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Block Multipath Matching Pursuit

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Abstract. In this paper, we propose a new reconstruction algorithm called Block Multipath Matching pursuit (BMMP), which is an extension of multipath matching pursuit (MMP) in block compressed sensing. Then the reconstruction condition of BMMP based on restricted isometric property (RIP) is established. In addition, we also provides a guarantee of reconstruction in the case of noise measurement. Finally, the effectiveness and advancement of the BMMP algorithm are verified by numerical experiments.

Keywords. Multipath matching pursuit (MMP), Block orthogonal matching pursuit(BOMP), Compressed sensing (CS), Restricted isometry property (RIP),

1. Introduction

Compressed sensing (CS) is widely used to reconstruct the K-sparse signal $\mathbf{x} \in \mathbb{R}^N$ from Eq.(1), and it has received extensive attention in the recent decade [1, 2].

$$\mathbf{y} = \Phi\mathbf{x}, \tag{1}$$

where $\Phi \in \mathbb{R}^{M\times N}(N \gg M)$ is the measurement matrix. An original method to reconstruct the sparse signal $\mathbf{x}$ from Eq.(1) is to solve a sparsity promoting optimization problem:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \|\mathbf{x}\|_0\text{: subject to } \|\mathbf{y} - \Phi\mathbf{x}\|_2 \le \varepsilon, \tag{2}$$

However, solving Eq.(2) is NP-hard. Recently, greedy algorithm has obtained huge interest due to their high computational efficiency, such as orthogonal matching pursuit (OMP) algorithm [3]. We know that multipath matching pursuit (MMP) [4] based on multipath selection investigates multiple promising candidates by arranging the correlation between the Φ column and the residual, which is is an improved algorithm of OMP in reconstruction performance with computational higher complexity. For the reconstruction of block sparse signal, using the non-zero component distribution of signal to design the algorithm can have better reconstruction performance. For example, block orthogonal matching tracking (BOMP) [5, 6] algorithm is more suitable than OMP algorithm because of its better performance for block sparse signals. Our research mainly focus on reconstructing block sparse signals [7, 8] in this paper.

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In this paper, we introduce MMP algorithm into the block compressed sensing to further improve the reconstruction performance. We propose a algorithm called block multipath matching pursuit (BMMP), which identifies the index of the Φ based on the degree of correlation between the block and residuals, and uses the tree-searching strategy to store these candidates for generating L subpaths on each iteration. BMMP can significantly improve the reconstruction performance, and has obvious advantages over the traditional MMP algorithm in running time.

This paper is structured as follows: Section 2 describes notations. Section 3 introduces the implementation process of the algorithm. In Section 4, we analyze the conditions for accurate sparse signal reconstruction based on RIP. Section 5 compares our work with relevant algorithms for sparse signal reconstruction. In Section 6 we give the conclusions of this paper.

2. Notions

In this section, notations are summarized. Let $D = N/d$ denotes the number of blocks, the initial value of Ω be $\{1,2,\dots,D\}$. Let $\mathbf{x}$ be a K-sparse signal, $T \in \Omega$ is its true block support set, and $|T| \leq K$. $\Omega \backslash T$ denotes the block index set contained in Ω but not in T, where $|T|$ is the cardinality of T. Φ' denotes the transpose of Φ. Let T^c and S^c be the complementary set of T and S, i.e., $T^c = \Omega \setminus T$ and $S^c = \Omega \setminus S$. $\Phi[S]$ is the submatrix of Φ that contains only the column with block indices by S. $\mathbf{x}[S]$ is a subvector that guarantees only $\mathbf{x}$ with a block index of S, and ϕ_j denotes the j-th column of Φ, x_i is the i-th entry of $\mathbf{x}$. At the k-th iteration, let S^k be the set of all candidate sets, and s_i^k be the i-th candidate. L denotes the number of paths in each iteration. $\mathbf{r}_i^k$ denotes the residual of candidate s_i^k. For any full column rank matrix $\Phi[S]$, $\Phi^\dagger[S] = (\Phi'[S]\Phi[S])^{-1}\Phi'[S]$ is the Moore-Penrose pseudoinverse of $\Phi[S]$. Let $\mathbf{P}[S] = \Phi[S]\Phi^\dagger[S]$ denote the projection of span$(\Phi[S])$, and $\mathbf{P}^\perp[S] = I - \mathbf{P}[S]$ denote the orthogonal complement of $\mathbf{P}[S]$.

3. Proposed Algorithms

In general, the block K-sparse signal $\mathbf{x} \in \mathbb{R}^N$ can be modeled as

$$\mathbf{x} = [\underbrace{x_1,\cdots,x_d}_{\mathbf{x}[1]},\underbrace{x_{d+1},\cdots,x_{2d}}_{\mathbf{x}[2]},\cdots,\underbrace{x_{N-d+1},\cdots,x_N}_{\mathbf{x}[D]}]', \tag{3}$$

where $\|\mathbf{x}\|_{2,0} = \sum_{i=1}^{D} I(\|\mathbf{x}[i]\|_2 > 0) \leq K$, where $\|\mathbf{x}\|_{2,0}$ is a mixed ℓ_2/ℓ_0-norm and $I(\cdot)$ is a indicator function. The mixed ℓ_2/ℓ_p-norm (where $p = 1,2,\infty$) denotes $\|\mathbf{x}\|_{2,p} = \|\mathbf{w}\|_p$, where $\mathbf{w} \in \mathbb{R}^D$ with $\mathbf{w}_\ell = \|\mathbf{x}[\ell]\|_2$ for $1 \leq \ell \leq D$. We found the block-sparsity will reduce to normal sparsity for $d = 1$. Therefore, a signal with block sparsity K can be regarded as a traditional sparse signal which has a special distribution of non-zero components, whose sparsity is dK. Similarly, for $\Phi \in \mathbb{R}^{M\times N}$ we have

$$\Phi = [\underbrace{\phi_1,\cdots,\phi_d}_{\Phi[1]},\underbrace{\phi_{d+1},\cdots,\phi_{2d}}_{\Phi[2]},\cdots,\underbrace{\phi_{N-d+1},\cdots,\phi_N}_{\Phi[D]}], \tag{4}$$

Multipath matching pursuit (MMP) is an algorithm that performs tree search under greedy strategy. Compared with OMP, it has better reconstruct performance but higher computational complexity. We introduce the multipath search idea of MMP algorithm into the block sparse signal reconstruction algorithm. Then we summarize BMMP algorithm in Algorithm 1.

Algorithm 1 THE BMMP ALGORITHM.

Input: $\mathbf{y} \in \mathbb{R}^M$, $\Phi \in \mathbb{R}^{M\times N}$, L, K, D
Output: $\hat{\mathbf{x}}$;
Initialize: $S^0 := \{\emptyset\}$, $k := 0$, $\mathbf{r}_1^0 = \mathbf{y}$;
  while $k < K$ **do**
    $k := k+1$, $v := 0$, $S^k := \emptyset$
    for $j = 1$ **to** $|S^{k-1}|$ **do**
      $\tilde{\pi} = \arg\max_{1\le i\le D, |\pi|=L} \left\| \left(\Phi'[i]\mathbf{r}_j^{k-1} \right)_\pi \right\|_2^2$
      for $l = 1$ **to** L **do**
        $s_{temp} := s_j^{k-1} \cup \tilde{\pi}(l)$
        if $s_{temp} \notin S^k$ **then**
          $v := v+1$
          $s_v^k := s_{temp}$
          $S^k := S^k \cup \{s_v^k\}$
          $\hat{\mathbf{x}}_v^k = \Phi^\dagger[s_v^k]\mathbf{y}$
          $\mathbf{r}_v^k := \mathbf{y} - \Phi[s_v^k]\hat{\mathbf{x}}_v^k$
        end if
      end for
    end for
  end while
  $i^* = \arg\min \left\|\mathbf{r}_i^K\right\|_2^2$
  $\hat{\mathbf{x}} = \begin{cases} \Phi^\dagger[s_{i^*}^k]\mathbf{y}, & \text{on the estimated block-support sets } s_{i^*}^k, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$

4. Perfect reconstruction condition of BMMP

First, we need introduce a natural extension of classic restricted isometry property (RIP), block-RIP [7]. Φ is said to satisfy the block-RIP

$$(1-\delta_B)\left\|\mathbf{x}\right\|_2^2 \leqslant \left\|\Phi\mathbf{x}\right\|_2^2 \leqslant (1+\delta_B)\left\|\mathbf{x}\right\|_2^2 \tag{5}$$

for all block K-sparse signal $\mathbf{x} \in \mathbb{R}^N$, where $\delta_B \in (0,1)$ is a constant. Restricted isometry constant (RIC) δ_K is the minimum of δ_B. Then we list several useful lemmas.

Lemma 1 *[9] Let Φ satisfy the block-RIP with both order K_1 and K_2. If $K_1 \le K_2$, then $\delta_{K_1} \le \delta_{K_1}$.*

Lemma 2 *[9] Let sets S_2, S_1 satisfy $|S_2 \setminus S_1| \ge 1$ and Φ satisfy the block-RIP of order $|S_2 \cup S_1|$. Then for any $x \in \mathbb{R}^{|S_2\setminus S_1|\times d}$*

$$\left(1-\delta_{|S_2\cup S_1|}\right)\|\mathbf{x}\|_2^2 \leqslant \left\|\mathbf{P}^{\perp}[S_1]\Phi[S_2\setminus S_1]\mathbf{x}\right\|_2^2 \leqslant \left(1+\delta_{|S_2\cup S_1|}\right)\|\mathbf{x}\|_2^2. \tag{6}$$

Next we state our main result.

Theorem 1 *Consider the system model in Eq.(1), The BMMP is said to exactly reconstructs any block K-sparse signal* $\mathbf{x}\in\mathbb{R}^N$ *if* Φ *satisfies the Block-RIP of order* $K+L$ *with the RIC*

$$\delta_{K+L} < \sqrt{\frac{L}{K+L}}, \tag{7}$$

Proof: Assume these is a candidate $s_{i_0}^k$ that $|T\cap s_{i_0}^k| = |s_{i_0}^k| = k(0\le k<K)$ at the k-th iteration. Then we will prove that BMMP can find at least one correct block index at the $(k+1)$-th iteration under Eq.(7). For this purpose, we need to compare the maximum element β_1^{k+1} in $\{\left\|\Phi'[j]\mathbf{r}_{i_0}^k\right\|_2 : j\in T\setminus s_{i_0}^k\}$ with the L-th largest element α_L^{k+1} in $\{\left\|\Phi'[j]\mathbf{r}_{i_0}^k\right\|_2 : j\in T^C\}$. If $\beta_1^{k+1} > \alpha_L^{k+1}$, then the index corresponding to β_1^{k+1} will be chosen.

Noting that $|s_{i_0}^k| = k$, then $|T\setminus s_{i_0}^k| = K-k$ we have

$$\begin{aligned}\beta_1^{k+1} &= \max_{j\in T\setminus s_{i_0}^k}\left\|\Phi[j]'\mathbf{r}_{i_0}^k\right\|_2 = \left\|\Phi'[T\setminus s_{i_0}^k]\mathbf{r}_{i_0}^k\right\|_{2,\infty}\\ &\overset{(a)}{\geq} \frac{\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2}{\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2},\end{aligned} \tag{8}$$

where (a) follows $\|\mathbf{x}\|_{2,\infty} \geq \frac{\|\mathbf{x}\|_2}{\sqrt{\|\mathbf{x}\|_{1,0}}}$ and $\left|T\setminus s_{i_0}^k\right| = K-k$. Then we can need to obtain the upper bound of α_L^{k+1}:

$$\begin{aligned}\alpha_L^{k+1} &\leq \frac{1}{L}\sum_{j=1}^{L}\alpha_j^{k+1} = \frac{1}{L}\sum_{j=1}^{L}\left\|\Phi'[t_j]\mathbf{r}_{i_0}^k\right\|_2 = \frac{1}{L}\left\|\Phi'[S]\mathbf{r}_{i_0}^k\right\|_{2,1}\\ &\overset{(a)}{=} \frac{\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_{2,1}}{L},\end{aligned} \tag{9}$$

where $S := \{t_1,t_2,...,t_L\}\subset T^C$. Using Eqs.(8) and (9), we have

$$\beta_1^{k+1} - \alpha_L^{k+1} \geq \frac{\eta}{\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2} \tag{10}$$

where

$$\eta = \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2$$

$$-\frac{\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_{2,1}}{L} \tag{11}$$

Thus, to show $\beta_1^{k+1} > \alpha_L^{k+1}$, it suffices to show $\eta > 0$. Then we let

$$\theta = \frac{K-k}{L} \text{ and } \sigma = \frac{1-\sqrt{\theta+1}}{\sqrt{\theta}}. \tag{12}$$

and

$$\gamma[j] = \begin{cases} 0 & j \notin S, \\ \frac{\sigma}{\sqrt{L}}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2 \operatorname{sgn}\left(\Phi'[j]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right) & j \in S, \end{cases} \tag{13}$$

where $\operatorname{sgn}(\cdot)$ is the signum function. For Eq.(12), we have

$$\frac{1+\sigma^2}{1-\sigma^2} = \sqrt{\theta+1} \text{ and } \frac{2\sigma}{1-\sigma^2} = -\sqrt{\theta}. \tag{14}$$

Now, we consider the right-hand side of Eq.(11),

$$\begin{aligned}\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\mathbf{x}+\gamma\right)\right\|_2^2 &= \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2 + \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\gamma\right\|_2^2 \\ &\quad + \frac{2\sigma\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_{2,1}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2}{\sqrt{L}},\end{aligned} \tag{15}$$

$$\begin{aligned}\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\sigma^2\mathbf{x}-\gamma\right)\right\|_2^2 &= \sigma^4\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2 + \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\gamma\right\|_2^2 \\ &\quad - \frac{2\sigma^3\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_{2,1}}{\sqrt{L}}.\end{aligned} \tag{16}$$

Combining Eqs.(15) and (16) yields

$$\begin{aligned}&\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\mathbf{x}+\gamma\right)\right\|_2^2 - \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\sigma^2\mathbf{x}-\gamma\right)\right\|_2^2 \\ &\overset{Eq.(14)}{=} \left(1-\sigma^4\right)\left(\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2\right. \\ &\qquad \left. -\sqrt{\frac{\theta}{L}}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_{2,1}\right).\end{aligned} \tag{17}$$

Now, recalling Eq.(11) where

$$\eta = \left\|\mathbf{r}_{i_0}^k\right\|_2^2 - \frac{\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2\left\|\Phi'[S]\mathbf{r}_{i_0}^k\right\|_{2,1}}{L}$$

and $\mathbf{r}_{i_0}^k = \mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]$, along with Eq.(17) and Eq.(12)

$$\eta = \frac{\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\mathbf{x}+\gamma\right)\right\|_2^2 - \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\sigma^2\mathbf{x}-\gamma\right)\right\|_2^2}{1-\sigma^4}. \tag{18}$$

Applying Eq.(18) to Eq.(10), we further have

$$\begin{aligned}
\beta_1^{k+1}-\alpha_L^{k+1} &\geq \frac{\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\mathbf{x}+\gamma\right)\right\|_2^2 - \left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi\left(\sigma^2\mathbf{x}-\gamma\right)\right\|_2^2}{\left(1-\sigma^4\right)\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2} \\
&\overset{Lemma 2}{\geq} \frac{\left(1-\delta_{K+L}\right)\left\|\mathbf{x}[T\setminus s_{i_0}^k]+\gamma\right\|_2^2 - \left(1+\delta_{K+L}\right)\left\|\sigma^2\mathbf{x}[T\setminus s_{i_0}^k]-\gamma\right\|_2^2}{\left(1-\sigma^4\right)\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2} \\
&\overset{Eq.(14)}{=} \frac{\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2}{\sqrt{K-k}}\left(1-\delta_{K+L}\sqrt{\frac{K+L-k}{L}}\right)
\end{aligned} \tag{19}$$

Therefore, $\beta_1^{k+1} > \alpha_L^{k+1}$ will be guaranteed when

$$\delta_{K+L} < \sqrt{\frac{L}{K+L-k}}. \tag{20}$$

Eq. 20 holds when Eq.(7) is satisfied for $0 \leq k < K$. □

Remark 1 *When $L = 1$, Theorem 1 becomes an exact reconstruction condition for the BOMP algorithm, which is just [9, Corollary 1].*

Remark 2 *When $d = 1$, Theorem 1 becomes an exact reconstruction condition for the MMP algorithm, which is just [10, Theorem 1].*

Next consider the noisy scenario $\mathbf{y} = \Phi\mathbf{x}+\mathbf{e}$, we present the reconstruction guarantee of BMMP to get true block support set T.

Theorem 2 *For $\mathbf{y} = \Phi\mathbf{x}+\mathbf{e}$, Φ has unit ℓ_2-norm columns and and $\mathbf{e}$ is an ℓ_2-bounded noise. Let the RIC of Φ satisfy $\delta_{K+L} < \frac{\sqrt{L}}{\sqrt{K+L}}$ and*

$$\min_{i\in T}\|x[i]\|_2 \geq \frac{\sqrt{2L(1+\delta_{K+L})}}{\sqrt{L}-\sqrt{K+L}\delta_{K+L}}\|\mathbf{e}\|_2 \tag{21}$$

BMMP algorithm will identify all correct block support of $\mathbf{x}$.

Proof: We assume there is a candidate $s_{i_0}^k$ that $|T \cap s_{i_0}^k| = |s_{i_0}^k| = k (0 \le k < K)$ at the k-th iteration. Then we will prove that BMMP successfully performs the $(k+1)$-th iteration under Eq.(7) and Eq.(21). Therefore, we compare the maximum element β_1^{k+1} in $\{\|\Phi'[j]\mathbf{r}_k\|_2 : j \in T \setminus s_{i_0}^k\}$ with the L-th largest element α_L^{k+1} in $\{\left\|\Phi'[j]\mathbf{r}_{i_0}^k\right\|_2 : j \in T^C\}$. If $\beta_1^{k+1} > \alpha_L^{k+1}$, then the index corresponding to β_1^{k+1} will be chosen.

Noting that $|s_{i_0}^k| = k$, then $|T \setminus s_{i_0}^k| = K - k$ we have

$$\begin{aligned}
\beta_1^{k+1} &= \max_{j \in T \setminus s_{i_0}^k} \left\|\Phi'[j]\mathbf{r}_i^k\right\|_2 = \left\|\Phi'[T \setminus s_{i_0}^k]\mathbf{r}_i^k\right\|_{2,\infty} = \left\|\Phi'[T \setminus s_{i_0}^k]\left(\mathbf{P}^\perp[s_{i_0}^k]\mathbf{y} + \mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right)\right\|_{2,\infty} \\
&\ge \left\|\Phi'[T \setminus s_{i_0}^k]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{y}\right\|_{2,\infty} - \left\|\Phi'[T \setminus s_{i_0}^k]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right\|_{2,\infty} \\
&\overset{Eq.(8)}{\ge} \frac{\left\|\mathbf{P}^\perp[s_{i_0}^k]\Phi[T \setminus s_{i_0}^k]\mathbf{x}[T \setminus s_{i_0}^k]\right\|_2^2}{\sqrt{K-k}\left\|\mathbf{x}[T \setminus s_{i_0}^k]\right\|_2} - \left\|\Phi'[T \setminus s_{i_0}^k]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right\|_{2,\infty}
\end{aligned} \tag{22}$$

Next, let's analyze α_L^{k+1}

$$\begin{aligned}
\alpha_L^{k+1} &\overset{(a)}{\le} \frac{1}{L}\left\|\Phi'[S]\mathbf{r}_{i_0}^k\right\|_{2,1} = \frac{1}{L}\left\|\Phi'[S]\left(\mathbf{P}^\perp[s_{i_0}^k]\mathbf{y} + \mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right)\right\|_{2,1} \\
&\le \frac{1}{L}\left\|\Phi'[S]\mathbf{P}^\perp[s_{i_0}^k]\Phi[T \setminus s_{i_0}^k]\mathbf{x}[T \setminus s_{i_0}^k]\right\|_{2,1} + \frac{1}{L}\left\|\Phi'[S]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right\|_{2,1}
\end{aligned} \tag{23}$$

where $S := \{t_1, t_2, ..., t_L\} \subset T^C$. By relating Eq.(22) and Eq.(23), it is clear that $\beta_1^{k+1} > \alpha_L^{k+1}$ is guaranteed when

$$\begin{aligned}
&\frac{\left\|\mathbf{P}^\perp[s_{i_0}^k]\Phi[T \setminus s_{i_0}^k]\mathbf{x}[T \setminus s_{i_0}^k]\right\|_2^2}{\sqrt{K-k}\left\|\mathbf{x}[T \setminus s_{i_0}^k]\right\|_2} - \left\|\Phi'[T \setminus s_{i_0}^k]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right\|_{2,\infty} \\
&\qquad \ge \frac{\left\|\Phi'[S]\mathbf{P}^\perp[s_{i_0}^k]\Phi[T]\mathbf{x}[T]\right\|_{2,1} + \left\|\Phi'[S]\mathbf{P}^\perp[s_{i_0}^k]\mathbf{e}\right\|_{2,1}}{L}.
\end{aligned} \tag{24}$$

For (24), equivalently,

$$\begin{aligned}
&\frac{\left\|\mathbf{P}^\perp[s_i^k]\Phi[T \setminus s_i^k]\mathbf{x}[T \setminus s_i^k]\right\|_2^2}{\sqrt{K-k}\left\|\mathbf{x}[T \setminus s_i^k]\right\|_2} - \frac{\left\|\Phi'[S]\mathbf{P}^\perp[s_i^k]\Phi[T]\mathbf{x}[T]\right\|_{2,1}}{L} \\
&\qquad \ge \frac{\left\|\Phi'[S]\mathbf{P}^\perp[s_i^k]\mathbf{e}\right\|_{2,1}}{L} + \left\|\Phi'[T \setminus s_i^k]\mathbf{P}^\perp[s_i^k]\mathbf{e}\right\|_{2,\infty}.
\end{aligned} \tag{25}$$

First, we simplify the left-hand side of Eq.(25) with the result of Eq.(19).

$$
\begin{aligned}
&\frac{\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T\setminus s_{i_0}^k]\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2^2}{\sqrt{K-k}\left\|\mathbf{x}[T\setminus s_{i_0}^k]\right\|_2}-\frac{\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\Phi[T]\mathbf{x}[T]\right\|_{2,1}}{L}\\
&\geq\left(1-\delta_{K+L}\sqrt{\frac{K+L-k}{L}}\right)\frac{\left\|\mathbf{x}[T\setminus s_i^k]\right\|_2}{\sqrt{K-k}}\\
&\geq\left(1-\delta_{K+L}\sqrt{\frac{K-k+L}{L}}\right)\min_{i\in T}\|x[i]\|_2. \qquad (26)
\end{aligned}
$$

Let $j_1:=\arg\max\limits_{j\in S}\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi'[T\setminus s_{i_0}^k]\mathbf{e}\right\|_2$ and $j_2:=\arg\max\limits_{j\in T}\left\|\mathbf{P}^{\perp}[s_{i_0}^k]\Phi'[T\setminus s_{i_0}^k]\mathbf{e}\right\|_2$, the right-hand side of Eq.(25) can be simplified as

$$
\begin{aligned}
&\frac{\left\|\Phi'[S]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\right\|_{2,1}}{L}+\left\|\Phi'[T\setminus s_{i_0}^k]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\right\|_{2,\infty}\\
&\leq\left\|\Phi'[j_1]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\right\|_{2,1}+\left\|\Phi'[j_2]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\right\|_{2,1}\\
&\overset{(a)}{\leq}\sqrt{2}\left\|\Phi'[j_1\cup j_2]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\right\|_2\\
&\overset{(b)}{\leq}\sqrt{2(1+\delta_{K+L})}\,\|\mathbf{e}\|_2 \qquad (27)
\end{aligned}
$$

where (a) because $\Phi'[j_1\cup j_2]\mathbf{P}^{\perp}[s_{i_0}^k]\mathbf{e}\in\mathbb{R}^2$, (b) is due to Lemma 2 and Lemma 1. Combining Eq.(26) and Eq.(27), we have

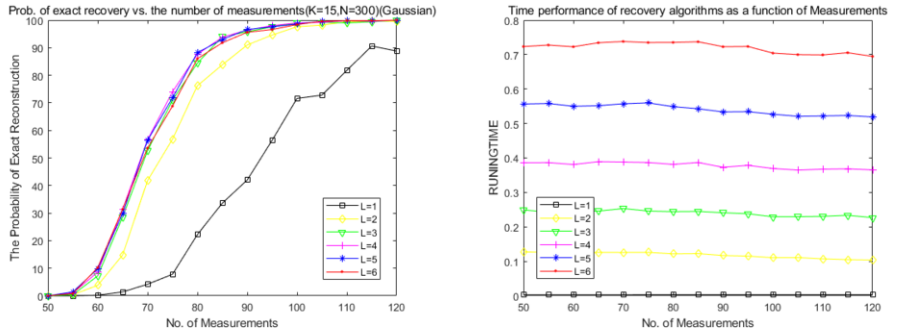
$$
\min_{i\in T}\|x[i]\|_2\geq\frac{\sqrt{2L(1+\delta_{K+L})}}{\sqrt{L}-\sqrt{K+L-k}\,\delta_{K+L}}\|\mathbf{e}\|_2 \qquad (28)
$$

which is just Eq.(21). Thus Theorem 2 is proved. □

5. Numerical experiments

In this section, we evaluate the reconstruction performance of BMMP through numerical experiments. In our numerical experiments, we use a random measurement matrix Φ of size 150×300, whose entries are selected independently of the Gaussian distribution $N(0,1/M)$. Then we generate a block K-sparse signal signal $\mathbf{x}$, whose non-zero block indices are randomly selected, and all nonzero elements are taken from $N(0,1)$. We conducted 1000 independent experiments to observe the reconstruction performance of BMMP algorithm.

In Figure.1 (a), we observe the relationship between probability of exact reconstructing block K-sparse signal and the number of measurement M. It can be seen that no matter what value L takes, the reconstruction probability increases with the increase of M. The reconstruction probability becomes better with the increase of L from 1 to 3,

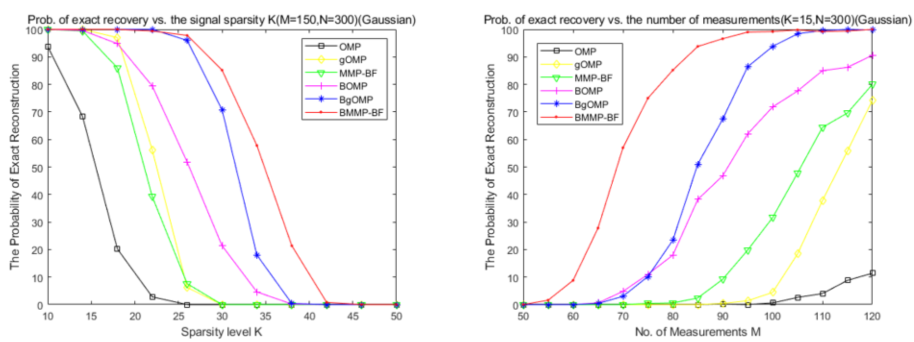


(a) The exact reconstruction probability (b) Average Running time

Figure 1. The performance of BMMP for different L.

while the increase of L hardly improves the reconstruction performance for $L \geq 3$. Then Figure.1 (b) shows that the average running time grows equally as L varies from 1 to 6. Thus, we choose $L = 3$ in the later experiments.

Then the exact reconstruction probability and running time of several reconstruction algorithms are compared, which including (1)OMP algorithm [3], gOMP algorithm [11] with $S = 3$, MMP-BF algorithm [4] with $L = 3$, BOMP algorithm [5], BgOMP algorithm [12] with $S = 3$ and BMMP algorithm with $L = 3$.



(a) Sparsity K (b) The number of measurement M

Figure 2. The performance of several reconstruction algorithms with different K.

In Figure 2 (a), we show a function of the exact reconstruction probability and block sparsity level K for several reconstruction algorithms. We can see that the reconstruction performance of BMMP algorithm is significantly better than other algorithms, and the critical block sparsity of signals that cannot be exactly reconstructed is larger. Figure.2 (b) describes the effect of measurement M on reconstruction probability, where the block sparsity K is fixed to 15. As can be seen, with the increase of measurement M, the reconstruction probability of all algorithms is significantly improved. The critical number of measurement M that enables BMMP to exactly reconstruct the block sparse signal is significantly smaller than other algorithms.

6. Conclusion

The block sparse signal reconstruction algorithm we describe in this paper, BMMP, can be considered as an extension of the MMP algorithm for block sparse systems. Because there are more than one candidate can be kept each time, the BMMP algorithm has better reconstruction performance. Numerical experiments show that BMMP algorithm has excellent performance in terms of exact reconstruction probability compared with many existing algorithms. In addition, we use block-RIP to study the reconstruction condition of BMMP algorithm. In noise-free case, BMMP algorithm accurately reconstructs K-block sparse signal $\mathbf{x}$ within K iterations under $\delta_{K+L} < \frac{\sqrt{L}}{\sqrt{K+L}}$. Finally, in the noisy scenario $\mathbf{y} = \Phi\mathbf{x} + \mathbf{e}$, if the RIC of Φ satisfy $\delta_{K+L} < \frac{\sqrt{L}}{\sqrt{K+L}}$ and $\min_{i\in T} \|x[i]\|_2 \geq \frac{\sqrt{2L(1+\delta_{K+L})}}{\sqrt{L}-\sqrt{K+L}\delta_{K+L}} \|\mathbf{e}\|_2$, BMMP algorithm can reconstruct true supports T.

References

[1] Candes E J, Tao T. Decoding by linear programming[J]. IEEE transactions on information theory, 2005, 51(12): 4203-4215.

[2] Donoho D L. For most large underdetermined systems of linear equations, the minimal ℓ_1 solution is also the sparsest solution[J]. Communications on pure and applied mathematics, 2006, 59(6): 797-829.

[3] Tropp J A, Gilbert A C. Signal recovery from random measurements via orthogonal matching pursuit[J]. IEEE Transactions on information theory, 2007, 53(12): 4655-4666.

[4] Kwon S, Wang J, Shim B. Multipath matching pursuit[J]. IEEE Transactions on Information Theory, 2014, 60(5): 2986-3001.

[5] Eldar Y C, Kuppinger P, Bolcskei H. Block-sparse signals: Uncertainty relations and efficient recovery[J]. IEEE Transactions on Signal Processing, 2010, 58(6): 3042-3054.

[6] Swirszcz G, Abe N, Lozano A C. Grouped orthogonal matching pursuit for variable selection and prediction[J]. Advances in Neural Information Processing Systems, 2009, 22.

[7] Eldar Y C, Mishali M. Robust recovery of signals from a structured union of subspaces[J]. IEEE Transactions on Information Theory, 2009, 55(11): 5302-5316.

[8] Mishali M, Eldar Y C. Blind multiband signal reconstruction: Compressed sensing for analog signals[J]. IEEE Transactions on signal processing, 2009, 57(3): 993-1009.

[9] Wen J, Zhou Z, Liu Z, et al. Sharp sufficient conditions for stable recovery of block sparse signals by block orthogonal matching pursuit[J]. Applied and Computational Harmonic Analysis, 2019, 47(3): 948-974.

[10] Li H, Wang J, Yuan X. On the fundamental limit of multipath matching pursuit[J]. IEEE Journal of Selected Topics in Signal Processing, 2018, 12(5): 916-927.

[11] Wang J, Kwon S, Shim B. Generalized orthogonal matching pursuit[J]. IEEE Transactions on signal processing, 2012, 60(12): 6202-6216.

[12] Qi R, Yang D, Zhang Y, et al. On recovery of block sparse signals via block generalized orthogonal matching pursuit[J]. Signal Processing, 2018, 153: 34-46.