On the Properties of the Priority Deriving Procedure in the Pairwise Comparisons Method

Konrad Kułakowski

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Abstract The pairwise comparisons method is a convenient tool used when the relative order of preferences among different concepts (alternatives) needs to be determined. There are several popular implementations of this method, including the Eigenvector Method, the Least Squares Method, the Chi Squares Method and others. Each of the above methods comes with one or more inconsistency indices that help to decide whether the consistency of input guarantees obtaining a reliable output, thus taking the optimal decision.

This article explores the relationship between inconsistency of input and discrepancy of output. A global ranking discrepancy describes to what extent the obtained results correspond to the single expert's assessments. On the basis of the inconsistency and discrepancy indices, two properties of the weight deriving procedure are formulated. These properties are proven for Eigenvector Method and Koczkodaj's Inconsistency Index. Several estimates using Koczkodaj's Inconsistency Index for a principal eigenvalue, Saaty's inconsistency index and the Condition of Order Preservation are also provided.

1 Introduction

The first documented uses of comparisons in pairs date back to the thirteenth century [4]. Later, the method was developed by *Fechner* [6], *Thurstone* [30] and Saaty [25]. The latter proposed the Analytic Hierarchy Process (*AHP*) extension to the pairwise comparisons (herein abbreviated as *PC*) theory, the framework allowing dealing with a large number of criteria.

At the beginning of the twentieth century the method was used in psychometrics and psychophysics [30]. Now the method is considered part of decision theory [26]. Its utility has been confirmed in numerous examples [31,11,21,29]. Despite its long existence the area still prompts researchers to further exploration. Examples of such exploration are the *Rough Set* approach [10], voting systems [5], fuzzy *PC* relation handling [22,32], incomplete *PC* relation [2,7], non-numerical rankings [14], nonreciprocal *PC* relation properties [9], rankings with the reference set of alternatives [18,19] and others. Further references can be found in [28,12].

The aim of the *PC* method is to determine the weights of objects, so that the more important object is, the higher (or the lower) weight it has. The method provides the user with a number of specific techniques for deriving weights from the *PC* matrix M [3]. With every specific technique an appropriate inconsistency index is associated that describes to what extent M is inconsistent. The value of an inconsistency index is perceived as a kind of quality determinant for M - the input data to the deriving weight procedure. Following the popular adage "garbage in, garbage out" one could say that when the inconsistency is high (consistency is low) the result must be poor. Indeed, for various inconsistency indices there exist thresholds of acceptability, above which the obtained results are considered to be unreliable.

However, poor and unreliable results may be also due to the deriving method itself. One way to learn something about the heuristic procedure is to compare how far the data on input are from the ideal input with the resulting quality on output. In the Pairwise Comparisons (PC) Method the input quality is determined by inconsistency indices. In the present article the global ranking discrepancy [20] is proposed as a method of determining the output quality in the pairwise comparisons method. Based on both the inconsistency index and the global ranking discrepancy two basic

AGH University of Science and Technology,

al. Mickiewicza 30, Kraków, Poland,

konrad.kulakowski@agh.edu.pl

properties of the weights deriving procedure are formulated (Sec. 4). Both postulated properties are proven for the pair: the eigenvalue based method and Koczkodaj's inconsistency index. The consequence of this fact are four theorems describing the relationship between Koczkodaj's Index and Saaty's index [27], the principal eigenvalue of the matrix *M*, the first and the second condition of order preservation introduced by Bana e Costa and Vansninck [1] (Sec. 6).

2 Preliminaries

2.1 Pairwise comparisons method

The input data for the *PC* method is a *PC* matrix $M = [m_{ij}]$, where $m_{ij} \in \mathbb{R}_+$ and $i, j \in \{1, ..., n\}$, that expresses a quantitative relation *R* over the finite set of concepts $C \stackrel{df}{=} \{c_i \in \mathcal{C} \land i \in \{1, ..., n\}\}$. The set \mathcal{C} is a non empty universe of concepts and $R(c_i, c_j) = m_{ij}, R(c_j, c_i) = m_{ji}$. The values m_{ij} and m_{ji} indicate the relative importance of concepts c_i and c_j , so that according to the best knowledge of experts who provide the matrix *M* the importance of c_i equals the importance of c_j multiplied by factor m_{ij} .

Definition 1 A matrix *M* is said to be reciprocal if $\forall i, j \in \{1, ..., n\}$: $m_{ij} = \frac{1}{m_{ji}}$ and *M* is said to be consistent if $\forall i, j, k \in \{1, ..., n\}$: $m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

Since the *PC* matrix is usually created by humans (experts), the information contained therein may be inconsistent. That is, there may be a triad of values m_{ij}, m_{jk}, m_{ki} from *M* for which $m_{ij} \cdot m_{jk} \cdot m_{ki} \neq 1$. In other words, different ways of estimating the value of a concept may lead to different results. This fact leads to the concept of an inconsistency index describing the extent to which the matrix *M* is inconsistent. There are a number of inconsistency indexes associated with the pairwise comparisons deriving methods, including *Eigenvector Method* [25], *Least Squares Method*, *Chi Squares Method* [3], *Koczkodaj's distance based inconsistency index* [15] and others. The two best-known indexes are defined below.

Definition 2 The eigenvalue based consistency index (*Saaty's index*) of $n \times n$ reciprocal matrix M is equal to:

$$\mathscr{S}(M) = \frac{\lambda_{max} - n}{n - 1} \tag{1}$$

where λ_{max} is the principal eigenvalue of *M*.

The value $\lambda_{max} \ge n$ and $\lambda_{max} = n$ only if *M* is consistent [27]. The more accurate estimate of the value λ_{max} is shown in (Theorem 4).

Definition 3 Koczkodaj's inconsistency index \mathcal{K} of $n \times n$ and (n > 2) reciprocal matrix M is equal to:

$$\mathscr{K}(M) = \max_{i,j,k \in \{1,...,n\}} \left\{ \min\left\{ \left| 1 - \frac{m_{ij}}{m_{ik}m_{kj}} \right|, \left| 1 - \frac{m_{ik}m_{kj}}{m_{ij}} \right| \right\} \right\}$$
(2)

where i, j, k = 1, ..., n and $i \neq j \land j \neq k \land i \neq k$.

The result of the pairwise comparisons method is a ranking - a mapping that assigns values to the concepts. Formally, it can be defined as the following discrete function.

Definition 4 The ranking function for *C* (the ranking of *C*) is a function $\mu : C \to \mathbb{R}_+$ that assigns to every concept from $C \subset \mathcal{C}$ a positive value from \mathbb{R}_+ .

In other words, $\mu(c)$ represents the ranking value for $c \in C$. The μ function is usually written in the form of a vector of weights i.e. $\mu \stackrel{df}{=} [\mu(c_1), \dots, \mu(c_n)]^T$. One popular method assumes that μ is $\mu_{max} = [\mu_{max}(c_1), \dots, \mu_{max}(c_n)]^T$ the principal eigenvector of M and rescale them so that the sum of its elements is 1, i.e.

$$\mu_{ev} = \left[\frac{\mu_{max}(c_1)}{s_{ev}}, \dots, \frac{\mu_{max}(c_n)}{s_{ev}}\right]^T \text{ where } s_{ev} = \sum_{i=1}^n \mu_{max}(c_i)$$
(3)

where μ_{ev} - is the ranking function. Due to the *Perron-Frobenius* theorem [23,25] μ_{max} exists, because a real square matrix with positive entries has a unique largest real eigenvalue such that the associated eigenvector has strictly positive components.

2.2 Ranking Discrepancy

Following the *PC* matrix definition their entries represent the relative importance of concepts, so that one would expect that for $M = [m_{ij}]$ holds that $m_{ij} = \mu(c_i)/\mu(c_j)$. Since m_{ij} is the results of a subjective expert judgment, it is subject to error. Therefore, in practice m_{ij} only approximates the ratio $\mu(c_i)/\mu(c_j)$ i.e. $m_{ij} \approx \mu(c_i)/\mu(c_j)$. Let us define the difference between m_{ij} and $\mu(c_i)/\mu(c_j)$ formally.

Definition 5 Let the ranking error ϵ be the value:

$$\epsilon(i,j,\mu) \stackrel{\text{df}}{=} m_{ji} \frac{\mu(c_i)}{\mu(c_j)} = \frac{1}{m_{ij}} \frac{\mu(c_i)}{\mu(c_j)} \tag{4}$$

where *M* is the *PC* matrix and μ is the ranking function over concepts represented by *M*. Whenever the parameter μ is known or irrelevant to the conducted reasoning the expression $\epsilon(i, j, \mu)$ will be shortened to $\epsilon(i, j)$.

The value $\epsilon(i, j, \mu) = \epsilon(i, j)$ determines how much m_{ji} - a single expert judgment differs from $\mu(c_i)/\mu(c_j)$ - the ranking result. In the ideal case the expert judgment should perfectly correspond to the ranking results. Thus, for every $i, j \in \{1, ..., n\}$ it should hold that $\epsilon(i, j) = 1$. Unfortunately, the ranking is usually not perfect. Therefore, depending upon whether an expert has underestimated or overestimated the relative value of c_i with respect to c_j , the value $\epsilon(i, j)$ may be above or below 1. Of course the same applies to the ranking μ . In other words it may be the case that actually the judgment given as m_{ij} is correct, whilst the ranking μ is constructed defectively. The reciprocity of M implies that the value $\epsilon(i, j)$ is also reciprocal, i.e. $\epsilon(i, j) = \frac{1}{\epsilon(j, i)}$. Therefore, either $\epsilon(i, j) \ge 1$ or $\frac{1}{\epsilon(i, j)} \ge 1$. This observation allows one to formulate the following local ranking error \mathscr{E} definition.

Definition 6 Let the local ranking discrepancy *&* be the value:

$$\mathscr{E}(i,j,\mu) \stackrel{df}{=} \max\{\epsilon(i,j,\mu) - 1, \frac{1}{\epsilon(i,j,\mu)} - 1\}$$
(5)

Whenever the parameter μ is known or irrelevant to the conducted reasoning the expression $\mathscr{E}(i, j, \mu)$ will be shorten to $\mathscr{E}(i, j)$.

Any other value of $\mathscr{E}(i, j)$ than 0 means that the expert judgement given as m_{ij} differs from the ratio $\mu(c_i)/\mu(c_j)$ an appropriate number of times. For instance $\mathscr{E}(i, j) = 0.5$ would mean that the expert judgment m_{ij} is half the time more or half the time less than

 $\mu(c_i)/\mu(c_j)$. In other words, in this particular case the local discrepancy between the ranking result and the expert judgement reaches 50%.

Definition 7 Let the global ranking discrepancy $\mathcal{D}(M,\mu)$ for the pairwise comparisons matrix M, and the ranking μ , be the maximal value of $\mathcal{E}(i, j, \mu)$ for i, j = 1, ..., n, i.e.

$$\mathscr{D}(M,\mu) \stackrel{df}{=} \max_{i,j \in \{1,\dots,n\}} \mathscr{E}(i,j,\mu) \tag{6}$$

Thus, the ranking discrepancy [20] represents the largest local ranking error and reveals to the users the worst case of discrepancy in the ranking μ and the matrix M.

3 Sources of discrepancy

As it was indicated in the previous section, there can be two main reasons why $\epsilon(i, j) \neq 1$. The first is the poor quality of judgements provided by experts. Expert estimates may be inaccurate, flawed or disturbed in some other way¹. The method for assessing the quality of expert estimates is to determine the input data inconsistency level, hence to calculate an inconsistency index. Most inconsistency indices depend on both the results of paired comparisons, and the ranking result given as a vector μ [3]. Thus, their applicability is limited to a specific ranking method. For instance, the most popular *Saaty's* inconsistency index \mathscr{S} uses the principal eigenvalue of M, thus, indirectly depends on the eigenvector μ . The exception is *Koczkodaj's* inconsistency index \mathscr{K} , defined only on the basis of M. It does not depend on any particular priorities deriving method. This property makes it universal and suitable for any weights calculation scheme². For further

¹ Some researchers argue that one cause of inconsistency (thus also discrepancy) is the judgement scale [13]. There are some theoretical reasons [9,8] for using a judgement scale smaller than the Fülöp constant i.e. $\sqrt{\frac{1}{2}(11+5\sqrt{5})} \approx 3.330191$.

² A comparative analysis of both $\mathcal{S}(M)$ and $\mathcal{K}(M)$ can be found in [3].

consideration we assume that the inconsistency index *IC* depends only on the *PC* matrix *M* having in mind that some *IC* makes sense only in the context of some specific priorities deriving methods. Thus, the inconsistency index could be written in the form:

$$IC: \mathscr{M}_{\mathbb{R}_+}(n) \to \mathbb{R}_+ \cup \{0\}$$

$$\tag{7}$$

where $\mathcal{M}_{\mathbb{R}_+}(n)$ is the set of all reciprocal matrices $n \times n$ over \mathbb{R}_+ . The inconsistency index IC(M) equals 0 when $M \in \mathcal{M}_{\mathbb{R}_+}(n)$ is consistent, i.e. following [25], for every triad m_{ik}, m_{kj} and m_{ij} of entries from M holds that $m_{ik}m_{kj} = m_{ij}$. It is assumed that the more consistent (the less inconsistent) M the smaller IC(M). It is worth noting that inconsistency index \mathcal{K} do not use the entire set of real numbers as a set of their values. Instead values $\mathcal{K}(M) \in [0, 1)$. The work [17] contains a detailed proposal (axiomatization) of what an inconsistency index should be.

The second reason for discrepancy is the way in which the ranking was created. In other words, the ranking discrepancy may be a side effect of the ranking procedure. If that is so, it is worth considering what properties should meet the priorities deriving method. To answer this question let us define the ranking method more formally.

Definition 8 Let the priority deriving procedure for the pairwise comparisons method be represented by the following mapping *P*:

$$P: \mathscr{M}_{\mathbb{R}_{+}}(n) \to L \tag{8}$$

where $L = \{\mu | \mu : C \to \mathbb{R}_+\}$ is the space of ranking functions over $C = \{c_1, ..., c_n\}$, and $\mathcal{M}_{\mathbb{R}_+}(n)$ is the set of all reciprocal matrices $n \times n$ over \mathbb{R}_+ . Let the set of all priority deriving procedures be denoted by \mathcal{P} . The specific ranking designated by P(M) will be denoted as $\mu_{P(M)}$ or μ_P (if M is known), i.e. $P(M) = \mu_{P(M)} = \mu_P$. Of course, although P is written with the help of the functional notation (which may suggest that P is a function) in practice it might be implemented as an arbitrarily complicated procedure.

4 Properties of the Priority Deriving Procedure

The principal assumption of the pairwise comparisons method is that for $M = [m_{ij}]$ holds m_{ij} reflects the relative importance of concepts. Hence the situation in which $m_{ij} \neq \mu(c_i)/\mu(c_j)$, can be caused only by the inconsistency of M regardless of how μ has been chosen. Conversely, for a consistent M one would expect that $m_{ij} = \mu(c_i)/\mu(c_j)$ for all $i, j \in \{1, ..., n\}$ and for any $\mu \in L$ obtained by the procedure $P \in \mathcal{P}$. In other words it is required that for the regular ranking procedure P the existence of inconsistency in M is the direct cause of the existence of discrepancy (Def. 7). This common-sense postulate can be written as follows:

Proposition 1 The priority deriving procedure $P \in \mathcal{P}$ is said to be regular if

$$IC(M) = 0 \Rightarrow \mathcal{D}(M, \mu_{P(M)}) = 0$$
(9)

for the given (applicable) inconsistency index IC and the PC matrix $M \in \mathcal{M}_{\mathbb{R}_+}(n)$.

The proposition formulated above can be seen as a kind of relationship between *IC* and *P*. Indeed, when considering an *IC* that depends only from *M* (e.g. Koczkodaj's inconsistency index \mathcal{K}), since \mathcal{D} is defined and fixed, the first proposition describes an inherent property of *P*.

The matrix *M* is usually created as the result of the hard work of experts in the field of the relation *R*. Individuals, including experts, are often inconsistent in their judgements [24,27]. However, there is a level of tolerable inconsistency beyond which judgments would appear to be uninformed, random, or arbitrary [27]. Thus, the question arises of how to deal with excessive inconsistency. The literature [11,27] advises revising the matrix *M* so that the new version of the matrix is more (sufficiently) consistent. Another method is to find the closest (in the geometrical sense) consistent approximation of *M* [16]. The purpose of the inconsistency reduction is to make the values m_{ij} and $m_{ik}m_{kj}$ closer to each other, i.e. (having in mind that $m_{ik} \approx \mu^{(c_i)}/\mu^{(c_k)}$ and $m_{kj} \approx \mu^{(c_k)}/\mu^{(c_j)}$) minimizing the discrepancy. Hence, it is natural to expect that decreasing inconsistency leads to a discrepancy reduction. This postulate can be formulated as follows:

Proposition 2 It is said that the priority deriving procedure $P \in \mathcal{P}$ follows the inconsistency if

$$IC(M) \to 0 \Rightarrow \mathcal{D}(M, \mu_{P(M)}) \to 0 \tag{10}$$

and there exists a reasonably small $0 < \kappa < IC(M)$ such that:

$$IC(M) \ge IC(M') + \kappa \Rightarrow \mathcal{D}(M, \mu_{P(M)}) > \mathcal{D}(M', \mu_{P(M)})$$
(11)

for the given (applicable) inconsistency index IC, where M is the PC matrix from $\mathscr{M}_{\mathbb{R}_+}(n)$.

This proposition 2 meets the expectations that improving the quality of input data eventually brings the expected results. In other words if the inconsistency index for the given matrix M is appropriately reduced then there is a guarantee that discrepancy will also be reduced.

Both are defined in the context of any fixed inconsistency index *IC*. Therefore, when determining if the procedure has the given property a suitable index *IC* needs to be taken into account.

5 Properties of the Eigenvalue based Priority Deriving Procedure

Of course the value $\mathcal{D}(M, \mu_{P(M)}) = 0$ is optimal from the perspective of the decision maker (there is no doubt what to choose since every single expert judgement perfectly matches the ranking result). Since, decreasing the inconsistency IC(M) entails decreasing

 $\mathscr{D}(M, \mu_{P(M)})$ down to 0, decreasing *IC*(*M*) makes sense. Therefore, meeting the properties presented in (Sec. 4) provides arguments for decreasing inconsistency in *M*. It also provides strong arguments for using the given pair *IC* and *P* together. In this section the most popular eigenvalue based priority deriving procedure P_{ev} and two applicable inconsistency indices \mathscr{H} and \mathscr{S} are examined. Both properties, as proposed in (Sec. 4), are confirmed for the pair (\mathscr{K}, P_{ev}). For the pair (S, P_{ev}) the first property is proven.

To demonstrate that for the first pair (\mathcal{K}, P_{ev}) both properties hold, and the one property implies the other, first let us prove the following auxiliary theorem.

Lemma 1 For every pairwise comparisons matrix $M \in \mathcal{M}_{\mathbb{R}_+}(n)$ and the eigenvalue based pairwise comparisons procedure P_{ev} it holds that:

$$\alpha \le \epsilon(i, j, \mu_{P_{ev}(M)}) \le \frac{1}{\alpha}$$
(12)

where $\alpha \stackrel{df}{=} 1 - \mathscr{K}(M)$.

Proof Following the equation 2, Koczkodaj's distance inconsistency index $\mathcal{K}(M)$, in short \mathcal{K} , means that the maximal local inconsistence for some maximal triad m_{pq} , m_{qr} and m_{pr} is \mathcal{K} . Thus, in the case of any triad in form of m_{ik} , m_{kj} , m_{ij} it must hold that:

$$\mathscr{K} \ge \min\left\{ \left| 1 - \frac{m_{ij}}{m_{ik} m_{kj}} \right|, \left| 1 - \frac{m_{ik} m_{kj}}{m_{ij}} \right| \right\}$$
(13)

This means that either:

$$m_{ij} \le m_{ik} m_{kj}$$
 implies $\mathscr{K} \ge 1 - \frac{m_{ij}}{m_{ik} m_{kj}}$ (14)

or

$$m_{ik}m_{kj} \le m_{ij} \text{ implies } \mathscr{K} \ge 1 - \frac{m_{ik}m_{kj}}{m_{ij}}$$
 (15)

is true. Let us denote $\alpha \stackrel{df}{=} 1 - \mathcal{K}$. The statements above can then be written in the form:

$$m_{ij} \le m_{ik} m_{kj}$$
 implies $m_{ij} \ge \alpha \cdot m_{ik} m_{kj}$ (16)

$$m_{ik}m_{kj} \le m_{ij} \text{ implies } \frac{1}{\alpha} \cdot m_{ik}m_{kj} \ge m_{ij}$$
 (17)

Since $\alpha \leq 1$, both cases (16) and (17) lead independently to the common conclusion that:

$$\alpha \cdot m_{ik} m_{kj} \le m_{ij} \le \frac{1}{\alpha} m_{ik} m_{kj} \tag{18}$$

for every $i, j, k \in \{1, ..., n\}$.

On the other hand, following the P_{ev} procedure [27] the vector $\mu \stackrel{df}{=} \mu_{P_{ev}(M)}$ is the principal eigenvector of *M*. Thus, it satisfies the equation:

$$M\mu = \lambda_{max}\mu \tag{19}$$

where λ_{max} is the principal eigenvalue of *M*. The i-th equation of (19) has the form:

$$m_{i1}\mu(c_1) + \ldots + m_{in}\mu(c_n) = \lambda_{max} \cdot \mu(c_i)$$
⁽²⁰⁾

In other words ϵ can be written as:

$$\epsilon(i,j,\mu) \stackrel{df}{=} \frac{1}{m_{ij}} \cdot \frac{\mu(c_i)}{\mu(c_j)} = \frac{1}{m_{ij}} \cdot \frac{m_{i1}\mu(c_1) + \dots + m_{in}\mu(c_n)}{m_{j1}\mu(c_1) + \dots + m_{jn}\mu(c_n)}$$
(21)

Applying (18) to the left side of (20), we obtain

$$m_{i1}\mu(c_1) + \ldots + m_{in}\mu(c_n) \le \frac{1}{\alpha} \left(m_{ij} m_{j1}\mu(c_1) + \ldots + m_{ij} m_{jn}\mu(c_n) \right)$$
(22)

and accordingly:

$$\alpha \left(m_{ij} m_{j1} \mu(c_1) + \dots + m_{ij} m_{jn} \mu(c_n) \right) \le m_{i1} \mu(c_1) + \dots + m_{in} \mu(c_n)$$
(23)

Thus (22) means that:

$$\frac{1}{m_{ij}} \cdot \frac{\mu(c_i)}{\mu(c_j)} \le \frac{1}{m_{ij}} \cdot \left(\frac{1}{\alpha} \cdot \frac{m_{ij} \left(m_{j1} \mu(c_1) + \dots + m_{jn} \mu(c_n) \right)}{m_{j1} \mu(c_1) + \dots + m_{jn} \mu(c_n)} \right) = \frac{1}{\alpha}$$
(24)

and accordingly (23) implies that

$$\alpha = \frac{1}{m_{ij}} \cdot \left(\alpha \cdot \frac{m_{ij} \left(m_{j1} \mu(c_1) + \dots + m_{jn} \mu(c_n) \right)}{m_{j1} \mu(c_1) + \dots + m_{jn} \mu(c_n)} \right) \le \frac{1}{m_{ij}} \cdot \frac{\mu(c_i)}{\mu(c_j)}$$
(25)

Both the above inequalities lead to the conclusion that:

$$\alpha \le \epsilon(i, j, \mu) \le \frac{1}{\alpha} \tag{26}$$

which is the desired assertion.

Equipped with the Lemma 1, we can easily prove the properties of P_{ev} with respect to the inconsistency index \mathcal{H} .

Theorem 1 The eigenvalue based pairwise comparisons procedure P_{ev} is regular and follows the inconsistency with respect to Koczkodaj's inconsistency index \mathcal{K} .

Proof Let $\mu \stackrel{df}{=} P_{ev}(M)$ be fixed and known. Thus, from Lemma 1

$$\alpha - 1 \le \epsilon(i, j) - 1 \le \frac{1}{\alpha} - 1 \tag{27}$$

for all $i, j \in \{1, ..., n\}$. The same applies to $\epsilon(j, i)$. Thus, due to the reciprocity of ϵ also

$$\alpha - 1 \le \frac{1}{\epsilon(i,j)} - 1 \le \frac{1}{\alpha} - 1 \tag{28}$$

It is easy to observe that

$$0 \le \max\{\epsilon(i,j) - 1, \frac{1}{\epsilon(i,j)} - 1\} \le \frac{1}{\alpha} - 1$$
(29)

In other words:

$$0 \le \mathscr{E}(i,j) \le \frac{1}{\alpha} - 1 \tag{30}$$

Since the above (30) is true for all $i, j \in \{1, ..., n\}$ then, due to (Def. 7), holds that:

$$0 \le \mathscr{D}(M,\mu) \le \frac{1}{\alpha} - 1 \tag{31}$$

Since $\alpha \stackrel{df}{=} 1 - \mathcal{K}(M)$, then

$$\mathscr{K}(M) \to 0 \Rightarrow \alpha \to 1 \Rightarrow \left(\frac{1}{\alpha} - 1\right) \to 0$$
(32)

Thus, due to (31)

$$\mathscr{K}(M) \to 0 \text{ implies } \mathscr{D}(M,\mu) \to 0$$
 (33)

which proves the first postulate (10) of the second property. The estimation (31) also allows one to prove the first property. Simply, for $\mathcal{K}(M) = 0$ means that $\left(\frac{1}{\alpha} - 1\right) = 0$ therefore

$$0 \le \mathcal{D}(M,\mu) \le 0$$
 implies $\mathcal{D}(M,\mu) = 0$ (34)

An important question during the process of inconsistency reduction is whether the improvement is large enough to have an actual impact on the final ranking. Let us consider two matrices *M* and *M'* where the second one was obtained from the first one as the result of an inconsistency reduction process (judgment revision [11], inconsistency reduction algorithm [16]). The question of how much (at least) the inconsistency should be reduced to be important for the all ranked concepts boils down to the question of the reasonably small $\kappa \in \mathbb{R}_+ \setminus \{0\}$. To answer this question look at (31). In particular the inequality (11) is met if $\mathcal{K}(M')$ is so that

$$\mathscr{D}(M',\mu) \le \left(\frac{1}{1-\mathscr{K}(M')} - 1\right) < \mathscr{D}(M,\mu)$$
(35)

Thus, in particular we demand that

$$\mathscr{K}(M') < 1 - \frac{1}{\mathscr{D}(M,\mu) + 1} \tag{36}$$

hence,

$$\mathscr{K}(M) - \mathscr{K}(M') > \mathscr{K}(M) + \frac{1}{\mathscr{D}(M,\mu) + 1} - 1$$
(37)

Therefore, the suitable κ candidate is:

$$\kappa \stackrel{df}{=} \mathscr{K}(M) + \frac{1}{\mathscr{D}(M,\mu) + 1} - 1 \tag{38}$$

It is clear that due to the (37) $\kappa < \mathscr{K}(M) < 1$. It is also $0 \le \kappa$ ($0 < \kappa$, when $\mathscr{K}(M) > 0$). The fact that κ is not negative is a simple conclusion from (31). It holds that:

$$\mathscr{D}(M,\mu) \le \frac{1}{1 - \mathscr{K}(M)} - 1 \tag{39}$$

Thus, it is easy to see that:

$$\mathscr{K}(M) + \frac{1}{\mathscr{D}(M,\mu) + 1} \ge 1 \tag{40}$$

hence the right side of the equation (38) is greater or equal to ³ 0. Therefore, due to the way in which it was constructed, κ as proposed in (38) satisfies the conditions of the second property.

Lemma 1 immediately implies another useful assertion that allow us to use Koczkodaj's inconsistency to relatively estimate one entry of the principal eigenvector by another. This leads to the following corollary.

Corollary 1 Every entry in the principal eigenvector bounds each other according to the following inequality

$$\alpha m_{ij}\mu(c_j) \le \mu(c_i) \le \frac{1}{\alpha} m_{ij}\mu(c_j) \tag{41}$$

where $\alpha \stackrel{df}{=} 1 - \mathcal{K}(M)$ and $M\mu = \lambda_{max}\mu$.

The situation of discrepancy in which the expert judgment m_{ij} differs from the ranking result $\mu(c_i)/\mu(c_j)$ is not comfortable for people interested in the ranking results. The high discrepancy may be the cause of complaints, doubts as to the result, lack of a sense of justice and so on. In other words, the high discrepancy may cause customer satisfaction with the results of the ranking to be low. Reversely, when the discrepancy is small, customers are likely to be more satisfied than when the discrepancy is high. Although some customers will probably never be totally satisfied with the ranking (this particularly applies to those who support the concept of lower values μ) in general it can be assumed that the higher the discrepancy the lower the satisfaction, and conversely, the lower the discrepancy the higher the satisfaction. Of course, the most attention-grabbing and emotive situations are those of the highest discrepancy. Reducing the inconsistency $\mathcal{K}(M)$ by (at least) κ allows each time to reduce the most severe and unsatisfying cases of discrepancy. With help comes the second property that leads to the following conclusion.

³ Indeed, for the fully consistent M_1 ($\mathcal{K}(M_1) = \mathcal{D}(M_1, \mu) = 0$), an κ candidate is 0

Corollary 2 For the eigenvalue based pairwise comparisons procedure P_{ev} , the inconsistency index \mathcal{K} , and the given PC matrix M, it is recommended to reduce the inconsistency $\mathcal{K}(M)$ by such κ that guarantees reduction of the discrepancy $\mathcal{D}(M,\mu)$. One possible κ is:

$$\kappa \stackrel{\text{df}}{=} \mathscr{K}(M) + \frac{1}{\mathscr{D}(M,\mu) + 1} - 1 \tag{42}$$

Theorem 2 The eigenvalue based pairwise comparisons procedure P_{ev} is regular with respect to Saaty's inconsistency index S.

Proof The proof of the first property immediately results from [25], i.e. it is easy to see that $\mathscr{S}(M) = 0$ if and only if $\lambda_{max} = n$. Hence, according to [25, theorem 1], the equality $\lambda_{max} = n$ implies that $m_{ij} = \mu(c_i)/\mu(c_j)$ for every $i, j \in \{1, ..., n\}$. Thus, every $\mathscr{E}(i, j) = 0$, hence, $\mathscr{D}(M, \mu) = 0$.

6 Properties of Koczkodaj's inconsistency index

The theorems proven in (Sec. 5) allow the use of $\mathcal{K}(M)$ for the effective estimation of other quantities. Some of the inequalities involving $\mathcal{K}(M)$ will be presented below in the form of appropriate assertions.

Theorem 3 For any PC matrix M holds that

$$\mathscr{S}(M) \le \frac{1}{1 - \mathscr{K}(M)} - 1 \tag{43}$$

where $\mathcal{S}(M)$, $\mathcal{K}(M)$ are Saaty's and Koczkodaj's inconsistency indices respectively.

Proof It is known [25,20] that the following inequality holds:

$$\frac{1}{(n-1)}\sum_{i=1,i\neq j}^{n} \left(\epsilon(i,j) - 1\right) = \mathscr{S}(M)$$
(44)

On the other hand from the Lemma 1 ($\mu = \mu_{P_{ev}(M)}$ is omitted) we know that

$$\alpha \le \epsilon(i,j) \le \frac{1}{\alpha} \tag{45}$$

hence,

$$\alpha - 1 \le \epsilon(i, j) - 1 \le \frac{1}{\alpha} - 1 \tag{46}$$

Therefore,

$$(n-1)(\alpha-1) \le \sum_{i=1, i \ne l}^{n} (\epsilon(l,i)-1) \le (n-1)\left(\frac{1}{\alpha}-1\right)$$
(47)

Then by dividing both sides by (n-1) and applying (44) we get:

$$\alpha - 1 \le \mathscr{S}(M) \le \frac{1}{\alpha} - 1 \tag{48}$$

which is the desired assertion.

The above theorem allows one to estimate the maximum value of Saaty's inconsistency index using \mathcal{K} . In particular it is easy to see that $\mathcal{K}(M) = 0.09(09)$ implies that $\mathcal{S}(M) < 0.1$. In other words a Koczkodaj inconsistency \mathcal{K} smaller than 0.909(09) guarantees meeting the consistency criteria proposed by Saaty [27]. Hence every *PC* matrix *M* for which $\mathcal{K}(M) < 0.90909$ is also consistent enough in Saaty's sense. Of course, in such a case due to the Lemma 1 we can also expect some regularity in discrepancy among different paired comparisons.

The above theorem immediately suggest the estimation for the principal eigenvalue of M.

Theorem 4 For any PC matrix M its principal eigenvalue λ_{max} is bounded as follows:

$$(n-1)(\alpha-1) + n \le \lambda_{max} \le (n-1)\left(\frac{1}{\alpha} - 1\right) + n \tag{49}$$

where $\alpha \stackrel{df}{=} 1 - \mathscr{K}(M)$.

Proof Due to (Def. 1) and (48) we obtain

$$\alpha - 1 \le \frac{\lambda_{max} - n}{n - 1} \le \frac{1}{\alpha} - 1 \tag{50}$$

thus

$$(n-1)(\alpha-1) + n \le \lambda_{max} \le (n-1)\left(\frac{1}{\alpha} - 1\right) + n \tag{51}$$

which is the desired inequality.

Since $-1 < (\alpha - 1) \le 0$, then $-(n - 1) < (n - 1)(\alpha - 1)$. Therefore, $0 < (n - 1)(\alpha - 1) + n$, which confirms the well known fact [25] that λ_{max} is positive. Since the above theorem shows that Koczkodaj's inconsistency index \mathscr{K} can be used to estimate the principal eigenvalue, the uses of \mathscr{K} extend beyond the *PC* method.

The theorems proven in [20] allow for the formulation of the more general Conditions of Order Preservation (See appendix A) initially introduced by *Bana e Costa* and *Vansnick* [1]. Thus, the first *POP (the preservation of order preference condition)* leads to the following new theorem.

Theorem 5 For the eigenvalue based pairwise comparisons procedure P_{ev} , the PC matrix $M = [m_{ij}]$, expressing the quantitative relationships R between concepts $c_1, \ldots, c_n \in C$, and the ranking $\mu_{P_{ev}(M)} = \mu$, holds that

$$m_{ij} > \frac{1}{1 - \mathcal{K}(M)} \quad implies \ \mu(c_i) > \mu(c_j) \tag{52}$$

Proof From [20, Theorem 2] it holds that

$$\left\{ \left(\mathscr{D}(M,\mu) \le \delta \right) \Rightarrow \left(m_{ij} > \delta + 1 \right) \right\} \Rightarrow \left\{ \left(m_{ij} > 1 \right) \Rightarrow \left(\mu(c_i) > \mu(c_j) \right) \right\}$$
(53)

for some $\delta \in \mathbb{R}_+$. Due to the (31) the right side of (53) is

$$\left(\mathscr{D}(M,\mu) \le \frac{1}{1 - \mathscr{K}(M)} - 1\right) \Rightarrow \left(m_{ij} > \frac{1}{1 - \mathscr{K}(M)}\right)$$
(54)

and since $\frac{1}{1-\mathcal{K}(M)} > 1$ then also

$$m_{ij} > \frac{1}{1 - \mathscr{K}(M)}$$
 implies $\mu(c_i) > \mu(c_j)$ (55)

which is the desired assertion.

Similarly the second *POIP* (*the preservation of order of intensity of preference condition*) leads to the new interesting theorem.

Theorem 6 For the eigenvalue based pairwise comparisons procedure P_{ev} , the PC matrix $M = [m_{ij}]$, expressing the quantitative relationships R between concepts $c_1, \ldots, c_n \in C$, and the ranking $\mu_{P_{ev}(M)} = \mu$, holds that:

$$\frac{m_{ij}}{m_{kl}} > \left(\frac{1}{1 - \mathscr{K}(M)}\right)^2 \quad implies \quad \frac{\mu(c_i)}{\mu(c_j)} > \frac{\mu(c_k)}{\mu(c_l)} \tag{56}$$

Proof From [20, Theorem 3] holds that

$$\left\{ \left(\mathscr{D}(M,\mu) \le \delta \right) \Rightarrow \frac{m_{ij}}{m_{kl}} > (\delta+1)^2 \right\} \Rightarrow \left\{ m_{ij} > m_{kl} > 1 \Rightarrow \frac{\mu(c_i)}{\mu(c_j)} > \frac{\mu(c_k)}{\mu(c_l)} \right\}$$
(57)

for some $\delta \in \mathbb{R}_+$. Due to the (31) the right side of (57) is:

$$\left(\mathscr{D}(M,\mu) \le \frac{1}{1 - \mathscr{K}(M)} - 1\right) \Rightarrow \frac{m_{ij}}{m_{kl}} > \left(\frac{1}{1 - \mathscr{K}(M)} - 1 + 1\right)^2$$
(58)

and since $\frac{1}{1-\mathcal{K}(M)} > 1$ then also

$$\frac{m_{ij}}{m_{kl}} > \left(\frac{1}{1 - \mathcal{K}(M)}\right)^2 \Rightarrow \frac{\mu(c_i)}{\mu(c_j)} > \frac{\mu(c_k)}{\mu(c_l)} \tag{59}$$

which is the desired assertion.

The above two theorems show a direct relationship between the level of inconsistency and the conditions of order preservations. They may be used as a quick criterion for assessing whether in a certain case a given condition will be met. The theorems show that the lower the inconsistency the easier the left sides of (52) and (56) could be satisfied. Thus, in practice the more consistent the *PC* matrix the more often *POP* and *POIP* conditions for randomly selected matrices are satisfied.

7 Summary

This paper presents a new perspective on the pairwise comparisons method. Following the proposed approach the weight deriving method is a heuristic procedure that transforms the input (a *PC* matrix) into the output (a weight vector). Hence, after determining the quality of the input and the quality of the output it is possible to discuss the quality of the weight deriving method. As the output quality indicator global ranking discrepancy [20] has been proposed. Following the *PC* method theory the input quality is determined with the help of an inconsistency index. With the help of these two indices two properties of a good weight deriving method have been proposed. The first stipulates that when a *PC* matrix is consistent then there should be no discrepancy between expert judgments and the ranking results. The second one refers to the reasonableness of reducing inconsistency. It requires that the appropriately significant decrease in inconsistency (if it is greater than 0) always leads to a decrease in discrepancy. In other words even if it is not possible to reduce inconsistency to 0, making a significant inconsistency reduction must result in a discrepancy reduction. The properties were proven for *Koczkodaj's* inconsistency index \mathcal{K} and the *Eigenvector based* method P_{ev} . The presented reasoning results in four further claims revealing relationships between \mathcal{K} and the principal eigenvalue, *Saaty's* inconsistency index and two conditions of order preservation.

The proposed properties are general, and thus relate to each pairwise comparisons weight deriving method and each applicable inconsistency index. Therefore, despite the fact that the presented approach allows showing the relationship between \mathcal{K} and P_{ev} from a new perspective, many questions, especially regarding the other weight deriving methods and the inconsistency indices, remain. The answers to these questions may bring new interesting results, which allow us to enrich our understanding of the *PC* method.

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A Conditions of Order Preservations

In [1] Bana e Costa and Vansnick formulate two (COP) conditions of order preservations. The first, the preservation of order preference condition (POP), claims that the ranking result in relation to the given pair of concepts (c_i, c_j) should not break with the expert judgement, i.e. for a pair of concepts $c_1, c_2 \in C$ such that c_1 dominates c_2 i.e. $m_{1,2} > 1$ it should hold that:

$$\mu(c_1) > \mu(c_2) \tag{60}$$

The second one *the preservation of order of intensity of preference condition (POIP)*, claims that if c_1 dominates c_2 , more than c_3 dominates c_4 (for $c_1, \ldots, c_4 \in C$), i.e. if additionally $m_{3,4} > 1$ and $m_{1,2} > m_{3,4}$ then also

$$\frac{\mu(c_1)}{\mu(c_2)} > \frac{\mu(c_3)}{\mu(c_4)} \tag{61}$$