# Filtrations of Formal Languages by Arithmetic Progressions 

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#### Abstract

A filtration of a formal language $L$ by a sequence $s$ maps $L$ to the set of words formed by taking the letters of words of $L$ indexed only by $s$. We consider the languages resulting from filtering by all arithmetic progressions. If $L$ is regular, it is easy to see that only finitely many distinct languages result. By contrast, there exist CFL's that give infinitely many distinct languages as a result. We use our technique to show that the operation diag, which extracts the diagonal of words of square length arranged in a square array, preserves regularity but does not preserve context-freeness.


## 1 Introduction

Let $s=(s(i))_{i \geq 0}$ be an infinite strictly increasing sequence of non-negative integers. Berstel et al. [1] introduced the notion of filtering by $s$ : given a finite word $w=a_{0} a_{1} \cdots a_{n}$, we write $w[s]=a_{s(0)} a_{s(1)} \cdots a_{s(k)}$, where $k$ is the largest integer such that $s(k) \leq n<s(k+1)$. (If there is no such integer, then $w[s]=\epsilon$.) Given a language $L$, we define $L[s]=\{w[s]: w \in L\}$.

Example 1. If $w=$ theorem, and $s=0,2,4,6, \ldots$, the sequence of even integers, then $w[s]=$ term. If $t=1,3,5, \ldots$, the sequence of odd integers, then $w[t]=$ hoe.

Berstel et al. [1] proved a number of theorems about filters, and characterized those sequences $s$ that preserve regularity (i.e., $L[s]$ is always regular if $L$ is) and context-freeness.

In this note we revisit the concept of filtering from a slightly different point of view. Suppose we have an infinite set of filters $S=\left\{s_{1}, s_{2}, \ldots\right\}$. Given a language $L$, what can be said about the set of all filtered languages $\left\{L\left[s_{i}\right]: i \geq 1\right\}$ ? For example, is it finite?

In this note we are only concerned with filters $s$ that represent arithmetic progressions: there exist integers $a \geq 1, b \geq 0$ such that $s_{i}=a i+b$ for $i \geq 0$. We consider four different types of filter sets:
(a) $a \geq 1$ and $b=0$ : the weak arithmetic progressions
(b) $a \geq 1$ and $0 \leq b<a$ : the ordinary arithmetic progressions
(c) $a \geq 1$ and $b \geq 0$ : the strong arithmetic progressions
(d) $a=1$ and $b \geq 0$ : the shifts

If $L$ is regular, a simple argument (given below) shows that filtration by the strong arithmetic progressions produces only finitely many distinct languages (and hence the same is true for filtration by the weak and ordinary arithmetic progressions and shifts). By contrast, there exist context-free languages $L$ so that filtering only by the weak arithmetic progressions or the shifts produces infinitely many distinct languages (and hence the same is true for the ordinary and strong arithmetic progressions).

In Section 4 we introduce a natural operation on formal languages that is related to the results of Berstel et al. [1], but seemingly cannot be analyzed using their framework. We show that this operation preserves regularity, but does not preserve context-freeness.

We adopt the following notation: if $L$ is a language, and $s=\left(s_{i}\right)_{i \geq 0}$ is an arithmetic progression such that $s_{i}=a i+b$, then we define $L_{a, b}:=L[s]$. Similarly, if $w$ is a word, we define $w_{a, b}:=w[s]$.

## 2 The regular case

Theorem 2. If $L$ is regular, then filtering by the strong arithmetic progressions produces finitely many distinct languages.

Remark 3. It is easy to see that if $L$ is regular and $s$ is an arithmetic progression, then $L[s]$ is regular. Indeed, this follows immediately from the theorem that the regular languages are closed under applying a transducer, since it is easy to make a transducer that extracts the letters corresponding to indices in $s$. That is not the issue here; we need to see that among all the regular languages produced by filtering by a strong arithmetic progression, there are only finitely many distinct languages.

Proof. Let $A=\left(Q, \Sigma, \delta, q_{0}, F\right)$ be a DFA accepting $L$. Our proof is based on the boolean matrix interpretation of automata [3]. Let $M_{c}$ be the boolean incidence matrix of the underlying transition graph of the automaton corresponding to a transition on the symbol $c \in \Sigma$. That is, if $Q=\left\{q_{0}, q_{1}, \ldots, q_{n-1}\right\}$, then

$$
\left(M_{c}\right)_{i, j}= \begin{cases}1, & \text { if } \delta\left(q_{i}, c\right)=q_{j} \\ 0, & \text { otherwise }\end{cases}
$$

We also write $M=\bigvee_{c \in \Sigma} M_{c}$. By standard results about path algebra, the matrix $M^{n}$ has a 1 in row $i$ and column $j$ if and only if there is a length- $n$ path from $q_{i}$ to $q_{j}$.

Suppose $L=L(A)$. We show how to create a DFA $A=\left(Q^{\prime}, \Sigma, \delta^{\prime}, q_{0}^{\prime}, F^{\prime}\right)$ accepting $L_{a, b}$. The idea is that $w=c_{0} c_{1} \cdots c_{n-1}$ should be accepted if and only if there exists a word $x \in L$ such that

$$
x=x_{0} c_{0} x_{1} c_{1} \cdots x_{n-1} c_{n-1} x_{n}
$$

where $x_{0}, x_{1}, \ldots, x_{n}$ are words such that $\left|x_{0}\right|=b,\left|x_{i}\right|=a-1$ for $1 \leq i<n$, and $\left|x_{n}\right|<a$.
The state set is $Q^{\prime}=\left\{q_{0}^{\prime}\right\} \cup\{0,1\}^{n}$. Thus all states except $q_{0}^{\prime}$ are boolean vectors. We let $f$ be a boolean vector with 1 's in the positions corresponding to final states of $F$.

We define the transition function $\delta^{\prime}$ as follows:

$$
\begin{aligned}
\delta^{\prime}\left(q_{0}^{\prime}, c\right) & =\left[\begin{array}{ll}
1 & \overbrace{00 \cdots 0}^{n-1}
\end{array}\right] M^{b} M_{c} ; \\
\delta^{\prime}(q, c) & =q M^{a-1} M_{c},
\end{aligned}
$$

for all boolean vectors $q$ and symbols $c \in \Sigma$. Also define

$$
T=\left\{q: \text { there exists } i, 0 \leq i<a \text {, such that } q \cdot M^{i} \cdot f=1\right\}
$$

Finally, set

$$
F^{\prime}= \begin{cases}T \cup\left\{q_{0}^{\prime}\right\}, & \text { if } L \text { contains a word of length } \leq b ; \\ T, & \text { otherwise }\end{cases}
$$

An easy induction on $n$ now shows that if $\delta^{\prime}\left(q_{0}^{\prime}, c_{0} c_{1} \cdots c_{n-1}\right)=v$, then $v$ has 1 's in the positions corresponding to all states of the form $\delta\left(q_{0}, x_{0} c_{0} \cdots x_{n-1} c_{n-1}\right)$, where the words $x_{i}$ satisfy the inequalities mentioned previously. It follows that $L\left(A^{\prime}\right)=L_{a, b}$.

Note that $A^{\prime}$ has $2^{n}+1$ states, and this quantity does not depend on $a$ or $b$. There are only finitely many languages with this property.

## 3 The context-free case

Theorem 4. There exists a context-free language L such that filtering by the weak arithmetic progressions produces infinitely many distinct languages.

Proof. Consider the language

$$
L=\left\{10^{n} 2\left(0^{+} 3\right)^{n}: n \geq 1\right\} .
$$

Then it is easy to see that $L$ is context-free, as it is generated by the context-free grammar

$$
\begin{aligned}
& S \rightarrow 10 A B \\
& A \rightarrow 0 A B \mid 2 \\
& B \rightarrow 0 B \mid 03
\end{aligned}
$$

We claim that the languages $L_{a, 0}$ for $a \geq 2$ are all distinct. To see this, it suffices to show that $L_{a, 0} \cap 123^{+}=\left\{123^{a-1}\right\}$.

Clearly $123^{a-1}=z_{a, 0}$, where $z=10^{a-1} 2\left(0^{a-1} 3\right)^{a-1} \in L$.
Now suppose $x \in L_{a, 0} \cap 123^{+}$. Then $x=w_{a, 0}$ for some $w \in L$. Since each word in $L$ starts $10^{n} 2$ and contains no other 2's, we must have $n=a-1$. It follows that $w \in 10^{a-1} 2\left(0^{+} 3\right)^{a-1}$. But then $w$ contains only $a-13$ 's, so to get $a-13$ 's in $x$, each of them must be used. It follows that the exponent of 0 in each $0^{+} 3$ is $a-1$, and so $x=123^{a-1}$.

This completes the proof.
Theorem 5. There exists a context-free language such that $L$ filtered by the shifts results in infinitely many distinct languages.

Proof. Let $L=\left\{0^{n} 1^{n}: n \geq 0\right\}$. Then each of the languages $L_{1, b}$ is distinct, as for each $b \geq 0$, the word $1^{b}$ is the longest word of the form $1^{*}$ in $L_{1, b}$.

## 4 The operation diag

Inspired by [2], which considered the transposition of words arranged into square arrays, we introduce the following natural operation on words of length $n^{2}$ for some integer $n \geq 1$ : we arrange the letters of the word $w=a_{0} a_{1} \cdots a_{n^{2}-1}$ in row major order in a square array,

$$
\left[\begin{array}{cccc}
a_{0} & a_{1} & \cdots & a_{n-1} \\
a_{n} & a_{n+1} & \cdots & a_{2 n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n^{2}-n} & a_{n^{2}-n+1} & \cdots & a_{n^{2}-1}
\end{array}\right]
$$

and then take the diagonal $a_{0} a_{n+1} a_{2 n+2} \cdots a_{n^{2}-1}$. We call the result $\operatorname{diag}(w)$. Thus, for example, $\operatorname{diag}($ absorbent $)=$ art. Diagonals of matrices have long been studied in mathematics. We extend diag to languages $L$ as follows:

$$
\operatorname{diag}(L)=\left\{\operatorname{diag}(w): w \in L \text { and there exists } n \geq 1 \text { such that }|w|=n^{2}\right\}
$$

Theorem 6. If $L$ is regular then so is $\operatorname{diag}(L)$.
Proof. Given a DFA $A=\left(Q, \Sigma, \delta, q_{0}, F\right)$ accepting $L$, we construct an NFA $A^{\prime}=\left(Q^{\prime}, \Sigma, \delta^{\prime}, q_{0}^{\prime}, F^{\prime}\right)$ accepting $\operatorname{diag}(L)$. As in the proof of Theorem 2, we let $M_{c}$ be the $n \times n$ boolean incidence matrix of the underlying transition graph of the automaton corresponding to a transition on the symbol $c \in \Sigma$, and we define $M=\bigvee_{c \in \Sigma} M_{c}$.

The idea is that $w=a_{1} \cdots a_{t} \in L\left(A^{\prime}\right)$ if and only if there exists $x \in L(A)$ such that $x=a_{1} x_{1} \cdots a_{t-1} x_{t-1} a_{t}$ where $\left|x_{i}\right|=t$ for $1 \leq i<t$.

The states of $A^{\prime}$ are of the form $[v, V, W]$ where $v$ is a length- $n$ boolean vector and $V$ and $W$ are $n \times n$ boolean arrays. Let $i=\left[\begin{array}{ll}1 & \overbrace{00 \cdots 0}^{n-1}\end{array}\right]$ and $f$ be the boolean vector corresponding
to the final states of $A$. The transitions of $A^{\prime}$ are given by

$$
\begin{aligned}
\delta^{\prime}\left(q_{0}^{\prime}, c\right) & =\left\{\left[i \cdot M_{c}, M, X\right]: \exists n \geq 0 \text { such that } X=M^{n}\right\} \\
\delta^{\prime}([v, V, W], a) & =\left\{\left[v M_{a} W, V M, W\right]\right\} .
\end{aligned}
$$

for all $c \in \Sigma$, and boolean vectors $v$, and boolean matrices $V, W$. The final states of $A^{\prime}$ are

$$
F^{\prime}=\{[v, V, W]: v f=1 \text { and } V=W\} .
$$

We leave it to the reader to verify that $L\left(A^{\prime}\right)=\operatorname{diag}(L)$.
Theorem 7. There exists a context-free language $L$ such that $\operatorname{diag}(L)$ is not context-free.
Proof. For expository reasons, our example is over the alphabet $\{a, b, c, d, e, f, g, h, i, j, 0\}$ of 11 letters, although it is easy to reduce this.

Consider

$$
L=\left\{a 0^{3 m+1} b\left(0^{+} c\right)^{m-2} 0^{+} d 0^{3 n+1} e\left(0^{+} f\right)^{n-2} 0^{+} g 0^{3 p+1} h\left(0^{+} i\right)^{p-2} 0^{+} j: m, n, p \geq 3\right\} .
$$

It is clear that $L$ is context-free, as it is the concatenation $L_{1} L_{2} L_{3}$ of the three languages

$$
\begin{aligned}
& L_{1}=\left\{a 0^{3 m+1} b\left(0^{+} c\right)^{m-2} 0^{+}: m \geq 3\right\} \\
& L_{2}=\left\{d 0^{3 n+1} e\left(0^{+} f\right)^{n-2} 0^{+}: n \geq 3\right\} \\
& L_{3}=\left\{g 0^{3 p+1} h\left(0^{+} i\right)^{p-2} 0^{+} j: p \geq 3\right\}
\end{aligned}
$$

each of which is easily seen to be context-free.
We will show that $\operatorname{diag}(L)$ is not context-free by showing that

$$
L^{\prime}:=\operatorname{diag}(L) \cap a b c^{+} d e f^{+} g h i^{+} j
$$

is not context-free.
We claim that $L^{\prime}=\left\{a b c^{t} d e f^{t} g h i^{t} j: t \geq 1\right\}$. It is easy to see that every word of the form $a b c^{t} d e f^{t} g h i^{t} j$ for $t \geq 1$ is in $L^{\prime}$, since we can take $m=n=p=t+2$, and the exponent of 0 in each $0^{+}$term to be $3 m+1$.

It remains to see that these are the only words of the form $a b c^{+} d e f^{+} g h i^{+} j$ in $L^{\prime}$. Let $x \in L^{\prime}$, and let $y \in L$ such that $x=\operatorname{diag}(y)$. Then since the first two symbols of $x$ must be $a b$, and since they are separated by $3 m+10$ 's for some $m \geq 3$, it must be that $|y|=(3 m+1)^{2}$. Then $|x|=3 m+1$. We can repeat the argument with the letters $d, e$ and $g, h$ to get $m=n=p$. Removing the single occurrence of each letter $a, b, d, e, g, h, j$ from $x$ leaves $3 m-6$ letters, which must be chosen from $\{c, f, i\}$. But there are only $m-2$ possible occurrences of each of the letters $c, f, i$ in $y$, so each occurrence of these letters must appear on the diagonal of $y$ to get $x$. Then these letters must be separated by $3 m+10$ 's. Thus $x=a b c^{m-2} d e f^{m-2} g h i^{m-2} j$.

Now an easy argument from the pumping lemma shows that $L^{\prime}$ is not context-free. Hence $\operatorname{diag}(L)$ is not context-free.

## References

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