This is a preprint of a paper whose final and definite form is with
Fundamenta Informaticae, ISSN 0169-2968 (P), ISSN 1875-8681 (E)
`http://content.iospress.com/journals/fundamenta-informaticae`

Non-differentiable Solutions for Local Fractional Nonlinear Riccati Differential Equations

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Abstract. We investigate local fractional nonlinear Riccati differential equations (LFNRDE) by transforming them into local fractional linear ordinary differential equations. The case of LFNRDE with constant coefficients is considered and non-differentiable solutions for special cases obtained.

Keywords: nonlinear Riccati equations, non-differentiable functions, local fractional derivatives.

1. Introduction

Ordinary differential equations (ODE) via local fractional derivatives [1] have played an important role in applied science, such as in fractal damped vibrations [2], growth of populations [3], and fractal heat transfers [4]. The local fractional calculus is useful to describe diffusion [5–8], heat conduction [9], waves [10–12], and other phenomena [13]. In [14] the nonlinear Riccati differential equation (NRDE) is written in the form

$$\frac{d\Lambda\left(\mu\right)}{d\mu}=\varpi_0\left(\mu\right)+\varpi_1\left(\mu\right)\Lambda\left(\mu\right)+\varpi_2\left(\mu\right)\Lambda^2\left(\mu\right), \tag{1}$$

where $\varpi_0\left(\mu\right)\neq 0$ and $\varpi_2\left(\mu\right)\neq 0$. The fractional nonlinear Riccati differential equation (FNRDE), proposed in [15], takes the form

$$\frac{d^{\upsilon}\Theta\left(\mu\right)}{d\mu^{\upsilon}}=\varpi_0\left(\mu\right)+\varpi_1\left(\mu\right)\Theta\left(\mu\right)+\varpi_2\left(\mu\right)\Theta^2\left(\mu\right), \tag{2}$$

where υ is the order of the fractional operator [1, 16–18], $\varpi_0\left(\mu\right)\neq 0$ and $\varpi_2\left(\mu\right)\neq 0$. Here we consider the local fractional nonlinear Riccati differential equation (LFNRDE)

$$\frac{d^{\zeta}\Phi\left(\mu\right)}{d\mu^{\zeta}}=\varpi_0\left(\mu\right)+\varpi_1\left(\mu\right)\Phi\left(\mu\right)+\varpi_2\left(\mu\right)\Phi^2\left(\mu\right), \tag{3}$$

where ζ is the fractal dimension of the local fractional operator [1–13], $\varpi_0\left(\mu\right)\neq 0$ and $\varpi_2\left(\mu\right)\neq 0$. Our main aim is to study non-differentiable solutions of LFNRDE. The highlights of this paper are: we prove that the local fractional nonlinear equation can be converted into an equivalent linear equation with local fractional derivative, and that its analytical solution can be found by using the local fractional Laplace transform.

The article is organized as follows. Section 2 recalls the necessary tools of local fractional differentiation. In Section 3, transformation of the LFNRDE into a local fractional linear ODE is presented. The LFNRDE with constant coefficients is studied in Section 4. In Section 5, non-differentiable solutions for a LFNRDE are discussed. Finally, the conclusion is outlined in Section 6.

2. Mathematical tools

In this section, we introduce the basic theory of local fractional differentiation. Suppose that a function $\Phi\left(\mu\right)$ is local fractional continuous in the domain $I=(a,b)$. Then we write it as in [1–3]:

$$\Phi\left(\mu\right)\in C_{\zeta}\left(a,b\right). \tag{4}$$

Suppose that $\Phi\left(\mu\right)\in C_{\zeta}\left(a,b\right)$ and $0<\zeta\leq 1$. For $\delta>0$ and $0<|\mu-\mu_0|<\delta$, the limit

$$D^{(\zeta)}\Phi\left(\mu_0\right)=\left.\frac{d^{\zeta}\Phi\left(\mu\right)}{d\mu^{\zeta}}\right|_{\mu=\mu_0}=\lim_{\mu\to\mu_0}\frac{\Delta^{\zeta}\left[\Phi\left(\mu\right)-\Phi\left(\mu_0\right)\right]}{\left(\mu-\mu_0\right)^{\zeta}} \tag{5}$$

exists and is finite [1–3, 5], where $\Delta^{\zeta}\left[\Phi\left(\mu\right)-\Phi\left(\mu_0\right)\right]\cong\Gamma\left(1+\zeta\right)\left[\Phi\left(\mu\right)-\Phi\left(\mu_0\right)\right]$. If $\mu\in(a,b)$, then

$$D^{(\zeta)}\Phi\left(\mu\right)=\frac{d^{\zeta}\Phi\left(\mu\right)}{d\mu^{\zeta}}=\Phi^{(\zeta)}\left(\mu\right) \tag{6}$$

(see [1]). We say that $D_{\zeta}\left(a,b\right)$ is a ζ local fractional derivative set if (6) is valid for any point $\mu\in(a,b)$.

Theorem 2.1. (See [1–3, 5])
Let $\Phi_1(\mu)$, $\Phi_2(\mu) \in D_\zeta(a,b)$ be non-differentiable functions defined on fractal sets. The following local fractional differentiation rules hold:

(a) $D^{(\zeta)}[\Phi_1(\mu) \pm \Phi_2(\mu)] = D^{(\zeta)}\Phi_1(\mu) \pm D^{(\zeta)}\Phi_2(\mu)$;

(b) $D^{(\zeta)}[\Phi_1(\mu)\Phi_2(\mu)] = \left[D^{(\zeta)}\Phi_1(\mu)\right]\Phi_2(\mu) + \Phi_1(\mu)\left[D^{(\zeta)}\Phi_2(\mu)\right]$;

(c) $D^{(\zeta)}\left[\frac{\Phi_1(\mu)}{\Phi_2(\mu)}\right] = \frac{\left[D^{(\zeta)}\Phi_1(\mu)\right]\Phi_2(\mu) - \Phi_1(\mu)\left[D^{(\zeta)}\Phi_2(\mu)\right]}{\Phi_2^2(\mu)}$, provided $\Phi_2(\mu) \neq 0$.

3. From LFNRDE to linear ODE via local fractional derivatives

Taking the function

$$\chi(\mu) = \Phi(\mu)\varpi_2(\mu), \tag{7}$$

we find that

$$\begin{aligned} D^{(\zeta)}\chi(\mu) &= D^{(\zeta)}[\Phi(\mu)\varpi_2(\mu)] \\ &= \left[D^{(\zeta)}\Phi(\mu)\right]\varpi_2(\mu) + \Phi(\mu)\left[D^{(\zeta)}\varpi_2(\mu)\right]. \end{aligned} \tag{8}$$

Submitting Eq. (3) to Eq. (8), we have

$$\begin{aligned} D^{(\zeta)}\chi(\mu) &= \left[\varpi_0(\mu) + \varpi_1(\mu)\Phi(\mu) + \varpi_2(\mu)\Phi^2(\mu)\right]\varpi_2(\mu) + \Phi(\mu)\left[D^{(\zeta)}\varpi_2(\mu)\right] \\ &= \left[\varpi_0(\mu) + \varpi_1(\mu)\Phi(\mu) + \varpi_2(\mu)\Phi^2(\mu)\right]\varpi_2(\mu) + \left[\frac{D^{(\zeta)}\varpi_2(\mu)}{\varpi_2(\mu)}\right]\chi(\mu). \end{aligned} \tag{9}$$

Taking

$$\Omega_1(\mu) = \varpi_2(\mu)\varpi_0(\mu) \tag{10}$$

and

$$\Omega_2(\mu) = \varpi_1(\mu) + \frac{D^{(\zeta)}\varpi_2(\mu)}{\varpi_2(\mu)}, \tag{11}$$

Eq. (9) can be written as follows:

$$D^{(\zeta)}\chi(\mu) = \chi^2(\mu) + \Omega_2(\mu)\chi(\mu) + \Omega_1(\mu). \tag{12}$$

Set the function

$$\chi(\mu) := -\frac{D^{(\zeta)}\psi(\mu)}{\psi(\mu)}. \tag{13}$$

In the light of Eqs. (12)–(13), we have

$$\begin{aligned} D^{(\zeta)}\chi(\mu) &= D^{(\zeta)}\left[-\frac{D^{(\zeta)}\psi(\mu)}{\psi(\mu)}\right] \\ &= -\frac{D^{(2\zeta)}\psi(\mu)}{\psi(\mu)} + \left[\frac{D^{(\zeta)}\psi(\mu)}{\psi(\mu)}\right]^2 \\ &= -\frac{D^{(2\zeta)}\psi(\mu)}{\psi(\mu)} + \chi^2(\mu) \end{aligned} \tag{14}$$

so that

$$\begin{aligned}\frac{D^{(2\zeta)}\psi(\mu)}{\psi(\mu)} &= -\Omega_2(\mu)\chi(\mu) - \Omega_1(\mu)\\ &= \Omega_2(\mu)\frac{D^{(\zeta)}\psi(\mu)}{\psi(\mu)} - \Omega_1(\mu).\end{aligned} \tag{15}$$

Thus, we have

$$D^{(2\zeta)}\psi(\mu) - \Omega_2(\mu) D^{(\zeta)}\psi(\mu) + \Omega_1(\mu)\psi(\mu) = 0. \tag{16}$$

Note that Eq. (16) is a local fractional linear ODE. In view of Eqs. (7) and (13), we obtain the following non-differentiable solution of Eq. (3):

$$\Phi(\mu) = -\frac{D^{(\zeta)}\psi(\mu)}{\varpi_2(\mu)\psi(\mu)}. \tag{17}$$

4. LFNRDE with constant coefficients

We now consider the following LFNRDE with constant coefficients:

$$\frac{d^\zeta\Phi(\mu)}{d\mu^\zeta} = \varpi_0 + \varpi_1\Phi(\mu) + \varpi_2\Phi^2(\mu), \tag{18}$$

where $\varpi_0 \neq 0$ and $\varpi_2 \neq 0$. Following Eqs. (10), (11) and (16), we have

$$\Omega_1(\mu) = \varpi_2\varpi_0 \tag{19}$$

and

$$\Omega_2(\mu) = \varpi_1, \tag{20}$$

such that

$$D^{(2\zeta)}\psi(\mu) - \varpi_1 D^{(\zeta)}\psi(\mu) + \varpi_2\varpi_0\psi(\mu) = 0 \tag{21}$$

subject to the initial value conditions

$$D^{(\zeta)}\psi(0) = \alpha, \quad \psi(0) = \beta, \tag{22}$$

where

$$\Phi(\mu) = -\frac{D^{(\zeta)}\psi(\mu)}{\varpi_2\psi(\mu)}. \tag{23}$$

Taking the local fractional Laplace transform (LFLT) [1] of (21), we obtain

$$\psi(s) = \frac{\alpha s^\zeta + \beta(1+\varpi_1)}{s^{2\zeta} - \varpi_1 s^\zeta + \varpi_2\varpi_0},$$

where s^ζ is the local fractional Laplace operator. When $\Sigma = \varpi_1^2 - 4\varpi_2\varpi_0 > 0$, taking the inverse LFLT, we obtain

$$\psi(\mu) = A_0 E_\zeta\left(-C_0\mu^\zeta\right) + B_0 E_\zeta\left(-D_0\mu^\zeta\right) \tag{24}$$

such that

$$\Phi(\mu)=\frac{A_0C_0E_\zeta\left(-C_0\mu^\zeta\right)+B_0D_0E_\zeta\left(-D_0\mu^\zeta\right)}{\varpi_2\left[A_0E_\zeta\left(-C_0\mu^\zeta\right)+B_0E_\zeta\left(-D_0\mu^\zeta\right)\right]}, \tag{25}$$

where

$$A_0=\frac{\alpha}{2}+\frac{\beta\left(1+\varpi_1\right)-\frac{\alpha\varpi_1}{2}}{\sqrt{\varpi_1^2-4\varpi_2\varpi_0}}, \tag{26}$$

$$B_0=\frac{\alpha}{2}-\frac{\beta\left(1+\varpi_1\right)-\frac{\alpha\varpi_1}{2}}{\sqrt{\varpi_1^2-4\varpi_2\varpi_0}}, \tag{27}$$

$$C_0=\frac{\varpi_1+\sqrt{\varpi_1^2-4\varpi_2\varpi_0}}{2} \tag{28}$$

and

$$D_0=\frac{\varpi_1-\sqrt{\varpi_1^2-4\varpi_2\varpi_0}}{2}. \tag{29}$$

For $\Sigma=\varpi_1^2-4\varpi_2\varpi_0<0$, after taking the inverse LFLT, we have

$$\psi(\mu)=A_1E_\zeta\left(-C_1\mu^\zeta\right)+B_1E_\zeta\left(-D_1\mu^\zeta\right) \tag{30}$$

such that

$$\Phi(\mu)=-\frac{A_1C_1E_\zeta\left(-C_1\mu^\zeta\right)+B_1D_1E_\zeta\left(-D_1\mu^\zeta\right)}{\varpi_2\left[A_1E_\zeta\left(-C_1\mu^\zeta\right)+B_1E_\zeta\left(-D_1\mu^\zeta\right)\right]}, \tag{31}$$

where

$$A_1=\frac{\alpha}{2}+\frac{\beta\left(1+\varpi_1\right)-\frac{\alpha\varpi_1}{2}}{i^\zeta\sqrt{4\varpi_2\varpi_0-\varpi_1^2}}, \tag{32}$$

$$B_1=\frac{\alpha}{2}-\frac{\beta\left(1+\varpi_1\right)-\frac{\alpha\varpi_1}{2}}{i^\zeta\sqrt{4\varpi_2\varpi_0-\varpi_1^2}}, \tag{33}$$

$$C_1=\frac{\varpi_1+i^\zeta\sqrt{4\varpi_2\varpi_0-\varpi_1^2}}{2} \tag{34}$$

and

$$D_1=\frac{\varpi_1-i^\zeta\sqrt{4\varpi_2\varpi_0-\varpi_1^2}}{2} \tag{35}$$

with the fractal imaginary i^ζ. By setting $\Sigma:=\varpi_1^2-4\varpi_2\varpi_0=0$, and taking the inverse LFLT, we obtain

$$\Phi(\mu)=\alpha E_\zeta\left(-\frac{\varpi_1}{2}\mu^\zeta\right)+\left(\beta\left(1+\varpi_1\right)-\frac{\varpi_1\alpha}{2}\right)\frac{\mu^\zeta}{\Gamma(1+\zeta)}E_\zeta\left(-\frac{\varpi_1}{2}\mu^\zeta\right). \tag{36}$$

5. An illustrative example

Let us consider the LFNRDE with constant coefficients

$$\frac{d^{\zeta}\Phi(\mu)}{d\mu^{\zeta}} = 1 + 3\Phi(\mu) + \Phi^{2}(\mu) \tag{37}$$

subject to the initial condition

$$\Phi(0) = 1. \tag{38}$$

In light of Eqs. (21) and (23), Eq. (38) can be transformed into

$$D^{(2\zeta)}\psi(\mu) - 3D^{(\zeta)}\psi(\mu) + \psi(\mu) = 0, \tag{39}$$

where

$$D^{(\zeta)}\psi(0) = -\psi(0) = \xi. \tag{40}$$

By setting $D^{(\zeta)}\psi(0) = -\psi(0) = \beta = -\alpha$, we have

$$\Sigma = \varpi_1^2 - 4\varpi_2\varpi_0 = 5, \tag{41}$$

$$A_0 = \frac{\left(\sqrt{5} - 11\right)\alpha}{2\sqrt{5}}, \tag{42}$$

$$B_0 = \frac{\left(\sqrt{5} - 5\right)\alpha}{2\sqrt{5}}, \tag{43}$$

$$C_0 = \frac{3 + \sqrt{5}}{2} \tag{44}$$

and

$$D_0 = \frac{3 - \sqrt{5}}{2}. \tag{45}$$

Thus, making use of Eqs. (41)–(45), we have

$$\Phi(\mu) = \frac{\left(\sqrt{5} - 11\right)\left(3 + \sqrt{5}\right)E_{\zeta}\left(-\frac{3+\sqrt{5}}{2}\mu^{\zeta}\right) + \left(\sqrt{5} - 5\right)\left(3 - \sqrt{5}\right)E_{\zeta}\left(-\frac{3-\sqrt{5}}{2}\mu^{\zeta}\right)}{2\left(\sqrt{5} - 11\right)E_{\zeta}\left(-\frac{3+\sqrt{5}}{2}\mu^{\zeta}\right) + 2\left(\sqrt{5} - 5\right)E_{\zeta}\left(-\frac{3-\sqrt{5}}{2}\mu^{\zeta}\right)}$$

and its corresponding graph, when $\zeta = \ln 2/\ln 3$, is shown in Figure 1.

6. Conclusion

The nonlinear Riccati differential equation (NRDE) via local fractional derivative is first proposed in this work. With the help of the properties of the local fractional derivative, we can transform the LFNRDE into a local fractional ODE. The non-differentiable solution of the LFNRDE with constant coefficients is presented. An illustrative example with a Cantor-like function is given. We claim that our method may be applied to improve the performance in image classification, such as tea classification [19, 20], pathological brain detection [21], fruit classification [22, 23], and Alzheimer's disease detection [24, 25]. This is supported by the fact that fractional nonlinear equations are commonly used in image processing. Our method can help to improve the solution accuracy, thus increasing the image classification.

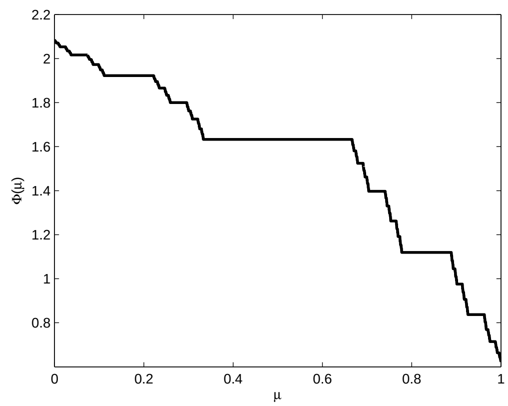

Figure 1. Solution $\Phi\left(\mu\right)$ of (37)–(38) when $\zeta = \ln 2/\ln 3$.

Acknowledgements

This research was partially supported by the Open Project Program of the State Key Lab of CAD&CG, Zhejiang University (A1616), and Natural Science Foundation of Jiangsu Province (BK20150983). Torres was supported by CIDMA and FCT within project UID/MAT/04106/2013.

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