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A Limit Theorem for Weighted Sums of Random Sets in Fuzzy Metric Space

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Abstract. Based on the definitions and properties of fuzzy metric for random sets. We considered the limit theory of weighted sums for random sets in the sense of fuzzy metric. The random sets are independent and compactly uniformly integrable, and the weights are more general constants. The convergence is in the sense of fuzzy metric induced by the Hausdorff metric.

Keywords. random sets, fuzzy metric, weighted sums

1. Introduction

For random sets, there are a lot of rich research on limit theory. For example, the strong laws of large numbers (SLLN) was firstly proved by [1], where they dealt with independent identically distributed compact random sets in finite-dimensional Euclidean space. Later many authors have obtained beautiful convergence results in different conditions, such as Puri and Ralescu in [2], Taylor and Inoue in [3], Adler, Rosalsky and Taylor in [4], Fu and Zhang in [5], Guan and Li in [6]. And the limit theory of random sets plays an important role in set-valued statistics inference.

Since 1965, Zadeh [7] introduce the fuzzy set theory. ”Fuzzy” have been a popular vocabulary. And the fuzzy theory have been applied in many fields, such as economics, mathematic finance, random cybernetics and so on. In practice, sometimes we will make decisions with uncertainty. For example, We want to judge which faces are similar among the many face photos. Since everyone’s judgment criteria are different, the conclusions are often different. Then the fuzzy metric is more appropriate to characterize this phenomenon. In 1994, George and Veeramani in [8] introduced the definition of fuzzy metric for single-valued variables and then discussed completeness and separability in fuzzy metric space. In 2005, Saadati and Vaezpour in [9] defined fuzzy normed space. Gregori and Morillas in [10] introduced examples of fuzzy metrics and its applications. In 2020, Ghasemi et al. in [11] discussed the laws of large numbers for fuzzy random variables in the sense of fuzzy metric, but they did not gave the definition of fuzzy metric for sets or fuzzy sets in detail. In 2021, Guan etc. in [12] introduced the definition of fuzzy metric for sets, discussed the properties and proved the laws of large numbers for random sets in

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the sense of fuzzy metric. In this paper, what we are concerned is the strong limit theory for weighted sums of compact random sets in the sense of fuzzy metric, where the fuzzy metric is induced by d_H.

This paper is organized as follows. In section 2, we shall briefly introduce some notations and definitions of random sets. In section 3, we shall prove a limit theorem for weighted sums of random sets in the sense of fuzzy metric induced by d_H.

2. Notations

In this paper, we assume $(\Omega, \mathscr{A}, \mu)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|)$ is a real separable Banach space, $\mathbf{K}(\mathfrak{X})$ $(\mathbf{K}_k(\mathfrak{X}), \mathbf{K}_c(\mathfrak{X}))$ is the family of all nonempty closed (compact and convex, respectively) subsets of $\mathfrak{X}$.

Let $A, B \in \mathbf{K}(\mathfrak{X})$ and $\lambda \in \mathbb{R}$. Define the Minkowski addition and scalar multiplication as the following:

$$A+B = \{x+y : x \in A,\ y \in B\}, \tag{2.1}$$

$$\lambda A = \{\lambda x :\ x \in A\}. \tag{2.2}$$

The Hausdorff metric on $\mathbf{K}(\mathfrak{X})$ is defined by

$$d_H(A,B) = max\{\underset{x\in A}{sup}\ \underset{y\in B}{inf}\|x-y\|, \underset{y\in B}{sup}\ \underset{x\in A}{inf}\|x-y\|\}. \tag{2.3}$$

for $A, B \in \mathbf{K}(\mathfrak{X})$. Then for an $A \in \mathbf{K}(\mathfrak{X})$, let $\|A\|_K = d_H(A, \{0\})$.

Let $\mathfrak{X}^*$ be the dual space of $\mathfrak{X}$, for $A \in \mathbf{K_{kc}}(\mathfrak{X})$, the support function is defined as

$$s(x^*, A) = \sup_{a\in A} < x^*, a >,\ \ x^* \in \mathfrak{X}^*, \tag{2.4}$$

where $< x^*, a >$ means the inner product.

Let $\mathbf{S}^*$ denote the unit sphere of $\mathfrak{X}^*$, $C(\mathbf{S}^*)$ the all continuous functions of $\mathbf{S}^*$, and the norm is defined as $\|v\|_C = \sup_{x^* \in S^*} |v(x^*)|$.

A set-valued mapping $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is called a random set if for each open subset O of $\mathfrak{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathscr{A}$. $coF : \Omega \to \mathbf{K}_{kc}(\mathfrak{X})$ is defined by $(coF)(\omega) = co(F(\omega))$ if $F : \Omega \to \mathbf{K}_k(\mathfrak{X})$ is a random set, and coF denotes the convex hull of the set F.

For each random set F, the expectation of F, denoted by $E[F]$, is defined by

$$E[F] = \int_\Omega F d\mu = \left\{ \int_\Omega f d\mu : f \in S_F \right\} \tag{2.5}$$

where $\int_\Omega f d\mu$ is the usual Bochner integral. Denote $L^1[\Omega; \mathfrak{X}]$ as the family of integrable $\mathfrak{X}$-valued random sets, and $S_F = \{f \in L^1(\Omega; \mathfrak{X}), f(\omega) \in F(\omega)\ \ a.e.(\mu)\}$.

Let $L^1[\Omega, \mathscr{A}, \mu; \mathbf{K}_k(\mathfrak{X})]$ denote the space of all integrably bounded compact random sets, $L^1[\Omega, \mathscr{A}, \mu; \mathbf{K}_{kc}(\mathfrak{X})]$) denote the compact and convex random sets. We can briefly denote them as $L^1[\Omega; \mathbf{K}_k(\mathfrak{X})]$ and $L^1[\Omega; \mathbf{K}_{kc}(\mathfrak{X})]$ respectively.

The Borel field of $\mathbf{K}(\mathfrak{X})$ is denoted by $\mathscr{B}(\mathbf{K}(\mathfrak{X}))$. We define a sub-$\sigma$-field $\mathscr{A}_F$ by $\mathscr{A}_F = \sigma\{F^{-1}(\mathscr{U}) : \mathscr{U} \in \mathscr{B}(\mathbf{K}(\mathfrak{X}))\}$, where $F^{-1}(\mathscr{U}) = \{\omega \in \Omega : F(\omega) \in \mathscr{U}\}$. Random sets $F_1, F_2, \cdots F_n$ are said to be independent if $\{\mathscr{A}_{F_n} : n \geq 1\}$ are independent.

Definition 2.1([13]) The binary mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a t-norm, if $\forall x, y, z, k \in [0,1]$, it satisfies the following conditions:

(1)$x * y = y * x$;

(2) $(x * y) * z = x * (y * z)$;

(3) if $x \leq z$ and $y \leq k$, then $x * y \leq z * k$;

(4) $x * 1 = x$.

When the mapping is continuous on $[0,1] \times [0,1]$, $*$ is said to be continuous t-norm.

Definition 2.2 ([12]) Let $*$ be a continuous t-norm. The 3-tuple $(\mathbf{K}(\mathfrak{X}), M, *)$ is said to be a fuzzy metric space for sets, if the mapping $M : \mathbf{K}(\mathfrak{X}) \times \mathbf{K}(\mathfrak{X}) \times (0, \infty)$ satisfies the following conditions, $\forall A, B, C \in \mathbf{K}(\mathfrak{X})$ and $t, s > 0$:

(1) $\forall t > 0$, $M(A, B, t) > 0$;

(2) $\forall t > 0, M(A, B, t) = 1 \Leftrightarrow A = B$;

(3) $M(A, B, t) = M(B, A, t)$;

(4) $M(A, B, t) * M(B, C, s) \leq M(A, C, t + s)$;

(5) $M(A, B, \cdot\,) : (0, \infty) \to [0,1]$ is continuous.

M is called a fuzzy metric on $\mathbf{K}(\mathfrak{X})$.

Definition 2.3 ([12]) Let $*$ be a continuous t-norm. If $\forall A, B \in \mathbf{K}(\mathfrak{X})$ and $t, s > 0$, N satisfies the following conditions:

(1) $\forall t > 0, N(A, t) > 0$;

(2) $N(A, t) = 1 \Leftrightarrow t = 0$;

(3) $\forall \alpha \neq 0, N(\alpha A, t) = N(A, \frac{t}{|\alpha|})$;

(4) $N(A, t) * N(B, s) \leq N(A + B, t + s)$;

(5) $N(A, \cdot\,) : (0, \infty) \to [0,1]$ is continuous;

(6) $\lim\limits_{t \to \infty} N(A, t) = 1$.

then $(\mathbf{K}(\mathfrak{X}), N, *)$ is said to be a fuzzy normed space for sets, N is called a fuzzy norm for sets on $\mathbf{K}(\mathfrak{X})$.

Lemma 2.1 (cf. [12]) Let N_{d_H} be a fuzzy norm, and M_{d_H} be a fuzzy metric induced by N_{d_H}. Then $\forall A, B, C \in \mathbf{K}_{kc}(\mathfrak{X})$ and scalar $\alpha \neq 0$:

(1) $M_{d_H}(A + C, B + C, t) = M_{d_H}(A, B, t)$;

(2) $M_{d_H}(\alpha A, \alpha B, t) = M_{d_H}(A, B, \frac{t}{|\alpha|})$.

Lemma 2.2 (cf. [12]) Let $\mathfrak{X}$ be a separable normed space. There exists a fuzzy normed space $C(S^*)$ and a function $j_0 : \mathbf{K}_{kc}(\mathfrak{X}) \to C(S^*)$ with the following properties: for $A, B \in \mathbf{K}_{kc}(\mathfrak{X})$, $t \geq 0$,

(1) $M_{d_H}(A, B, t) = M_d(j_0(A), j_0(B), t)$;

(2) $j_0(A + B) = j_0(A) + j_0(B)$;

(3) $\lambda \geq 0, j_0(\lambda A) = \lambda j_0(A)$,

where M_d is the fuzzy metric induced by metric d in embedding space. The lemma means $\mathbf{K}_{kc}(\mathfrak{X})$ can be embedded into a fuzzy normed space by $j_0(\cdot)$. We can take $j_0 : \mathbf{K}_{kc}(\mathfrak{X}) \to C(S^*)$ as $j_0(A) = s(\cdot, A)$.

3. Main Results

In this section, we shall prove a laws of large numbers for compactly uniformly integrable random sets in the sense of fuzzy metric. Before, we firstly gave the following two lemmas.

Lemma 3.1 ([11]) Let $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_{kc}(\mathfrak{X})]$ and j_0 be the isomorphic mapping provided by lemma 2.2.

$$E[j_0(V_n)] = j_0(E[V_n]). \tag{3.1}$$

Lemma 3.2 ([12]) Let $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_k(\mathfrak{X})]$. The fuzzy metric M_{d_H} is induced by d_H, then

$$M_{d_H}\left(\sum_{i=1}^{n} V_i, \sum_{i=1}^{n} coV_i, t\right) \geq \min_{1\leq i\leq n} M_{d_H}\left(coV_i, \{0\}, \frac{t}{\sqrt{p}}\right) \tag{3.2}$$

for any $t > 0$, where p is the dimension of $\mathfrak{X}$.

Lemma 3.3 ([12]) Let M be a fuzzy metric induced by fuzzy norm N. Then $\forall A, B \in \mathbf{K}_{kc}(\mathfrak{X})$,

$$\lim_{t\to\infty} M(A, B, t) = 1. \tag{3.3}$$

Definition 3.1(cf.[14]) The sequence $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_k(\mathfrak{X})]$ is said to be compactly uniformly integrable in the sense of d_H if $\forall \varepsilon > 0$, there is a compact subset $\mathscr{K}_\varepsilon \in \mathbf{K}_k(\mathfrak{X})$ such that

$$E[\|V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}}\|_K] < \varepsilon, \forall n \geq 1, \tag{3.4}$$

where $I_{\{V_n \notin \mathscr{K}_\varepsilon\}}$ denotes the characteristic function.

Then we have the following result which will be used later to prove the main result.

Theorem 3.1 Let $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_{kc}(\mathfrak{X})]$ be compactly uniformly integrable and j_0 be the embedding function. Then $\{j_0(V_n) : n \geq 1\}$ is also compactly uniformly integrable.

Proof. Since $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_{kc}(\mathfrak{X})]$ is compactly uniformly integrable, we know that $\forall \varepsilon > 0$, $\exists$ a compact $\mathscr{K}_\varepsilon \in \mathbf{K}_k(\mathfrak{X})$ such that $E[\|V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}}\|_K] < \varepsilon, \forall n \geq 1$. Since j_0 is a isometric isomorphic mapping, $j_0(co\mathscr{K}_\varepsilon)$ is also compact.

$$\begin{aligned}
E[\|j_0(V_n)I_{\{j_0(V_n)\notin j_0(co\mathscr{K}_\varepsilon)\}}\|_C] &= E[\|j_0(V_n)I_{\{V_n \notin co\mathscr{K}_\varepsilon\}}\|_C] \\
&\leq E[\|j_0(V_n)I_{\{V_n \notin \mathscr{K}_\varepsilon\}}\|_C] \\
&= E[\|j_0(V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}})\|_C] \\
&= E[\|j_0(V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}}) - j_0(\{0\})\|_C] \\
&= E[d_H(V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}}, \{0\}] \\
&= E[\|V_n I_{\{V_n \notin \mathscr{K}_\varepsilon\}}\|_K] \\
&< \varepsilon.
\end{aligned}$$

That means $\{j_0(V_n) : n \geq 1\}$ is compactly uniformly integrable.

Theorem 3.2 Let $\{V_n : n \geq 1\} \subset L^1[\Omega; \mathbf{K}_k(\mathfrak{X})]$ be an independent and compactly uniformly integrable random sets. Let $\{b_n : n \geq 1\}$ be a sequence of positive constants with $b_n \uparrow$ and $n = O(b_n)$. If

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|V_n\|_K^p] < \infty, 1 \leq p \leq 2. \tag{3.5}$$

Then in the metric M_{d_H}, we have the following convergence:

$$\frac{1}{b_n} \sum_{i=1}^{n} V_i \longrightarrow \frac{1}{b_n} \sum_{i=1}^{n} E[coV_i] \ \ a.e. \tag{3.6}$$

that is for $t > 0$,

$$M_{d_H}\left(\frac{1}{b_n} \sum_{i=1}^{n} V_i, \frac{1}{b_n} \sum_{i=1}^{n} E[coV_i], t\right) \longrightarrow 1 \ \ a.e.. \tag{3.7}$$

Proof. *Step 1*. Let $V_n : \Omega \to \mathbf{K}_{kc}(\mathfrak{X})$ be independent random sets and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|V_n\|_K^p] < \infty, 1 \leq p \leq 2. \tag{3.8}$$

By theorem 3.1 we know that $\{j_0(V_n) : n \geq 1\}$ is also compactly uniformly integrable. Then $\{j_0(V_n) : n \geq 1\}$ are independent $C(S^*)$-valued random elements and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|j_0(V_n)\|_C^p] &= \sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|j_0(V_n) - j_0(\{0\})\|_C^p] \\
&= \sum_{n=1}^{\infty} \frac{1}{b_n^p} E[d_H^p(V_n, \{0\})] \\
&= \sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|V_n\|_K^p] \\
&< \infty, \ \ 1 \leq p \leq 2.
\end{aligned}$$

By a standard SLLN in Banach space([15]), it follows that

$$\frac{1}{b_n} \sum_{i=1}^{n} j_0(V_i) \longrightarrow \frac{1}{b_n} \sum_{i=1}^{n} E\left[j_0(V_i)\right] \ \ a.e. \tag{3.9}$$

By theorem 3.1 and lemma 3.1 we know that

$$E[j_0(V_i)] = j_0(E[V_i]), \tag{3.10}$$

$$\frac{1}{b_n}\sum_{i=1}^{n} j_0(V_i) = j_0(\frac{1}{b_n}\sum_{i=1}^{n} V_i). \tag{3.11}$$

Then

$$j_0\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i\right) \longrightarrow j_0\left(\frac{1}{b_n}\sum_{i=1}^{n} E[V_i]\right) \;\; a.e. \tag{3.12}$$

It follows from the embedding theorem that

$$M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i, \frac{1}{b_n}\sum_{i=1}^{n} E[V_i], \frac{t}{2}\right) \longrightarrow 1 \;\; a.e.. \tag{3.13}$$

Step 2. Consider the general case. $V_n : \Omega \to \mathbf{K}_k(\mathfrak{X})$, so $coV_n : \Omega \to \mathbf{K}_{kc}(\mathfrak{X})$. Since $\mathbf{K}_{kc}(\mathfrak{X}) \subseteq \mathbf{K}_k(\mathfrak{X})$, so $\{coV_n : n \geq 1\}$ is also compactly uniformly integrable and

$$\sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|coV_n\|_K^p] \leq \sum_{n=1}^{\infty} \frac{1}{b_n^p} E[\|V_n\|_K^p] < \infty, 1 \leq p \leq 2. \tag{3.14}$$

It follows from step 1 that

$$M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} coV_i, \frac{1}{b_n}\sum_{i=1}^{n} E[coV_i], \frac{t}{2}\right) \longrightarrow 1 \;\; a.e. \tag{3.15}$$

and by lemma 3.2 and lemma 3.3, we can have

$$\begin{aligned} M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i, \frac{1}{b_n}\sum_{i=1}^{n} coV_i, \frac{t}{2}\right) &\geq \min_{1\leq i\leq n} M_{d_H}\left(\frac{coV_i}{b_n}, \{0\}, \frac{t}{2\sqrt{p}}\right) \\ &= \min_{1\leq i\leq n} M_{d_H}\left(coV_i, \{0\}, \frac{tb_n}{2\sqrt{p}}\right) \\ &\to 1, \;\; n \to \infty \;\; a.e. \end{aligned}$$

From the triangle inequality, it follows that

$$\begin{aligned} M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i, \frac{1}{b_n}\sum_{i=1}^{n} E[coV_i]), t\right) \geq M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i, \frac{1}{b_n}\sum_{i=1}^{n} coV_i, \frac{t}{2}\right) * \\ M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} coV_i, \frac{1}{b_n}\sum_{i=1}^{n} E[coV_i], \frac{t}{2}\right), \end{aligned}$$

the right terms in above tend to 1. Then we have

$$M_{d_H}\left(\frac{1}{b_n}\sum_{i=1}^{n} V_i, \frac{1}{b_n}\sum_{i=1}^{n} E[coV_i], t\right) \longrightarrow 1 \;\; a.e. \tag{3.17}$$

The result is proved.

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References

[1] Artstein Z, Vitale RA. A strong law of large numbers for random compact sets. The Ann of Prob. 1975; 3: 879–882.

[2] Puri ML, Ralescu DA. Strong law of large numbers for Banach space valued random sets. The Ann of Prob. 1983. 11: 222–224.

[3] Taylor RL, Inoue H. Convergence of weighted sums of random sets. Stoch. Ana and Appli. 1985; 3: 379–396.

[4] Adler A, Rosalsky A, Taylor RL. Strong laws of large numbers for weighted sums of random elements in Normed Linear Spaces. Inter J of Math and Math Sci. 1989; 12: 507–530.

[5] Fu KA, Zhang LX. Strong limit theorems for random sets and fuzzy random sets with slowly varying weights. Inf Sci. 2008; 178: 2648–2660.

[6] Guan L, Li S. A law of large numbers for exchangeable random variables on nonadditive measures. Synergies of Soft Computing and Statistics for Intelligent Data Analysis. 2013; 190: 145–152.

[7] Zadeh LA. Fuzzy Sets. Inf and Con. 1965; 8: 338-353.

[8] George A, Veeramani P. On some results in fuzzy metric spaces. F S and Sys. 1994; 64: 395–399.

[9] Saadati R, Vaezpour S. Some results on fuzzy Banach spaces. J of App Math and Comp. 2005; 17: 475–484.

[10] Gregori V, Morillas S. Examples of fuzzy metrics and applications. F S and Sys. 2011; 170: 95–111.

[11] Ghasemi R, Nezakati A, Rabiei MR. A strong law of large numbers for fuzzy random sets in fuzzy Banach space. Adv in F Sys. 2020; 1–10.

[12] Guan L. Wei J. Min H, and Zhang J. The strong laws of large numbers for set-valued random variables in Fuzzy metric space. Math. 2021, 9(11): 1192. https://doi.org/10.3390/math9111192.

[13] Klement EP, Mesiar R, and Pap E. Triangular norms. Kluwer Academic Publishers. 2000;

[14] Guan L, and Li S. Laws of large numbers for weighted sums of fuzzy set-valued random variables. Inter. J. of Uncer, F and Knowledge-based Sys. 2004; 12: 811825.

[15] Adler A, Rosalsky A, Taylor RL. Some strong laws of large numbers for sums of random elements. Bull. Inst. Math. Acad. Sinica. 1992; 20: 335–357.