

Proving the power of postselection^{*}

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Abstract. It is a widely believed, though unproven, conjecture that the capability of postselection increases the language recognition power of both probabilistic and quantum polynomial-time computers. It is also unknown whether polynomial-time quantum machines with postselection are more powerful than their probabilistic counterparts with the same resource restrictions. We approach these problems by imposing additional constraints on the resources to be used by the computer, and are able to prove for the first time that postselection does augment the computational power of both classical and quantum computers, and that quantum does outperform probabilistic in this context, under simultaneous time and space bounds in a certain range. We also look at postselected versions of space-bounded classes, as well as those corresponding to error-free and one-sided error recognition, and provide classical characterizations. It is shown that NL would equal RL if the randomized machines had the postselection capability.

keywords: postselection, quantum Turing machines, probabilistic Turing machines, space-bounded computation, one-sided error, zero error

1 Introduction

The notion of postselection as a mode of computation was introduced by Aaronson [Aar05]. Postselection is the (unrealistic) capability of discarding all branches of a computation in which a specific event does not occur, and focusing on the surviving branches for the final decision about the membership of the input string in the recognized language. Aaronson examined PostBQP, the class of languages recognized with bounded error by polynomial-time quantum computers with postselection, and showed it to be identical to the well-known classical complexity class PP. The corresponding class for probabilistic polynomial-time computers with postselection is known to equal BPP_{path} . It is, however, still an open question whether postselection adds anything to the power of quantum or classical polynomial-time computation, since we do not know whether the standard classes (without postselection) for these models, that is, BQP and BPP, respectively, equal their postselected versions or not. It is also not known whether $BPP_{\text{path}} = PP$, or polynomial-time quantum machines with postselection outperform their classical counterparts, as is conjectured to be the case between the standard versions. All these classes sit between P and PSPACE, and any proof of inequality between them would therefore be as hard as proving $P \neq PSPACE$.

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In this paper, we approach the problem of evaluating the power of postselection by imposing more resource restrictions on the computer. We demonstrate certain simultaneous space and time bounds, under which it is proven that postselection increases the power of both probabilistic and quantum machines, and that quantum computers with this capability outperform classical ones. More precisely, we prove that polynomial-time and $o(\log \log n)$ -space probabilistic computers with postselection are more powerful than their standard versions. To handle quantum machines, we constrain the time bound further to allow only real-time computation, where the computer has just time enough for a single left-to-right scan of the input. Under this restriction, we prove that quantum computers with postselection that are allowed to use only $O(1)$ space outperform those without postselection, and that probabilistic machines with postselection are inferior to quantum ones for any common space bound that is sublogarithmic.

We also examine postselected versions of classes that are only space-bounded, as well as those corresponding to error-free and one-sided error recognition, and provide several classical characterizations. It turns out that postselection adds nothing to purely space-bounded machines, and the error-free and one-sided error classes for probabilistic machines with postselection for any space bound $s(n) = \Omega(\log n)$ are identical to each other, and to $\text{NSPACE}(s(n))$. We show that allowing one-sided bounded error to be committed by a small-space quantum machine with postselection enlarges the class of recognized languages, in contrast to the probabilistic case. We also show that NL would equal RL if the randomized machines had the postselection capability.

Aaronson’s definition is based on quantum circuits, whereas we use a model of postselection based on Turing machines (TMs), which facilitates dealing with space bounds. Lāce *et al.* were the first to use machines, rather than circuits, for modeling postselection, in their groundbreaking work [LSF09] on the effect of this capability on quantum finite automata. We show that an idiosyncratic feature in their model makes it strictly more powerful than ours, which is more in alignment with Aaronson’s original definition.

A key tool used in our proofs is the *Turing machine with restart*, which is simply an ordinary TM that has been augmented with the capability of resetting itself to the initial configuration in a single step. We demonstrate that this restart action is equivalent in effect to postselection. The model of TM with restart is obtained by generalizing the recently introduced real-time finite automata with restart [YS10c].

The rest of the paper is structured as follows: Section 2 recalls the standard definitions of probabilistic and quantum machines, and introduces the notation that will be used later. Machines with restart are defined in Section 3. Section 4 contains our definition of postselection. Our proofs of the superiority of machines with postselection over their standard versions, and that of the quantum variant over the classical one, are presented in Section 5. Characterizations of classes of languages recognized by these machines exactly, or with one-sided error, can be found in Section 6. The model of Lāce *et al.* is compared with ours in Section 7. Section 8 is a conclusion. Additional details on some of the discussed points are provided in the Appendices.

2 Preliminaries

Since our discussion will involve both space and time complexity issues, we will use resource-bounded versions of Turing machines as our models of computation. We now present quick

definitions of standard probabilistic and quantum TMs, which will serve as templates for the new models to be introduced in the subsequent sections. We assume some familiarity with probabilistic and quantum computation, and the reader is referred to [YS11b] for a wider coverage of the basics.

All TMs we will consider have a read-only input tape, which contains the input string, sandwiched between the two special end-marker symbols \clubsuit and $\$$. The two-way input tape head is initially positioned on the left end-marker \clubsuit , and is not allowed to leave the area delimited by the end-markers. Space complexity is measured by the maximum number of cells that are ever visited with non-zero probability (or amplitude, in the quantum case) by the read/write head of the single work tape of the machine, as a function of the length of the input string. A configuration of a TM is a collection of its internal state, positions of the input and work tape heads, and the contents of the work tape.

A probabilistic Turing machine (PTM) is a 6-tuple

$$\mathcal{P} = (Q, \Sigma, \Gamma, \delta, q_1, \Delta),$$

where Q , Σ , Γ , and q_1 denote the set of internal states, the input alphabet, the work tape alphabet, and the initial state, respectively.

The transition function δ is specified such that

$$\delta(q, \sigma, \gamma, q', d_i, \gamma', d_w) \in \tilde{\mathbb{R}}$$

is the probability that the PTM will change its internal state to q' , write γ' on the work tape, and update the positions of the input and work tape heads with respect to d_i and d_w , respectively, where $d_i, d_w \in \langle \rangle = \{left, right, stationary\}$, if it scans σ and γ on the input and work tapes, respectively, when originally in internal state q . $\tilde{\mathbb{R}}$ is the set consisting of $p \in \mathbb{R}$ such that there is a deterministic algorithm that computes p to within 2^{-n} in time polynomial in n .

For each input string $w \in \Sigma^*$, δ defines a unique configuration transition matrix, A^w . A PTM is *well-formed* if all columns of A^w are stochastic vectors. This constraint defines the following local conditions for PTM well-formedness that δ must obey: For each $q \in Q$, $\sigma \in \tilde{\Sigma} = \Sigma \cup \{\clubsuit, \$\}$, and $\gamma \in \Gamma$,

$$\sum_{q', d_i, \gamma', d_w} \delta(q, \sigma, \gamma, q', d_i, \gamma', d_w) = 1,$$

where $q' \in Q$, $\gamma' \in \Gamma$, and $d_i, d_w \in \langle \rangle$.

In order to make the presentation of the essential differences among the various machine models to be discussed in the paper easier, we decree that all TM definitions include the item $\Delta = \{\tau_1, \dots, \tau_k\}$, which is the set of “move outcomes,” that summarize the overall condition of the computation after each step. In standard (probabilistic and quantum) TMs, $\Delta = \{c, a, r\}$, and Q , the usual finite set of internal states of the machine, is partitioned to three corresponding subsets Q_c , Q_a , and Q_r , called the sets of continuing (non-halting), accepting, and rejecting states, respectively. The computation is terminated, and the input is accepted (resp., rejected) if the TM enters a state belonging to Q_a (resp., Q_r). The machine continues with the next move otherwise. We use the name Q_h (the set of halting states) to refer to all states at which the computation is terminated, and we naturally have $Q_h = Q_a \cup Q_r$ for all standard TM variants.

For ease in the modeling of space-efficient quantum computation, our quantum Turing machines (QTMs) are assumed to contain an additional component, namely, a finite register, which is used in the observation of the move outcomes. The set of different values that this register can contain is denoted by Ω , and is also partitioned into $|\Delta|$ subsets (Ω_c , Ω_a , and Ω_r in the case of standard QTMs), corresponding to the different types of move outcomes. A QTM is then defined as a 7-tuple

$$\mathcal{M} = (Q, \Sigma, \Gamma, \Omega, \delta, q_1, \Delta),$$

where Ω_c is required to contain a special *initial symbol* ω_1 , and new conditions to be described below are imposed on the transition function δ .

The transition function of a QTM is specified so that

$$\delta(q, \sigma, \gamma, q', d_i, \gamma', d_w, \omega) \in \tilde{\mathbb{R}}$$

is the amplitude with which the QTM will change its internal state to q' , write γ' on the work tape and ω in the finite register, and update the positions of the input and work tape heads with respect to d_i and d_w , respectively, where $d_i, d_w \in \langle \triangleright, \triangleleft \rangle$, if it scans σ and γ on the input and work tapes, respectively, when originally in internal state q . (The finite register always contains ω_1 at the beginning of every move.)

After each transition, the finite register is measured to see which one of the sets Ω_a , Ω_r , or Ω_c the current register symbol belongs to, and the following actions are associated with the measurement outcomes:

- “c”: the computation continues;
- “a”: the computation halts, and the input is accepted;
- “r”: the computation halts, and the input is rejected.

The finite register is irreversibly reinitialized to ω_1 before the next transition takes place.

Any superiority that quantum computers have over their probabilistic counterparts can be traced to the fact that machine configurations can have negative as well as positive amplitudes, sometimes allowing parallel computational branches to interfere with each other in a way that is impossible in classical computation. The amount with which a particular symbol will contribute to the probability of measurement of the associated outcome is the modulus squared of the corresponding amplitude at the time of observation. Appendix A contains a description of the well-formedness conditions that quantum machines must satisfy.

Any sufficiently general quantum model can simulate the corresponding probabilistic model (subject to the same space and time restrictions) exactly, with essentially no overhead [Wat09,YS11b]. So the question faced when comparing such a pair of models is always whether they are equivalent in power, or the quantum version can outperform the probabilistic one.

We will examine the language recognition of different types of machines under several error regimes. The terminology to be used in this regard is summarized below.

The language $L \subseteq \Sigma^*$ recognized by machine \mathcal{M} with (*strict*) *cutpoint* $\lambda \in \mathbb{R}$ is defined as

$$L = \{w \in \Sigma^* \mid \text{P}(\mathcal{M} \text{ accepts } w) > \lambda\}.$$

The language $L \subseteq \Sigma^*$ recognized by machine \mathcal{M} with *nonstrict cutpoint* $\lambda \in \mathbb{R}$ is defined as [BJKP05]

$$L = \{w \in \Sigma^* \mid \text{P}(\mathcal{M} \text{ accepts } w) \geq \lambda\}.$$

The two cases described above comprise recognition with (*two-sided*) *unbounded error*, where every member of the recognized language L is accepted with a probability greater than every nonmember of L .

Machines that recognize a language with cutpoint 0, i.e. those that accept a string with nonzero probability if and only if that string is a member of the language, are said to be *nondeterministic*.

The language $L \subseteq \Sigma^*$ is said to be recognized by machine \mathcal{M} with (*two-sided*) *bounded error* if there exists an error bound ϵ ($0 \leq \epsilon < \frac{1}{2}$) such that

- $P(\mathcal{M} \text{ accepts } w) \geq 1 - \epsilon$ for all $w \in L$, and,
- $P(\mathcal{M} \text{ rejects } w) \geq 1 - \epsilon$ for all $w \notin L$.

Recognition with *one-sided bounded error* is defined as recognition by a nondeterministic machine with bounded error. A bounded-error machine whose error bound equals 0 is said to be performing *error-free* (or *exact*) computation.

Table 1 lists some of the language classes that we will mention. Most of the terminology here is standard, but our definition of the EQSPACE classes is different from that of Watrous, who used this designation for the first time [Wat99] in terms of QTMs that were allowed to fail to halt with probability 1 for input strings that are not members of the language to be recognized. Our definition, paralleling the definition of EQTime in [BV97], corresponds to the class EQ_{AS}SPACE in [Wat99]. The reader should also note that the EPTIME(t) classes in Table 1 clearly equal TIME(t) for any t ; we include the name here only because the postselected version of these classes will be studied in Section 6.

Table 1. The standard classes of languages recognized by PTMs and QTMs

<i>space-bounded classes</i>						
machine type	unbounded-error	nondeterministic	bounded-error	one-sided	bounded-error	error-free
PTM	PrSPACE	NSPACE	BSPACE	RSPACE		EPSPACE
QTM	PrQSPACE	NQSPACE	BQSPACE	RQSPACE		EQSPACE
<i>time-bounded classes</i>						
machine type	unbounded-error	nondeterministic	bounded-error	one-sided	bounded-error	error-free
PTM	PrTIME	NTIME	BPTIME	RTIME		EPTIME
QTM	PrQTIME	NQTIME	BQTIME	RQTIME		EQTIME

Simultaneous time-space bounds for nondeterministic, probabilistic, and quantum machines have been studied in, for instance, [BM80,DvM06,vMW08], respectively, from which we take the following class definitions:

BPTISP(t, s), RTISP(t, s), and NTISP(t, s) are the classes of languages recognized with (two-sided) bounded error, one-sided bounded error, and with cutpoint zero, respectively, by PTMs running in time t , and using space s . BQTISP is the quantum counterpart of BPTISP. We define PrTISP and PrQTISP to be the unbounded-error counterparts of BPTISP and BQTISP.

We should clarify a potential source of confusion about randomized space-bounded classes [Sak96]. RSPACE(s), which is defined to be the class of languages recognized with one-sided error by PTMs using space s , turns out to be identical to NSPACE(s), i.e. the class of languages

recognized by nondeterministic TMs that use space s [Gil77]. The designation RL, which was originally a shorthand for $\text{RSPACE}(\log n)$, is now used to denote the more interesting class of languages that are recognized with positive one-sided error by logspace PTMs with polynomial time bounds [Gol08], that is, $\text{RTISP}(\text{poly}(n), \log n)$.

Some of our results involve *real-time* machines, i.e., those that are restricted to move the input head to the right in every step of the computation, forcing them to have a runtime of $n + 2$ steps, where n is the length of the input.¹ Inspired by Bruda’s definition [Bru02] of the $\text{rt-SPACE}(s)$ classes, we define $\text{RTBSPACE}(s)$ as the class of languages recognized with (two-sided) bounded error by real-time PTMs using space s . RTBQSPACE is the quantum counterpart of RTBSPACE .

It is well known that, for machines with constant space usage, one can remove the work tape altogether, at the cost of having a longer program, without changing the recognized language. This specialization of TMs yields the well-known finite automata [Sip06, Rab63a, YS11b]. One quirk in the literature that we should be careful about is the fact that the transition probabilities and amplitudes of probabilistic and quantum finite automata (PFAs and QFAs, respectively) are allowed to be arbitrary real numbers (including uncomputable ones) of absolute value at most 1, enabling these machines to recognize many Turing-undecidable languages [Rab63a], and we will use different names (Table 2) to denote the classes arising from this different range of δ .

Table 2. Classes of languages recognized by real-time finite automata

<u>machine type</u>	<u>unbounded-error</u>	<u>nondeterministic</u>
real-time PFA	$S \cup \text{co}S = \text{u}S$	REG
real-time QFA	$\text{QAL} \cup \text{coQAL} = \text{uQAL}$	NQAL

S and QAL are the classes of languages recognized with cutpoint $\frac{1}{2}$ by real-time PFAs [Rab63a] and QFAs [YS11b], respectively. REG is the class of regular languages.

3 Probabilistic and quantum machines with restart

In this section, we introduce the effects of adding the capability of “restarting” the computation, that is, restoring the internal state, input head position, and the work tape to their initial settings in a single move, to the set of allowed actions of several Turing machine variants and specializations. This simple and seemingly useless action turns out to be important in our analysis of postselection in the rest of the paper. Finite automata with restart were introduced and analyzed in [YS10c], we generalize the concept to TMs here.

Using our general framework of TM definitions mentioned in the previous section, we can define a (probabilistic or quantum) TM with restart simply by stating that the set of possible move outcomes, Δ , contains an additional element, rs , and the overall set of states, Q , is now correspondingly partitioned into four subsets: The usual Q_c , Q_a , and Q_r , and the set Q_{rs} ,

¹ It is well-known [Rab63b, Aan74] that increasing the number of work tapes increases the language recognition power of standard versions of real-time machines. Our models have a single work tape, but all our results regarding real-time machines with postselection remain valid when multiple work tapes are allowed.

namely, the set of restarting states. Any transition to a state in Q_{rs} results the machine to restart from the initial configuration in the next move, as explained above.

A segment of computation of a TM with restart which begins with a (re)start, and ends with a halting or restarting state will be called a *round*. For any time bound t , a *t-time TM with restart* is a TM with restart with the restriction that the runtime of no single round is greater than t . We will be focusing on *real-time machines with restart*, that is, TMs in which the input head is forbidden to make single-step leftward or stationary moves, but restarts are allowed. Note that the *overall* expected runtime of a t -time TM with restart can be much more than t ; for instance, [YS10c] contains several examples of real-time TMs with restart that have exponential runtime.

Let $p_{\mathcal{R}}^a(w)$ ($p_{\mathcal{R}}^r(w)$) be the probability that w is accepted (rejected) in a single round of a TM with restart named \mathcal{R} . For a given input string $w \in \Sigma^*$, the overall acceptance and rejection probabilities of w can be calculated as shown in the following lemma [YS10c].

Lemma 1. $P(\mathcal{R} \text{ accepts } w) = \frac{p_{\mathcal{R}}^a(w)}{p_{\mathcal{R}}^a(w) + p_{\mathcal{R}}^r(w)}$ and $P(\mathcal{R} \text{ rejects } w) = \frac{p_{\mathcal{R}}^r(w)}{p_{\mathcal{R}}^a(w) + p_{\mathcal{R}}^r(w)}$.

Proof.

$$\begin{aligned} P(\mathcal{R} \text{ accepts } w) &= \sum_{i=0}^{\infty} (1 - p_{\mathcal{R}}^a(w) - p_{\mathcal{R}}^r(w))^i p_{\mathcal{R}}^a(w) \\ &= p_{\mathcal{R}}^a(w) \left(\frac{1}{1 - (1 - p_{\mathcal{R}}^a(w) - p_{\mathcal{R}}^r(w))} \right) \\ &= \frac{p_{\mathcal{R}}^a(w)}{p_{\mathcal{R}}^a(w) + p_{\mathcal{R}}^r(w)} \end{aligned}$$

$P(\mathcal{R} \text{ rejects } w)$ is calculated in the same way. □

4 Turing machines with postselection

We are now ready to present our model of computation with postselection.

Turing machines with postselection are defined such that the overall state set is partitioned into four subsets, namely, the sets of continuing (Q_c), postselection accept (Q_{pa}), postselection reject (Q_{pr}), and nonpostselection halting (Q_{nh}) states. Correspondingly, we set $\Delta = \{c, pa, pr, nh\}$. For these machines, $Q_h = Q_{pa} \cup Q_{pr} \cup Q_{nh}$. The computation is terminated whenever the machine enters a state in Q_h . For every possible input, a Turing machine with postselection always halts in a state in $Q_{pa} \cup Q_{pr}$ with nonzero probability.

Probabilistic and quantum real-time finite automata with postselection (PFAPs and QFAPs) are obtained by specializing the TM version in the manner described in Section 2; see Appendix B for detailed definitions.

Let $p_{\mathcal{P}}^a(w)$ (resp. $p_{\mathcal{P}}^r(w)$) be the probability that a TM with postselection named \mathcal{P} reaches a state in Q_{pa} (resp. Q_{pr}) when run on an input w .² For each such $w \in \Sigma^*$, the overall acceptance and rejection probabilities of w are obtained by normalization, and are given by

$$P(\mathcal{P} \text{ accepts } w) = \frac{p_{\mathcal{P}}^a(w)}{p_{\mathcal{P}}^a(w) + p_{\mathcal{P}}^r(w)}, \quad (1)$$

² Note that we are using notation identical to that introduced in the discussion for machines with restart for these probabilities; the reason will be evident shortly.

and

$$P(\mathcal{P} \text{ rejects } w) = \frac{p_{\mathcal{P}}^r(w)}{p_{\mathcal{P}}^a(w) + p_{\mathcal{P}}^r(w)}. \quad (2)$$

“Postselection” is the name given to this process, where any computational path ending with a transition to a state in Q_{nh} is simply discarded, and only the ones ending with a state in $Q_{pa} \cup Q_{pr}$ are “selected”.

For every language class \mathbf{C} defined using a resource-bounded probabilistic or quantum machine model under a particular error regime, we define the class \mathbf{PostC} of the languages recognized under the same error regime by the corresponding type of machines with postselection.

Some other classes that will be studied are presented in Table 3. (We need to name these separately, since it is easy to see that the standard versions of all these classes correspond to the same standard class, namely, the regular languages. The postselected versions are not identical, as will be apparent in the subsequent sections.)

Table 3. Classes of languages recognized by real-time constant-memory machines with postselection under different error regimes

<u>machine type</u>	<u>two-sided bounded-error</u>	<u>one-sided bounded-error</u>	<u>error-free</u>
PFAP	PostBS	PostRS	PostES
QFAP	PostBQAL	PostRQAL	PostEQAL
real-time PTM	PostRTBPPSPACE(1)	PostRTRSPACE(1)	PostRTEPPSPACE(1)
real-time QTM	PostRTBQSPACE(1)	PostRTRQSPACE(1)	PostRTEQSPACE(1)

Since none of the results on real-time constant-space machines in this paper are sensitive to the existence of uncomputable numbers among the transition probabilities of the program, we will mostly use the shorter class names in the top two rows of Table 3 in the subsequent sections, with the implication that the same relationship is valid among the corresponding classes in the bottom two rows.

Trivially, every postselected class contains its standard version with the same resource bounds, and we are interested in finding out whether the inclusion is proper or not.

Although we will not have much to say about classes defined solely in terms of time bounds, the reader will note that Aaronson’s $\mathbf{PostBQP}$ [Aar05] has an equivalent definition in terms of our model, as the class of languages recognized with bounded error by polynomial-time QTMs with postselection.³

5 The power of postselection

It is evident from the similarity of the statement of Lemma 1 and Equations 1 and 2 that there is a close relationship between machines with restart and those with postselection. This is set out in the following theorem.

³ A detailed treatment of the equivalence between the TM and circuit models of quantum computation can be found in [Yao93].

Theorem 1. *For any time bound t and space bound s , the class of languages recognized by t -time and s -space PTMs (resp. QTMs) with postselection is identical to the class of languages recognized by t -time PTMs (resp. QTMs) with restart using space s . The same equality is also valid for the real-time, in particular, finite memory, versions of these models.*

Proof. Given a (probabilistic or quantum) machine with postselection called \mathcal{P} , we can construct a corresponding machine with restart called \mathcal{R} , which is identical to \mathcal{P} , except that all nonpostselection halting states of \mathcal{P} are designated as restart states in \mathcal{R} . \mathcal{R} 's accept and reject states correspond precisely to \mathcal{P} 's postselection accept and reject states, respectively.

Given a machine with restart \mathcal{R} , we construct a corresponding machine with postselection \mathcal{P} by starting with an exact copy of \mathcal{R} , designating the old accept and reject states as the postselection accept and reject states of \mathcal{P} , respectively, and converting the restart states to nonpostselection halting states.

By Lemma 1 and Equations 1 and 2, the machines before and after these conversions recognize the same language, with the same error bound. \square

Theorem 1 will be useful in our analyses of the language classes corresponding to machines with postselection. One immediate corollary is that the postselected versions of classes that are defined solely in terms of space bounds (for machines with two-way input heads) are equal to the corresponding classes for standard machines.

So the additional power brought by the capability of postselection is nil for space-bounded machines, and is probably difficult to prove for time-bounded machines. We therefore consider machines operating under simultaneous time-space bounds, and are now able to demonstrate that postselection increases the recognition power of both probabilistic and quantum computers. For a given string w , let $|w|_\sigma$ denote the number of occurrences of symbol σ in w .

Theorem 2. *PTMs with postselection that use $o(\log \log n)$ space and that have polynomial expected runtime are strictly more powerful than their standard versions.*

Proof. As proven by Dwork and Stockmeyer, standard polynomial-time PTMs that use $o(\log \log n)$ space recognize precisely the regular languages [DS90]. The nonregular language $L_{eq} = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$ can be recognized by a real-time PTM with restart using $O(1)$ space [YS10c]. By Theorem 1, $L_{eq} \in \text{PostRTBPSPACE}(1)$. We conclude that the BPTISP classes are properly contained in the PostBPTISP classes for these time-space bounds. \square

If the PTMs in question are further restricted so that their input tape heads can never move left, we do not need an expected value to exist for the runtime:

Theorem 3. *One-way PTMs with postselection that use $o(\log \log n)$ space are strictly more powerful than their standard versions.*

Proof. One-way PTMs that use $o(\log \log n)$ space are known to recognize precisely the regular languages [Fre85]. The remainder follows the proof of Theorem 2. \square

A similar advantage can also be demonstrated for QTMs, albeit under more severe bounds. w^r denotes the reverse of string w .

Theorem 4. $\text{RTBQSPACE}(1) \subsetneq \text{PostRTBQSPACE}(1) \subseteq \text{PostBQAL}$.

Proof. It is known [KW97,Boz03,Jea07] that $\text{RTBQSPACE}(1) = \text{REG}$. The nonregular language $L_{pal} = \{w \in \{a,b\}^* \mid w = w^r\}$ can be recognized by a real-time QTM with restart using $O(1)$ space [YS10c]. By Theorem 1, $L_{pal} \in \text{PostRTBQSPACE}(1)$, leading us to conclude that QTMs with postselection outperform their standard counterparts under these time-space bounds. \square

We are also able to show that quantum postselection machines outperform their classical counterparts. This follows easily from the well-known relation

$$\text{BSPACE}(s) \subsetneq \text{BQSPACE}(s) \quad (3)$$

for any space bound $s = o(\log n)$, ($L_{pal} \in \text{BQSPACE}(1)$ [AW02], but $L_{pal} \notin \text{BSPACE}(s)$ for such s [FK94]) since we now know that these classes equal their postselected versions. We wish to find a setup where an advantage of a standard quantum model over its classical counterpart has not been shown, and demonstrate the superiority of postselected quantum over postselected probabilistic in that context. The real-time restriction is again seen to be useful in this regard. It is presently not known whether $\text{RTBSPACE}(s) = \text{RTBQSPACE}(s)$ or not for any space bound $s = \omega(1)$. However, the fact that we have a real-time QFA with postselection for recognizing L_{pal} (Theorem 4), combined with the argument above for Equation 3, lead us to conclude that

$$\text{PostRTBSPACE}(s) \subsetneq \text{PostRTBQSPACE}(s) \text{ for all } s = o(\log n).$$

Aaronson [Aar05] showed that any PostBQP computation can be repeated a polynomial number of times to reduce the error probability to $2^{-p(n)}$ for any desired polynomial p , and this applies easily to our generalized classes, with one exception: Since it is not allowed to increase the runtime of real-time machines, error reduction for the PostRTBSPACE and PostRTBQSPACE classes has to be performed by repeating the original computation not sequentially, but parallelly, essentially by increasing the size of the program. We know how to do this parallelization only for $O(1)$ -space machines; see Appendix C.

Theorem 5. *PostBS, PostBQAL, and the classes PostBPTIME(t) and PostBQTIME(t) (for every time bound t) are closed under complementation, union, and intersection. Furthermore the classes PostRTBSPACE(s) and PostRTBQSPACE(s) are closed under complementation for any space bound s .*

Proof. As above, Aaronson’s proof of PostBQP’s closure properties can be adapted easily for our classes, except for real-time machines using non-constant space. (See Appendix D.) \square

Theorem 6. *For any time bound t and space bound s ,*

- $\text{PostBPTISP}(t, s) \subseteq \text{PrTISP}(t, s)$, and,
- $\text{PostBQTISP}(t, s) \subseteq \text{PrQTISP}(t, s)$.

Furthermore, $\text{PostBS} \subseteq \text{S}$, and $\text{PostBQAL} \subseteq \text{QAL}$.

Proof. A given machine \mathcal{P} with postselection can be converted to a machine in the corresponding standard model (without postselection) as follows:

- All transitions to nonpostselection halting states of \mathcal{P} at the end of computation are replaced by two equiprobable transitions to accept and reject states.
- All postselection accept states of \mathcal{P} are designated as accept states in the new machine.

Therefore, only the strings which are members of the original machine’s language are accepted with probability exceeding $\frac{1}{2}$ by the new machine. \square

For probabilistic machines, the result above can be strengthened so that bounded-error postselected probabilistic space is shown to be properly contained in standard unbounded-error probabilistic space, without requiring simultaneous time bounds, for all sublogarithmic space bounds. We recall the ineffectiveness of postselection for space bounded machines, and use the following fact:

Fact 1. *For any space bound $s \in o(\log n)$,*

$$\text{BSPACE}(s) \subsetneq \text{PrSPACE}(s).$$

Moreover, $\text{BSPACE}(1) \subsetneq \text{RTPrSPACE}(1)$.

Proof. As mentioned above, $L_{pal} \notin \text{BSPACE}(s)$ for any such s , but $L_{pal} \in \text{PrSPACE}(1)$. For the second relation, we also use the fact that $\text{PrSPACE}(1) = \text{RTPrSPACE}(1)$ [Kan91]. \square

Recalling Aaronson’s celebrated result stating the equality of bounded-error postselected quantum polynomial time to standard unbounded-error probabilistic polynomial time, we ask whether the same relationship holds for real-time automata. The answer turns out to be negative.

Theorem 7. $\text{PostBQAL} \subsetneq \text{S}$.

Proof. By Theorem 6, we have $\text{PostBQAL} \subseteq \text{QAL}$. It is known [YS11b] that $\text{QAL} = \text{S}$, and that S is not closed under union and intersection [Fli72,Fli74,Lap74,Tur82]. We conclude by Theorem 5 that the containment must be proper. \square

For instance, for any triple of integers u, v, w , where $0 < u < v < w$, the languages $L_1 = \{a^m b^k c^n \mid m^u > k^v > 0\}$ and $L_2 = \{a^m b^k c^n \mid k^v > n^w > 0\}$ are in $\text{RTPrSPACE}(1)$, whereas $L_1 \cup L_2$ is not in S [Tur82]. It must therefore be the case that at least one of L_1 and L_2 is not in PostBQAL .

6 Machines with postselection under other error regimes

Whether the permission to commit two-sided, rather than one-sided, error enlarges the class of languages recognized by probabilistic programs is an open question for a wide range of resource bounds. For standard quantum programs, we do not even know if error-free computation is equivalent to deterministic computation or not [ADH97]. In this section, we examine these issues for machines with postselection.

We start with a characterization of the error-free classes.

Theorem 8. *For every time bound t ,*

- $\text{PostEPTIME}(t) = \text{NTIME}(t) \cap \text{coNTIME}(t)$,

– $\text{PostEQTIME}(t) = \text{NQTIME}(t) \cap \text{coNQTIME}(t)$.

Proof. We can convert a machine with postselection \mathcal{M} that recognizes its language L with no error to an equivalent nondeterministic machine \mathcal{M}' which operates within the same time and space bounds by simply designating all nonpostselection halting states of \mathcal{M} as reject states (in addition to any more reject states inherited from the original definition) in \mathcal{M}' . If in addition to this transformation, we also switch the designations of the original postselection halting states, we obtain a nondeterministic machine recognizing the complement of L .

For the inclusion in the other direction, let \mathcal{M}_1 and \mathcal{M}_2 be two probabilistic (resp. quantum) TMs recognizing a language L , and its complement, respectively, with cutpoint zero in time t . We build a $O(t)$ -time PTM (resp. QTM) with postselection named \mathcal{R} that runs \mathcal{M}_1 and \mathcal{M}_2 separately on its input. If \mathcal{M}_1 accepts, \mathcal{R} accepts. Otherwise \mathcal{R} runs \mathcal{M}_2 on the input. If \mathcal{M}_2 accepts, \mathcal{R} rejects. Otherwise, \mathcal{R} halts in a nonpostselection state. It is easy to see that \mathcal{R} makes no error, and halts with probability 1. \square

This construction can be modified easily to apply to the real-time finite automata cases as well, and we can conclude

Corollary 1. $\text{PostES} = \text{REG}$, and $\text{PostEQAL} = \text{NQAL} \cap \text{coNQAL}$.

(Note that it is still open whether $\text{NQAL} \cap \text{coNQAL}$ contains a nonregular language or not, though we do know that $\text{NQAL} \neq \text{coNQAL}$ [YS10a].)

Theorem 8 implies, for instance, that since the decision version of the integer factorization problem is in $\text{NP} \cap \text{coNP}$, there exists an error-free polynomial-time probabilistic TM with postselection which recognizes that language, whereas the only known standard (quantum) algorithm with worst-case polynomial-time for this problem [Sho97] commits bounded error.⁴

The reader will note that the transformations in the proof of Theorem 8 do not increase the space usage of the machines in question. Using the equivalence of space-bounded machines with and without postselection, we can therefore view those constructions as providing an alternative proof for the facts

$$\text{EPSPACE}(s) = \text{NSPACE}(s) \cap \text{coNSPACE}(s)$$

and

$$\text{EQSPACE}(s) = \text{NQSPACE}(s) \cap \text{coNQSPACE}(s)$$

for all s . Making use of the Immerman-Szelepcsényi theorem, we conclude

$$\text{EPSPACE}(s) = \text{NSPACE}(s) = \text{RSPACE}(s)$$

for all $s = \Omega(\log n)$, meaning that error-free probabilistic machines (with or without postselection) are equivalent those operating with one-sided error under these space bounds.⁵

⁴ After the completion of this paper, it came to our attention that Brun and Wilde [BW11] made similar remarks in the context of computation using postselected closed timelike curves.

⁵ Note that $\text{EPSPACE}(s) = \text{NSPACE}(s)$ also follows for all $s = \Omega(\log n)$ by a modification of Gill's proof [Gil77] of $\text{RSPACE}(s) = \text{NSPACE}(s)$ to take the Immerman-Szelepcsényi theorem into account, without the need to talk about postselection. We suspect that this is a well-known fact, but we have not seen it stated anywhere.

The error-free and one-sided-error modes of computation are also equivalent for PFAPs, since every language in PostRS obviously has a two-way PFA recognizing it with one-sided error, and those machines are equivalent to deterministic finite automata:

$$\text{PostRS} = \text{REG} = \text{PostES}. \quad (4)$$

Things change in the quantum case.

Theorem 9. $\text{PostEQAL} \subsetneq \text{PostRQAL} \subseteq \text{NQAL}$.

Proof. The complement of L_{pal} can be recognized with one-sided bounded error by a real-time QFA with restart [YS10c], and is therefore in PostRQAL by Theorem 1. But the same language is known to be outside $\text{NQAL} \cap \text{coNQAL}$ [YS10a]. The proper containment follows using Corollary 1. The second subset relationship is given by a simplification of the proof of Theorem 8. \square

For greater t , the same simplification to the proof of Theorem 8 mentioned above yields

$$\text{PostRTIME}(t) \subseteq \text{NTIME}(t),$$

and

$$\text{PostRQTIME}(t) \subseteq \text{NQTIME}(t),$$

which is not enough to say whether one-sided error is useful in this range.

An open problem regarding classical nondeterministic space is whether $\text{NL} = \text{RL}$ or not. (It is known that $\text{RSPACE}(s) = \text{NSPACE}(s)$ for all s , but the proof of this fact [Gil77] involves the construction of a randomized machine that runs in time that is double exponential in terms of s . Recall from Section 2 that $\text{RL} \equiv \text{RTISP}(\text{poly}(n), \log n)$.) We prove the equality in the case where postselection is allowed:

Theorem 10. *For every space constructible function s , $\text{NSPACE}(s) = \text{PostRTISP}(2^{O(s)}, s)$.*

Proof. We first note that every nondeterministic machine which uses space $s(n) = \Omega(\log n)$ must have certificates of length at most $l = 2^{O(s(n))}$ for every accepted string of length n , and every such machine where s is space-constructible can incorporate a counter that helps cut the execution off if the runtime exceeds l , so $\text{NSPACE}(s) = \text{NTISP}(2^{O(s)}, s)$.

By Theorem 1, $\text{PostRTISP}(2^{O(s)}, s)$ is contained in $\text{RSPACE}(s)$, which, as already mentioned, equals $\text{NSPACE}(s)$ for every s . In the other direction, for any nondeterministic TM \mathcal{N} that uses space s , we build a PTM with postselection \mathcal{A} as follows: Upon reading an input of length n , \mathcal{A} first computes the maximum length l , mentioned in the previous paragraph, of the shortest certificate that \mathcal{N} could possibly use for such an input. \mathcal{A} then rejects the input with probability 2^{-2l} . With the remaining probability, \mathcal{A} randomly guesses a certificate of length at most l for the input, checking it in at most $2^{O(s)}$ time in space s , accepting if it is a valid certificate, and halting in a nonpostselection state otherwise. \mathcal{A} rejects nonmembers of the language of \mathcal{N} with probability 1, and accepts members with probability greater than $\frac{2}{3}$. \square

Finally, we would like to see whether the permission to commit two-sided, rather than one-sided, error gives any advantage to machines with postselection. For any language L , let \overline{L} denote the complement of L .

Theorem 11. $\text{PostRQAL} \subsetneq \text{PostBQAL}$.

Proof. The language $L_{eq\bar{e}q} = \{aw_1 \cup bw_2 \mid w_1 \in L_{eq}, w_2 \in \overline{L_{eq}}\}$ is not a member of NQAL [YS10a], and therefore also not in PostRQAL, by Theorem 9. We will construct a PFAP \mathcal{P} that recognizes $L_{eq\bar{e}q}$ with two-sided bounded error.

Since $L_{eq} \in \text{PostBS}$ (Theorem 2), its complement, $\overline{L_{eq}}$, is also in PostBS (Theorem 5). Let the corresponding PFAPs be called \mathcal{M}_1 and \mathcal{M}_2 , respectively. \mathcal{P} handles inputs shorter than two symbols deterministically. For longer inputs, it passes control to \mathcal{M}_1 or \mathcal{M}_2 , depending on whether the first symbol is an a or a b .

Since $L_{eq\bar{e}q} \in \text{PostBS} \subsetneq \text{PostBQAL}$, the statement has been proven. \square

In the probabilistic case, the superiority of finite automata with two-sided error follows from Theorem 2 and Equation 4, but we can do better than this: No nondeterministic TM (and therefore, no PTM with postselection that commits one-sided bounded error) using $o(\log n)$ space can recognize a nonregular deterministic context-free language [AGM92]. Since L_{eq} (Theorem 2) is such a language, for which there exists a bounded-error constant-space PTM, we conclude that the advantage given by the flexibility to err both ways extends to all sublogarithmic bounds for two-way classical machines (with or without postselection).

7 Riga quantum finite automata with postselection

The first automaton-based model of computation with postselection was presented by Lāce, Scegulnaja-Dubrovskaja and Freivalds, [LSF09] who were interested exclusively in quantum finite automata, that is, real-time, constant-space QTMs. The main difference between Lāce *et al.*'s way of modeling postselection and our approach is that the transitions of a *Riga QFA with postselection* (RQFAP), as we name their model, are not assumed to lead the machine to at least one postselection state with nonzero probability. RQFAPs have the additional unrealistic capability of detecting if the total probability of postselection states is zero at the end of the processing of the input, and accumulating all probability in a single output in such a case.

Although the motivation for this feature is not explained in [LSF09,SLF10], such an approach may be seen as an attempt to compensate for some fundamental weaknesses of finite automata. In many computational models with bigger space bounds, one can modify a machine employing the Riga approach without changing the recognized language so that the postselection state set will have nonzero probability for any input string. This is achieved by just creating some computational paths that end up in the postselection set with sufficiently small probabilities so that their inclusion does not change the acceptance probabilities of strings that lead the original machine to the postselection set significantly. These paths can be used to accept or to reject the input as desired whenever there is zero probability of observing the other postselection states. Unfortunately, we do not know how to implement this construction in quantum finite automata with arbitrary amplitudes, so we prefer our model, in which the only nonstandard capability conferred to the machines is postselection, to the Riga version.

In this section, we will prove that the Riga model is not equivalent to ours in computational power. More results on the properties of these machines can be found in Appendix E.

We will consider Riga finite automata with postselection⁶ as finite-state machines of the type introduced in Section 4 (also see Appendix B) with an additional component $\chi \in \{A, R\}$, such that whenever the postselection probability is zero for a given input string $w \in \Sigma^*$,

- w is accepted with probability 1 if $\chi = A$,
- w is rejected with probability 1 if $\chi = R$.

The related language classes are named by prefixing the letter \mathfrak{R} to the corresponding class name from Table 3.

In the classical case, Riga machines are equal in power to ours, that is, $\mathfrak{R}\text{PostBS} = \text{PostBS}$ and $\text{PostES} = \mathfrak{R}\text{PostES}$ (see Appendix E for a proof).

Recall from Section 5 that $\text{PostEQAL} = \text{NQAL} \cap \text{coNQAL}$. We will now show that the corresponding class for RQFAPs is larger.

Theorem 12. $\mathfrak{R}\text{PostEQAL} = \text{NQAL} \cup \text{coNQAL}$.

Proof. For $L \in \text{NQAL}$, designate the accepting states of the QFA recognizing L with cutpoint zero as postselection accepting states with $\chi = R$. (There are no postselection reject states.)

For $L \in \text{coNQAL}$, designate the accepting states of the QFA recognizing the complement of L with cutpoint zero as postselection rejecting states with $\chi = A$. (There are no postselection accept states.)

Finally, let L be a member of $\mathfrak{R}\text{PostEQAL}$ and \mathcal{M} be a RQFAP recognizing L with zero error. If $\chi = R$, we have that, for all $w \in L$, $p_{\mathcal{M}}^a(w)$ is nonzero, and $p_{\mathcal{M}}^r(w) = 0$, and for all $w \notin L$, $p_{\mathcal{M}}^a(w) = 0$. Thus, we can design a real-time QFA recognizing L with cutpoint zero. If $\chi = A$, we can similarly design a real-time QFA recognizing the complement of L with cutpoint zero. \square

By using the fact⁷ that $L_{pal} \in \text{coNQAL} \setminus \text{NQAL}$ [YS10a], we can state that RQFAPs are strictly more powerful than our version of real-time QFAs with postselection, at least in the error-free mode:

Corollary 2. $\text{PostEQAL} \subsetneq \mathfrak{R}\text{PostEQAL}$.

In the bounded-error case, it is not known whether RQFAPs can outperform QFAPs or not.

8 Concluding remarks

Figure 1 summarizes our results on constant-memory computers. Dotted arrows indicate subset relationships, and unbroken arrows represent the cases where it is known that the inclusion is proper. Note that the subscript \mathbb{R} appearing in the TM-related class names indicates that the transition probabilities/amplitudes are allowed to be unrestricted real numbers, making them correspond to two-way finite automata. In comparisons involving probabilistic computers, we were able to demonstrate the superiority results in the figure for some larger time and space bounds.

⁶ The original definitions of RQFAPs in [LSF09] are based on weaker QFA variants. Replacing those with the machines of Appendix B does not change the model [YS11a]). The classical probabilistic versions of these machines are studied here for the first time.

⁷ L_{pal} was proven to be in $\mathfrak{R}\text{PostEQAL}$ for the first time in [LSF09].

Using his QTM model mentioned above, Watrous [Wat99] also proved for all space-constructible $s = \Omega(\log n)$ that $\text{NQSPACE}(s) = \text{coC}_{=} \text{SPACE}(s)$. Žák’s proof of the nondeterministic time hierarchy [Ž83,FS06] is general enough to apply to all of the $\text{C}_{=} \text{SPACE}$, $\text{coC}_{=} \text{SPACE}$, $\text{C}_{=} \text{TIME}$, and $\text{coC}_{=} \text{TIME}$ classes, where the last two families of classes are straightforward generalizations of $\text{C}_{=} \text{P}$ and $\text{coC}_{=} \text{P}$.⁸ Finally, it is easy to generalize the proof that $\text{NQP} = \text{coC}_{=} \text{P}$ [YY99] to hold for the families NQTIME and $\text{coC}_{=} \text{TIME}$ as well.

Hierarchy theorems for intersection classes like the ones we have in the characterizations of EPTIME , EQTIME , and EQSPACE given by Theorem 8 are not presently known. Another important open question concerns the relationships among the various TIME and QTIME classes, and their postselected versions.

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A Well-formedness of quantum machines

For any input string x , let \mathcal{C}_x be a suitably ordered list of all the reachable configurations of the space-bounded QTM under consideration. The transition function δ (see the definition in Section 2) induces a set of $|\mathcal{C}_x| \times |\mathcal{C}_x|$ configuration transition matrices, $\{E_\omega \mid \omega \in \Omega\}$, where the $(i, j)^{\text{th}}$ entry of E_ω , the amplitude of the transition from c_j to c_i by writing $\omega \in \Omega$ on the register, is defined by δ whenever the j th configuration c_j is reachable from the i th configuration c_i in one step, and is zero otherwise. The QTM is said to be well-formed if

$$\sum_{\omega \in \Omega} E_\omega^\dagger E_\omega = I. \quad (5)$$

As described in [YS11b], an easy way of checking whether a QTM is well-formed is to verify if the columns of the $|\mathcal{C}_x| |\Omega| \times |\mathcal{C}_x|$ -dimensional matrix \mathbf{E} (Figure 2) obtained by concatenating all the E_ω s one under the other, form an orthonormal set. In the figure, and in the rest of this section, we use the convention that $\Delta = \{c, \tau_2, \tau_3, \dots, \tau_k\}$ for $k > 2$, c is sometimes referred to as τ_1 , and the elements of Ω are ordered so that those corresponding to the same τ value are grouped together for each $\tau \in \Delta$.

The QTM definition in [YS11b] only requires that the finite register alphabet Ω is partitioned in terms of the possible outcomes in Δ , and imposes no such condition on the state set Q , as we have been doing in this paper. We will now show that our additional restrictions do not cause a decrease of power, by describing a procedure for building a new QTM \mathcal{M}' which obeys these restrictions and recognizes the same language as a given QTM $\mathcal{M} = (Q, \Sigma, \Gamma, \Omega, \delta, q_1, \Delta)$, defined in the more relaxed format of [YS11b]:

Let $\mathcal{M}' = \{Q', \Sigma, \Gamma, \Omega, \delta', (q_1, \omega_1), \Delta\}$, where

⁸ We thank Lance Fortnow for explaining this to us.

		c_1	c_2	\dots	$c_{ C_x }$
$\tau_1 = c$	c_1	E_{ω_1}			
	c_2				
	\vdots				
	$c_{ C_x }$				
τ_2	c_1	$E_{\omega_{ \Omega_c +1}}$			
	c_2				
	\vdots				
	$c_{ C_x }$				
τ_k	c_1	$E_{\omega_{ \Omega }}$			
	c_2				
	\vdots				
	$c_{ C_x }$				

Fig. 2. Matrix E

- $Q' = Q \times \Delta$,
- $Q'_\tau = \{(q, \tau) \mid q \in Q\}$ for all $\tau \in \Delta$, and,
- ω_1 is the initial register symbol of both \mathcal{M} and \mathcal{M}' .

We wish to define δ' so that it will be guaranteed that whenever a value τ is observed as the result of a measurement of the register, all configurations with nonzero amplitudes will have as their state component the value (q, τ) for some $q \in Q$. Thus, we can have a partition of the state set as well according to Δ .

In all the Turing machine variants in this paper, the only member of Δ that does *not* indicate that the computation has halted is c . Transitions from configurations that the machine is in when any move outcome other than c is observed will never actually be performed, since the machine will not be running any longer. Of course, the amplitudes corresponding to those transitions must still be selected so that Equation 5 is satisfied. We make sure that the transitions from configurations with state component (q, c) for some $q \in Q$ mimic the corresponding transitions in \mathcal{M} while entering correctly categorized states of \mathcal{M}' , and fill in the rest of δ' so that \mathcal{M}' is well-formed, as described below.

For every $\tau \in \Delta$, if \mathcal{M} contains the transition

$$\delta(q, \sigma, \gamma, q', d_i, \gamma', d_w, \omega) = \alpha,$$

where $\omega \in \Omega_\tau$, we add the transition

$$\delta'((q, c), \sigma, \gamma, (q', \tau), d_i, \gamma', d_w, \omega) = \alpha$$

to \mathcal{M}' .

Consider the matrix of Figure 2 as representing the transitions of \mathcal{M} on an input x . For the new machine \mathcal{M}' , we will have a bigger matrix, since the new list of reachable configurations is $|\Delta|$ times longer than that of \mathcal{M} . Ordering the configurations so that those with the same τ value in their state components are grouped together under the name $\mathcal{C}_x(Q_\tau)$, we obtain the template of Figure 3, in which only the transitions from states (q, c) for $q \in Q$ are filled in, according to the specification presented above.

		$\mathcal{C}_x(Q_c)$	$\mathcal{C}_x(Q_{\tau_2})$	\cdots	$\mathcal{C}_x(Q_{\tau_k})$	
$\tau_1 = c$	$\mathcal{C}_x(Q_c)$	E_{ω_1}				
	$\mathcal{C}_x(Q_{\tau_2})$	0				
	\vdots	\vdots				
	$\mathcal{C}_x(Q_{\tau_k})$	0				
	$\mathcal{C}_x(Q_c)$	E_{ω_2}				
	$\mathcal{C}_x(Q_{\tau_2})$	0				
	\vdots	\vdots				
	$\mathcal{C}_x(Q_{\tau_k})$	0				
	\vdots	\vdots				
	\vdots	\vdots				
	τ_2	$\mathcal{C}_x(Q_c)$	0			
		$\mathcal{C}_x(Q_{\tau_2})$	$E_{\omega_{ \Omega_c +1}}$			
$\mathcal{C}_x(Q_{\tau_3})$		0				
\vdots		\vdots				
$\mathcal{C}_x(Q_{\tau_k})$		0				
\vdots		\vdots				
τ_k	\vdots	\vdots				
	\vdots	\vdots				
	$\mathcal{C}_x(Q_c)$	0				
	\vdots	\vdots				
	$\mathcal{C}_x(Q_{\tau_{k-1}})$	0				
	$\mathcal{C}_x(Q_{\tau_k})$	$E_{\omega_{ \Omega }}$				

Fig. 3. Partially filled configuration transition matrix of \mathcal{M}' on input x

The remainder of δ' is set as follows to ensure well-formedness:
 For every $\tau_j, \tau_l \in \Delta$, if \mathcal{M} contains the transition

$$\delta(q, \sigma, \gamma, q', d_i, \gamma', d_w, \omega) = \alpha,$$

where $\omega \in \Omega_{\tau_l}$, we add the transition

$$\delta'((q, j), \sigma, \gamma, (q', \tau_{j+l-1 \pmod{|\Delta|}}), d_i, \gamma', d_w, \omega) = \alpha$$

to \mathcal{M}' .

This procedure yields a transition function that satisfies Equation 5, since it distributes exact copies of the $|\Omega|$ nonzero submatrices seen in the first “column”⁹ of the matrix of Figure 3 to $|\Omega|$ “boxes” in each of the other columns, while ensuring that each “row” contains exactly one such nonzero submatrix. The orthonormality of the actual set of column vectors follows from the well-formedness of \mathcal{M} .

B Definitions of postselection finite automata

A real-time PFA with postselection (PFAP) is a 5-tuple $\mathcal{P} = (Q, \Sigma, \{A_\sigma \mid \sigma \in \tilde{\Sigma}\}, q_1, \Delta)$, where Q, q_1, Σ have the same semantics as in our previous definitions, the A_σ 's are transition matrices, whose columns are stochastic vectors, such that A_σ 's $(j, i)^{th}$ entry, denoted $A_\sigma[j, i]$, is the probability of the transition from state q_i to state q_j when reading symbol σ , and $\Delta = \{pa, pr, nh\}$.

The computation of a PFAP can be traced by a stochastic state vector, say v , whose i^{th} entry, denoted $v[i]$, corresponds to state q_i . For a given input string $w \in \Sigma^*$ (the string read by the machine is $\tilde{w} = \#w\$$),

$$v_i = A_{\tilde{w}_i} v_{i-1},$$

where \tilde{w}_i denotes the i^{th} symbol of \tilde{w} , $1 \leq i \leq |\tilde{w}|$, and v_0 is the initial state vector, whose first entry is 1. ($|\tilde{w}|$ denotes the length of \tilde{w} .) A PFAP must satisfy the inequality

$$\sum_{q_i \in \{Q_{pa} \cup Q_{pr}\}} v_{|\tilde{w}|}[i] > 0$$

for all inputs w .

The *acceptance and rejection probabilities of input string w by PFAP \mathcal{P} before postselection* are defined as

$$p_{\mathcal{P}}^a(w) = \sum_{q_i \in Q_{pa}} v_{|\tilde{w}|}[i] \quad \text{and} \quad p_{\mathcal{P}}^r(w) = \sum_{q_i \in Q_{pr}} v_{|\tilde{w}|}[i].$$

A realtime QFA with postselection (QFAP) is a 5-tuple $\mathcal{M} = (Q, \Sigma, \{\mathcal{E}_\sigma \mid \sigma \in \tilde{\Sigma}\}, q_1, \Delta)$, where Q, Σ, q_1 , and Δ are as defined above for PFAPs, and \mathcal{E}_σ is a collection of state transition matrices¹⁰ $\{E_{\sigma,1}, \dots, E_{\sigma,k}\}$ for some $k \in \mathbb{Z}^+$ satisfying

$$\sum_{i=1}^k E_{\sigma,i}^\dagger E_{\sigma,i} = I.$$

⁹ Note that each empty box in the matrix of Figure 3 corresponds to a $|\mathcal{C}_x| \times |\mathcal{C}_x|$ matrix.

¹⁰ Note the difference from Appendix A, where these were *configuration* transition matrices.

Additionally, we define the operator

$$P = \{P_{\tau \in \Delta} \mid P_{\tau} = \sum_{q \in Q_{\tau}} |q\rangle\langle q|\}$$

representing the single measurement of the state type at the end of the computation.¹¹

For a given input string $w \in \Sigma^*$ (the string read by the machine is $\tilde{w} = \text{\textcircled{c}}w\text{\textcircled{\$}}$), the overall state of the machine can be traced by

$$\rho_j = \mathcal{E}_{\tilde{w}_j}(\rho_{j-1}) = \sum_{i=1}^k E_{\tilde{w}_j, i} \rho_{j-1} E_{\tilde{w}_j, i}^{\dagger},$$

where $1 \leq j \leq |\tilde{w}|$ and $\rho_0 = |q_1\rangle\langle q_1|$ is the initial density matrix [NC00]. A QFAP must satisfy the inequality

$$\text{tr}(P_{pa}\rho_{|\tilde{w}|}) + \text{tr}(P_{pr}\rho_{|\tilde{w}|}) > 0$$

for all inputs w .

The *acceptance and rejection probabilities of input string w by QFAP \mathcal{M} before postselection* are defined as

$$p_{\mathcal{M}}^a(w) = \text{tr}(P_{pa}\rho_{|\tilde{w}|}) \quad \text{and} \quad p_{\mathcal{M}}^r(w) = \text{tr}(P_{pr}\rho_{|\tilde{w}|}).$$

C Error reduction for postselection finite automata

Let \mathcal{M} be a machine with postselection (or, equivalently, a machine with restart). We have the following relation [YS10c].

Lemma 2. *The language $L \subseteq \Sigma^*$ is recognized by \mathcal{M} with error bound ϵ if and only if $\frac{p_{\mathcal{M}}^r(w)}{p_{\mathcal{M}}^a(w)} \leq \frac{\epsilon}{1-\epsilon}$ when $w \in L$, and $\frac{p_{\mathcal{M}}^a(w)}{p_{\mathcal{M}}^r(w)} \leq \frac{\epsilon}{1-\epsilon}$ when $w \notin L$.*

Proof. This follows from Lemma 1, since, for all $\epsilon \in [0, \frac{1}{2})$,

$$\text{P}(\mathcal{M} \text{ accepts } w) = \frac{p_{\mathcal{M}}^a(w)}{p_{\mathcal{M}}^a(w) + p_{\mathcal{M}}^r(w)} = \frac{1}{1 + \frac{p_{\mathcal{M}}^r(w)}{p_{\mathcal{M}}^a(w)}} \geq 1 - \epsilon \Leftrightarrow \frac{p_{\mathcal{M}}^r(w)}{p_{\mathcal{M}}^a(w)} \leq \frac{\epsilon}{1 - \epsilon}.$$

The argument for $w \notin L$ is identical. □

Lemma 3. *If L is recognized by QFAP (resp., PFAP) \mathcal{M} with error bound $\epsilon \in (0, \frac{1}{2})$, then there exists a QFAP (resp., PFAP), say \mathcal{M}' , recognizing L with error bound ϵ^2 .*

Proof. We give a proof for QFAPs, which can be adapted easily to PFAPs. \mathcal{M}' can be obtained by taking the tensor product of k copies of \mathcal{M} , where the new postselection accept (resp., reject) states, Q'_{pa} (resp., Q'_{pr}), are $\otimes_{i=1}^k Q_{pa}$ (resp., $\otimes_{i=1}^k Q_{pr}$), where Q_{pa} (resp., Q_{pr}) are the postselection accept (resp., reject) states of \mathcal{M} .

¹¹ Multiple measurements of the register are not required in real-time computation [YS11b]. That is why we do not need any “continuing” states in the partition induced by Δ ; the computation stops only at the end of the input.

Let $\rho_{\tilde{w}}$ and $\rho'_{\tilde{w}}$ be the density matrices of \mathcal{M} and \mathcal{M}' , respectively, after reading \tilde{w} for a given input string $w \in \Sigma^*$. By definition, we have

$$p_{\mathcal{M}}^a(w) = \sum_{q_i \in Q_{pa}} \rho_{\tilde{w}}[i, i], \quad p_{\mathcal{M}'}^a(w) = \sum_{q_{i'} \in Q'_{pa}} \rho_{\tilde{w}}[i', i']$$

and

$$p_{\mathcal{M}}^r(w) = \sum_{q_i \in Q_{pr}} \rho_{\tilde{w}}[i, i], \quad p_{\mathcal{M}'}^r(w) = \sum_{q_{i'} \in Q'_{pr}} \rho_{\tilde{w}}[i', i'].$$

By using the equality $\rho'_{\tilde{w}} = \otimes_{i=1}^k \rho_{\tilde{w}}$, the following can be obtained with a straightforward calculation:

$$p_{\mathcal{M}'}^a(w) = (p_{\mathcal{M}}^a(w))^k$$

and

$$p_{\mathcal{M}'}^r(w) = (p_{\mathcal{M}}^r(w))^k.$$

We examine the case of $w \in L$ (the case $w \notin L$ is symmetric). Since L is recognized by \mathcal{M} with error bound ϵ , we have (due to Lemma 2)

$$\frac{p_{\mathcal{M}}^r(w)}{p_{\mathcal{M}}^a(w)} \leq \frac{\epsilon}{1 - \epsilon}.$$

If L is recognized by \mathcal{M}' with error bound ϵ^2 , we must have

$$\frac{p_{\mathcal{M}'}^r(w)}{p_{\mathcal{M}'}^a(w)} \leq \frac{\epsilon^2}{1 - \epsilon^2}.$$

Thus, any k satisfying

$$\left(\frac{\epsilon}{1 - \epsilon} \right)^k \leq \frac{\epsilon^2}{1 - \epsilon^2}$$

provides the desired machine \mathcal{M}' due to the fact that

$$\frac{p_{\mathcal{M}'}^r(w)}{p_{\mathcal{M}'}^a(w)} = \left(\frac{p_{\mathcal{M}}^r(w)}{p_{\mathcal{M}}^a(w)} \right)^k.$$

By solving this equation, we get the best value

$$k = 1 + \left\lceil \frac{\log\left(\frac{1}{\epsilon} + 1\right)}{\log\left(\frac{1}{\epsilon} - 1\right)} \right\rceil.$$

Therefore, for any $0 < \epsilon < \frac{1}{2}$, we can find a value for k . □

Theorem 13. *If L is recognized by QFAP (resp., PFAP) \mathcal{M} with error bound $0 < \epsilon < \frac{1}{2}$, then there exists a QFAP (resp., PFAP), say \mathcal{M}' , recognizing L with error bound $\epsilon' < \epsilon$ such that ϵ' can be arbitrarily close to 0.*

D Closure properties of PostBS and PostBQAL

Theorem 14. *PostBS and PostBQAL are closed under complementation, union, and intersection.*

Proof. For any language recognized by a postselection finite automaton with bounded error, we can obtain a new machine recognizing the complement of that language with bounded error, by just swapping the designations of the postselection accept and reject states. Therefore, both classes are closed under complementation.

Let L_1 and L_2 be members of PostBQAL (resp., PostBS). Then, there exist two QFAPs (resp., PFAPs) \mathcal{P}_1 and \mathcal{P}_2 recognizing L_1 and L_2 with error bound $\epsilon \leq \frac{1}{4}$, respectively. Moreover, let Q_{pa_1} and Q_{pr_1} (resp., Q_{pa_2} and Q_{pr_2}) represent the sets of postselection accept and reject states of \mathcal{P}_1 (resp., \mathcal{P}_2), respectively, and let $Q_{p_1} = Q_{pa_1} \cup Q_{pr_1}$ and $Q_{p_2} = Q_{pa_2} \cup Q_{pr_2}$. By taking the tensor products of \mathcal{P}_1 and \mathcal{P}_2 , we obtain two new machines, say \mathcal{M}_1 and \mathcal{M}_2 , and set their definitions so that

- the sets of the postselection accept and reject states of \mathcal{M}_1 are $Q_{p_1} \otimes Q_{p_2} \setminus Q_{pr_1} \otimes Q_{pr_2}$ and $Q_{pr_1} \otimes Q_{pr_2}$, respectively, and
- the sets of the postselection accept and reject states of \mathcal{M}_2 are $Q_{pa_1} \otimes Q_{pa_2}$ and $Q_{p_1} \otimes Q_{p_2} \setminus Q_{pa_1} \otimes Q_{pa_2}$, respectively.

Thus, the following inequalities can be verified for a given input string $w \in \Sigma^*$:

- if $w \in L_1 \cup L_2$, $P(\mathcal{M}_1 \text{ accepts } w) \geq \frac{15}{16}$;
- if $w \notin L_1 \cup L_2$, $P(\mathcal{M}_1 \text{ accepts } w) \leq \frac{1}{16}$;
- if $w \in L_1 \cap L_2$, $P(\mathcal{M}_2 \text{ accepts } w) \geq \frac{9}{16}$;
- if $w \notin L_1 \cap L_2$, $P(\mathcal{M}_2 \text{ accepts } w) \leq \frac{1}{16}$.

We conclude that both classes are closed under union and intersection. \square

E More results on Riga postselection finite automata

Theorem 15. $\mathfrak{R}PostBS = PostBS$, $\mathfrak{R}PostRS = PostRS$, and $PostES = \mathfrak{R}PostES$.

Proof. We give a proof of the first equality. The same technique can also be used for the remaining equalities.

Since $PostBS \subseteq \mathfrak{R}PostBS$ is trivial, we show that $\mathfrak{R}PostBS \subseteq PostBS$. Let L be in $\mathfrak{R}PostBS$, and let \mathcal{P} with state set Q , postselection states $Q_p = Q_{pa} \cup Q_{pr}$, and $\chi \in \{A, R\}$ be the Riga PFA with postselection recognizing L with error bound $\epsilon < \frac{1}{2}$. Suppose that L' is the set of strings that lead \mathcal{P} to the postselection set with zero probability. By designating all postselection states as accepting states and removing the probability values of transitions, we obtain a real-time nondeterministic finite automaton which recognizes the complement of L' . Thus, there exists a real-time deterministic finite automaton, say \mathcal{D} , recognizing L' . Let $Q_{\mathcal{D}}$ and $A_{\mathcal{D}}$ be the overall state set, and the set of accept states of \mathcal{D} , respectively.

We combine \mathcal{P} and \mathcal{D} with a tensor product to obtain a PFAP \mathcal{P}' . The postselection state set of \mathcal{P}' is $((Q \setminus Q_p) \otimes A_{\mathcal{D}}) \cup (Q_p \otimes (Q_{\mathcal{D}} \setminus A_{\mathcal{D}}))$. The postselection accept states of \mathcal{P}' are:

$$\begin{cases} ((Q \setminus Q_p) \otimes A_{\mathcal{D}}) \cup (Q_{pa} \otimes (Q_{\mathcal{D}} \setminus A_{\mathcal{D}})), & \text{if } \chi = "A", \\ Q_{pa} \otimes (Q_{\mathcal{D}} \setminus A_{\mathcal{D}}) & , \text{if } \chi = "R". \end{cases}$$

\mathcal{P}' is structured so that if the input string w is in L' , the decision is given deterministically with respect to χ , and if $w \notin L'$, (that is, the probability of postselection by \mathcal{P} is nonzero,) the decision is given by the standard postselection procedure. Therefore, L is recognized by \mathcal{P}' with the same error bound as \mathcal{P} , meaning that $L \in \text{PostBS}$. \square

Theorem 16. $\mathfrak{R}\text{PostEQAL} \subsetneq \mathfrak{R}\text{PostRQAL}$.

Proof. Recall the language $L_{eq\bar{e}q} = \{aw_1 \cup bw_2 \mid w_1 \in L_{eq}, w_2 \in \overline{L_{eq}}\}$ from Theorem 11. $L_{eq\bar{e}q}$ is not a member of $\text{NQAL} \cup \text{coNQAL}$ [YS10a], which equals $\mathfrak{R}\text{PostEQAL}$ by Theorem 12.

We know that $\overline{L_{eq}}$ is a member of PostRQAL ([YS10c] and Theorem 1). Let \mathcal{M}_1 be the corresponding machine. We first build a QFAP called \mathcal{M}_2 by converting all postselection accept states of \mathcal{M}_1 to postselection reject states, and setting all remaining states of \mathcal{M}_1 to be nonpostselection halting states.

Based on these machines, we construct a RQFAP \mathcal{R} recognizing $L_{eq\bar{e}q}$ with one-sided bounded error, as follows:

If the input length is less than two, \mathcal{R} gives memorized answers. Therefore, we assume that length of the input is greater than 1 in the remainder. Let u be the postfix of the input obtained by deleting the first symbol. If the first input symbol is a b , \mathcal{R} passes control to \mathcal{M}_1 . Thus if $u \in \overline{L_{eq}}$, the input is accepted with a probability greater than $\frac{1}{2}$, and if $u \notin \overline{L_{eq}}$, the input is rejected with certainty.

If the first input symbol is an a , \mathcal{R} passes control to \mathcal{M}_2 . Thus, if $u \notin L_{eq}$, then the input is rejected with certainty, and if $u \in L_{eq}$, the computation ends in some nonpostselection halting states with probability 1. Therefore, the decision is given by the value of χ . So, if we set the χ of \mathcal{R} to “A”, $L_{eq\bar{e}q}$ is recognized with one-sided bounded error, as required. \square

Theorem 17. $\mathfrak{R}\text{PostBQAL}$ is closed under complementation.

Proof. If a language is recognized by a RQFAP with bounded error, by swapping the accepting and rejecting postselection states and by setting χ to $\{A, R\} \setminus \chi$, we obtain a new RQFAP recognizing the complement of the language with bounded error. Therefore, $\mathfrak{R}\text{PostBQAL}$ is closed under complementation. \square

Theorem 18. $\mathfrak{R}\text{PostBQAL} \subseteq \text{uQAL} = \text{uS}$.

Proof. The equality has been shown in [YS11b]. The rest of the proof is similar to the proof of Theorem 6, with the exception that

- if $\chi = A$, we have recognition with nonstrict cutpoint;
- if $\chi = R$, we have recognition with strict cutpoint.

\square

It was shown in [SF10] that the language L_{say} , i.e. $\{w \mid \exists u_1, u_2, v_1, v_2 \in \{a, b\}^*, w = u_1bu_2 = v_1bv_2, |u_1| = |v_2|\}$, cannot be recognized by a RQFAP. Since $L_{say} \notin \text{uS}$ [FYS10], the same result follows easily from Theorem 18.

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