Modern Management Based on Big Data IV
A.J. Tallón-Ballesteros (Ed.)

doi:10.3233/FAIA230178

New Type of Chaotic Attractors and Their Applications

Inese BULA [a,b,1], Michael A. RADIN [c]
[a] *Department of Mathematics, University of Latvia*
[b] *Institute of Mathematics and Computer Science of University of Latvia*
[c] *Rochester Institute of Technology, School of Mathematical Sciences, Rochester, New York 14623, U.S.A.*

Abstract. We study the periodic solutions of a discrete neuron model for period two or period three of the parameter (internal decay rate) $(\beta_n)_{n=0}^{\infty}$. The novelty of this research is finding a chaotic attractor for certain interval, outside the defined interval the solution goes to positive infinity or to negative infinity. The investigation can be useful in the design of chaos-based neural networks architecture.

Keywords. neuron model, discrete dynamical system, difference equation, chaotic attractor

1. Introduction

In [1], the authors investigated the delayed differential equation

$$x'(t) = -g(x(t-\tau)), \tag{1}$$

that is used to model a single neuron, where $g\colon \mathbf{R} \to \mathbf{R}$ is signal function and $\tau \le 0$ is a synaptic transmission delay. A more historical insight into this equation can be found in [2]. From Eq. (1) a model for a single neuron is obtained $x'(t) = -g(x[t]))$, where $[t]$ denote a greatest integer function. When we integrate this equation from n to $t \in [n, n+1]$ we get $x(t) = x(n) - \int_n^t g(x([s]))\,ds = x(n) - g(x(n))(t-n)$. By letting $t \to n+1$ and denoting $x(n) = x_n$ a difference equation is obtained $x_{n+1} = x_n - g(x_n)$. This equation is generalized for a discrete-time network of a single neuron model ([3]):

$$x_{n+1} = \beta x_n - g(x_n),\ n = 0, 1, 2, ..., \tag{2}$$

where $\beta > 0$ is an internal decay rate and the signal function g is the following piecewise constant function with McCulloch-Pitts nonlinearity:

$$g(x) = \begin{cases} 1, x \ge 0, \\ -1, x < 0. \end{cases} \tag{3}$$

[1]Corresponding Author: Inese Bula, Department of Mathematics, Faculty of Physics, Mathematics and Optometry, University of Latvia, Jelgavas 3, Riga, and Institute of Mathematics and Computer Science of University of Latvia, Raina boulv. 29, Riga, Latvia; E-mail: inese.bula@lu.lv

Several authors investigated Eq. (2) ([3,4,5,6,7,8], etc.). Difference equations have been used as mathematical models for applications including neurons (see [9]).

The novelty of the authors of this article is the proposal to view the internal decay rate β as a sequence of periodic numbers $(\beta_n)_{n=0}^{\infty}$. In [10,11,12], the authors investigated the periodic solutions of a discrete neuron model when $(\beta_n)_{n=0}^{\infty}$ is periodic with periods two and three. The existence of periodic points is different for sequences with period two (even number) and three (odd number). In [13], the authors consider the situation when the sequence is periodic with period 2 and show that at certain values of the coefficients β_0 and β_1 a chaotic attractor is formed. It could be said that we deliberately create such data (big data) that could be used to achieve certain goals.

In this article, we will focus on the sequences with period 3. We study the following non-autonomous piecewise linear difference equation:

$$x_{n+1} = \beta_n x_n - g(x_n), \tag{4}$$

where $(\beta_n)_{n=0}^{\infty}$ is a periodic sequence with period three where

$$\beta_n = \begin{cases} \beta_0, & \text{if } n = 3k, \\ \beta_1, & \text{if } n = 3k+1, \; k = 0,1,2,\dots, \\ \beta_2, & \text{if } n = 3k+2, \end{cases}$$

$\beta_0 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, all β_i, $i = 1,2,3$, are not equal, with function g in form Eq. (3).

If we consider the right side of difference Eq. (4) as a function $h : \mathbf{R} \to \mathbf{R}$ and let $x_n = h^n(x_0)$, $x_0 \in \mathbf{R}$, $n = 1,2,\dots$, then we obtain the first order difference equation $x_{n+1} = h(x_n)$ with initial condition $x_0 \in \mathbf{R}$. From the definition of Eq. (4), it follows that first iteration of function h is in the form: $x_1 = h(x_0) = \begin{cases} \beta_0 x_0 - 1, & x_0 \geq 0, \\ \beta_0 x_0 + 1, & x_0 < 0. \end{cases}$ Depending on the circumstance, sometimes it is more convenient to describe the dynamics more easily with the behavior of a function, and at other times with a difference equation.

In general, we consider a first order difference equation (see [14]) $x_{n+1} = f(x_n)$, where $f : \mathbf{R} \to \mathbf{R}$. Then the *orbit of a point* $x_0 \in \mathbf{R}$ is defined to be the set of points $\{x_0, x_1 = f(x_0), x_2 = f(f(x_0)) = f^2(x_0), \dots, x_n = f^n(x_0), \dots\}$. A point x^* is said to be a *fixed point* of the map f or an *equilibrium point* of equation $x_{n+1} = f(x_n)$ if $f(x^*) = x^*$.

The concept of periodicity is one of the most important notion in the field of dynamical systems. Its importance follows from the fact that many physical phenomena have certain patterns that repeat themselves (for example, the motion of a pendulum, the motion of planets, the population size of blowflies or other insects at time n, the price of commodity at time n).

Let $\bar{x}$ be in the domain of a mapping f. A point $\bar{x}$ is said to be a *periodic point* of f with period k if $f^k(\bar{x}) = \bar{x}$ for some positive integer k. Note that $\bar{x}$ is a periodic point with period k if it is a fixed point of the map f^k.

We organize our paper as follows. In the next section we present results about Eq. (4). In section 3 we analyze the Lyapunov exponent and find this exponent for our dynamical system. We show that for certain values of coefficients β_i there exists a chaotic attractor. At the end we give some concluding remarks, applications and future ideas.

2. Some results about difference equations with period three coefficients

In this chapter, we refer to some of the most important results of article [11].

We remark that Eq. (4) with g in form Eq. (3) has no equilibrium points.

Theorem 1 *The equation (4) has no periodic orbits with periods* $3n+1$ *and* $3n+2$, $n = 0,1,2,\ldots$.

The number of periodic orbits depends on the relationship between the parameters β_0, β_1 and β_2. In case with two periodic coefficients what both less than 1 exist only periodic solution with period two but in case with three periodic coefficients what all three are less than 1 not exist periodic solution with period three, in this case exist only periodic solution with period six.

If the product of the coefficients $\beta_1\beta_2\beta_3$ is strictly greater than 1 then there are always solutions with period three.

Theorem 2 *If* $\beta_0\beta_1\beta_2 > 1$ *then initial conditions*

$$x_0 = \frac{\beta_1\beta_2 + \beta_2 + 1}{\beta_0\beta_1\beta_2 - 1} \text{ and } x_0 = -\frac{\beta_1\beta_2 + \beta_2 + 1}{\beta_0\beta_1\beta_2 - 1}$$

form periodic solutions of equation (4) with period three; all points of orbit are positive in first case and negative in second case and both orbits are unstable.

If $\beta_0\beta_1\beta_2 > 1$, then we have observed that our difference equation exhibits unbounded solutions. Observe that in the conditions of the next theorem, the inequalities include the initial points of a cycle with period three (Theorem 2).

Theorem 3 *If* $\beta_0\beta_1\beta_2 > 1$ *and* $x_0 > \dfrac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2 - 1}$, *then* x_0 *forms unbounded solutions of Eq. (4) - going to* $+\infty$. *If* $\beta_0\beta_1\beta_2 > 1$ *and* $x_0 < -\dfrac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2 - 1}$, *then* x_0 *forms unbounded solutions of Eq. (4) - going to* $-\infty$.

3. Chaotic attractor

In this chapter, we will prove that the Eq. (4) (we can also say the function h) forms a chaotic attractor in the set $[-1,\,1]$ under the condition $1 < \beta_i \le 2$, $i = 0,1,2$.

First, our aim is to determine the invariant interval (a set $I \subset X$ is said to be invariant under the map $f : X \to X$ if $f(I) = I$).

The invariant interval must contain the entire interval $[-1,\,1]$.

Now suppose that $x_0 \in [-1,\,1]$. Then the following statements hold true:

a) if $0 \le x_0 \le 1$, then $-1 = 0-1 \le h(x_0) = \beta_i x_0 - 1 \le 2\cdot 1 - 1 = 1$, $i = 0,1,2$,

b) if $-1 \le x_0 < 0$, then $-1 = 2\cdot(-1)+1 \le h(x_0) = \beta_i x_0 + 1 < 0+1 = 1$, $i = 0,1,2$.

Definition 1 *The Lyapunov exponent* $\lambda(x_0)$ *of the orbit* $\{x_0, x_1, x_2, \ldots\}$ *is defined as*

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(x_k)|,$$

provided that the limit exists.

In [14], the authors showed that if the Lyapunov exponent $\lambda > 0$, then the sensitivity dependence on initial conditions exists. The Lyapunov exponent at a point x measures the growth in error per iteration. As the Lyapunov exponent becomes larger, the magnification of error becomes greater.

Theorem 4 *If $\beta_0\beta_1\beta_2 > 1$, then function h have a positive Lyapunov exponent.*

Proof. For every $\beta_0 > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and arbitrary initial point x_0 (what is not point of discontinuity) the Lyapunov exponent is

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|h'(x_k)| =$$

$$= \lim_{n\to\infty} \frac{1}{n}(\ln\beta_0 + \ln\beta_1 + \ln\beta_2 + \ldots + \ln\beta_0 + \ln\beta_1 + \ln\beta_2 + i_1\cdot\ln\beta_0 + i_2\cdot\ln\beta_1) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{\ln(\beta_0\beta_1\beta_2)\cdot(n-i_1-i_2)}{3} + i_1\cdot\ln\beta_0 + i_2\cdot\ln\beta_1\right) = \frac{\ln(\beta_0\beta_1\beta_2)}{3} > \frac{\ln 1}{3} = 0,$$

where

$$i_1 = \begin{cases} 0, \text{ if } n = 3m, \\ 1, \text{ if } n = 3m+1 \text{ or } 3m+2, \end{cases} \quad i_2 = \begin{cases} 0, \text{ if } n = 3m \text{ or } 3m+1, \\ 1, \text{ if } n = 3m+2, \end{cases} \quad m \in \mathbf{N}.$$

■

We will show that for certain values of the coefficients β_1, β_2 and β_2 Eq. (4) forms a chaotic system.

A Discrete Dynamical System, denoted by DDS for short, is the description of an evolutive phenomenon in terms of a map f whose image is contained in its domain X. Then the pair $\{X, f\}$ is called DDS.

Definition 2 *([15], see [16]) A set $A \subset I$ is called an attractor for a DDS $\{I, f\}$ if the following conditions hold:*
1) A is closed;
2) A is invariant;
3) there exists $\eta > 0$ such that, for any $x \in I$ fulfilling $dist(x,A) < \eta$, we have $\lim_{k\to\infty} dist(f^k(x),A) = 0$;
4) A is a minimal, that is there are no proper subsets of A fulfilling 1), 2) and 3).

Definition 3 *([15]) If A is an attractor of function f, then the set*

$$\left\{x \in \mathbf{R} \,|\, \lim_{k\to\infty} f^k(x) \in A\right\}$$

is called an attraction basin of attractor A.

Definition 4 *([16]) An invariant set A is called a chaotic attractor provided it is an attractor and f has sensitive dependence on initial conditions on A (or f have a positive Lyapunov exponent on A).*

Theorem 5 *Let* $1 < \beta_0 \le 2,\ 1 < \beta_1 \le 2,\ 1 < \beta_2 \le 2$ *and at least two from them are different. Then* $[-1, 1]$ *is a chaotic attractor of function h and attraction basin is* $\left]-\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}, \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}\right[$.

Proof. In case $1 < \beta_0 \le 2, 1 < \beta_1 \le 2, 1 < \beta_2 \le 2$ and at least two from them are different, the interval $[-1, 1]$ is an invariant set for the function h and the Lyapunov exponent is positive by Theorem 4. Thus $[-1, 1]$ is a chaotic attractor of function h.

Since $1 < \beta_0 \le 2, 1 < \beta_1 \le 2, 1 < \beta_2 \le 2$ and

$$\beta_1\beta_2 + \beta_2 + 1 \ge \beta_0\beta_1\beta_2 - 1 \Leftrightarrow 2 \ge \beta_0\beta_1\beta_2 - \beta_1\beta_2 - \beta_2 = \beta_2(\beta_1(\beta_0 - 1) - 1)$$

then $\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} \ge 1$. Similarly, it can be proved that all points of a cycle with period three are greater than 1, that is, also $\frac{\beta_0\beta_2+\beta_0+1}{\beta_0\beta_1\beta_2-1} \ge 1$ and $\frac{\beta_0\beta_1+\beta_1+1}{\beta_0\beta_1\beta_2-1} \ge 1$.

Let $\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} > 1$. Our aim is to show that for all

$$x_0 \in \left]-\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}, \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}\right[\setminus [-1, 1]$$

the orbit by the function h eventually falls in the interval $[-1, 1]$. We consider only the case when $1 < x_0 < \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}$. The case when $-\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} < x_0 < -1$ is similar and will be omitted.

If $1 < x_0 < \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}$, then

$$0 < \beta_0 - 1 < x_1 = \beta_0 x_0 - 1 < \frac{\beta_0(\beta_1\beta_2+\beta_2+1)}{\beta_0\beta_1\beta_2-1} - 1 = \frac{\beta_0\beta_2+\beta_0+1}{\beta_0\beta_1\beta_2-1}.$$

If $0 < x_1 \le 1$, then the proof is complete. If this is not the case, then $1 < x_1 < \frac{\beta_0\beta_2+\beta_0+1}{\beta_0\beta_1\beta_2-1}$ and therefore

$$0 < \beta_1 - 1 < x_2 = \beta_1 x_1 - 1 < \frac{\beta_1(\beta_0\beta_2+\beta_0+1)}{\beta_0\beta_1\beta_2-1} - 1 = \frac{\beta_0\beta_1+\beta_1+1}{\beta_0\beta_1\beta_2-1}.$$

If $0 < x_2 \le 1$, then the proof is complete. If this is not the case, then $1 < x_2 < \frac{\beta_0\beta_1+\beta_1+1}{\beta_0\beta_1\beta_2-1}$ and therefore

$$0 < \beta_2 - 1 < x_3 = \beta_2 x_2 - 1 \frac{\beta_2(\beta_0\beta_1+\beta_1+1)}{\beta_0\beta_1\beta_2-1} - 1 = \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}.$$

Provided that $x_n \notin [-1, 1]$, by induction we then conclude that

$$\begin{aligned}
&1 < x_{3k} = \beta_0^k\beta_1^k\beta_2^k x_0 - \beta_0^{k-1}\beta_1^k\beta_2^k - \beta_0^{k-1}\beta_1^{k-1}\beta_2^k - \ldots - \beta_2 - 1 < \tfrac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1},\\
&1 < x_{3k+1} = \beta_0^{k+1}\beta_1^k\beta_2^k x_0 - \beta_0^k\beta_1^k\beta_2^k - \beta_0^k\beta_1^{k-1}\beta_2^k - \ldots - \beta_0 - 1 < \tfrac{\beta_0\beta_2+\beta_0+1}{\beta_0\beta_1\beta_2-1},\\
&1 < x_{3k+2} = \beta_0^{k+1}\beta_1^{k+1}\beta_2^k x_0 - \beta_0^k\beta_1^{k+1}\beta_2^k - \beta_0^k\beta_1^k\beta_2^k - \ldots - \beta_1 - 1 < \tfrac{\beta_0\beta_1+\beta_1+1}{\beta_0\beta_1\beta_2-1},\\
&k = 0, 1, 2, \ldots.
\end{aligned}$$

Next note that the difference between the iterations x_{3k} and x_{3k+3} is

$$x_{3k}-x_{3k+3}=\beta_0^k\beta_1^k\beta_2^k x_0-(\beta_0^{k+1}\beta_1^{k+1}\beta_2^{k+1}x_0-\beta_0^k\beta_1^{k+1}\beta_2^{k+1}-\beta_0^k\beta_1^k\beta_2^{k+1}-\beta_0^k\beta_1^k\beta_2^k)=$$
$$=(\beta_0\beta_1\beta_2)^k(\beta_0\beta_1\beta_2-1)\left(\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}-x_0\right),\ k=0,1,2,\ldots.$$

Since all multipliers are positive and $\lim\limits_{k\to\infty}(\beta_0\beta_1\beta_2)^k=+\infty$ then the difference between x_{3k} and x_{3k+3} increases and we then get $x_0>x_3>x_6>\ldots>x_{3k}>x_{3k+3}>\ldots$. Thus we conclude that there exists $k\in\mathbf{N}$ such that $x_{3k}\leq 1$.

Similarly, the difference between iterations x_{3k+1} and x_{3k+4}, x_{3k+2} and x_{3k+5} increases and we get $x_1>x_4>x_7>\ldots>x_{3k+1}>x_{3k+4}>\ldots$ and $x_2>x_5>x_8>\ldots>x_{3k+2}>x_{3k+5}>\ldots$. Thus we conclude that there exists $k_1\in\mathbf{N}$ such that $x_{3k_1+1}\leq 1$ and there exists $k_2\in\mathbf{N}$ such that $x_{3k_2+2}\leq 1$. ■

Example 1 Suppose that $\beta_0=1.92$, $\beta_1=1.26$ and $\beta_2=1.5$. We obtain the following period-3 cycle $\{1.669963481, 2.206329884, 1.779975654\}$ and the following basin of attraction $]-1.669963481, 1.669963481[$. If we start with initial condition $x_0=1.66$ (a point close to the boundary of the interval), then we observe the situation described in Theorem 5, where the first ten iterations of the solution are greater than 1 (see Fig. 1). Then $x_{11}=0.628189142<1$ and all other points of the solution are in the interval $[-1, 1]$. In Fig. 1 we see that

$$x_0>x_3>x_6>x_9>x_{12},\quad x_1>x_4>x_7>x_{10}>x_{13},\quad x_2>x_5>x_8>x_{11}.$$

The behavior of the other points cannot be clearly described, but all other points are located in the invariant interval (attractor) $[-1, 1]$.

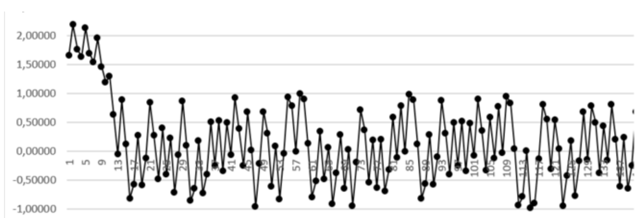

Figure 1. First 140 values of solution of difference equation (4) if $\beta_0=1.92$, $\beta_1=1.26$ and $\beta_2=1.5$, and $x_0=1.66$.

4. Conclusion

In this article, we have shown a new example of a chaotic attractor. The novelty is based on the use of periodicity. It could be that at certain values of the parameters β_i, $i=0,1,2$, which are not all in the interval $]1;2]$, there exist some other chaotic attractors. The case with $\beta_0=1$, $\beta_1=2$ and $\beta_2=3$ mentioned in the article [11] is interesting. Although there could be an invariant interval $[-2, 2]$ here, numerical experiments show that all solutions are periodic or eventually periodic.

The properties of chaos (sensitivity to initial conditions) can be used in random number generation (see, for example, [17]) as well as in cryptography (see, for example, [18]). An important role here is played the uniform distribution of elements of the solution. For example, the histogram of the solution of the difference equation (4) with $\beta_0 = 2$, $\beta_1 = 1.8$ and $\beta_2 = 2$ with the initial condition $x_0 = 0.6$ is shown in Fig. 2. The larger the values of β_i, $i = 0, 1, 2$, the greater the sensitivity to the initial conditions (larger Lyapunov exponent), the more appropriate the histogram looks.

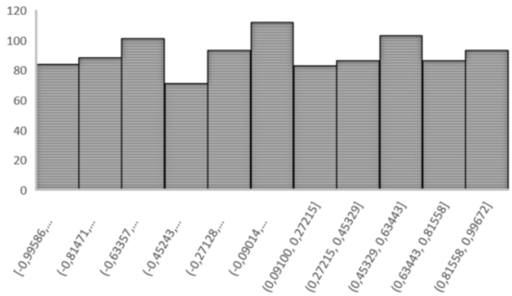

Figure 2. The histogram of the solution (first 1000 values) of the difference equation (4) with $\beta_0 = 2$, $\beta_1 = 1.8$, $\beta_2 = 2$ and initial condition $x_0 = 0.6$.

References

[1] Wu J. Introduction to Neural Dynamics and Signal Transmission Delay. Berlin: De Gruyter; 2001.
[2] Halanay A, Yorke JA. Some New Results and Problems in the Theory of Differential-Delay Equations. SIAM Review. 1971; 13:55-80.
[3] Chen Y. All solutions of a class of difference equations are truncated periodic. Appl. Math. Lett. 2002; 15:975-9.
[4] Bin H, Huang L, Zhang G. Convergence and periodicity of solutions for a class of difference systems. Adv. Difference Equ. 2006; 2006: [1]-10.
[5] Wei Z, Huang L, Meng Y. Unboundedness and periodicity of solutions for a discrete-time network model of three neurons. Appl. Math. Model. 2008; 32:1463-74.
[6] Zhou Z. Periodic orbits on discrete dynamical systems. Comput. Math. Appl. 2003; 45:1155-61.
[7] Zhu H, Huang L. Dynamics of a class of nonlinear discrete-time neural networks. Comput. Math. Appl. 2004; 48:85-94.
[8] Yi T, Zhou Z. Periodic solutions of difference equations. J. Math. Anal. Appl. 2002; 286:220-29.
[9] Ibaz B, Casado JM, Sanjuan MA. Map-based models in neuronal dynamics. Phys. Rep. 2011; 501:1-74.
[10] Bula I, Radin MA. Periodic orbits of a neuron model with periodic internal decay rate. Appl. Math. Comput. 2015; 266:293-303.
[11] Bula I, Radin MA, Wilkins N. Neuron model with a period three internal decay rate. Electron. J. Qual. Theory Differ. Equ. 2017; 46:[1]-19.
[12] Bula I, Radin MA. Eventually periodic solutions of single neuron model. Nonlinear Anal. Model. Control. 2020; 25:903-18.
[13] Bula I, Radin MA. Chaotic single neuron model with periodic coefficients with period two. Unpublished.
[14] Elaydi SN. Discrete Chaos: with Applications in Science and Engineering. Sec.ed., Boca Raton: ChapmanHall/CRC; 2008.
[15] Salinelli E, Tomarelli F. Discrete Dynamical Models. Unitext, 76, Switzerland: Springer; 2014.
[16] Robinson C. Dynamical Systems: Stability, Symbolic Dynamics, and Chaos. Florida: CRS Press; 1995.
[17] Addabbo T, Fort A, Rocchi S, Vignoli V. Digitized chaos for pseudo-random number generation in cryptography. In: Kocarev L, Lian S, editors. Chaos-based Cryptography. Theory, Algorithms and Applications. Berlin, Heidelberg: Springer-Verlag; 2011. p. 67-97.
[18] Schmitz R. Use of chaotic dynamical systems in cryptography. Journal of The Franklin Institute. 2001; 338:429-41.