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# Derivation Algebras of Restricted Hom-Lie Triple Systems

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**Abstract.** In this paper, we investigate the properties of derivations of restricted Hom-Lie triple systems. By giving the concepts of derivations, generalized derivations, centroids and Hom-$p$-subsystems, we obtain some good results on generalized derivation algebras and their subalgebras, which including the subspace $\mho$ satisfying some conditions in the linear transformation $\mathrm{End}(L)$ is a restricted Hom-Lie triple system, the generalized derivation is a Hom-$p$-subsystem of $\mho$ and the central derivation is the intersection of the centroid and the derivation. The novelty of this paper is to discuss the derivation properties of Hom-Lie triple systems on the field $\mathbb{F}$ of characteristic $p$, which makes the derivation structure of restricted Hom-Lie triple systems more complete.

**Keywords.** derivations, restricted Hom-Lie triple systems, Hom-$p$-subsystems

## 1. Introduction

Lie triple system first appeared in Cartan's study of Riemannian geometry. As the object of algebraic research, it was introduced by Jacobson in 1949, which is mainly used to study the subspace of closed associative algebras under the triple commutator $[[a,b],c]$. Through Lie triple system, the theory of symmetric space and algebraic theory can be closely linked, so as to establish a bridge between algebraic problems and geometric problems. Therefore, as a new algebraic system, Lie triple system has attracted the attention and extensive research of scholars.

This paper is based on two aspects. On the one hand, Terrell.L Hodge [1] gave the concept of restricted Lie triple systems in 2001. Dong [2] studied some properties and relations of restricted Lie triple systems and restrictable Lie triple systems. In [3], Liu and Chen developed the Frattini theory of groups to restricted Lie triple systems, and obtained the properties of Frattini-subsystems and Frattini $p$-subsystems of restricted Lie triple systems. On this basis, Chen et al. [4] gave some conditions for the commutative property of restricted Lie triple systems, and characterized some features of $p$-mappings and semi-elements of restricted Lie triple systems. In 2023, Liu [5] studied the relationship among derivation structures of restricted Lie triple systems.

On the other hand, in the 1990s, Hom-type algebras emerged in physics so as to find quantum deformations of some algebras on vector fields. Hom-type algebras are closely related to theoretical physics, Yang-Baxter equations, braid groups and quantum groups,

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and many important results have been obtained. In 2012, Yau [6] first discussed Hom-Lie triple systems. In [7], Zhou et al. studied the properties of generalized derivation algebras and centroids of Hom-Lie triple systems and the relationship between them. Then, Chen et al. [8] studied the product structure and complex structure of Hom-Lie triple systems from the algebraic point of view, and gave the corresponding decomposition. In 2022, B-hutia et al. [9] studied equivariant cohomolgy theory, one-parameter formal deformation and central extensions of Hom-Lie triple systems in the case of equivariant.

At present, the discussion on the restricted Hom-structure has attracted more attention from scholars at home and abroad. In 2015, Guan [10] studied the structure of restricted Hom-Lie algebras. In [11], Shaqaqha introduced the restricted Hom-Lie Superalgebras. Although many achievements have been made in such research, the study on the problem of derivation structure is not complete, and there are still many problems that need to be further discussed. In [12], Yara gave the definition of restricted Hom-Lie triple system. Inspired by the above literatures, this paper mainly studies the derivation problem under the premise of restricted Hom-Lie triple system. In particular, it is proved that the subspace $\mho$ satisfying certain conditions in the linear transformation $\mathrm{End}(M)$ is restricted Hom-Lie triple system.

This paper is structured as follows. Firstly, Section 2 mainly contains some related work. Secondly, in Section 3, we mainly present some fundamental definitions about restricted Hom-Lie triple systems, derivations and linear transformations satisfying certain conditions. Then, in Section 4, we prove some properties between them. Finally, we summarize the article and further prospects. In this paper, we always assume that the characteristic of $\mathbb{F}$ is $p$, $p$ is prime and $p \geq 2$.

## 2. Related Work

In [7], Zhou et al. studied the properties of generalized derivation algebras and centroids of Hom-Lie triple systems and the relationship between them. In 2023, Liu [5] studied the structural relationship between derivations of restricted Lie triple systems. Recently, the definition of restricted Hom-Lie triple systems is given. However, the properties of the derivation structure in this context have not been studied as far as I know.

## 3. Preliminaries

**Definition 3.1.** *[7] Let $(M,[\cdot,\cdot,\cdot],\alpha=(\alpha_1,\alpha_2))$ is a triple, where $M$ is a vector space over a field $\mathbb{F}$, $[\cdot,\cdot,\cdot]: M\times M\times M\to M$ is a trilinear map, and twisted maps $\alpha_i: M\to M(i=1,2)$, such that for all $a,b,c,u,v\in M$,*

$$[a,a,c]=0;$$

$$[a,b,c]+[b,c,a]+[c,a,b]=0;$$

$$[\alpha_1(u),\alpha_2(v),[a,b,c]]=[[u,v,a],\alpha_1(b),\alpha_2(c)]+[\alpha_1(a),[u,v,b],\alpha_2(c)]+[\alpha_1(a),\alpha_2(b),[u,v,c]],$$

*then $(M,[\cdot,\cdot,\cdot],\alpha=(\alpha_1,\alpha_2))$ is a Hom-Lie triple system.*

**Definition 3.2.** *[12] Let $(M,[\cdot,\cdot,\cdot],\alpha=(\alpha_1,\alpha_2))$ is a Hom-Lie triple system,*

*(1)if $\alpha=\alpha_1=\alpha_2$ and $\alpha([a,b,c])=[\alpha(a),\alpha(b),\alpha(c)]$, for all $a,b,c\in M$, then we call $M$ is a multiplicative Hom-Lie triple system;*

*(2)if $M$ is a multiplicative Hom-Lie triple system and $\alpha$ is invertible, then $M$ is called regular.*

**Definition 3.3.** *Let $(M,[\cdot,\cdot,\cdot],\alpha)$ be a regular Hom-Lie triple system, $H$ is the subspace of $M$, if $\alpha(H)\subseteq H$, for all $a,b,c\in H$, $[a,b,c]_M\in H$, then $H$ is a Hom-subsystem of $M$.*

From Definition 3.3, it is easy to prove that if $M_1=\{a\in M \mid \alpha(a)=a\}$, then $M_1$ is a Hom-subsystem of $M$.

Let $M$ is a regular Hom-Lie triple system, we use $L(M,M)$ to represent the space linearly spanned by $L(a,b)$ and define $L_s(M)=L(M,M)\oplus M$.

**Definition 3.4.** *[12] Let $(M,[\cdot,\cdot,\cdot],\alpha)$ be a regular Hom-Lie triple system, if there is a map $[p]:M_1\to M_1, x\mapsto x^{[p]}$, satisfying the following conditions:*

$$(ka)^{[p]}=k^p a^{[p]},\forall a\in M_1,k\in\mathbb{F},$$

$$(a+b)^{[p]}=a^{[p]}+b^{[p]}+\sum_{i=1}^{p-1}s_i(a,b),\forall a,b\in M_1,$$

$$[\alpha^{-1}(a),\alpha^{-1}(b),c^{[p]}]=(a,b,c,\cdots,c)(p\ copies\ of\ c),\forall a,b\in M,c\in M_1,$$

$$[\alpha^{-1}(a),b^{[p]},\alpha^{-1}(c)]=(a,b,\cdots,b,c)(p\ copies\ of\ b),\forall a,c\in M,b\in M_1,$$

*where is$s_i(a,b)$ is the coefficient of $\lambda^{i-1}$ in $(\mathrm{ad}(\lambda a+b))^{p-1}(a)\in L_s(M)$, $(a,b,\cdots,b,c)=[[[[a,b,b],b,b],\cdots],b,c]$ and $(a,b,c,\cdots,c)=[[[[a,b,c],c,c],\cdots],c,c]$, then $(M,[\cdot,\cdot,\cdot],\alpha,[p])$ is a restricted Hom-Lie triple system.*

When $\alpha=id_M$, we call restricted Hom-Lie triple system is the restricted Lie triple system. From Definition 3.4, for all $a\in M_1$, we have $a^{[p]}\in M_1$. Then $\alpha(a^{[p]})=a^{[p]}=(\alpha(a))^{[p]}$, that is $\alpha\circ[p]=[p]\circ\alpha$.

**Definition 3.5.** *Let $(M,[\cdot,\cdot,\cdot],\alpha,[p])$ be a restricted Hom-Lie triple system over $\mathbb{F}$, define the subspace $\Omega$ of $\mathrm{End}(M)$ and linear transformation $\widetilde{\alpha}$ on $\Omega$ which satisfy*

$$\Omega=\{K\in\mathrm{End}(M)\mid K\alpha=\alpha K\}$$

*and*

$$\widetilde{\alpha}:\Omega\to\Omega;\widetilde{\alpha}(K)=\alpha K,$$

*where $\widetilde{\alpha}$ is multiplicative and $\mathrm{End}(M)$ is composed of linear maps $K$ on $M$.*

By Definition 3.5, $\Omega$ is a restricted Hom-Lie triple system with respect to the bracket operation $[K_1,K_2]=K_1K_2-K_2K_1,\forall K_1,K_2\in\Omega$ (see Theorem 4.1).

**Definition 3.6.** *Let $M$ be a restricted Hom-Lie triple system. $H$ is the subspace of $M$. If $\alpha(H)\subseteq H,[H,H,H]_M\subseteq H$, for all $a\in H_1$, there is $a^{[p]}\in H_1$, in which $H_1=\{a\in H\mid \alpha(a)=a\}$, then $H$ is the Hom-p-subsystem of $M$.*

**Definition 3.7.** *Let $M$ be a restricted Hom-Lie triple system. $K \in \mathrm{End}(M)$ is called to be an $\alpha^k$-derivation of $M$ (where $(k \geq 0, k \in N)$), if it satisfies for all $a,b,c \in M$,*

$$K\alpha = \alpha K,$$
$$K([a,b,c]) = [K(a), \alpha^k(b), \alpha^k(c)] + [\alpha^k(a), K(b), \alpha^k(c)] + [\alpha^k(a), \alpha^k(b), K(c)].$$

We use $\mathrm{Der}_{\alpha^k}(M)$ to represent all $\alpha^k$-derivations and define $\mathrm{Der}(M) = \bigoplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(M)$, then $\mathrm{Der}(M)$ about the bracket operation and linear transformation

$$\widetilde{\alpha} : \mathrm{Der}(M) \to \mathrm{Der}(M); \widetilde{\alpha}(K) = K\alpha$$

is the Hom-$p$-subsystem of $\Omega$, which is called the derivation algebra of $M$.

**Definition 3.8.** *Let $M$ be a restricted Hom-Lie triple system. A linear map $K : M \to M$ is called to be a generalized $\alpha^k$-derivation of $M$, if there exist $K', K'', K''' \in \mathrm{End}(M)$, such that for all $a,b,c \in M$,*

$$K\alpha = \alpha K, K'\alpha = \alpha K', K''\alpha = \alpha K'', K'''\alpha = \alpha K''',$$
$$K'''([a,b,c]) = [K(a), \alpha^k(b), \alpha^k(c)] + [\alpha^k(a), K'(b), \alpha^k(c)] + [\alpha^k(a), \alpha^k(b), K''(c)].$$

Let $\mathrm{GDer}_{\alpha^k}(M)$ represent the set of generalized $\alpha^k$-derivation and define $\mathrm{GDer}(M) = \bigoplus_{k\geq 0} \mathrm{GDer}_{\alpha^k}(M)$.

**Definition 3.9.** *Let $M$ be a restricted Hom-Lie triple system. $K \in \mathrm{End}(M)$ is called to be an $\alpha^k$-centroid of $M$, if it satisfies for all $a,b,c \in M$,*

$$K\alpha = \alpha K,$$
$$K([a,b,c]) = [K(a), \alpha^k(b), \alpha^k(c)] = [\alpha^k(a), K(b), \alpha^k(c)] = [\alpha^k(a), \alpha^k(b), K(c)].$$

Let $\mathrm{C}_{\alpha^k}(M)$ be the set of $\alpha^k$-centroid. Define $\mathrm{C}(M) = \bigoplus_{k\geq 0} \mathrm{C}_{\alpha^k}(M)$.

**Definition 3.10.** *Let $M$ be a restricted Hom-Lie triple system. A linear map $K : M \to M$ is said to be an $\alpha^k$-central derivation of $M$, if it satisfies for all $a,b,c \in M$,*

$$K\alpha = \alpha K,$$
$$K([a,b,c]) = [K(a), \alpha^k(b), \alpha^k(c)] = 0.$$

Let $\mathrm{ZDer}_{\alpha^k}(M)$ be the set of $\alpha^k$-central derivation, then $\mathrm{ZDer}(M) = \bigoplus_{k\geq 0} \mathrm{ZDer}_{\alpha^k}(M)$.

## 4. Main Results

**Theorem 4.1.** *Let $M$ be a restricted Hom-Lie triple system and $\Omega$ defined as above, then $\Omega$ is a restricted Hom-Lie triple system.*

*Proof.* Since Lie triple system is the natural ternary extension of Lie algebra, we only need to prove that $\Omega$ is a Hom-Lie algebra. For $K_1, K_2, K_3 \in \Omega$, by direct calculation, we can get that

$$[K_1, K_2] = -[K_2, K_1],$$
$$[\widetilde{\alpha}(K_1), [K_2, K_3]] + [\widetilde{\alpha}(K_2), [K_3, K_1]] + [\widetilde{\alpha}(K_3), [K_1, K_2]] = 0.$$

Therefore, $\Omega$ is a regular Hom-Lie triple system.

Next, let $\Omega_1$ satisfy $\Omega_1 = \{K_1' \in \Omega \mid \widetilde{\alpha}(K_1') = K_1'\}$, it is obvious that $\Omega_1$ is the Hom-subsystem of $\Omega$. Define a map $[p] : \Omega_1 \to \Omega_1; K_1' \mapsto (K_1')^p$. For $K_1, K_2 \in \Omega, K_1', K_2' \in \Omega_1, k \in \mathbb{F}$, in view of Definition 3.4, we have

$$(kK_1')^{[p]} = (kK_1')(kK_1')\cdots(kK_1') = k^p(K_1')^p = k^p(K_1')^{[p]},$$
$$(K_1'+K_2')^{[p]} = (K_1')^{[p]} + (K_2')^{[p]} + \sum_{i=1}^{p-1} s_i(K_1', K_2').$$

To verify $[\widetilde{\alpha}^{-1}(K_1), (K_1')^{[p]}, \widetilde{\alpha}^{-1}(K_2)] = (K_1, K_1', \cdots, K_1', K_2)$, it is necessary to prove that

$$[K_1, \widetilde{\alpha}((K_1')^{[p]}), K_2] = [K_1, (K_1')^{[p]}, K_2] = [[[\widetilde{\alpha}(K_1), K_1', K_1'], \cdots], K_1', \widetilde{\alpha}(K_2)].$$

The following mathematical induction proves that when $n \geq 3$, we have

$$[[[\widetilde{\alpha}(K_1), K_1', K_1'], \cdots], K_1', \widetilde{\alpha}(K_2)] = [\sum_{i=0}^{n}(-1)^i C_n^i (K_1')^i K_1 (K_1')^{n-i}, K_2].$$

In fact, for $n = 3$,

$$[[\widetilde{\alpha}(K_1), K_1', K_1', K_1', \widetilde{\alpha}(K_2)] = [\sum_{i=0}^{3}(-1)^i C_3^i (K_1')^i K_1 (K_1')^{3-i}, K_2].$$

Assume that the conclusion holds when $n = m$, that is

$$[[[\widetilde{\alpha}(K_1), K_1', K_1'], \cdots], K_1', \widetilde{\alpha}(K_2)] = [\sum_{i=0}^{m}(-1)^i C_m^i (K_1')^i K_1 (K_1')^{m-i}, K_2].$$

When $n = m+1$,

$$\begin{aligned}
[[[\widetilde{\alpha}(K_1), K_1', K_1'], \cdots], K_1', \widetilde{\alpha}(K_2)] &= [\sum_{i=0}^{m}(-1)^i C_m^i (K_1')^i K_1 (K_1')^{m-i}, K_1', \widetilde{\alpha}(K_2)] \\
&= [\sum_{i=0}^{m+1}(-1)^i C_{m+1}^i (K_1')^i K_1 (K_1')^{m+1-i}, K_2].
\end{aligned}$$

In particular, when $n = p$,

$$[[[\widetilde{\alpha}(K_1),K_1',K_1'],\cdots],K_1',\widetilde{\alpha}(K_2)]=[\sum_{i=0}^{p}(-1)^iC_p^i(K_1')^iK_1(K_1')^{p-i},K_2]$$
$$=[K_1(K_1')^p-(K_1')^pK_1,K_2]=[K_1,(K_1')^p,K_2]=[K_1,(K_1')^{[p]},K_2].$$

From the proof, we can obtain $[\widetilde{\alpha}^{-1}(K_1),(K_1')^{[p]},\widetilde{\alpha}^{-1}(K_2)]=(K_1,K_1',\cdots,K_1',K_2)$. Repeating the above arguments, we can get that $[\widetilde{\alpha}^{-1}(K_1),\widetilde{\alpha}^{-1}(K_2),(K_1')^{[p]}]=(K_1,K_2,K_1',\cdots,K_1')$, so we will omit its proof.

Therefore, $\Omega$ is a restricted Hom-Lie triple system. □

**Theorem 4.2.** *Let $M$ be a restricted Hom-Lie triple system, then* $\mathrm{GDer}(M)$ *is a Hom-p-subsystem of* $\Omega$.

*Proof.* For $K_1\in\mathrm{GDer}_{\alpha^k}(M),K_2\in\mathrm{GDer}_{\alpha^s}(M),K_3\in\mathrm{GDer}_{\alpha^m}(M),a,b,c\in M$, we get $\widetilde{\alpha}(K_1)\in\mathrm{GDer}_{\alpha^{k+1}}(M)$, that is $\widetilde{\alpha}(\mathrm{GDer}(M))\subseteq\mathrm{GDer}(M)$.

Next, by calculation, we have

$$\begin{aligned}&[[K_1,K_2,K_3](a),\alpha^{k+s+m}(b),\alpha^{k+s+m}(c)]\\&=[K_1''',K_2''',K_3'''](\left[a,b,c\right])-[\alpha^{k+s+m}(a),[K_1',K_2',K_3'](b),\alpha^{k+s+m}(c)]\\&\quad-[\alpha^{k+s+m}(a),\alpha^{k+s+m}(b),[K_1'',K_2'',K_3''](c)].\end{aligned}$$

Therefore, $\mathrm{GDer}(M)$ is a Hom-subsystem of $\Omega$.

For any $K\in\mathrm{GDer}(M)'$, $\mathrm{GDer}(M)'=\{K\in\mathrm{GDer}(M)\mid\widetilde{\alpha}(K)=K\}$, we have $\widetilde{\alpha}(K^{[p]})=\widetilde{\alpha}(K^p)=\alpha K\cdots K=\widetilde{\alpha}(K)K^{p-1}=K^p=K^{[p]}$. Hence, $\mathrm{GDer}(M)$ is the Hom-$p$-subsystem of $\Omega$. □

**Theorem 4.3.** *Let $M$ be a restricted Hom-Lie triple system over $\mathbb{F}$. And the characteristic of field $\mathbb{F}$ is not equal to 2. The following holds that* $\mathrm{ZDer}(M)=\mathrm{C}(M)\cap\mathrm{Der}(M)$.

*Proof.* On the one hand, for $K\in\mathrm{C}(M)\cap\mathrm{Der}(M)$, $a,b,c\in M$, by $K\in\mathrm{C}_{\alpha^k}(M)$ and $K\in\mathrm{Der}_{\alpha^k}(M)$, we have $3K([a,b,c])=K([a,b,c])$. As the characteristic is $p$ and $p>2$, $K([a,b,c])=0=[K(a),\alpha^k(b),\alpha^k(c)]$. Hence $K\in\mathrm{ZDer}_{\alpha^k}(M)$.

On the other hand, for any $K\in\mathrm{ZDer}_{\alpha^k}(M)$, we can get $K\in\mathrm{Der}_{\alpha^k}(M)$. From $K([a,b,c])=[K(a),\alpha^k(b),\alpha^k(c)]=0$, we have $[K(a),\alpha^k(b),\alpha^k(c)]=-[\alpha^k(b),K(a),\alpha^k(c)]=0$. As the arbitrariness of $a,b,c$, we have $[\alpha^k(a),K(b),\alpha^k(c)]=[\alpha^k(a),\alpha^k(b),K(c)]=0$. Then $K\in\mathrm{C}_{\alpha^k}(M)$, that is $K\in\mathrm{C}_{\alpha^k}(M)\cap\mathrm{Der}_{\alpha^k}(M)$. Hence, $\mathrm{ZDer}(M)\subseteq\mathrm{C}(M)\cap\mathrm{Der}(M)$.

Consequently the conclusion holds. □

## 5. Conclusion and Future Direction

In this paper, we give the concepts of derivations, generalized derivations, centroids and Hom-$p$-subsystems, we prove the subspace $\Omega$ satisfying some conditions in the linear transformation $\mathrm{End}(M)$ is a restricted Hom-Lie triple system, the generalized derivation is a Hom-p-subsystem of $\Omega$ and the central derivation is the intersection of the centroid and the derivation. About the further research, we can consider the properties of quasi-derivations and quasi-centroids, and try to construct a special restricted Hom-Lie triple system to improve the content.

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