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Stability and Longitudinal Resonances of the Kapitsa Pendulum

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Abstract. The generalized Kapitsa problem of stabilizing the upper position of a deformable pendulum under the action of small vertical oscillations of its base in a gravity field is solved. The presence of a small parameter of the problem allows us to carry out averaging and obtain approximate equations of motion of the pendulum. Two models of a pendulum are considered and compared: a flexible inextensible rod and a flexible tensile rod. The influence of each parameter of the problem on stability is studied. The limits of applicability of the model of a flexible inextensible pendulum are obtained.

Keywords. Kapitsa's pendulum, inverted pendulum, stability conditions, two-scale expansion, average motion, oscillations, resonance, longitudinal oscillations

1. Introduction

The history of the problem of stabilizing an inverted pendulum in a gravity field under the influence of small vertical vibrations of the base dates back to the beginning of the 20th century, with the work of A. Stephenson [1]. Small deviations of a given pendulum from the vertical are described by the Mathieu equation. In 1951 P.L. Kapitsa [2], relying on the asymptotic theory of N.N. Bogoliubov for nonlinear systems, published more detailed studies of the problem. Currently, the theory and practice of generalizing and applying problems of stabilization of inverted pendulums, as well as other nonlinear problems of dynamics, is actively developing. New robust models for modeling and control are appearing [3].

This work continues the tasks considered in [4-5]. The pendulum model is a flexible tensile rod, described by the Bernoulli-Euler beam equations. As in the classical formulation, we assume that the amplitude of the base oscillations is small relative to the length of the pendulum.

Following the classical solution of P.L. Kapitsa, using the theory of two-scale expansions of N.N. Bogoliubov and Y.A. Mitropolski [6], an asymptotically approximate system of equations was obtained that describes the averaged movements of the pendulum.

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The averaging carried out in this work differs from that published in the monograph by I.I. Blekhman [7], since we take into account the initial phase of oscillations of the base of the pendulum, which affects the amplitude of the averaged movements.

The work pays special attention to the issue of the admissibility of adopting a model in which the speed of propagation of longitudinal waves is considered equal to infinity. The conditions for this simplification are considered, as well as the conditions under which the presence of longitudinal waves fundamentally changes the dynamics and stability of the system. This occurs, in particular, when the vibration frequency of the base approaches the resonant frequencies of the longitudinal elastic vibrations of the rod.

2. Equations of dynamics of a tensile deformable rod

Let a pendulum of length L be hinged at the lower end, and let its lower point move along the vertical axis according to the law $a\sin\left(\omega\tilde{t}+\beta\right)$, $\varepsilon = a/L << 1$. In the reference position, the axis of the pendulum is straight and directed vertically (see Figure 1). Let us denote by $\tilde{u}\left(\tilde{x},\tilde{t}\right)$ and $\tilde{w}\left(\tilde{x},\tilde{t}\right)$, $\tilde{x}\in\left(0,L\right)$ small displacements of the pendulum axis points in the longitudinal and transverse directions, respectively. Small deviations of the pendulum in dimensionless variables $x=\tilde{x}/L$, $t=\omega\tilde{t}$ and functions $u=\tilde{u}/L$, $w=\tilde{w}/L$ in the moving coordinate system associated with the anchor point are described by the equations (see [4], eqs. (5) and (10); [5], eq. (2.36))

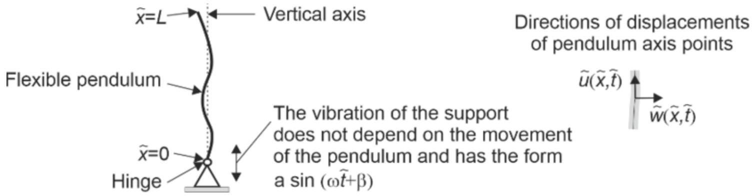

Figure 1. Model of the Kapitsa pendulum.

$$\frac{\partial^2 u}{\partial t^2}+\frac{g}{L\omega^2}-\varepsilon\sin\left(t+\beta\right)=\frac{E}{L^2\rho\,\omega^2}\frac{\partial^2 u}{\partial x^2},\quad u(0,t)=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=0, \tag{1}$$

$$\frac{\partial^4 w}{\partial x^4}+\frac{\partial}{\partial x}\left(p\left(x,t\right)\frac{\partial w}{\partial x}\right)+\frac{P_D\,\zeta}{\varepsilon^2}\frac{\partial^2 w}{\partial t^2}=0,\quad w(0,t)=\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=1}=\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=1}=0, \tag{2}$$

where ω is the frequency of vibrations of the base, g is the acceleration of free fall, ε and β are the small dimensionless amplitude and the initial phase of vibrations of the base, respectively, E – Young's modulus, ρ – volumetric density of the rod, dimensionless constants have the form $P_D=\dfrac{P_0L^2}{D}$, $\zeta=\dfrac{a^2\omega^2}{gL}$, P_0 is the weight of the rod, D is the bending rigidity of the rod. Dimensionless force of longitudinal compression of the rod axis $p(x,t)$ must be found from Eq. (1) (see [4], eq. (11))

$$p(x,t) = -\frac{L^2}{D} ES \frac{\partial u}{\partial x} = P_D \left((1-x) - \frac{\zeta}{\varepsilon} \frac{(\cos \nu x \tan \nu - \sin \nu x)}{\nu} \sin(t+\beta) \right), \quad (3)$$

where $\nu = \frac{L\omega}{c}$ is the dimensionless oscillation frequency and $c = \sqrt{E/\rho}$ is the velocity of propagation of the longitudinal wave, S is the cross-sectional area. For an incompressible rod we have $(\cos \nu x \tan \nu - \sin \nu x)/\nu \underset{\nu \to 0}{\to} (1-x)$.

To solve problem (2), similar to [5] (eq. (2.39)), an auxiliary boundary value problem is introduced, and then solution (2) is sought in the form of a series in the eigenfunctions Ψ_n of problem (4)

$$w(x,t) = \sum_{n=1}^{N} \Psi_n(x)\, w_n(t), \qquad \frac{d^4\Psi_n}{dx^4} + \lambda_n \frac{d}{dx}\left((1-x) \frac{d\Psi_n}{dx} \right) = 0 \quad (4)$$

$$\Psi_n(0) = \Psi_n''(0) = \Psi_n''(1) = \Psi_n'''(1) = 0, \quad \int_0^1 (1-x) \frac{d\Psi_k}{dx} \frac{d\Psi_n}{dx} dx = 0, \quad k \neq n. \quad (5)$$

Note that the critical values λ_n of problem (4) are the Eulerian critical values of the problem of static deflection of a rod by a longitudinal force. Integrating equation (4) with boundary conditions (5), we arrive at the Airy equation $V''(x,\lambda) + \lambda(1-x)V(x,\lambda) = 0$, $\Psi_k(x) = \int_0^x V(x,\lambda_k)dx$ and its solution in the form of a series $V(x,\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda(1-x)^3)^k}{a_k}, \quad a_0 = 1, \quad a_k = 3k(3k-1)a_{k-1}$.

Problem (2) is reduced to the system

$$\sum_{k=1}^{N} a_{nk} \frac{d^2 w_k}{dt^2} + \frac{\varepsilon^2 b_n}{\zeta} \left(\frac{\lambda_n}{P_D} - 1 \right) w_n + \sum_{k=1}^{N} \varepsilon c_{nk} \sin(t+\beta) w_k = 0, \qquad n = 1,\dots,N. \quad (6)$$

By introducing the vector $\{w_k\}_{k=1,N}^T$, it is convenient to write system (6) in matrix form

$$\mathbf{A} \cdot \frac{d^2\mathbf{W}}{dt^2} + \frac{\varepsilon^2}{\zeta} \mathbf{P} \cdot \mathbf{W} + \varepsilon \mathbf{C} \cdot \mathbf{W} \sin(t+\beta) = 0, \qquad \varepsilon = \frac{a}{L}, \qquad \zeta = \frac{a^2\omega^2}{gL}, \quad (7)$$

where the coefficient matrices have the form $a_{nk} = \int_0^1 \Psi_n(x)\Psi_k(x)dx$, $P_{nn} = b_n \left(\frac{\lambda_n}{P_D} - 1 \right)$, $b_n = \int_0^1 (1-x) \left(\frac{d\Psi_n}{dx} \right)^2 dx$, $c_{nk} = \int_0^1 \frac{(\cos \nu x \tan \nu - \sin \nu x)}{\nu} \frac{d\Psi_n}{dx} \frac{d\Psi_k}{dx} dx$. Note that $\lambda_1 = 0$, which means $P_{11} < 0$, therefore the second term in the equation (7) will contribute to the destabilizing effect on the movement of the pendulum. The first form of oscillation of a pendulum is the movement of the pendulum as a rigid whole, $w_1(x)=x$. The third term in system (7), as in the Mathieu equation, will have a stabilizing effect. Let us evaluate this influence using the averaging method.

3. Averaged equations of motion of a pendulum

Let us set the initial conditions of system (7) in the following form $\mathbf{W}(0)=\mathbf{W}_0,\ \ d\mathbf{W}/dt|_{t=0}=0$, that is, the rod is deflected without an initial velocity. System (7) has a small parameter; let us introduce slow time $\tau=\varepsilon t$ and present the solution in the form of an asymptotic expansion [6] in a series in powers of a small parameter ε, where $\mathbf{U}_m(\tau)$ are slowly changing functions of time, $\mathbf{V}_m(t,\tau)$ are rapidly changing functions that have a zero average value for the period

$$\mathbf{W}(t,\tau,\varepsilon)=\sum_{m=0}^{\infty}\left(\mathbf{U}_m(\tau)+\mathbf{V}_m(t,\tau)\right)\varepsilon^m,\quad \frac{1}{2\pi}\int_0^{2\pi}\mathbf{V}_m(t,\tau)dt=0,\quad m=0,1,.... \tag{8}$$

Substituting series (8) into system (7) and successively equating terms at equal powers of the small parameter ε, taking into account $\frac{d^2\mathbf{W}}{dt^2}=\frac{\partial^2\mathbf{W}}{\partial t^2}+2\varepsilon\frac{\partial^2\mathbf{W}}{\partial t\,\partial\tau}+\varepsilon^2\frac{\partial^2\mathbf{W}}{\partial\tau^2}$, we obtain in the zero asymptotic approximation a system of equations and initial conditions for a slowly varying function $\mathbf{U}_0(\tau)$

$$\mathbf{A}\cdot\frac{d^2\mathbf{U}_0}{d\tau^2}+\mathbf{D}\cdot\mathbf{U}_0=0,\qquad \mathbf{D}=\frac{1}{\zeta}\mathbf{P}+\frac{1}{2}\mathbf{C}\cdot\mathbf{A}^{-1}\cdot\mathbf{C}, \tag{9}$$

$$\mathbf{U}_0(\tau)\big|_{\tau=0}=\mathbf{W}_0,\quad \left.\frac{d\mathbf{U}_0}{d\tau}\right|_{\tau=0}=\mathbf{A}^{-1}\cdot\mathbf{C}\cdot\mathbf{U}_0\cos\beta. \tag{10}$$

The condition for the stability of the solution of the averaged system will be the condition of positive definiteness of the matrix $\mathbf{D}$. The matrix $\mathbf{A}$ is positive definite.

4. Examples. The discussion of the results

Let's consider an example of a steel pendulum with Young's modulus equal to $E=2.04\cdot10^{11}$ N/m^2 and density $\rho=7850$ kg/m^3 in a gravity field with acceleration $g=9.81$ m/c^2. Let the pendulum have length L and a square cross-section with thickness h, $h/L\le 0.1$. We will calculate the stability condition for various amplitudes of vibrations of the base a depending on the angular velocity ω. To check the correctness of the result, we will compare the solution to the problem with the known solution for a non-deformable rod.

The expression for the matrix $\mathbf{C}$ includes terms containing $\tan\nu$, which, in the absence of friction, will grow unlimitedly at $\nu \to (k+0.5)\pi,\ k=0,1,\ldots$ and these frequencies are resonant. Note that for the model of an inextensible rod, the longitudinal force, instead of formula (3), will be calculated by the formula

$$p(x,t) = P_D(1-x)\big(1-(\zeta/\varepsilon)\sin(t+\beta)\big) \tag{11}$$

and taking into account the orthogonality (5) in the formula for $\mathbf{C}$ only diagonal terms will remain, independent of the frequency ν, and $\mathbf{C} \to \mathbf{B}$ at $\nu \to 0$.

For a given pendulum length L and amplitude a for a model of a non-deformable pendulum, we can calculate the critical value ω^* – the minimum frequency of base oscillations that ensures the stability of the Kapitsa pendulum in the upper position, $\omega^* = \frac{4gL}{3a^2}$. We can compare ω^* with the first resonant value of the base oscillation frequency $\omega_1 = \frac{\nu_1 c}{L} = \frac{\pi}{2}\frac{c}{L}$.

In the numerical solution, the first three natural modes of vibration were taken into account.

In Table 1 shows the critical values of oscillation frequencies ω^* and $\nu^* = L\sqrt{\rho/E}\,\omega^*$, as well as limiting maximum frequency values ω^{up}, ν^{up} at which it is still possible to use the model of an inextensible rod. The estimation was carried out using the first minor of the matrix $\mathbf{D}$, calculated for two models of the rod. We will require that the inequality $\delta = \left|D_I^{ex} - D_I^{non}\right| / \left|D_I^{non}\right| < 0.05$, where D_I^{non} and D_I^{ex} are the values of the first minor, calculated using the models of an inextensible and tensile rod, respectively. Table 1 shows that for short pendulums the range of applicability of the inextensible rod model is very wide, but for sufficiently long pendulums this range completely disappears.

Table 1. Values of the stability boundaries ν^*, ω^* of the tensile rod problem and values of upper limits of applicability of the inextensible rod model ν^{up}, ω^{up} at $h = 0.01L,\ a = 0.01L.$

L	ν^*	ν^{up}	ω^*	ω^{up}
0.1	0.002	0.25	102	12700
0.32	0.0115	0.25	183	4000
1.	0.0635	0.25	324	1270
1.5	0.116	0.22	394	750
2.5	0.25	0.20	510	410

Graphs of the value δ for the first three minors of the matrix $\mathbf{D}$ for different pendulum lengths are shown in Figure 2. Graphs of the functions of the first three minors of the matrix $\mathbf{D}$, calculated using the models of an inextensible and tensile rod, are shown in Figure 3, where the growth pattern of the values of D_I, D_{II}, D_{III} at resonances is visible. Thus, for an extensible pendulum, in contrast to an inextensible one, at resonances the frequency of stable oscillations near the upper equilibrium position increases, and between resonances, on the contrary, it decreases, as can be seen in Figure 3.

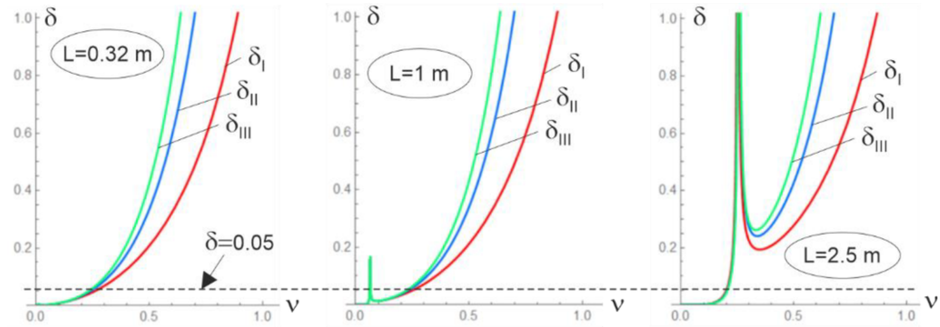

Figure 2. Graphs of δ for different pendulum lengths at $h = 0.01L,\ a = 0.01L$.

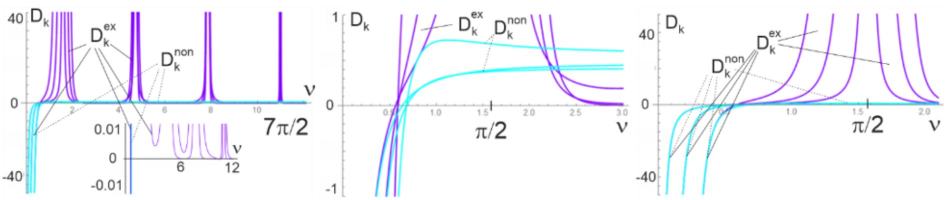

Figure 3. First minors of the matrix $\mathbf{D}$ for $L = 1\,\mathrm{m},\ h = a = 0.001L$.

5. Conclusion

The problem of the dynamics of a flexible tensile rod with a hinged lower end and a free upper end under the action of a given vibration of the lower fixing point in a gravity field has been solved analytically and numerically. The boundary value problem is reduced to a system of ordinary differential equations, where, as in the case of the classical Mathieu equation, there is a term that rapidly changes in time and is associated with vibrations of the support. An averaged approximate system of equations is obtained. The conditions for the stability of the averaged system are found. The limits of applicability of the model of an inextensible flexible pendulum are obtained.

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