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Periodic Travelling Waves of the Delayed Nicholson's Blowflies Model with Diffusion

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Abstract. In this paper, we investigate the existence of periodic travelling waves with large wave speed for the diffusion Nicholson's blowflies model with delay effect by applying the perturbation technique. The proof depends on transforming the differential equation with the wave profile into integral equation, combining the implicit function theorem and the Liapunov-Schmidt reduction.

Keywords. Nicholson's blowflies model, Diffusion, Perturbation method, Periodic travelling wave solutions

1. Introduction

The Nicholson's blowflies equation is a classical model, which is widely used to research the population dynamics of some insect pests. Based on the delayed growth of Nicholson's blowflies, in order to overcome the discrepancy in estimating the delay value, Gueney *et al.* [1] proposed the following delay Nicholson's blowflies model

$$\dot{N}(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}. \tag{1}$$

Here $N(t)$ denotes the population density at time $t \geq 0$; δ is the adult mortality rate; τ is a maturation delay; p is the highest average daily egg production rate; $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate; a, δ, p and τ are positive constants. Kulenović and Ladas [2] proved the solution of equation (1) oscillated about its equilibrium state solution. Moreover, Kulenović et al. [3] established sufficient conditions for the global attractivity of the positive equilibrium of equation (1). See [4,5,6,7] for more progress on various types of solutions of system (1).

On account of equation (1), So and Yang [8] proposed the diffusion form of Nicholson's blowflies equation

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \delta u(x,t) + pu(x,t-\tau)e^{-au(x,t-\tau)}, \tag{2}$$

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which described the population growth of Nicholson's blowflies with spatial diffusion. $u(x,t)$ denotes the population density at time $t \geq 0$ and location $x \in \mathbb{R}^N$; Δ is Laplacian operator and $d > 0$ is the diffusion rate. In [8], the stability of the steady-state solutions for equation (2) was investigated. Whereafter, So and Zou [9] researched the existence of travelling wavefront of equation (2) by upper-lower solution method. The existence of nonmonotone travelling waves was investigated [10]. Stability of travelling wavefronts was investigated by weighted energy method in [11,12]. Furthermore, Yang [13] investigated the properties of periodic travelling wave solutions bifurcating from the zero equilibrium by Hopf bifurcation theory.

Inspired by the work in [13], in this paper, we will investigate the existence of periodic travelling waves bifurcating from the non-zero equilibrium for equation (2) by applying the perturbation method. Our approach is different from the method in [13], which depends on transforming the differential equation with the wave profile into integral equation, combining the implicit function theorem and the Liapunov-Schmidt method. Moreover, we associate the existence of periodic travelling wave solutions for equation (2) with the existence of periodic solutions for related delay differential equation. Similar idea also has appeared in the literature (see [14,15]).

Throughout this paper, $C_\rho = \{g \in C(\mathbb{R},\mathbb{R}) : g(t+\rho) = g(t), \rho \in \mathbb{R}\}$ denotes the Banach space of all ρ-periodic functions equipped with the supremum norm $||g||_{C_\rho} = \sup\{||g(t)||_{\mathbb{R}} : t \in [0,\rho]\}$. For $\Phi \in C(\mathbb{R},\mathbb{R})$ and $s \in \mathbb{R}$, define $\Phi_s \in C([-\tau,0],\mathbb{R})$ by $\Phi_s(r) = \Phi(s+r)$, $r \in [-\tau,0]$. For $\Psi \in C(\mathbb{R},\mathbb{R})$ and $\theta \in \mathbb{R}$, $\Psi^s \in C([0,\tau],\mathbb{R})$ denotes a translation of Ψ defined by $\Psi^s(\theta) = \Psi(s+\theta)$, $\theta \in [0,\tau]$. Furthermore, for $P \in C(\mathbb{R},\mathbb{R})$ and $M \in C_\rho$, define $\langle P, M\rangle = \int_0^\rho P(s)M(s)\mathrm{d}s$.

2. Derivation of the integral equation

The aim of this part is to convert the differential equation with the wave profile into an integral equation. Let $u(t,x) = u^\star(s)$ and $s = \nu \cdot x + ct \in \mathbb{R}$, where ν is a unit vector in $\mathbb{R}^N$. Substituting it into system (2), we have

$$d\ddot{u}^\star(s) - c\dot{u}^\star(s) - \delta u^\star(s) + pu^\star(s-c\tau)e^{-au^\star(s-c\tau)} = 0. \tag{3}$$

Let $\mathcal{U}(s) = u^\star(cs)$ and $\varepsilon = \frac{1}{c^2}$, system (3) can be rewritten as

$$\varepsilon d\ddot{\mathcal{U}}(s) - \dot{\mathcal{U}}(s) - \delta\mathcal{U}(s) + p\mathcal{U}(s-\tau)e^{-a\mathcal{U}(s-\tau)} = 0. \tag{4}$$

Equation (4) is a singular perturbation of system (3). For sufficiently large c, as will be the case later in this work, ε is sufficiently small. When $\varepsilon = 0$, from system (4), we have

$$\dot{u}(s) = -\delta u(s) + pu(s-\tau)e^{-au(s-\tau)}, \tag{5}$$

which has equilibrium $u_* = \frac{1}{a}\ln\frac{p}{\delta} \in \mathbb{R}_+$ as $p > \delta$. For equation (5), the following result can be obtained.

Lemma 2.1 [16] *When $p > \delta e^2$, system (5) admits a periodic solution branch parameterized by τ, bifurcating from positive equilibrium u_*.*

Next, we derive the integral equation on account of Lemma 2.1. Introducing the variable $U(s) = \mathcal{U}((1+\iota)s)$ and $\epsilon = \frac{\varepsilon}{1+\iota}$, where $\iota \in [-\iota^*, \iota^*]$ for a constant $0 < \iota^* < 1$. Moreover, we suppose that $u^*(s)$ is a ρ-periodic hyperbolic solution of equation (5) in Lemma 2.1 (For the proof of hyperbolicity, refer to [15]). Substituting them into system (4), we obtain

$$\epsilon d\ddot{U}(s) - \dot{U}(s) - \delta(1+\iota)U(s) + p(1+\iota)U\Big(s - \frac{\tau}{1+\iota}\Big)e^{-aU\left(s-\frac{\tau}{1+\iota}\right)} = 0. \tag{6}$$

Set $v(s) = U(s) - u^*(s)$, for $s \in \mathbb{R}$, then equation (6) can be rewritten as follows:

$$\epsilon d\ddot{v}(s) - \dot{v}(s) - v(s) + [\mathcal{I}v](s) + \mathcal{M}(\varepsilon, \iota, s, v) = 0, \tag{7}$$

where $\mathcal{I} : C_\rho \to C_\rho$ and $\mathcal{M} : (0,\infty) \times \mathbb{R} \times \mathbb{R} \times C_\rho \to \mathbb{R}$ are defined as

$$[\mathcal{I}v](s) = v(s) - \delta v(s) + \Big[pe^{-au^*(s-\tau)} - apu^*(s-\tau)e^{-au^*(s-\tau)}\Big]v(s-\tau),$$

and

$$\begin{aligned}\mathcal{M}(\varepsilon, \iota, s, v) = {} & d\epsilon\ddot{u}^*(s) - \Big[-\delta u^*(s) + pu^*(s-\tau)e^{-au^*(s-\tau)}\Big] - \delta(1+\iota)[v(s) + u^*(s)] \\ & + p(1+\iota)\left[v + u^*\right]\Big(s - \frac{\tau}{1+\iota}\Big)e^{-a[v+u^*]\left(s-\frac{\tau}{1+\iota}\right)} + \delta v(s) \\ & - \Big[pe^{-au^*(s-\tau)} - apu^*(s-\tau)e^{-au^*(s-\tau)}\Big]v(s-\tau).\end{aligned}$$

Since the equation $\epsilon d\lambda^2 - \lambda - 1 = 0$ always has real solutions $\lambda_1(\epsilon) = \frac{1-\sqrt{1+4\epsilon d}}{2\epsilon d} < 0$, $\lambda_2(\epsilon) = \frac{1+\sqrt{1+4\epsilon d}}{2\epsilon d} > 0$. Let $p_i(\epsilon, t) = \frac{e^{\lambda_i(\epsilon)t}}{\sqrt{1+4\epsilon d}}$, for each $i \in \{1, 2\}$ and $\epsilon > 0$. Then equation (7) has a ρ-periodic solution $v(s)$ when and only when $v(s)$ is the solution of the integral equation

$$[Lv](s) = \mathcal{P}(\varepsilon, \iota, s, v), \qquad s \in \mathbb{R}, \tag{8}$$

where L: $C_\rho \to C_\rho$ and $\mathcal{P}$: $(0,\infty) \times \mathbb{R} \times \mathbb{R} \times C_\rho \to \mathbb{R}$ are defined by

$$[Lv](s) = v(s) - \int_{-\infty}^{s} e^{-(s-t)}\,[\mathcal{I}v](t)\mathrm{d}t, \tag{9}$$

and

$$\begin{aligned}\mathcal{P}(\varepsilon, \iota, s, v) = {} & \int_{-\infty}^{s}\Big[p_1(\epsilon, s-t) - e^{-(s-t)}\Big][\mathcal{I}v](t) + p_1(\epsilon, s-t)\mathcal{M}(\varepsilon, \iota, t, v)\mathrm{d}t \\ & + \int_{s}^{+\infty} p_2(\epsilon, s-t)\,[[\mathcal{I}v](t) + \mathcal{M}(\varepsilon, \iota, t, v)]\,\mathrm{d}t.\end{aligned}$$

Our objective of the remaining part is to derive the existence of solution with ρ-periodic for equation (8).

3. The properties of L and $\mathcal{P}$

Before we study the properties of L, let's provide some results of a functional differential equation. Linearizing equation (5) at $u^*(s)$, we yield the following equation

$$\dot{v}(s) = B_1(s)v_s(0) + B_2(s)v_s(-\tau) \triangleq \mathcal{B}(s)v_s, \tag{10}$$

where $B_1(s) = -\delta$, $B_2(s) = pe^{-au^*(s-\tau)} - apu^*(s-\tau)e^{-au^*(s-\tau)}$. For system (10), we have the following results on account of the theory of delay differential equations [17].

Proposition 3.1 *(i) There exists a unique nonzero ρ-periodic solution $\dot{u}^*$ in system (10).*
(ii) System (10) has the adjoint equation $\dot{\chi}(s) + B_1(s)\chi^s(0) + B_2(s+\tau)\chi^s(\tau) = 0$, which admits a sole nonzero ρ-periodic solution $\chi_*(s) \in C(\mathbb{R},\mathbb{R})$, satisfying $\langle \chi_*, \dot{u}^* \rangle \equiv 1$.
(iii) For any function $F \in C_\rho$, the equation $\dot{v}(s) = \mathcal{B}(s)v_s + F(s)$ exists a solution with periodic ρ when and only when $\langle \chi_*, F \rangle = 0$.

Moreover, for system (10), we obtain the following consequence.

Lemma 3.1 *Set $W(s) = \dot{u}^*(s) + \tau B_2(s)\dot{u}^*_s(-\tau)$, $s \in \mathbb{R}$. Then the nonhomogeneous system $\dot{v}(s) = \mathcal{B}(s)v_s + W(s)$ has no ρ-periodic solution.*

Proof. Suppose that equation $\dot{v}(s) = \mathcal{B}(s)v_s + W(s)$ has a solution $\varrho(s) \in C_\rho$. Setting $\varpi(s) = \varrho(s) - s\dot{u}^*(s)$, $s \in \mathbb{R}$, we have $\dot{\varpi}(s) = \dot{\varrho}(s) - \dot{u}^*(s) - s\ddot{u}^*(s) = \mathcal{B}(s)\varrho_s + \tau B_2(s)\dot{u}^*_s(-\tau) - s\mathcal{B}(s)\dot{u}^*_s$. Furthermore, let $R_s(0) = R(s) = s\dot{u}^*(s)$, then $\mathcal{B}(s)\varpi_s = \mathcal{B}(s)\varrho_s - \mathcal{B}(s)R_s = \mathcal{B}(s)\varrho_s - s\mathcal{B}(s)\dot{u}^*_s + \tau B_2(s)\dot{u}^*_s(-\tau)$. Hence, we yield $\dot{\varpi}(s) = \mathcal{B}(s)\varpi_s$ for all $s \in \mathbb{R}$. It is a contradiction on account of Proposition 3.1. □

Now we investigate the properties of L. Using $T(L)$ to represent the null space of L, we obtain the following lemma.

Lemma 3.2 *$T(L)$ is spanned by $\dot{u}^*$.*

Proof. Suppose that $v \in C_\rho$ is a solution of $[Lv](s) = 0$. By differentiating $[Lv](s) = 0$, we have $\dot{v}(s) = \mathcal{B}(s)v_s$. It follows Proposition 3.1 that $v = k_1\dot{u}^*$ for $k_1 \in \mathbb{R}$. Moreover, if $v = k\dot{u}^*$, then $v \in C_\rho$, and $\dot{v}(s) = \mathcal{B}(s)v_s = -v(s) + [\mathcal{I}v](s)$. We yield $v(s) = \int_{-\infty}^{s} e^{-(s-t)}[\mathcal{I}v](t)\mathrm{d}t$. Hence, we have $v = k\dot{u}^* \in T(L)$. □

Furthermore, for L, there is the following result.

Lemma 3.3 *For any function $D \in C_\rho$, the nonhomogeneous equation*

$$[Lv](s) = D(s), \tag{11}$$

exists a ρ-period solution when and only when $\langle \chi_* - \dot{\chi}_*, D \rangle = 0$.

Proof. Set $Q(s) = v(s) - D(s)$, substituting it into equation (11), we have

$$Q(s) = \int_{-\infty}^{s} e^{-(s-t)}[\mathcal{I}Q](t)\mathrm{d}t + \int_{-\infty}^{s} e^{-(s-t)}[\mathcal{I}D](t)\mathrm{d}t. \tag{12}$$

Differentiating equation (12), we yield

$$\dot{Q}(s) = \mathcal{B}(s)Q_s + [\mathcal{I}D](s). \tag{13}$$

From Proposition 3.1, we know that there is a ρ-period solution of system (13) when and only when

$$\langle \chi_*, [\mathcal{I}D]\rangle = \int_0^\rho \chi_*(s)\,[D(s) + B_1(s)D(s) + B_2(s)D(s-\tau)]\,\mathrm{d}s = 0. \tag{14}$$

Substituting $\gamma = s - \tau$ into $\int_0^\rho \chi_*(s)B_2(s)D(s-\tau)\mathrm{d}s$, then

$$\int_0^\rho \chi_*(s)B_2(s)D(s-\tau)\mathrm{d}s = \int_0^\rho \chi_*(s+\tau)B_2(s+\tau)D(s)\mathrm{d}s.$$

So, we have

$$\begin{aligned}
&\int_0^\rho \chi_*(s)\,[B_1(s)D(s) + B_2(s)D(s-\tau)]\,\mathrm{d}s \\
=&\int_0^\rho [\chi_*(s)B_1(s) + \chi_*(s+\tau)B_2(s+\tau)]\,D(s)\mathrm{d}s \\
=&-\int_0^\rho \dot{\chi}_*(s)D(s)\mathrm{d}s.
\end{aligned} \tag{15}$$

In view of systems (14) and (15), there is

$$\langle \chi_*, [\mathcal{I}D]\rangle = -\langle \dot{\chi}_*, D\rangle + \langle \chi_*, D\rangle = \langle \chi_* - \dot{\chi}_*, D\rangle = 0. \tag{16}$$

It implies that as equation (16) sets up, there is a solution $v \in C_\rho$ of equation (11). □

From Lemma 3.3, we know that system (8) has a solution v with periodic ρ when and only when $\mathcal{P}(\varepsilon, \iota, s, v) \in \mathcal{R}(L) \triangleq \{D \in C_\rho : \langle \chi_* - \dot{\chi}_*, D\rangle = 0\}$. Noticing that for any function $v \in C_\rho$, we can't ensure that $\mathcal{P}(\varepsilon, \iota, s, v) \in \mathcal{R}(L)$. As a result, the following consequences are needed.

Lemma 3.4 *$C_\rho = T(L) \oplus G_\rho$, where $G_\rho = \left\{H \in C_\rho : \langle \chi_*, H\rangle = 0\right\}$.*

Proof. For any function $v \in C_\rho$, set $v_1 = \langle \chi_*, v\rangle \dot{u}^*$ and $v_2 = v - v_1$. It follows Lemma 3.2 that $v_1 \in T(L)$. From $\langle \chi_*, \dot{u}^*\rangle = 1$, there is

$$\langle \chi_*, v_2\rangle = \langle \chi_*, v - v_1\rangle = \langle \chi_*, v\rangle - \langle \chi_*, v_1\rangle = \langle \chi_*, v\rangle - \langle \chi_*, v\rangle\langle \chi_*, \dot{u}^*\rangle = 0.$$

Furthermore, suppose that $H \in T(L) \cap G_\rho$. It follows $H \in T(L)$ that $H = g_1\dot{u}^*$ for $g_1 \in \mathbb{R}$. Moreover, from $H \in G_\rho$, we have $0 = \langle \chi_*, H\rangle = g_1\langle \chi_*, \dot{u}^*\rangle = g_1$. Then $H = 0$. □

Lemma 3.5 *$L : G_\rho \to \mathcal{R}(L)$ is a bijection.*

Proof. Noticing that $L : G_\rho \to \mathcal{R}(L)$ is onto on account of $L(G_\rho) = L(T(L) \oplus G_\rho) = L(C_\rho) = \mathcal{R}(L)$. Choosing $H_1, H_2 \in G_\rho$ such that $LH_1 = LH_2$, we have $H_1 - H_2 \in T(L)$. From Proposition 3.1, we obtain $H_1 - H_2 = g_2\dot{u}^*$ for $g_2 \in \mathbb{R}$. Furthermore, from Lemma 3.4, we have $\langle \chi_*, H_1\rangle = \langle \chi_*, H_2\rangle = 0$. $0 = \langle \chi_*, H_1 - H_2\rangle = g_2\langle \chi_*, \dot{u}^*\rangle = g_2$. Then $H_1 = H_2$. □

Finally, we give the following result for the nonlinear operator $\mathcal{P}$.

Lemma 3.6 *Extending* $\mathcal{P}(\varepsilon,\iota,s,v)$ *at* $\varepsilon=0$ *by* $\mathcal{P}(0,\iota,s,v)=\int_{-\infty}^{s}e^{-(s-t)}\mathcal{N}(0,\iota,t,v)dt,\ s\in\mathbb{R}$, *then* $\mathcal{P}(\varepsilon,\iota,\cdot,v)$ *and* $\frac{\partial}{\partial v}(\mathcal{P}(\varepsilon,\iota,\cdot,v))$ *are both continuous functions on* (ε,ι,v). *Moreover, there exist* $\mathcal{P}(0,0,\cdot,0)=0$ *and* $\frac{\partial}{\partial v}(\mathcal{P}(0,0,\cdot,0))=0$.

Proof. For $s\in\mathbb{R}$, we yield $\int_{-\infty}^{s}\left[p_1(\epsilon,s-t)-e^{-(s-t)}\right]v(t)\mathrm{d}t\le\frac{1-[2+\lambda_1(\epsilon)]\sqrt{1+4\epsilon d}}{\lambda_1(\epsilon)\sqrt{1+4\epsilon d}}\|v\|_{C_\rho}$. Noticing that $\epsilon=\frac{\varepsilon}{1+\iota}\to 0$ and $\lambda_1(\epsilon)\to -1$ as $\varepsilon\to 0^+$, we have $1-[2+\lambda_1(\epsilon)]\sqrt{1+4\epsilon d}\to 0$ as $\varepsilon\to 0^+$. Moreover, we obtain $\int_{s}^{+\infty}p_2(\epsilon,s-t)v(t)\mathrm{d}t\le\frac{1}{\lambda_2(\epsilon)\sqrt{1+4\epsilon d}}\|v\|_{C_\rho}$. It follows from $\lambda_2(\epsilon)\to\infty$ as $\varepsilon\to 0^+$ that $\frac{1}{\lambda_2(\epsilon)\sqrt{1+4\epsilon d}}\to 0$ as $\varepsilon\to 0^+$. So, we yield $\mathcal{P}(0,\iota,s,v)=\int_{-\infty}^{s}e^{-(s-t)}\mathcal{N}(0,\iota,t,v)\mathrm{d}t$. According to the definition of $\mathcal{M}(\varepsilon,\iota,\cdot,v)$, $\mathcal{M}(\varepsilon,\iota,\cdot,v)$ and $\frac{\partial}{\partial v}(\mathcal{M}(\varepsilon,\iota,\cdot,v))$ are continuous. Furthermore, there exist $\mathcal{M}(0,0,\cdot,0)=0$ and $\frac{\partial}{\partial v}(\mathcal{M}(0,0,\cdot,0))=0$. According to the definition of $\mathcal{P}(\varepsilon,\iota,\cdot,v)$, $\mathcal{P}(\varepsilon,\iota,\cdot,v)$ and $\frac{\partial}{\partial v}(\mathcal{P}(\varepsilon,\iota,\cdot,v))$ are continuous and satisfy $\mathcal{P}(0,0,\cdot,0)=0$ and $\frac{\partial}{\partial v}(\mathcal{P}(0,0,\cdot,0))=0$. □

4. Periodic travelling waves

Next, we prove the existence of ρ-periodic solution for system (8). From Lemma 3.4, for any $v\in C_\rho$, we have $v=\varsigma\dot{u}^*+H$, where $\varsigma\in\mathbb{R}$ and $H\in G_\rho$. Substituting it into equation (8), we yield

$$[LH](s)=\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^*+H). \tag{17}$$

Next, we need to guarantee $\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^*+H)\in\mathcal{R}(L)$ on account of solve equation (17). Introducing the operator $\mathcal{F}:C_\rho\to C_\rho$ be defined by

$$[\mathcal{F}v](s)=\frac{1}{\langle\chi_*,\chi_*^T\rangle}\langle\chi_*-\dot{\chi}_*,v\rangle\chi_*^T(s),\qquad s\in\mathbb{R}.$$

For any function $v\in C_\rho$, we have

$$\langle\chi_*-\dot{\chi}_*,v-\mathcal{F}v\rangle=\frac{1}{\langle\chi_*,\chi_*^T\rangle}\langle\chi_*-\dot{\chi}_*,v\rangle\langle\dot{\chi}_*,\chi_*^T\rangle.$$

Noticing that $\langle\dot{\chi}_*,\chi_*^T\rangle=\int_0^\rho\dot{\chi}_*(s)\chi_*^T(s)\mathrm{d}s=\frac{1}{2}\chi_*(s)\chi_*^T(s)|_0^\rho=0$, we yield $\langle\chi_*-\dot{\chi}_*,v-\mathcal{F}v\rangle=0$ and $(I-\mathcal{F})v\in\mathcal{R}(L)$. Thus system (17) reduces to

$$[LH](s)=(I-\mathcal{F})\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^*+H),\qquad H\in G_\rho, \tag{18}$$

and

$$\mathcal{F}\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^*+H)=0,\qquad H\in G_\rho. \tag{19}$$

It follows Lemma 3.5 that system (18) becomes

$$H=L^{-1}(I-\mathcal{F})\mathcal{H}(\varepsilon,\iota,\cdot,\varsigma\dot{u}^*+H). \tag{20}$$

Define $\Phi : \mathbb{R}\times[-\iota^*,\iota^*]\times\mathbb{R}\times G_\rho$ by $\Phi(\varepsilon,\iota,\varsigma,H) = H - L^{-1}(I-\mathcal{F})\mathcal{P}(\varepsilon,\iota,\cdot,\varsigma\dot{u}^* + H)$. System (20) is equivalent to $\Phi(\varepsilon,\iota,\varsigma,H) = 0$. Form Lemma 3.6, we have $\Phi(0,0,0,0) = 0$ and $\frac{\partial}{\partial H}(\Phi(0,0,0,0)) = I - L^{-1}(I-\mathcal{F})\frac{\partial}{\partial v}(\mathcal{P}(0,0,\cdot,0)) = I$. Hence, there exist a number $0 < a^* \leq \iota^*$ and a continuous function $H : [0,a^*]\times[-a^*,a^*]\times[-a^*,a^*] \to G_\rho$ such that $H(0,0,0) = 0$, $\Phi(\varepsilon,\iota,\varsigma,H(\varepsilon,\iota,\varsigma)) = 0$ and $[LH(\varepsilon,\iota,\varsigma)](s) = (I-\mathcal{F})\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^* + H(\varepsilon,\iota,\varsigma))$. Substituting $H(\varepsilon,\iota,\varsigma)$ into system (19), we obtain

$$\mathcal{F}\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^* + H(\varepsilon,\iota,\varsigma)) = 0, \tag{21}$$

which is equivalent to

$$\int_0^\rho \Big[\chi_*(s) - \dot{\chi}_*(s)\Big]\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^* + H(\varepsilon,\iota,\varsigma))\mathrm{d}s = 0. \tag{22}$$

Define $\Psi : [0,a^*]\times[-a^*,a^*]\times[-a^*,a^*] \to \mathbb{R}$ by $\Psi(\varepsilon,\iota,\varsigma) = \int_0^\rho \Big[\chi_*(s) - \dot{\chi}_*(s)\Big]\mathcal{P}(\varepsilon,\iota,s,\varsigma\dot{u}^* + H(\varepsilon,\iota,\varsigma)))\mathrm{d}s$. Then we have

$$\Psi(\varepsilon,\iota,\varsigma) = 0. \tag{23}$$

It follows $H(0,0,0) = 0$ and $\mathcal{P}(0,0,\cdot,0) = 0$ that $\Psi(0,0,0) = 0$. Moreover, we need to prove that $\frac{\partial}{\partial\iota}(\Psi(0,0,0)) \neq 0$. Suppose that $\frac{\partial}{\partial\iota}(\Psi(0,0,0)) = 0$, we obtain

$$\frac{\partial}{\partial\iota}(\mathcal{F}\mathcal{P}(0,\iota,\cdot,H(0,\iota,0)))\Big|_{\iota=0} = 0. \tag{24}$$

In view of systems (18) and (24), we yield

$$L\left(\frac{\partial}{\partial\iota}(H(0,\iota,0))\right) = \frac{\partial}{\partial\iota}(\mathcal{P}(0,\iota,\cdot,H(0,\iota,0)))\Big|_{\iota=0}. \tag{25}$$

From Lemma 3.6 and the expression of $\mathcal{M}$, there are

$$\mathcal{P}(0,\iota,s,v) = \int_{-\infty}^{s} e^{-(s-t)}\mathcal{M}(0,\iota,t,v)\mathrm{d}t, \tag{26}$$

and

$$\frac{\partial}{\partial\iota}(\mathcal{M}(0,\iota,t,H(0,\iota,0)))\Big|_{\iota=0} = W(t). \tag{27}$$

From equations (25), (26) and (27), we yield

$$\begin{aligned}\left[L\left(\frac{\partial}{\partial\iota}(H(0,\iota,0))\right)\right](s) &= \frac{\partial}{\partial\iota}(H(0,\iota,0))(s) - \int_{-\infty}^{s} e^{-(s-t)}\left[\mathcal{I}\left(\frac{\partial}{\partial\iota}(H(0,\iota,0))\right)\right](t)\mathrm{d}t \\ &= \int_{-\infty}^{s} e^{-(s-t)}W(t)\mathrm{d}t.\end{aligned}$$

Differentiating the above equation, we have $\frac{\partial}{\partial\iota}(H(\dot{0},\iota,0))(s) = \mathcal{B}(s)\left[\frac{\partial}{\partial\iota}(H(0,\iota,0))\right]_s +$ $W(s)$. Because $\frac{\partial}{\partial\iota}(H(0,\iota,0))(s)$ is not a function with ρ-periodic, it leads to a contradiction. Hence, we have $\frac{\partial}{\partial\iota}(\Psi(0,0,0)) \neq 0$. For system (23), we yield that there exist a number $0 < \varepsilon_* = \frac{1}{{c_*}^2} \leq a^* \leq \iota^*$ and a continuous function $\iota : [0,\varepsilon_*]\times[-\varepsilon_*,\varepsilon_*] \to \mathbb{R}$ such that $\iota(0,0) = 0$, $\Psi(\varepsilon,\iota(\varepsilon,\varsigma),\varsigma) = 0$, $(\varepsilon,\varsigma) \in [0,\varepsilon_*]\times[-\varepsilon_*,\varepsilon_*]$. Hence, the following consequence can be obtained.

Theorem 4.1 *When $p > \delta e^2$, there exists a sufficiently large number $c_* > 0$ such that, for each wave speed $c > c_*$, equation (2) has a $[1+\iota(\varepsilon,\varsigma)]\rho$-periodic travelling wave solution*

$$D(x,t) = u^*\left(\frac{\nu\cdot x+ct}{c[1+\iota(\varepsilon,\varsigma)]}\right)+\varsigma\dot{u}^*\left(\frac{\nu\cdot x+ct}{c[1+\iota(\varepsilon,\varsigma)]}\right)+H(\varepsilon,\iota(\varepsilon,\varsigma),\varsigma)\left(\frac{\nu\cdot x+ct}{c[1+\iota(\varepsilon,\varsigma)]}\right).$$

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