where

$$\begin{aligned}
& f_2(k,d,\alpha,\Delta) \\
= & \sigma^2\left[tr(WS^{-1}W' - F_{k,d}S^{-1}F_{k,d}') - tr(WAW' - F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\
& +2G_{q+2,n-p}(l_1;\Delta)\beta'(B_1'W - B'F_{k,d})\eta + \beta'(B_1'B_1 - B'B)\beta
\end{aligned} \tag{35}$$

Then using the theorem in Anderson [25], we can get

$$\lambda_p\left((W'W - F_{k,d}'F_{k,d})S^{-1}\right) \leq \frac{\eta'(W'W - F_{k,d}'F_{k,d})\eta}{\eta' S\eta} \leq \lambda_1\left((W'W - F_{k,d}'F_{k,d})S^{-1}\right) \tag{36}$$

So if $\Delta > \Delta_2$, then $MSE\left(\hat{\beta}_{PTRE}(k)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$, where

$$\Delta_2 = \frac{f_2(k,d,\alpha,\Delta)}{\lambda_1\left((W'W - F_{k,d}'F_{k,d})S^{-1}\right)\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \tag{37}$$

Then we have

Theorem 3.3 When the null hypothesis satisfy, when $k > max\left(d, \frac{d(\theta_i^2\lambda_i+\sigma^2)+2\sigma^2\lambda_i}{2\theta_i^2\lambda_i}\right) > 0$, the PTLTE is superior to the PTRE in the MSE critrion.

Theorem 3.4 Under the alternative hypothesis, when $k > d > 0$, and $\Delta > \Delta_2$, then the PTLTE is superior to the PTRE in the MSE criterion.

3.3. Comparison of the PTLTE and PTLE

When the null hypothesis satisfy, we discuss the following difference:

$$\begin{aligned} & MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) \\ = & \sum_{i=1}^{p} \frac{\sigma^2(k-1)(\lambda_i+d)^2(2\lambda_i+k+1)}{\lambda_i(\lambda_i+k)^2(\lambda_i+1)^2}\left[1-\tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0)\right] \\ & +\sum_{i=1}^{p}\frac{[(1+k-2d)\lambda_i+2k-d-dk][(1-k)\lambda_i+d-dk]}{(\lambda_i+k)^2(\lambda_i+1)^2}\theta_i^2 \end{aligned} \tag{38}$$

Though $\lambda_i > 0$ and $0 < \tilde{a}_{ii}\lambda_i G_{q+2,n-p}(l_1;0) < 1$, then when $k > d > \frac{1+2k+k\lambda_i}{2\lambda_i+1+k} > 0$, $MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$.

Under the alternative hypothesis, we can consider

$$\begin{aligned} & MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) \\ = & \sigma^2\left[tr(F_dS^{-1}F_d') - tr(WAW')G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta' F_d'F_d\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\ & +2G_{q+2,n-p}(l_1;\Delta)\beta' B_2' F_d\eta + \beta' B_2' B_2\beta \\ & -\{\sigma^2\left[tr(F_{k,d}S^{-1}F_{k,d}') - tr(F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta' F_{k,d}'F_{k,d}\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\ & +2G_{q+2,n-p}(l_1;\Delta)\beta' B' F_{k,d}\eta + \beta' B' B\beta\} \\ = & \sigma^2\left[tr(F_dS^{-1}F_d' - F_{k,d}S^{-1}F_{k,d}') - tr(F_dAF_d' - F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\ & +\eta'(F_d'F_d - F_{k,d}'F_{k,d})\eta\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right] \\ & +2G_{q+2,n-p}(l_1;\Delta)\beta'(B_2'F_d - B'F_{k,d})\eta + \beta'(B_2'B_2 - B'B)\beta \end{aligned} \tag{39}$$

Since $G_{q+2,n-p}(l_1;\Delta) > 0$ and $2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta) > 0$, then

$$MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$$

if and only

$$\eta'(F_d'F_d - F_{k,d}'F_{k,d})\eta \geq \frac{f_3(k,d,\alpha,\Delta)}{\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \tag{40}$$

where

$$\begin{aligned} & f_3(k,d,\alpha,\Delta) \\ = & \sigma^2\left[tr(F_dS^{-1}F_d' - F_{k,d}S^{-1}F_{k,d}') - tr(F_dAF_d' - F_{k,d}AF_{k,d}')G_{q+2,n-p}(l_1;\Delta)\right] \\ & +2G_{q+2,n-p}(l_1;\Delta)\beta'(B_2'F_d - B'F_{k,d})\eta + \beta'(B_2'B_2 - B'B)\beta \end{aligned} \tag{41}$$

Then using the theorem in Anderson [25], we can get

$$\lambda_p\left((F_d'F_d - F_{k,d}'F_{k,d})S^{-1}\right) \leq \frac{\eta'(F_d'F_d - F_{k,d}'F_{k,d})\eta}{\eta' S\eta} \leq \lambda_1\left((F_d'F_d - F_{k,d}'F_{k,d})S^{-1}\right) \quad (42)$$

So if $\Delta > \Delta_2$, then $MSE\left(\hat{\beta}_{PTLE}(d)\right) - MSE\left(\hat{\beta}_{PTLTE}(k,d)\right) > 0$, where

$$\Delta_3 = \frac{f_3(k,d,\alpha,\Delta)}{\lambda_1\left((W'W - F_{k,d}'F_{k,d})S^{-1}\right)\left[2G_{q+2,n-p}(l_1;\Delta) - G_{q+4,n-p}(l_2;\Delta)\right]} \quad (43)$$

Now we have

Theorem 3.5 When the null hypothesis satisfy, when $k > d > \frac{1+2k+k\lambda_i}{2\lambda_i+1+k} > 0$, the PTLTE is superior to the PTLE in the MSE criterion.

Theorem 3.6 Under the alternative hypothesis, when $k > d > 0$, and $\Delta > \Delta_3$, then the PTLTE is better than the PTLE in the MSE criterion.

4. Conclusion

In this paper we proposed a preliminary test Liu-type estimator in linear regression model with linear restrictions. We also show that under certain conditions the new estimator is superior to the PTE, PTRE and PTLE in the MSE sense.

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