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Invariants for Curves in Equiform Galilean Space

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Abstract. The paper studies invariants for curves in three dimensional equiform Galilean geometry. We obtain Lie algebra structure for equiform Galilean space and concrete expressions of curve invariants using Fels-Olver's moving frame method. The corresponding relationship between Galilean invariants and equiform Galilean invariants for curves is received, the Frenet formula for curves in equiform Galilean space is also showed.

Keywords. equiform Galilean space, curve invariant, moving frame method, Frenet formula

1. Introduction

The Galilean space G_3 is a three dimensional projective space, and was dealt with in detail in [1,2]. Galilean motion has important applications in dynamics, physics, control theory and other mathematical fields. Which has attracted more and more scholars to study the curves in Galilean geometry. For instance, Ogrenmis, Ergut and Bektas [3] considered helix in three dimensional Galilean space, gained its characterizations for a curve about Frenet frame. Öğrenmis and Yeneroğlu [4] considered inextensible curve flows in Galilean space and represented the curve flow as a partial differential equation about the curvature and torsion. Yilmaz [5] obtained Frenet-Serret frame and its equations of a curve in four dimensional Galilean space and proved that tangent vector of a curve satisfied a vector differential equation of fourth order. Akar, Yüce and Kuruoğlu [6] studied one-parameter planar motions in Galilean plane. Mahdipour-Shirayeh [7] demonstrated a classiflcation of spacetime curves under Galilean motions using Cartans method of equivalence. Ozturk, Cengiz and Koc Ozturk [8] established relationship between the curve motions in Galilean space, its similarity geometry and the inviscid, viscous Burgers equations.

The equiform geometry preserves angles between planes and lines, respectively. Some basic notions of equiform differential geometry of curves in G_3 was given in [9], the basic invariants, Frenet's formulas and curves of constant curvatures was obtained. Yoon [10] discussed an inextensible curve flow in the equiform geometry of Galilean space, and received a set of partial differential equations characterizing the flow. Aydin and Ergüt [11] studied equiform differential geometry of curves in four dimensional Galilean space, get the angle between the equiform Frenet vectors and their derivatives

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and characterized generalized helices about their equiform curvatures. Yoon, Lee and Lee [12] considered osculating curves and equiform osculating curves in four dimensional Galilean space and characterize such curves. Bozok, Sepet and Ergüt [13] found that the Frenet formulas and curvatures of inextensible curve flows and its equiformly invariant vector fields and intrinsic quantities were independent of time.

Now, we want to know: Is there any other way to find the spectific representation of curve invariants in three dimensional equiform Galilean space? What is the relation between curve invariants in equiform Galilean space and curve invariants in Galilean space? This paper will answer these questions. The paper's outline is given as follows: Section 2 considers curves in three dimensional equiform Galilean space. First, we give some notions about three dimensional equiform Galilean space. Second, we obtain conscrete expression of infinitesimal generators of equiform Galilean group in three dimensional Galilean space. Third, we compute the moving frame and differential invariants for curves in three dimensional equiform Galilean geometry using Fels-Olver's moving frame method [14,15]. The relationship between Galilean invariants and equiform Galilean invariants for curves is also deductived. Fourth, we establish specific representation of Frenet frame for curves in three dimensional equiform Galilean space. Section 3 is a conclusion.

2. Curves in the equiform Galilean space

First, let's give a brief introduction to basic notions from the equiform Galilean space. The transformation group in the equiform Galilean space is given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ \lambda a_1 & \lambda cos\phi & \lambda sin\phi \\ \lambda a_2 & -\lambda sin\phi & \lambda cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{1}$$

where $\lambda, a_1, a_2, \phi, a, b, c$ are real numbers.

The equiform Galilean scalar product of vectors $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$ is given in this form

$$< \alpha, \beta >= \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\ y_1 y_2 + z_1 z_2, & \text{if } x_1 = 0 \text{ and } x_2 = 0. \end{cases} \tag{2}$$

The distance between α, β is defined by the formula

$$d_{\alpha\beta} = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2. \end{cases} \tag{3}$$

Let $C : I \subset \mathbb{R} \longrightarrow G_3$ be a curve in Galilean space G_3, has the following form

$$C(x) = (x, y(x), z(x)). \tag{4}$$

2.1. The structure of Lie algebra

In three dimensional equiform Galilean geometry, the equiform Galilean transformation is given by

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix}=\begin{pmatrix}\lambda & 0 & 0\\ \lambda a_1 & \lambda cos\phi & \lambda sin\phi\\ \lambda a_2 & -\lambda sin\phi & \lambda cos\phi\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\begin{pmatrix}a\\ b\\ c\end{pmatrix}.$$

According to above expressions, the corresponding infinitesimal generators of the three dimensional equiform Galilean group in local coordinate (x,y,z) are given by

$$\begin{aligned}\mathbf{v_1}=\partial_x,\ \mathbf{v_2}=\partial_y,\ \mathbf{v_3}=\partial_z,\ \mathbf{v_4}=x\partial_y,\ \mathbf{v_5}=x\partial_z,\\ \mathbf{v_6}=z\partial_y-y\partial_z,\ \mathbf{v_7}=x\partial_x+y\partial_y+z\partial_z.\end{aligned} \tag{5}$$

For the infinitesimal generators (5), we have the following commutator table:

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{v_4}$	$\mathbf{v_5}$	$\mathbf{v_6}$	$\mathbf{v_7}$
$\mathbf{v_1}$	0	0	0	$\mathbf{v_2}$	$\mathbf{v_3}$	0	$\mathbf{v_1}$
$\mathbf{v_2}$	0	0	0	0	0	$-\mathbf{v_3}$	$\mathbf{v_2}$
$\mathbf{v_3}$	0	0	0	0	0	$\mathbf{v_2}$	$\mathbf{v_3}$
$\mathbf{v_4}$	$-\mathbf{v_2}$	0	0	0	0	$-\mathbf{v_5}$	$\mathbf{v_4}$
$\mathbf{v_5}$	$-\mathbf{v_3}$	0	0	0	0	$\mathbf{v_4}$	0
$\mathbf{v_6}$	0	$\mathbf{v_3}$	$-\mathbf{v_2}$	$\mathbf{v_5}$	$-\mathbf{v_4}$	0	0
$\mathbf{v_7}$	$-\mathbf{v_1}$	$-\mathbf{v_2}$	$-\mathbf{v_3}$	$-\mathbf{v_4}$	0	0	0

2.2. Moving frame and invariants for curves

In this subsection, we compute moving frame and invariants for curves in three dimensional equiform Galilean geometry using Fels-Olver's moving frame method [14,15].

For a parametrized curve (4), the prolonged group transformations of equiform Galilean transformation (1) are given by

$$\begin{aligned}
\bar{y}_{\bar{x}}&=\frac{d\bar{y}}{d\bar{x}}=a_1+y_x cos\phi+z_x sin\phi,\\
\bar{z}_{\bar{x}}&=\frac{d\bar{z}}{d\bar{x}}=a_2-y_x sin\phi+z_x cos\phi,\\
\bar{y}_{\bar{x}\bar{x}}&=\frac{d^2\bar{y}}{d\bar{x}^2}=\frac{1}{\lambda}(y_{xx}cos\phi+z_{xx}sin\phi),\\
\bar{z}_{\bar{x}\bar{x}}&=\frac{d^2\bar{z}}{d\bar{x}^2}=\frac{1}{\lambda}(-y_{xx}sin\phi+z_{xx}cos\phi),\\
\bar{y}_{\bar{x}\bar{x}\bar{x}}&=\frac{d^3\bar{y}}{d\bar{x}^3}=\frac{1}{\lambda^2}(y_{xxx}cos\phi+z_{xxx}sin\phi),\\
\bar{z}_{\bar{x}\bar{x}\bar{x}}&=\frac{d^3\bar{y}}{d\bar{x}^3}=\frac{1}{\lambda^2}(-y_{xxx}sin\phi+z_{xxx}cos\phi),\\
&\cdots\cdots
\end{aligned} \tag{6}$$

which are received by utilizing the differential operator

$$\frac{d}{d\bar{x}} = \frac{d}{\lambda dx} \tag{7}$$

successively to $\bar{y}$ and $\bar{z}$. Now, we select the cross-section

$$\mathscr{K} = \{0,0,0,0,0,0,1\},$$

and solve the above equations for $\lambda,\ a_1,\ a_2,\ \phi,\ a,\ b,\ c$ generates the right equivariant moving frame

$$\begin{aligned}
\lambda &= \sqrt{y_{xx}^2 + z_{xx}^2},\\
a &= -x\sqrt{y_{xx}^2 + z_{xx}^2},\\
b &= x(y_x z_{xx} - y_{xx} z_x) - y z_{xx} + z y_{xx},\\
c &= x(y_x y_{xx} + z_x z_{xx}) - y y_{xx} - z z_{xx},\\
\phi &= -arctan\frac{y_{xx}}{z_{xx}},\\
a_1 &= \frac{-y_x z_{xx} + z_x y_{xx}}{\sqrt{y_{xx}^2 + z_{xx}^2}},\\
a_2 &= \frac{-y_x y_{xx} - z_x z_{xx}}{\sqrt{y_{xx}^2 + z_{xx}^2}}.
\end{aligned} \tag{8}$$

The differential invariants are

$$\begin{aligned}
d\bar{x} &\longmapsto \sqrt{y_{xx}^2 + z_{xx}^2}dx := d\theta,\\
\bar{z}_{\bar{x}\bar{x}\bar{x}} &\longmapsto = \frac{y_{xxx} y_{xx} + z_{xxx} z_{xx}}{\sqrt{y_{xx}^2 + z_{xx}^2}^3} := \bar{\kappa},\\
\bar{y}_{\bar{x}\bar{x}\bar{x}} &\longmapsto = \frac{y_{xxx} z_{xx} - y_{xx} z_{xxx}}{\sqrt{y_{xx}^2 + z_{xx}^2}^3} := \bar{\tau},
\end{aligned} \tag{9}$$

where $\theta,\ \bar{\kappa},\ \bar{\tau}$ are the arclength, curvature and torsion of the curves in three dimensional equiform Galilean geometry, respectively. curvature $\bar{\kappa}$, torsionand $\bar{\tau}$ and their derivatives about the arclength θ form a complete system of differential invariants of curves in equiform Galilean space. Any geometric invariants in equiform Galilean space are invariant under the equiform Galilean transformation (1).

As we all know, the transformation group in three dimensional Galilean space G_3 is provided by [1]

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & cos\phi & sin\phi \\ a_2 & -sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

For the given curve (4), we repeat the above Fels-Olver's moving frame method, the corresponding arclength, curvature and torsion of the curve in three dimensional Galilean geometry are provided by

$$\begin{aligned} d\bar{x} &\longmapsto dx := ds, \\ \bar{z}_{\bar{x}\bar{x}} &\longmapsto = \sqrt{y_{xx}^2 + z_{xx}^2}^{\,3} := \kappa, \\ \bar{y}_{\bar{x}\bar{x}\bar{x}} &\longmapsto = \frac{y_{xxx}z_{xx} - z_{xxx}y_{xx}}{\sqrt{y_{xx}^2 + z_{xx}^2}} := \tau, \end{aligned} \tag{10}$$

where $s,\ \kappa,\ \tau$ are the arclength, curvature and torsion of the curves in three dimensional Galilean geometry, respectively.

Compare (9) and (10) to find the following relation between Galilean invariants and equiform Galilean invariants for curves:

$$d\theta = \kappa ds,\ \bar{\kappa} = -\frac{\kappa_s}{\kappa^2} = (\frac{1}{\kappa})_s,\ \bar{\tau} = \frac{\tau}{\kappa}. \tag{11}$$

2.3. *Frenet formulas for curves*

In this subsection, let's establish specific representation of Frenet frame for curves in equiform Galilean space.

The Frenet vectors of the curve (4) are defined by

$$\begin{aligned} \mathbf{t} &= \frac{dr}{dx} = (1,\ y_x,\ z_x), \\ \mathbf{n} &= \frac{\mathbf{t_x}}{\kappa} = \frac{1}{\kappa}(0,\ y_{xx},\ z_{xx}), \\ \mathbf{b} &= \frac{1}{\kappa}(0,\ -z_{xx},\ y_{xx}), \end{aligned} \tag{12}$$

where $\mathbf{t},\ \mathbf{n},\ \mathbf{b}$ are called the unit tangent vector, unit normal vector and binormal vector of the curve r in the Galilean space, respectively. Then the Frenet formulas of the curve r in three dimensional Galilean geometry can be the form as

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_x = \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{13}$$

Letting $\mathbf{T} = \frac{dr}{d\theta} = \frac{dr}{dx}\frac{dx}{d\theta} = \frac{1}{\kappa}(1,\ y_x,\ z_x) = \frac{\mathbf{t}}{\kappa}$, then the Frenet frame vectors in equiform Galilean space can have the following form

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{t}}{\kappa} = \frac{1}{\kappa}(1,\ y_x,\ z_x), \\ \mathbf{N} &= \frac{\mathbf{n}}{\kappa} = \frac{1}{\kappa^2}(0,\ y_{xx},\ z_{xx}), \\ \mathbf{B} &= \frac{\mathbf{b}}{\kappa} = \frac{1}{\kappa^2}(0,\ -z_{xx},\ y_{xx}). \end{aligned} \tag{14}$$

The corresponding Frenet formula in three dimensional equiform Galilean space is represented as

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}_\theta = \begin{pmatrix} \bar{\kappa} & 1 & 0 \\ 0 & \bar{\kappa} & \bar{\tau} \\ 0 & \bar{\tau} & \bar{\kappa} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}. \tag{15}$$

3. Conclusion

The paper considers invariants for curves in three dimensional equiform Galilean geomtry. We establish the relation between Galilean invariants, Frenet formulas in Galilean space and equiform Galilean invariants, Frenet formulas in equiform Galilean space. Whether there is a similar relationship between curves in higer dimensional Galilean space and in higer dimensional Equiform Galilean space is the focus of our future study.

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