# Approximate Hotspots of Orthogonal Trajectories

#### Ali Gholami Rudi\*

#### Abstract

We study the problem of finding hotspots, i.e. regions, in which a moving entity spends a significant amount of time, for polygonal trajectories. The fastest exact algorithm, due to Gudmundsson, van Kreveld, and Staals (2013) finds an axis-parallel square hotspot of fixed side length in  $O(n^2)$  for a trajectory with n edges. Limiting ourselves to the case in which the entity moves in a direction parallel either to the x or to the y-axis, we present an approximation algorithm with the time complexity  $O(n \log^3 n)$  and approximation factor 1/2.

**Keywords**: Trajectory, Hotspot, Kinetic tournament, Geometric algorithms

#### 1 Introduction

Tracking technologies like GPS gather huge and growing collections of trajectory data, for instance for cars, mobile devices, and animals. The analysis of these collections poses many interesting problems, which has been the subject of much attention recently [1]. One of these problems is the identification of the region, in which an entity spends a large amount of time. Such regions are usually called stay points, popular places, or hotspots in the literature.

We focus on polygonal trajectories, in which the trajectory is obtained by linearly interpolating the location of the moving entity, recorded at specific points in time (the assumption of polygonal trajectories is very common in the literature; see for instance [2, 3, 4, 5]). Gudmundsson et al. [6] define several problems about trajectory hotspots and present an  $O(n^2)$  exact algorithm to solve the following: defining a hotspot as an axis-aligned square of fixed side length, the goal is to find a placement of such a square that maximizes the time the entity spends inside it for a trajectory with n edges (there are other models and assumptions about hotspots, for a brief survey

<sup>\*</sup>Department of Electrical and Computer Engineering, Bobol Noshirvani University of Technology, Babol, Mazandaran, Iran. Email: gholamirudi@nit.ac.ir.

of which, the reader may consult [6]; e.g. the assumption of pre-defined potential regions [7], counting only the number of visits or the number of visits from different entities [2], or based on the sampled locations only [8]). To solve this problem, they first show that the function that maps the location of the square to the duration the trajectory spends inside it, is piecewise linear and its breakpoints happen when a side of the square lies on a vertex, or a corner of the square on an edge of the trajectory. Based on this observation, they subdivide the plane into  $O(n^2)$  faces and test each face for the square with the maximum duration.

In this paper, we limit ourselves to trajectories whose edges are parallel to the axes of the coordinate system (we call these trajectories orthogonal). One possible application of this problem is finding regions in a (possibly multi-layer, 3-dimensional) VLSI chip, with high wire density considering their current, to identify potential chip hot spots. The algorithm presented by Gudmundsson et al. [6] finds an exact solution for this problem in  $O(n^2)$ ; we are not aware of a faster exact algorithm. Our contribution is to provide a faster approximation algorithm with constant approximation factor. A c-approximate hotspot of a trajectory is a square, in which the entity spends no less than c times the time it spends in the optimal hotspot. We present an algorithm for this problem with an approximation ratio of 1/2 and the time complexity  $O(n \log^3 n)$ , in which n is the number of trajectory edges. In this algorithm we combine kinetic tournaments [9] with segment trees to maintain the maximum among sums of a set of piecewise linear functions. We also present a simpler  $O(n \log n)$  time algorithm for finding 1/4-approximate hotspots of orthogonal trajectories.

The rest of this paper is organized as follows. In Section 2, we introduce the notation used in this paper and also present a simple 1/4-approximation algorithm for finding hotspots. In Section 3, we present our main algorithm, and finally, we extend the algorithm to trajectories in  $\mathbb{R}^3$  in Section 4.

### 2 Preliminaries and Basic Results

A trajectory specifies the location of a moving entity through time. Therefore, it can be described as a function that maps each point in a specific time interval to a location in the plane. Similar to Gudmundsson et al. [6], we assume that trajectories are continuous and piecewise linear. The location of the entity is recorded at different points in time, which we call the vertices of a trajectory; when necessary we use the notation  $\tau(v)$  to denote the timestamp of vertex v. We assume that the entity moves in a straight line and at constant speed from a vertex to the next (this simplifying assumption is very common in the literature but there are other models for the movement of the entity between vertices [10]); we call the sub-trajectory connecting two contiguous vertices, an edge of the trajectory.

We represent a trajectory with its set of edges. This representation does not preserve the order of trajectory edges; in the problem studied in this paper the order of trajectory edges is insignificant and this representation is sufficient. In orthogonal trajectories, all trajectory edges are parallel either to the x or to the y-axis. In horizontal (similarly vertical) trajectories all edges are parallel to the x-axis (y-axis). We assign a weight to each edge to show how long the entity was moving on it (Definition 2.1).

**Definition 2.1.** To an edge e = uv, we assign a weight  $w_e$ , which denotes the duration of the sub-trajectory through its end points (the difference between the time recorded for its end points), therefore  $w_e = \tau(v) - \tau(u)$ , where  $\tau(w)$  is the timestamp of vertex w.

Unless otherwise stated, we assume that all squares mentioned in the rest of this paper to be axis-parallel and of side length s, which is an input and fixed during the algorithms.

**Definition 2.2.** Square (x,y) is a square whose lower left corner is at position (x,y) on the plane. The weight of a square r with respect to a trajectory T is the total duration in which the entity spends inside it. We denote it with  $w_T(r)$ , or if there is no confusion w(r). More formally, if the trajectory enters square r m times,  $w_T(r) = \sum_{i=1}^m (l_i - e_i)$ , in which  $e_i$  and  $l_i$  denote the time at which the entity enters and leaves the square in its i-th visit, respectively.

We now define two of the main concepts of this paper, i.e. hotspots and approximate hotspots (Definitions 2.3 and 2.4).

**Definition 2.3.** A hotspot is a square with the maximum possible weight. We denote the weight of a hotspot of trajectory T with h(T).

**Definition 2.4.** A c-approximate hotspot of a trajectory T, for  $c \le 1$ , is a square whose weight is at least c times the weight of a hotspot of T.

**Definition 2.5.** Let r be an axis-parallel square and H be a horizontal trajectory. The contribution rate of an edge e of H with respect to r is the rate at which the contribution of the weight of the edge to the weight of r increases, if r is moved to the right, in the positive direction of the x-axis (the contribution rate shows the slope of the curve in the square position-edge contribution plane). We denote the contribution rate of e as r(e).

It is not difficult to see that the absolute value of r(e) is either zero (for non-intersecting edges) or the ratio of its duration to its length. Note that if a square is moved to the left, the rate at which an edge e contributes to the weight of the square is -r(e). In Figure 1, the contribution rate of all edges except e and c are zero.

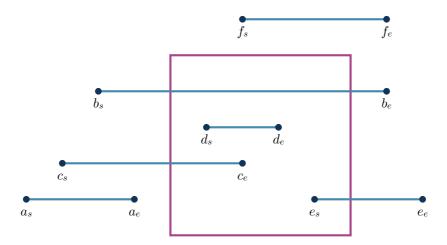


Figure 1: The contribution rate of e is positive, the contribution rate of c is negative, and the contribution rate of every other edge is zero

**Definition 2.6.** The contribution rate of a horizontal trajectory H with respect to square r, denoted as r(H), is defined as the sum of the contribution rates of all edges of H.

We now present some preliminary results about orthogonal trajectories.

**Lemma 2.7.** Let H and V be a partition of an orthogonal trajectory T, in which H contains its horizontal and V contains its vertical edges. Let h be the maximum of h(H) and h(V). Then, h is at least h(T)/2.

Proof. Let r be a hotspot in T. Every edge of T is either in H or in V and thus  $w_H(r) + w_V(r)$  equals h(T). Therefore, either  $w_H(r) \geq h(T)/2$  or  $w_V(r) \geq h(T)/2$ . Since  $h(H) \geq w_H(r)$  and  $h(V) \geq w_V(r)$ , we have  $\max(h(H), h(V)) \geq h(T)/2$  as required.

**Lemma 2.8.** Let H be a horizontal trajectory. There exists at least one square, whose weight equals h(H) such that one of its vertical sides contains a vertex of H.

*Proof.* Let r be a square with weight h(H) (and thus a hotspot) and suppose none of its vertical sides contains a vertex of H. Clearly, r(H) cannot be positive; otherwise, the weight of r increases by moving it to the right, which is impossible since it is a hotspot. Similarly, r(H) cannot be negative (otherwise, the weight of r increases by moving it to the left, which is again impossible). Therefore, r(H) is zero and by moving r to the right until one of its sides meets a vertex of H, its weight does not change.

**Lemma 2.9.** Let H be a horizontal trajectory. Among all squares with at least a corner coinciding with a vertex of H, let h be the weight of a square with the maximum weight. Then,  $h \ge h(H)/2$ .

Table 1: The contribution function of a horizontal edge  $e_i$  to their intersecting sweep squares  $(c_{e_i}(x) = a_i x + b_i)$ . The sweep line is at x, the right side of sweep squares is at x + s, the x-coordinate of the left end point of edge  $e_i$  is  $x_i$ , the x-coordinate of the right end point of edge  $e_i$  is  $x_i'$ , the duration of  $e_i$  is  $w_i$ , and the ratio of the duration of  $e_i$  to its length is  $m_i$   $(m_i = w_i/(x_i' - x_i))$ .

Case	Condition	$a_i$	$b_i$
1	$x_i \le x_i' \le x \le x + s$	0	0
2	$x_i \le x \le x_i' \le x + s$	$-m_i$	$m_i x_i - m_i w_i$
3	$x_i \le x \le x + s \le x_i'$	0	$m_i s$
4	$x \le x_i \le x_i' \le x + s$	0	$w_i$
5	$x \le x_i \le x + s \le x_i'$	$m_i$	$m_i s - m_i x_i$
6	$x \le x + s \le x_i \le x_i'$	0	0

Proof. Let r be the square with weight h(H), one of whose vertical sides contains a vertex v of H (such a square surely exists, as shown in Lemma 2.8). Suppose v is on the left side of r (the argument for the right side is similar). Let r' and r'' be the squares with side length s, whose lower left and upper left corners are on v, respectively. Given that the union of r' and r'' covers r, w(r') + w(r'') is at least h(H) and therefore  $\max(w(r'), w(r''))$  is at least h(H)/2. Since  $h \ge \max(w(r'), w(r''))$ , we have  $h \ge h(H)/2$ .

In Theorem 2.11, we show how to find a maximum weight square with a left corner on a trajectory vertex, for horizontal trajectories. The algorithm sweeps the plane horizontally. We call any square whose left side is on the sweep line, a sweep square. In Definition 2.10, we define the contribution function of an edge.

**Definition 2.10.** The contribution function  $c_{e_i}(x)$  of the *i*-th edge  $e_i$  of a horizontal trajectory H shows the contribution of  $e_i$  to its intersecting sweep squares, when the sweep line is at position x on the x-axis. The contribution function of an edge is piecewise linear. Let  $c_{e_i}(x) = a_i x + b_i$ , where  $a_i$  is the slope and  $b_i$  is the vertical intercept of  $c_{e_i}$ . Table 1 and Figure 2 show the definition of this function based on the relative position of an edge to the sweep line; edge  $e_i$  in Figure 2 corresponds to case i of Table 1.

**Theorem 2.11.** Among all axis-parallel squares with side length s and with a left corner on a vertex of a horizontal trajectory H with n edges, the square with the maximum weight can be found with the time complexity  $O(n \log n)$ .

*Proof.* Let  $\sigma = (e_1, e_2, ..., e_n)$  be the sequence of trajectory edges, ordered by their y-coordinates. We sweep the plane containing trajectory edges horizontally towards the positive direction of the x-axis. During the algorithm, for each edge  $e_i$  we maintain its contribution function,  $c_{e_i}(x)$ .

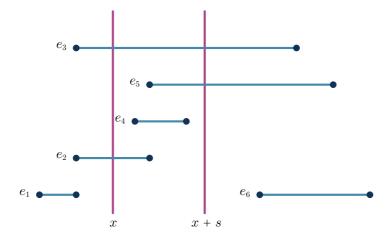


Figure 2: The left vertical line (at x) is the sweep line and the right vertical line (at x+s) is the line at which the right side of sweep squares lie. Representing the x-coordinate of the left end point of edge  $e_i$  with  $x_i$ , the duration of  $e_i$  with  $w_i$ , and the ratio of the duration of  $e_i$  to its length with  $m_i$ , we have  $c_{e_1}(x) = 0$ ,  $c_{e_2}(x) = -m_2 \cdot (x - x_2) + w_2$ ,  $c_{e_3}(x) = m_3 \cdot s$ ,  $c_{e_4}(x) = w_4$ ,  $c_{e_5}(x) = m_5 \cdot (x + s - x_5)$ , and  $c_{e_6}(x) = 0$ . Edge  $e_i$  corresponds to case i of Table 1.

We can store the function assigned to trajectory edges in two Fenwick trees [11]: A for storing the slope and B for storing the vertical intercept of the functions in the order specified by  $\sigma$ . More precisely, we store  $a_i$  as the i-th element of A and  $b_i$  as the i-th element of B. During the algorithm we need to compute the sum of the functions of a contiguous subsequence of the edges according to  $\sigma$ , like  $\sum_{q=i}^{j} c_{e_q}$ . To do so, we find the sum of the elements i to j in A to obtain its slope and the elements i to j in B to obtain its vertical intercept in  $O(\log n)$  using the Fenwick data structure.

The plane sweep algorithm processes four types of events: when the left or right end point of an edge, which we call an event point, meets the left or right side of sweep squares. At each of these events for edge e, the slope and vertical intercept of  $c_e$  is updated to reflect the current contribution of the edge to the weight of intersecting sweep squares.

To find a maximum weight square with a left corner on a trajectory vertex (as required), it suffices to compute the weight of sweep squares, when one of their left corners coincides with an event point during the algorithm. We do this as follows. For each event during the plane sweep algorithm for an edge  $e_i$ , we first update the value of  $c_{e_i}$ ; the slope and the vertical intercept of the function of  $e_i$  are updated based on the relative position of the sweep line and the edge. At an event for edge  $e_i$ , the value of  $a_i$  and  $b_i$  are updated according to Table 1 in A and B, respectively.

Let k be the smallest index such that the vertical distance between  $e_k$ 

and  $e_i$  (the difference between their y-coordinates) is at most s. Similarly, let j be the largest index such that the vertical distance between  $e_i$  and  $e_j$  is at most s; k and j can be found using binary search on  $\sigma$ . Then, the weight of the sweep square whose upper side is at e is  $\sum_{q=j}^{i} c_{e_q}$  and the weight of the sweep square whose lower side is at e is  $\sum_{q=i}^{k} c_{e_q}$ . These can be computed in  $O(\log n)$  as mentioned before.

Therefore, when the algorithm finishes after processing O(n) events, each with the time complexity  $O(\log n)$ , we can report the maximum weight square with a left corner on a trajectory vertex.

**Theorem 2.12.** There is an algorithm for finding an axis-parallel square of side length s for an orthogonal trajectory T, such that the weight of the square found by the algorithm is at least 1/4 of the weight of a hotspot.

Proof. Let T be an orthogonal trajectory. T can be partitioned into sets V and H containing the vertical and horizontal edges of T, respectively. Theorem 2.11 shows how to find a square  $r_H$  with the maximum possible weight, in which one of its corners is on a vertex of H (the algorithm can be performed twice, once after rotating the plane 180 degrees to find the maximum-weight squares with one of its right corners on a vertex of H). The same algorithm can obtain a square  $r_V$  with the maximum possible weight for V, after rotating the plane 90 degrees. By Lemma 2.9,  $w(r_H) \geq h(H)/2$  and  $w(r_V) \geq h(V)/2$ . Also, by Lemma 2.7,  $\max(h(H), h(V)) \geq h(T)/2$ , implying that  $\max(w(r_H), w(r_V)) \geq h(T)/4$ , as required.

## 3 A 1/2 Approximation Algorithm

In the proof of Theorem 2.11, to find a hotspot of a horizontal trajectory with n edges, as we moved a vertical sweep line to the right, we maintained the contribution of each edge to intersecting sweep squares (squares whose left side is on the sweep line) and computed the weight of sweep squares when necessary. Instead, to improve the approximation factor of the algorithm, in this section we maintain the weight of sweep squares (not just edge contributions) during the algorithm. Before presenting the details of the algorithm in Section 3.2, we provide an overview, and review kinetic tournament trees and segment trees in Section 3.1.

#### 3.1 Algorithm Overview

The weight of a sweep square is the sum of the contributions of trajectory edges to its weight, and thus, piecewise linear. Although there are infinitely many sweep squares, Lemma 3.1 implies that for finding a square with the maximum weight we can keep track of only n of them, that is exactly those whose upper side has the same height (y-coordinate) as a trajectory vertex.

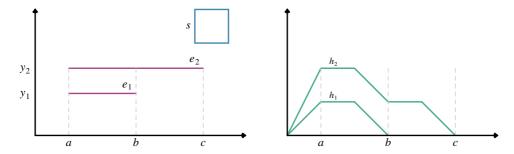


Figure 3: An example horizontal trajectory H (left) and the plane  $P_H$  (right) showing the weight of tracked squares

**Lemma 3.1.** Let H be a horizontal trajectory and let r be a square with a non-zero weight. There exists a square r' such that  $w(r') \geq w(r)$  and the y-coordinate of a vertex of H is equal to the y-coordinate of the upper side of r'.

*Proof.* If a vertex of H has the same y-coordinate as the upper side of r, we are done. Otherwise, we obtain r' by moving r downwards until the first vertex of H intersects the line containing the upper side of r. Clearly, any part of any edge in r is also in r', and therefore,  $w(r') \geq w(r)$ .

Therefore, our goal in this section is to maintain the weight of n specific squares as we move them with the sweep line in a plane sweep algorithm. More precisely, we maintain the weight of  $Square(x, y_i - s)$  for  $1 \le i \le n$ , where  $y_i$  is the y-coordinate of the i-th edge of the trajectory, as we move the sweep line (x is variable and denotes the position of the sweep line).

**Definition 3.2.** For a horizontal trajectory H, the i-th tracked square is  $Square(x, y_i - s)$ , in which  $y_i$  is the height of the i-th edge of the trajectory and x denotes the position of the sweep line. The function  $h_i \colon x \to w(Square(x, y_i - s))$  shows the weight of the i-th tracked square with respect to H, when the sweep line is at x.

We define a plane  $P_H$  for horizontal trajectory H, whose horizontal axis represents the position of the sweep line and whose vertical axis represents the weight of tracked squares. We add n curves to  $P_H$ : the i-th curve is for the function  $h_i$ . Suppose the highest point of the curves in  $P_H$  is for curve  $h_i$  at x = a. Obviously, the square with the maximum weight is  $Square(a, y_i - s)$ . Thus, to find a hotspot of the trajectory, we can compute the upper envelope of  $P_H$ . The plane  $P_H$  is shown for an example horizontal trajectory in Figure 3; the second tracked square at x = a achieves the maximum weight.

One solution for computing the upper envelope of  $P_H$  is using kinetic data structures. To use the common terminology of kinetic data structures,

let the x-axis in  $P_H$  denotes the time. We want to maintain the point with the maximum height in any of the curves as we move forward in time. For this purpose, we can use kinetic tournament trees [9], which we shall briefly describe as follows. We use an arbitrary binary tree with n leaves, in each of the nodes of which we store a function; let  $f_v$  denote the function stored at node v. We initialize the tree for time  $x = t_0$  as follows. The function describing the n curves of  $P_H$  ( $h_i$  for  $1 \le i \le n$ ) are stored in the leaves of the tree in an arbitrary order. For each non-leaf node v,  $f_v$  is recursively initialized as follows. Let  $u_1$  and  $u_2$  be the children of v. Without loss of generality, suppose  $f_{u_1}(t_0) \geq f_{u_2}(t_0)$ . We set  $f_v = f_{u_1}$  and call  $f_{u_1}$  the winner at v and give it a winning certificate. The winning certificate may not be valid indefinitely. Let  $t_1$  be the earliest time in the future  $(t_1 > t_0)$ , at which  $f_{u_2}(t_1)$  gets larger than  $f_{u_1}(t_1)$ . We say that the winning certificate of  $f_{u_1}$  fails at time  $t_1$ . At this point, the winner at v should be updated. The certificate failure times (at which failure events occur) of all non-leaf nodes of the tree are stored in a priority queue (the event queue) and are processed ordered by their time. When the next certificate fails, the winner and the certificate failure time of the corresponding node and its parent is updated. Therefore, the function with the maximum value is always stored at the root of the tree at any point of time.

To make the computation of certificate failure times more efficient, we store a linear function at each node. However, the functions  $h_i$  (for  $1 \le i \le n$ ) are piecewise linear (the break point happen when a trajectory edge enters or leaves  $Square(x, y_i - s)$ ). Consequently, the linear function assigned to the leaf corresponding to  $h_i$  should be updated at the break points of  $h_i$ ; the time at which these updates should be performed (update events) are also inserted into the event queue. At each update event for curve  $h_i$ , we need to update the function of the corresponding leaf; this may change the winner and the failure time of the winning certificate of the ancestors of the leaf. The functions can be updated as in dynamic and kinetic tournament trees [12].

The main challenge that we try to address in this section is reducing the cost of updating the tree. An edge may intersect  $Square(x, y_i - s)$  for O(n) different values of i. On the other hand, since there are O(n) edges, during the plane sweep algorithm we need to update the functions of the leaves of the tree  $O(n^2)$  times. This makes the complexity of the plane sweep algorithm  $\Omega(n^2)$ . To handle these updates more efficiently, we use a segment tree as the underlying data structure for the kinetic tournament tree. The details, correctness, and complexity of this algorithm is shown in the rest of this section.

We close this section with a brief introduction to segment trees. For a detailed introduction, the reader may consult classic texts such as [13]. A segment tree is a balanced binary tree that can be used for answering stabbing queries for a set of segments (or intervals) on a line. It is initialized with a set of n segments. The start and end points of these segments split the line into many elementary intervals. These intervals form the leaves of the segment tree in sorted order. To each node v, an interval is assigned Int(v), representing the union of the intervals of the leaves of the subtree rooted at v. Each node of the segment tree v also stores a subset of the input segments I(v). An example segment tree is shown in Figure 4. In this figure the interval near each vertex v is Int(v) and the set near each vertex is I(v) (empty sets are not shown).

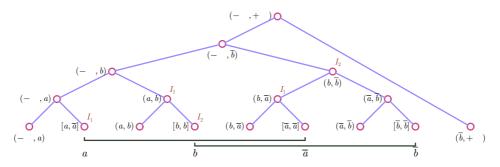


Figure 4: A segment tree for segments  $I_1 = [a, \overline{a}]$  and  $I_2 = [b, \overline{b}]$ , where  $a < b < \overline{a} < \overline{b}$ 

**Definition 3.3.** Let [a, a'] be a segment inserted into a segment tree. In one of the leaves v of the tree, we have Int(v) = [a, a]. We call this leaf, the segleaf of segment [a, a'].

To answer a stabbing query for value q (reporting every segment containing the value q), the nodes of the tree on a path from the root to a leaf are traversed and every segment in I(v) for every v in this path is reported; every output segment is reported exactly once and the complexity of answering a query is  $O(\log n + k)$ , where the size of the output (the number of intervals containing x) is k.

#### 3.2 The Algorithm

We use a segment tree T in our plane sweep algorithm. We assume that the input horizontal trajectory H has n edges and the y-coordinate of the i-th edge is  $y_i$ . T is initialized with n segments: the i-th segment  $g_i$  is  $(y_i - s, y_i)$ , corresponding to the i-th trajectory edge  $e_i$ . A sweep square Square(x,q) intersects  $s_i$  during the plane sweep algorithm, if q intersects  $g_i$ . This yields the following key observation.

**Observation 3.4.** A stabbing query for value y = q on T reports every edge intersecting the sweep square Square(x, q).

An example is shown in Figure 5. For each edge, a segment of length s is inserted into the segment tree. A stabbing query for y = q, yields the segments intersecting the sweep square whose lower side has height q.

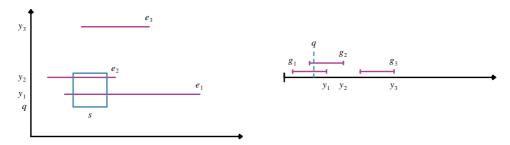


Figure 5: A horizontal trajectory (left) and the segments inserted into the segment tree (right)

For each node v, as in regular segment trees, we represent the set of segments stored at v as I(v) and the union of the intervals of the leaves of the subtree rooted at v as Int(v). For a node v of T,  $v_y$  denotes the midpoint of Int(v). With slight abuse of notation, for a leaf v we sometimes use v to refer to  $Square(x, v_y)$ , and by the weight of v, we mean  $w(Square(x, v_y))$ .

During the algorithm, we maintain the contribution function of each edge (Definition 2.10) and update it according to Table 1. We sometimes use  $c_{g_i}(x)$  to refer to the contribution function of  $e_i$ ,  $c_{e_i}(x)$ , and refer to it as the contribution function of segment  $g_i$ . We maintain the following attributes for each node v of T during the algorithm.

- $s_v(x)$ : The sum function  $s_v(x)$  is the sum of the functions assigned to the segments in I(v), i.e.  $s_v(x) = \sum_{g \in I(v)} c_g$ . Since  $s_v$  is linear, it is enough to store its slope and vertical intercept for each node v.
- $f_v(x)$ : The winner function  $f_v(x)$  is equal to  $s_v(x)$  for leaves and, for other nodes, is the sum of  $s_v(x)$  and the maximum function  $f_u(x)$  (for the current sweeping position x), among any child u of v. Like  $s_v$ ,  $f_v$  is the sum of linear functions, and thus, linear.
- winner(v): The winner leaf winner(v) for a leaf node v equals v itself. For other nodes, winner(v) equals winner(u), if u is the child of v with the maximum value of  $f_u$  for the current sweep line position (therefore,  $f_v = f_u + s_v$ ).

Sweeping starts at  $x = -\infty$ , at which the functions of all segments are zero. We use a priority queue Q to store sweeping events. There are two types of events: the failure of the winning certificate of a node of T (failure events) and a trajectory edge entering or leaving tracked squares (update events).

At a failure event for node v, the winner, the winner function, and the certificate failure time of v and some of its ancestors are updated, as in regular kinetic tournament trees. At an update event for edge  $e_i$ ,  $c_{e_i}$  changes according to Table 1, and the function  $s_u$  for every u such that  $g_i \in I(u)$  should be updated to reflect this change. After updating  $s_u$ ,  $f_u$  also needs to be updated. Since the updated function may change the winner and the failure time of u's parents, they should also be updated as in failure events.

**Lemma 3.5.** For a leaf v, let  $P = p_1 p_2 ... p_h$  be the path from the root of T to v, in which  $p_h = v$ . Then, the weight of the square at v is  $\sum_{i=1}^h s_{p_i}$  during the plane sweep algorithm.

*Proof.* Answering a stabbing query for a given value y, requires a traversal from the root of the tree to a leaf and reporting every segment stored in the nodes of the traversal. Therefore, a query for the value of  $v_y$  reports every segment in  $I(p_i)$  for  $1 \le i \le h$ . To compute the weight of v, we need to sum up the contribution of each intersecting segment. Since,  $s_{p_i}$  is the sum of the functions of the segments in  $I(p_i)$  and the label of every intersecting segment is stored in exactly one node of P,  $\sum_{i=1}^{h} s_{p_i}$  is the total contribution of the segments to the weight of the square.

**Lemma 3.6.** Let v be a leaf, u be one of its ancestors in T, and  $u_1u_2...u_h$  be the path from u to v, where  $u_1 = u$  and  $u_h = v$ . If winner(u) is v, we should have  $f_u = \sum_{i=1}^h s_{u_i}$ .

*Proof.* We use induction on h, the number of the nodes of the path from u to v. When h=1, u is a leaf and  $f_u=s_u$ . For h>1, let x and y be the two children of u. winner(u) is either winner(u) or winner(u). Therefore, either winner(u) or winner(u) is u. Without loss of generality, suppose winner(u) = u (this implies that u). Based on induction hypothesis, u0 by u1. Since u2 by u3 by u4 by u5 by u5 by u6 by u6 by u7 by u8 by u9 by u9

**Lemma 3.7.** A square at one of the leaves of T has the maximum weight among all sweep squares at any stage during the algorithm.

*Proof.* Lemma 3.1 implies that by moving a square downwards until its upper side has the same height as a trajectory vertex, its weight cannot decrease. Therefore, a tracked square  $Square(x, y_i - s)$  for some index  $i \ (1 \le i \le n)$  has the maximum weight. The segleaf of  $g_i$  (Definition 3.3), whose weight is  $Square(x, y_i - s)$ , surely appears as a leaf of T, as required.  $\square$ 

**Theorem 3.8.** T stores a sweep square with the maximum weight at its root during the plane sweep algorithm.

*Proof.* For the subtree rooted at any node v of T during the algorithm, we show that winner(v) is the square with the maximum weight among the squares, the height of whose lower side is in the interval Int(v) (that is,

we show that the statement is true for every node and not just the root). Lemma 3.7 implies that we need to consider only the leaves of the subtree rooted at v. Therefore, we show that a leaf with the maximum weight appears as the winner of v.

We use induction on the height (the distance from the leaves) of the nodes to show that the property holds for every node. For leaves, the statement is trivially true. Let v be a node with children x and y. We denote by  $T_w$  the subtree rooted at node w. Then, every leaf of  $T_v$  is a leaf in either  $T_x$  or  $T_y$ . By induction hypothesis, a leaf with the maximum weight in  $T_x$  and  $T_y$  appears as winner(x) and winner(x), respectively. Therefore, the square with the maximum weight in  $T_v$ , v', is either winner(x) or winner(y).

Let x' = winner(x) and y' = winner(y) and let  $p_x = x_1x_2...x_{h_x}$  and  $p_y = y_1y_2...y_{h_y}$  be the path from the root of T to x' and y', respectively  $(x_1 = y_1 \text{ is the root}, h_x \text{ is the depth of } x', h_y \text{ is the depth of } y', x_{h_x} = x'$  and  $y_{h_y} = y'$ ). Both  $p_x$  and  $p_y$  include v; let  $x_i = v$ . Since  $p_x$  and  $p_y$  diverge at v,  $x_j = y_j$  for every integer j such that  $1 \le j \le i$ . Based on Lemma 3.5, the weight of x' is  $\sum_{j=1}^{i} s_{x_j} + \sum_{j=i+1}^{h_x} s_{x_j}$  and the weight of y' is  $\sum_{j=1}^{i} s_{x_j} + \sum_{j=i+1}^{h_y} s_{y_j}$ . Also based on Lemma 3.6,  $f_x = \sum_{j=i+1}^{h_x} s_{x_j}$  and  $f_y = \sum_{j=i+1}^{h_y} s_{y_j}$ . Therefore, v' is x' if  $f_x > f_y$  and y', otherwise. This implies that v' is the same as winner(v), since winner(v) is chosen based on the value of  $f_x$  and  $f_y$ . This completes the proof.

The main challenge in the analysis of the sweeping algorithm is limiting the number of failure events. In a dynamic and kinetic tournament tree for movement functions with degree at most s, using a balanced binary tree and when implementing each update as a deletion followed by an insertion, the number of events is  $O(\frac{m}{n}\lambda_{s+2}(n)\log n)$ , where m is the number of updates and  $\lambda_s(n)$  is the maximum length of Davenport-Schinzel sequences of order s on n symbols [12]. For our problem, this yields a poor bound, since each update event may update the weight of O(n) leaves and thus  $m = O(n^2)$ , which implies that the total number of failure events is  $O(n^2\alpha(n)\log n)$ , in which  $\alpha(n)$  denotes the inverse Ackermann function. In Theorem 3.9 we present a tighter bound.

**Theorem 3.9.** The time complexity of the plane sweep algorithm for finding a hotspot of a horizontal trajectory with n edges is  $O(n \log^3 n)$ 

*Proof.* After a failure event for a node and updating its winner and winner function, we update its parent (unless it is the root). At each update for node v, we check if the winner at v needs to be changed. If so, we also update its parent. Otherwise, we stop. Therefore, instead of limiting the number of failure events, we can find an upper bound on the total number of winner changes at different nodes of T.

Let winner(v) = u at some point in the algorithm, in which u is a leaf. Since weight functions are linear, when winner(v) changes to a value w, where  $w \neq u$ , u can never become a winner at v, unless an update event updates the weight of w or v. Without the update events, therefore, the number of times a leaf can become a winner in its parent nodes is  $O(\log n)$ . This implies a total of  $O(n \log n)$  winner changes. It remains to limit the number of winner changes that can result from update events.

Suppose an update event for edge e updates the function assigned to segment g. Let S be the set of all nodes like v in T such that  $g \in I(v)$ . For every node v in S the sum and winner functions are updated. This change does not cause any winner change in  $T_v$  (the subtree rooted at v), because the relative weight of its leaves does not change. However, the new winner function of winner(v) may cause  $O(\log n)$  future winner changes in the ancestor of v. In segment trees one can show that the label of each segment appears in  $O(\log n)$  nodes (for details, see [13]) and thus the size of S is  $O(\log n)$ . Therefore, the number of winner changes by each update event is  $O(\log^2 n)$  and, since there are O(n) update events, the total number of winner changes induced by the update events is  $O(n\log^2 n)$ . Since for each winner change at node v, the failure time of v is updated in Q, the cost of performing each winner change is  $O(\log n)$ . Thus, the time complexity of the algorithm is  $O(n\log^3 n)$ .

**Theorem 3.10.** There exists an approximation algorithm with the approximation factor 1/2 and time complexity  $O(n \log^3 n)$  for finding a hotspot of an orthogonal trajectory with n edges.

*Proof.* Follows from Theorem 3.8, Theorem 3.9, and Lemma 2.7.

### 4 Extension to Three Dimensions

Any algorithm used for finding hotspots in 2-dimensions can be extended to find axis-parallel, cube hotspots of fixed side length for orthogonal trajectories in  $\mathbb{R}^3$ . We extend the definitions and notations presented in Section 2 to  $\mathbb{R}^3$ . The weight of a cube c with side length s with respect to trajectory T in  $\mathbb{R}^3$  is the total duration in which the entity spends inside it; we represent it as  $w_T(c)$ , as before. A hotspot of a trajectory T in  $\mathbb{R}^3$  is an axis-parallel cube (i.e. a cube whose faces are parallel to the planes defined by any pair of the axes of the coordinate system) of fixed side length s and the maximum weight, s

Let e be an edge parallel to the z-axis and let c be an axis-parallel cube. Exactly two faces of c are parallel to the xy-plane,  $Z_1(c)$  and  $Z_2(c)$ , with  $Z_1(c)$  appearing first (in the positive direction of the z-axis). The contribution rate of e with respect to c, denoted as  $c_z(e)$ , is the rate at which the contribution of the weight of e to the weight of e increases if e is moved in the positive direction of the e-axis. As in the 2-dimensional case, the absolute value of e is either zero or the ratio of the duration of e to its

length, which we denote as d(e). We define  $c_z(T)$  for orthogonal trajectory T as the sum of the contribution rates of all edges of T that are parallel to the z-axis. The following lemma extends Lemma 2.8 to three dimensions.

**Lemma 4.1.** Let T be a trajectory in  $\mathbb{R}^3$  with axis-parallel edges. For any axis-parallel cube like c, there is a cube with at least the same weight, such that a vertex of T is on one of the two planes formed by extending its xy-parallel faces.

Therefore, to find a hotspot of T, it suffices to search among the cubes with a vertex of T on one of the xy-parallel planes containing its xy-parallel faces. This observation suggests Theorem 4.2.

**Theorem 4.2.** Suppose algorithm A can find a c-approximate hotspot of any trajectory in  $\mathbb{R}^2$  containing n axis-aligned edges with the time complexity O(t(n)). For a trajectory T in  $\mathbb{R}^3$ , all of whose n edges are axis-aligned, there exists an algorithm with the time complexity  $O(n \cdot t(n))$  and approximation factor c to find an axis-aligned cube hotspot of T.

Proof. For each vertex v of T, let z(v) be the z-coordinate of v. Project all edges that are (maybe partially) between z=z(v) and z=z(v)+s to the plane z=z(v) to obtain an orthogonal 2-dimensional trajectory T'. Edges parallel to the z-axis are projected to an edge with length zero, whose weight denotes the duration of the portion between z=z(v) and z=z(v)+s. Perform algorithm A on T' to obtain a square s. Let c be the cube with  $Z_1(c)$  on s. It is not difficult to see that  $w_T(c)$  is equal to  $w_{T'}(s)$ . Record c, if it has the maximum weight so far. Repeat the preceding steps after reversing the direction of the z-axis to find cubes like c, with  $Z_2(c)$  on a vertex of T. Return the cube with the maximum weight. Lemma 4.1 implies that this cube is a c-approximate hotspot of T.

Theorem 4.2 and Theorem 3.10 imply Corollary 4.3.

Corollary 4.3. A 1/2-approximate cube hotspot of a three-dimensional trajectory with n edges can be found with the time complexity  $O(n^2 \log^3 n)$ .

It seems possible to generalize the result of Corollary 4.3 to  $R^d$  to obtain an algorithm with the time complexity  $O(n^{d-1}\log^3 n)$ .

## Acknowledgements

We are grateful to the anonymous reviewers for several suggestions to formalize the concepts of the paper and improve its presentation. We also wish to thank Marc van Kreveld for his valuable comments on an earlier version of this paper.

#### References

- [1] Y. Zheng. Trajectory data mining an overview. ACM Transactions on Intelligent Systems and Technology, 6(3):29:1–29:41, 2015.
- [2] M. Benkert, B. Djordjevic, J. Gudmundsson, and T. Wolle. Finding popular places. *International Journal of Computational Geometry & Applications*, 20(1):19–42, 2010.
- [3] M. Buchin, A. Driemel, M. J. van Kreveld, and V. Sacristán. Segmenting trajectories a framework and algorithms using spatiotemporal criteria. *Journal of Spatial Information Science*, 3(1):33–63, 2011.
- [4] B. Aronov, A. Driemel, M. J. van Kreveld, M. Löffler, and F. Staals. Segmentation of trajectories on nonmonotone criteria. *ACM Transactions on Algorithms*, 12(2):26:1–26:28, 2016.
- [5] A. G. Rudi. Looking for bird nests: Identifying stay points with bounded gaps. In *The Canadian Conference on Computational Geometry*, pages 334–339, 2018.
- [6] J. Gudmundsson, M. J. van Kreveld, and F. Staals. Algorithms for hotspot computation on trajectory data. In *International Conference* on Advances in Geographic Information Systems, pages 134–143, 2013.
- [7] L. O. Alvares, V. Bogorny, B. Kuijpers, J. A. F. de Macêdo, B. Moelans, and A. A. Vaisman. A model for enriching trajectories with semantic geographical information. In ACM International Symposium on Geographic Information Systems, page 22. ACM, 2007.
- [8] S. Tiwari and S. Kaushik. Mining popular places in a geo-spatial region based on gps data using semantic information. In Workshop on Databases in Networked Information Systems, pages 262–276. Springer, 2013.
- [9] J. Basch, L. J. Guibas, and J. Hershberger. Data structures for mobile data. *Journal of Algorithms*, 31(1):1–28, 1999.
- [10] H. J. Miller. Modelling accessibility using space-time prism concepts within geographical information systems. *International Journal of Ge*ographical Information Science, 5(3):287–301, 1991.
- [11] P. M. Fenwick. A new data structure for cumulative probability tables - an improved frequency-to-symbol algorithm. Software, Practice and Experience, 26(4):489–490, 1996.
- [12] P. K. Agarwal, H. Kaplan, and M. Sharir. Kinetic and dynamic data structures for closest pair and all nearest neighbors. ACM Transactions on Algorithms, 5(1):4:1–4:37, 2008.

[13] M. de Berg, O. Cheong, M. J. van Kreveld, and M. H. Overmars. *Computational Geometry - Algorithms and Applications*. Springer, third edition, 2008.