Fuzzy Systems and Data Mining VIII
A.J. Tallón-Ballesteros (Ed.)

doi:10.3233/FAIA220395

Fuzzy Henstock-Δ-Integral on Time Scales and Its Application

Yubing LI [a], Yabin SHAO [a,1], Zengtai GONG [a]
[a] *School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Abstract. In this paper, we define the fuzzy Henstock-Δ-integral and fuzzy Δ-derivative on time scales(or briefly FH-Δ-integral, Δ-derivative). Then, we give some convergence theorems for this kind of nonabsolute convergent integrals. Finally, we obtain an existence theorem of the global solutions for this kind of fuzzy generalized differential systems.

Keywords. fuzzy Henstock-Δ-integral, time scales, convergence theorems, fuzzy differential equations, existence theorem, global solutions

1. Introduction

To popularize and unify continuous and discontinuous dynamical systems, Hilger [1] first proposed the definition of time scales in 1988. For more detailed results, please see [2,3]. As we all know, the dynamic equations are important and interesting research in differential equations theories. These equations are widely used in mathematical biology, engineering and technology, mathematical economics and control theories (see [4,5]).

In 2015, Fard and Bidgoli [6] proposed the definitions of fuzzy Δ-integral and fuzzy Δ-derivative on $[a,b]_{\mathbb{T}}$. Therefore, the research of the initial value problem of solutions for fuzzy differential equations (FDEs) on $[a,b]_{\mathbb{T}}$ has attracted the attention of many scholars. After then, they [7] studied the qualitative problems to 2th FDEs on time scales.

To make the research of fuzzy differential equations on $[a,b]_{\mathbb{T}}$ more perfect, we define the FH-Δ-integral and fuzzy Δ-derivative. And then, we also give two convergence theorems for the FH-Δ-integral. Finally, as the application of this integral, we obtain an existence theorem of global continuous solutions for FDEs.

The paper is organized as follows. In Section 2, we introduce some preliminary results and basic notions of fuzzy numbers and time scales. In Section 3, we define the FH-Δ-integral and fuzzy Δ-derivative by using gH-difference on $[a,b]_{\mathbb{T}}$. Also, we give some convergence theorems of the FH-Δ-integral on $[a,b]_{\mathbb{T}}$. In Section 4, we give the existence theorem of the global solutions for FDEs. In Section 5, we give some conclusions.

[1]Corresponding Author: Yabin Shao, E-mail shaoyb@cqupt.edu.cn.

2. Preliminaries

2.1. Fuzzy Number Space

In this paper, $\mathbb{R}_{\mathscr{F}}$ denotes a set of fuzzy subsets. $\tilde{m}$ is a fuzzy number, if $\tilde{m} \in \mathbb{R}_{\mathscr{F}}$ satisfies $\tilde{m}$ is normal, convex, upper semi-continuous and $[\tilde{m}]_0 = \overline{\bigcup_{r\in(0,1]}[\tilde{m}]_r}$ is bounded.

Denote $[\tilde{m}]_r = \{x \in \mathbb{R} : m(x) \geq r, 0 < r \leq 1\}$. The Hausdorff distance of $\tilde{m}$ and $\tilde{n}$ is defined by

$$D(\tilde{m},\tilde{n}) = \sup\max\{| m_r^- - n_r^- |, | m_r^+ - n_r^+ |\},$$

where the cut-set is denoted by $[m_r^-, m_r^+]$ and $[n_r^-, n_r^+]$, respectively, $||\cdot|| = D(\cdot,\tilde{0})$ and $\tilde{0}$ is zero fuzzy number (see e.g. [8]).

Theorem 2.1 ([9]) *For $\tilde{m},\tilde{n},\tilde{p},\tilde{q} \in \mathbb{R}_{\mathscr{F}}$,*

1. $D(\tilde{m}+\tilde{p},\tilde{n}+\tilde{p}) = D(\tilde{m},\tilde{n})$.
2. $D(h\cdot\tilde{m},h\cdot\tilde{n}) = | h | D(\tilde{m},\tilde{n}), h \in \mathbb{R}$.
3. $D(\tilde{m}+\tilde{n},\tilde{p}+\tilde{q}) \leq D(\tilde{m},\tilde{p}) + D(\tilde{n},\tilde{q})$.

Definition 2.1 ([10,11]) *If $\tilde{l}$ is called the gH-difference of $\tilde{m}$ and $\tilde{n}$, then*

$$\tilde{m} \ominus_{gH} \tilde{n} = \tilde{l} \Longleftrightarrow \begin{cases} (a)\tilde{m} = \tilde{n} + \tilde{l}, \\ or \\ (b)\tilde{n} = \tilde{m} + (-1)\tilde{l}. \end{cases}$$

2.2. Time Scales

If $\mathbb{T}$ satisfies nonempty and closed subset of $\mathbb{R}$ (see e.g. [12]), then we say $\mathbb{T}$ is a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(q) = inf\{\tau \in \mathbb{T} | \tau > q, q \in \mathbb{T}\}$. Furthermore, if $\mu : \mathbb{T} \to [0,+\infty]$ is given by $\mu(q) = \sigma(q) - q$, then μ is called a graininess function.

Definition 2.2 ([12]) *t is called left (right) scattered if satisfies $\rho(q) < q(\sigma(q) > q)$. t is called left (right) dense if satisfies $\rho(q) = q(\sigma(q) = q)$. In addition, if t is both left-scattered(left-dense) and right-scattered(right-dense), then t is isolated(dense).*

Definition 2.3 ([7]) *Let $\tilde{f} : \mathbb{T} \to \mathbb{R}_{\mathscr{F}}$. If $\tilde{f}$ is continuous at each $q = \sigma(q)$ and the $\lim_{p\to q^-} \tilde{f}(p)$ exist (finite), for $q \in \mathbb{T}$ with $q = \rho(q)$ in $\mathbb{T}$, then we say $\tilde{f}$ is rd-continuous. In addition, denote by $C_{rd}(\mathbb{T},\mathbb{R}_{\mathscr{F}})$ the set of rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}_{\mathscr{F}}$.*

Definition 2.4 *Let $\pi \subset C_{rd}(\mathbb{T},\mathbb{R}_{\mathscr{F}})$. If π is equi-continuous at each $q = \sigma(q)$, then we say π is rd-equi-continuous.*

Note that we call π 'equi-continuous' instead of 'rd-equi-continuous' for convenience without confusion.

Definition 2.5 ([13]) *A partition P for $[a,b]_{\mathbb{T}}$ is a division of $[a,b]_{\mathbb{T}}$ denoted by*

$$P = \{a = q_0 \leq \xi_1 \leq q_1 \leq \cdots \leq q_{i-1} \leq \xi_i \leq q_i = b\}$$

for $q_i \in [a,b]_{\mathbb{T}}$ and $1 \leq i \leq n$. We shall call each $\xi_i \in [a,b]_{\mathbb{T}}$ a tag-point and each q_i an endpoint.

Definition 2.6 ([13]) *P is δ-fine, if δ is a Δ-gauge on $[a,b]_{\mathbb{T}}$, and for $i = 1,2,\cdots,n$ s.t.*

$$\xi_i \in [q_{i-1},q_i] \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)], \xi_i \in \mathbb{T}.$$

2.3. Embedding Theorem

Denoted by $X[a,b]_{\mathbb{T}}$ the set of continuous functions $f(q) : [a,b]_{\mathbb{T}} \to \overline{C}[0,1] \times \overline{C}[0,1], q \in [a,b]_{\mathbb{T}}$. $X[a,b]_{\mathbb{T}}$ is a Banach space with $\|f(q)\|_X = \sup \|f(q)\|_{\overline{C}[0,1]\times\overline{C}[0,1]}$.

Theorem 2.2 ([14]) *For $\tilde{m} \in \mathbb{R}_{\mathscr{F}}$, denote $j \circ \tilde{m} = (m^-, m^+)$. Then $j(\mathbb{R}_{\mathscr{F}})$ is a closed convex cone in $\overline{C}[0,1] \times \overline{C}[0,1]$ and $j : \mathbb{R}_{\mathscr{F}} \to \overline{C}[0,1] \times \overline{C}[0,1]$ satisfies following:*

(1) $j \circ (s\tilde{m} + t\tilde{n}) = s \cdot j \circ \tilde{m} + t \cdot j \circ \tilde{n}, (s,t \geq 0)$;
(2) $D(\tilde{m},\tilde{n}) = ||j \circ \tilde{m} - j \circ \tilde{n}||_X$.

Theorem 2.3 *Let j be the embedding operator from $(\mathbb{R}_{\mathscr{F}}, D)$ to $\overline{C}[0,1] \times \overline{C}[0,1]$, and the set of rd-continuous functions is represented by $C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_{\mathscr{F}})$, then $C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_{\mathscr{F}})$ is a closed convex subset on $X[a,b]_{\mathbb{T}}$.*

The dual of $C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_{\mathscr{F}})$ is represented by $C^*_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_{\mathscr{F}})$.

3. The Fuzzy Henstock-Δ-Integral and Derivative on $[a,b]_{\mathbb{T}}$

In this section, we define the FH-Δ-integral and fuzzy Δ-derivative on $[a,b]_{\mathbb{T}}$. In addition, we also give two convergence theorems for the FH-Δ-integral.

Definition 3.1 *Suppose $\tilde{f} : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$ and there exists $\tilde{F} : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$. If for $\forall\, \varepsilon > 0, \exists$ a Δ-gauge δ s.t. for any partitions $P = \{[q_{i-1},q_i]; \xi_i\}_{i=1}^n$, which is δ-fine*

$$D\big(\sum_{i=1}^n \tilde{f}(\xi_i)(q_i - q_{i-1}) + \tilde{F}(q_{i-1}), \tilde{F}(q_i)\big) < \varepsilon,$$

then we say $\tilde{f}$ is fuzzy Henstock-Δ-integrable on $[a,b]_{\mathbb{T}}$. We denote $\tilde{F}(q) = (FH)\int_a^q \tilde{f}(s)\Delta s$, $\tilde{f} \in FH_{[a,b]_{\mathbb{T}}}$.

Definition 3.2 *Suppose $\exists\, \tilde{f}^{\Delta}(p) \in \mathbb{R}_{\mathscr{F}}$ and given $\forall\, \varepsilon > 0$, $\exists\, U_{\mathbb{T}}$(i.e. $U_{\mathbb{T}} = (p-\iota, p+\iota) \cap [a,b]_{\mathbb{T}}$ for $\iota > 0$) s.t.*

$$D\big(\tilde{f}(\sigma(p)) \ominus_{gH} \tilde{f}(q), \tilde{f}^{\Delta}(p)(\sigma(p) - q)\big) \leq \varepsilon(\sigma(p) - q) \text{ for all } q \in U_{\mathbb{T}},$$

then $\tilde{f}$ is fuzzy Δ-differentiable. Denote by $\tilde{f}^{\Delta}(q)$ the fuzzy Δ-derivative.

Definition 3.3 *Suppose $\tilde{f} : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$. If for $\forall\, \varepsilon > 0, \exists\, \eta > 0$ s.t. for any disjoint finite interval families $\{[c_i,d_i]_{\mathbb{T}}, 1 \leq i \leq n\}$ satisfy $\sum\limits_{i=1}^n |d_i - c_i| < \eta$,*

$$\sum_{i=1}^n \omega(\tilde{f}, [c_i,d_i]_{\mathbb{T}}) < \varepsilon,$$

then we say $\tilde{f}$ is AC^*.

Definition 3.4 *The family of fuzzy functions is uniformly ACG^*, if $M = \bigcup\limits_{i=1}^n M_i$ and for $\forall\, i$, the family of fuzzy functions is uniformly AC^*.*

Definition 3.5 *Let $\tilde{A}_n,\tilde{A} \in \mathbb{R}_{\mathscr{F}}, n = 1,2,\cdots$, if for $\forall\, \varepsilon > 0, \exists\, N > 0$ s.t. $n > N$, $D(\tilde{A}_n,\tilde{A}) < \varepsilon$, then we say $\tilde{A}_n$ converges (strongly converges) to $\tilde{A}$. We write $\tilde{A}_n \longrightarrow \tilde{A}$.*

Definition 3.6 *Let $\tilde{A}_n,\tilde{A} \in \mathbb{R}_{\mathscr{F}}, n = 1,2,\cdots$, if for $\forall\, \varepsilon > 0, \exists\, N > 0$ s.t. $n > N$,*

$$| < \vartheta^*,\tilde{A}_n > - < \vartheta^*,\tilde{A} > | < \varepsilon$$

for all $\vartheta^* \in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$, where $< \vartheta^*,\tilde{A} >= \vartheta^*(\tilde{A}) \in \mathbb{R}$, then we say $\tilde{A}_n$ weakly converges to $\tilde{A}$. We write $\tilde{A}_n \xrightarrow{weakly} \tilde{A}$.

Definition 3.7 *Let $K : C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}}) \to C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ and $\tilde{\vartheta}_n \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ with $\tilde{\vartheta}_n \to \tilde{\vartheta}$, K is sequential weak continuous if for $\forall\, \varepsilon > 0, \exists\, N, n > N$, we have*

$$| < \vartheta^*,K(\tilde{\vartheta}_n) > - < \vartheta^*,K(\tilde{\vartheta}) > | < \varepsilon$$

for $\forall\, \vartheta^* \in C^*_{rd}([a,b]_{\mathbb{T}}$.

Definition 3.8 *Suppose $\tilde{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ belong to the class $\upsilon([a,b]_{\mathbb{T}} \times \mathbb{R}_{\mathscr{F}},\pi,S)$ if*

1. *$\pi : [a,b]_{\mathbb{T}} \to \mathbb{R}^+$ is continuous with $\|\pi\| \le \frac{1}{2U}, [a,b]_{\mathbb{T}} \subset [0,U]$.*
2. *Every $S > 0$,*
$$\{(FH)\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p, \|\tilde{\vartheta}\| \le S\} \subset C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$$
is weakly relatively compact and equi-continuous and uniformly ACG^*.
3. *Let every rd-continuous $\tilde{\vartheta} : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}, q \in [a,b]_{\mathbb{T}} \to \tilde{f}(q,\tilde{\vartheta}(q))$ is FH-Δ-integrable and*
$$\limsup_{s\to\infty}(\frac{1}{S}\sup_{\|\tilde{\vartheta}\|\le S}\|\tilde{f}(\cdot,\tilde{\vartheta}(\cdot))\|) < \frac{1}{2}.$$
4. *There is $\tilde{\vartheta}_n \longrightarrow \tilde{\vartheta}$ with respect to $C^*_{rd}([0,1]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ with $\tilde{f}(\cdot,\tilde{\vartheta}_n(\cdot)) \xrightarrow{weakly} \tilde{f}(\cdot,\tilde{\vartheta}(\cdot))$.*

Theorem 3.1 *Suppose $\tilde{f},\tilde{f}_n \in C_{rd}([0,1]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ is bounded. Then $\tilde{f}_n(q) \longrightarrow \tilde{f}(q)$ if and only if $\tilde{f}_n(q) \xrightarrow{weakly} \tilde{f}(q)$, $q \in [0,1]_{\mathbb{T}}$.*

Theorem 3.2 *Suppose $\tilde{f}(t),\tilde{f}_n : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$. If $\tilde{f}_n$ is FH-Δ-integrable, let $\tilde{F}_n$ is primitive of $\tilde{f}_n$ and satisfies:*

1. *$\vartheta^*\tilde{f}_n(q) \to \vartheta^*\tilde{f}(q)$ a.e. for all $\vartheta^* \in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$.*
2. *The set $\Pi = \{\vartheta^*\tilde{F}_n, n = 1,2,\cdots\}$ is ACG^* uniformly on $[a,b]_{\mathbb{T}}$, for $\forall\, \vartheta^* \in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$.*
3. *The set Π is equi-continuous and for $\forall\, \vartheta^* \in C^*_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$,*

then $\tilde{f} \in FH_{[a,b]_{\mathbb{T}}}$ and

$$\int_0^q \tilde{f}_n(p)\Delta p \to \int_0^q \tilde{f}(p)\Delta p.$$

Theorem 3.3 *Suppose $\tilde{f} \in FH_{[a,b]_{\mathbb{T}}}$, then the function*

$$\tilde{F}(q) = (FH)\int_0^q \tilde{f}(p)\Delta p$$

is rd-continuous at every $q \in [a,b]_{\mathbb{T}}$. Further, we get $\tilde{F}^{\Delta}(q) = \tilde{f}(q)$ for every point q of the rd-continuous of $\tilde{f}$.

4. Global Existence of Solution for FDEs on $[a,b]_{\mathbb{T}}$

We will study the FDEs:

$$\tilde{\vartheta}^{\Delta}(q)=\tilde{f}(q,\tilde{\vartheta}(q)),\quad \tilde{\vartheta}(0)=\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p, \forall\, q\in[a,b]_{\mathbb{T}}\subset[0,U]. \tag{1}$$

Definition 4.1 *If $\tilde{\vartheta}(q)$ is a global (i)-solution (or (ii)-solution) of (1) on $[0,U]$ iff it is rd-continuous and satisfies:*

$$\begin{cases}\tilde{\vartheta}(q)=\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p+\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\\ \tilde{\vartheta}(0)=\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p\end{cases} \tag{2}$$

or

$$\begin{cases}\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p=\tilde{\vartheta}(q)+(-1)\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\\ \tilde{\vartheta}(0)=\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p\end{cases} \tag{3}$$

respectively.

Theorem 4.1 *Let $\tilde{f}:[a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_{\mathscr{F}}\in\upsilon([a,b]_{\mathbb{T}}\times\mathbb{R}_{\mathscr{F}},\pi,S)$ and $[a,b]_{\mathbb{T}}\subset[0,U]$. Then there is a global (i)-solution and a global (ii)-solution $\tilde{\vartheta},\tilde{\vartheta}'$ of (1) for which $\tilde{\vartheta}(0)=\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p$.*

Proof. According to Definition 3.8 (3), we can find $S_0>0$ s.t. for $\forall\, S\geq S_0, q\in[a,b]_{\mathbb{T}}$,

$$\sup_{\|\tilde{\vartheta}\|\leq S}\|(FH)\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p\|<\frac{S}{2}.$$

Suppose $B_{S_0}\subset C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ is closed and we define the operators

$$K:B_{S_0}\to C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}}), T:B_{S_0}\to C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$$

by

$$K\tilde{\vartheta}(q)=(FH)\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p,\ T\tilde{\vartheta}(q)=(FH)\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p.$$

In addition, we use Arzela-Ascoli theorem, then

$$\{(FH)\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p, \tilde{\vartheta}\in B_{S_0}\}$$

is weakly relatively compact. Next, we testify the K is sequential weak continuous. Since $C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}_{\mathscr{F}})$ is a closed convex subset of $X[a,b]_{\mathbb{T}}$, we consider an arbitrary sequence $\tilde{\vartheta}_n\xrightarrow{weakly}\tilde{\vartheta}\in X[a,b]_{\mathbb{T}}$. By Definition 3.6, for every $\vartheta^*\in X[a,b]_{\mathbb{T}}$,

$$<\vartheta^*,\tilde{f}(p,\tilde{\vartheta}_n(p))>\longrightarrow<\vartheta^*,\tilde{f}(p,\tilde{\vartheta}(p))>.$$

In addition, $\{(FH)\int_0^q \tilde{f}(p,\tilde{\vartheta}_n(p))\Delta p, n\in\mathbb{N}\}$ is equi-continuous and by Theorem 3.2, we have

$$<\vartheta^*,\int_0^q \tilde{f}(p,\tilde{\vartheta}_n(p))\Delta p>\longrightarrow<\vartheta^*,\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p>.$$

According to Theorem 3.1,

$$\int_0^q \tilde{f}(p,\tilde{\vartheta}_n(p))\Delta p\longrightarrow\int_0^q \tilde{f}(p,\tilde{\vartheta}(p))\Delta p.$$

Thus, K is sequential weak continuous.

Next, we prove that the operator T is similar. Taking $\tilde{\vartheta}_n \subset B_{S_0} \xrightarrow{weakly} \tilde{\vartheta} \in X[a,b]_{\mathbb{T}}$, for $\forall\, \vartheta^* \in T^*$, then

$$\begin{aligned} &\| < \vartheta^* + \int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p, \int_0^U \pi(p)\tilde{\vartheta}_n(p)\Delta p > \| \\ &= \| \int_0^U \pi(p) < \vartheta^* + \tilde{\vartheta}(p), \tilde{\vartheta}_n(p) > \Delta p \| \\ &\leq \|\pi\| \int_0^U \| < \vartheta^* + \tilde{\vartheta}(p), \tilde{\vartheta}_n(p) > \| \Delta p. \end{aligned}$$

Since $\| < \vartheta^* + \tilde{\vartheta}(p), \tilde{\vartheta}_n(p) > \| \to 0$ and it is bounded, by convergence theorem that

$$\| < \vartheta^* + \int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p, \int_0^U \pi(p)\tilde{\vartheta}_n(p)\Delta p > \| \longrightarrow 0.$$

Thus, T is sequential weak continuous. Since

$$\|T(\tilde{m}) \ominus_{gH} T(\tilde{n})\| \leq \|\pi\| U \|\tilde{m} \ominus_{gH} \tilde{n}\|,$$

then T is a strict contraction. Consider $\tilde{\vartheta} \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_{\mathscr{F}})$,

$$\tilde{\vartheta}(q) = \int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p + (FH)\int_0^q \tilde{f}(p, \tilde{m}(p))\Delta p, \forall q \in [a,b]_{\mathbb{T}}$$

for $\tilde{m} \in B_{S_0}$. Then

$$\begin{aligned} \|\tilde{\vartheta}\| &\leq \| \int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p\| + \sup_{q\in[a,b]_{\mathbb{T}}} \|(FH)\int_0^q \tilde{f}(p,\tilde{m}(p))\Delta p\| \\ &\leq \|\pi\| U \|\tilde{\vartheta}\| + \frac{S_0}{2}. \end{aligned}$$

By known conditions, we have

$$\|\tilde{\vartheta}\| \leq \frac{S_0}{2(1-\|\pi\|U)} \leq S_0.$$

This shows that $\tilde{\vartheta} \in B_{S_0}$. According to Krasnosel'skii theorem, there is a $\tilde{\vartheta}$ s.t. $\tilde{\vartheta} = (K+T)\tilde{\vartheta}$, i.e.

$$\tilde{\vartheta}(q) = (K+T)(\tilde{\vartheta}(q)) = \int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p + \int_0^q \tilde{f}(p, \tilde{\vartheta}(p))\Delta p.$$

Thus, $\tilde{x}$ is a global (i)-solution.

Similarly, there is a $\tilde{\vartheta}$ s.t. $\tilde{\vartheta} = (\widehat{K}+\widehat{T})\tilde{\vartheta}$, when we define the mapping

$$\int_0^U \pi(p)\tilde{\vartheta}(p)\Delta p = (\widehat{K}+\widehat{T})\tilde{\vartheta} + (-1)\int_0^q \tilde{f}(p, \tilde{\vartheta}(p))\Delta p,$$

then $\tilde{\vartheta}$ is a global (ii)-solution.

5. Conclusions

To popularize the FDEs on time scales, we first propose the notions of FH-Δ-integral and fuzzy Δ-derivative on $[a,b]_{\mathbb{T}}$. Then, we obtain some convergence theorems of the FH-Δ-integral on $[a,b]_{\mathbb{T}}$. Finally, as the application, we obtain an existence theorem of the global (i)-solution and global (ii)-solution for fuzzy differential equations.

References

[1] Hilger S. Ein Maßkettenkalkül mit Anwendung auf Zentrumsma-nnigfaltigkeiten, Ph.D. Thesis, Universtät Würzburg, 1988.
[2] Hilger S. Analysis on measure chains-a unified approach to continuous and discrete calculus. Results in Mathematics. 1990 Feb;18:18-56.
[3] Bohner M, Peterson A. Dynamic equations on time scales. In An Introduction with Applications; Birkhäuser: Boston, MA, USA; 2001. 372p.
[4] Atici FM, Biles DC, Lebedinsky A. An application of time scales to economics. Mathematical and Computer Modelling. 2006 Aug;43(7-8):718-726.
[5] Liu G, Xiang X, Peng Y. Nonlinear integro-differential equations and optimal control problems on time scales. Computers & Mathematics with Applications. 2011 Oct;61(2):155-169.
[6] Fard OS, Bidgoli TA. Calculus of fuzzy functions on time scales (I). Soft Computing. 2015 Mar;19(2):293-305.
[7] Fard OS, Bidgoli TA. Existence and uniqueness of solutions to the second order fuzzy dynamic equations on time scales. Advances in Difference Equations. 2017 Aug;2017(1):1-17.
[8] Shao Y, Mou Q, Gong Z. On retarded fuzzy functional differential equations and nonabsolute fuzzy integrals. Fuzzy Sets and Systems. 2019 Feb;375:121-140.
[9] Dubois D, Prade H. Fuzzy sets and systems: theory and applications. New York: Academic press; 1980. 393 p.
[10] Bede B, Stefanini L. Generalized differentiability of fuzzy-valued functions. Fuzzy Sets and Systems. 2013 Nov;230:119-141.
[11] Stefanini L. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets and Systems. 2010 Jun;161(11):1564-1584.
[12] Amster P, De Npoli P, Tisdell CC. Variational methods for two resonant problems on time scales. International Journal of Difference Equations. 2007 Jan;2(1):1-12.
[13] Peterson A, Thompson B. Henstock-Kurzweil delta and nabla integrals. Journal of Mathematical Analysis and Applications. 2006 Nov;323(1):162-178.
[14] Wu CX, Ma M. Embedding problem of fuzzy number space: Part I. Fuzzy Sets and Systems. 1991 Apr;44:33-38.