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Co-Finiteness and Co-Emptiness of Reachability Sets in Vector Addition Systems with States

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Abstract. The boundedness problem is a well-known exponential-space complete problem for vector addition systems with states (or Petri nets); it asks if the reachability set (for a given initial configuration) is finite. Here we consider a dual problem, the co-finiteness problem that asks if the complement of the reachability set is finite; by restricting the question we get the co-emptiness (or universality) problem that asks if all configurations are reachable.

We show that both the co-finiteness problem and the co-emptiness problem are exponential-space complete. While the lower bounds are obtained by a straightforward reduction from coverability, getting the upper bounds is more involved; in particular we use the bounds derived for reversible reachability by Leroux (2013).

The studied problems were motivated by a result for structural liveness of Petri nets; this problem was shown decidable by Jančar (2017), without clarifying its complexity. The structural liveness problem is tightly related to a generalization of the co-emptiness problem, where the sets of

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initial configurations are (possibly infinite) downward closed sets instead of just singletons. We formulate the problems even more generally, for semilinear sets of initial configurations; in this case we show that the co-emptiness problem is decidable (without giving an upper complexity bound), and we formulate a conjecture under which the co-finiteness problem is also decidable.

Keywords: vector addition system, Petri net, co-finite reachability set, universal reachability set, structural liveness, complexity.

1. Introduction

Context. Analysis of behavioural properties of (models of) systems is a wide research area, in which the decidability and complexity questions constitute an important part. As the most relevant for us, we recall the reachability, coverability and liveness problems for Petri nets.

A concrete source of our motivation was the paper [1] that answered the decidability question for structural liveness in Petri nets positively; the open status of this question was previously recalled, e.g., in [2]. It is natural to continue with studying the computational complexity of this problem. Here we contribute indirectly to this topic by studying some related problems concerning reachability sets.

The algorithm in [1] reduces the structural liveness problem to the question if a Petri net with a downward closed set of initial markings is "universal", in the sense that every marking is reachable from (some of) the initial ones. This question has been solved by using the involved result proved in [3], namely that there is an algorithm that halts with a Presburger description of the reachability set when this set is semilinear. Since this approach is not constructive, it does not provide any complexity upper bound. This led us to consider the universality problem, which we call the *co-emptiness problem*, on its own. There is also a naturally related *co-finiteness problem* asking if a set of initial markings allows to reach all but finitely many markings; this problem can be thus seen as dual to the well-known boundedness problem that asks if the reachability set is finite.

Contributions. We formulate the co-emptiness and co-finiteness problems generally for semilinear sets of initial markings. Our results are summarized in Table 1. We show that the co-emptiness problem is decidable using a reduction to [3] that is similar to the above-mentioned approach used in [1] to decide the structural liveness problem. As before, no complexity upper bound can be derived from that approach. In the case of the co-finiteness problem we are even not sure with decidability, but we formulate a conjecture under which the problem is decidable. We then consider restrictions to the case with finite sets of initial markings and then in particular to the case with singleton sets of initial markings.

In the case of finite initial sets we show that the co-emptiness problem reduces in logarithmic space to the reachability problem, which is a famous non-elementary decidable problem (we can refer to [4] for the best known upper bound, and to [5] for the best known lower bound). The converse reduction (reachability to co-emptiness) is left open. We also show that the co-finiteness problem is decidable for finite initial sets (without relying on the above-mentioned conjecture).

In the case of singleton initial sets we show EXPSPACE-completeness for both co-emptiness and co-finiteness. This is the main technical result of the paper. While the lower bound is obtained by an

Initial Set	*Co-Emptiness*	*Co-Finiteness*
Semilinear	Decidable	Conjectured Decidable
Finite	$\leqslant_{\log}$ Reachability	Decidable
Singleton	**EXPSPACE-complete**	**EXPSPACE-complete**

Table 1. Contributions of the paper on the decidability and complexity of the co-emptiness and co-finiteness problems ($\leqslant_{\log}$ denotes logspace reducibility). The main technical results are indicated in boldface.

easy reduction from the coverability problem (a well-known EXPSPACE-complete problem, similarly as boundedness, cf. [6]), getting the upper bound is more involved. Using the bounds obtained for reversible reachability by Leroux in [7], we reduce the co-emptiness problem (with a single initial marking) to a large number of coverability questions in a large Petri net. The latter is bounded in such a way that the questions can still be answered in exponential space, using Rackoff's technique [6].

Though our results do not improve our knowledge about the complexity of structural liveness directly, we show that a related problem, namely the structural deadlock-freedom problem is tightly related (interreducible in logarithmic space) with the co-emptiness problem in the case of downward closed sets of initial markings.

We have found more convenient to present our results on the model of vector addition systems with states, or shortly VASSs. This model is equivalent to Petri nets and all our results, while proved for VASSs, also hold for Petri nets.

A preliminary version of this paper appeared, under the same title, in the proceedings of the 39th International Conference on Application and Theory of Petri Nets and Concurrency (PETRI NETS 2018), LNCS 10877, pp. 184–203, Springer 2018. Decidability of the co-finiteness problem for finite initial sets (see Theorem 3.12) was added subsequently, similarly as problem extensions comprising semilinear containment and projections.

Outline. In Section 2 we recall some preliminary notions, such as vector addition systems with states, and semilinear sets; we also show some straightforward extensions of the reachability problem that will be useful in later proofs. Section 3 defines the co-emptiness problem and the co-finiteness problem, and presents our partial decidability results for the general case (with semilinear initial sets) and for the restriction to finite sets of initial configurations. The main result is contained in Section 4 where we show the EXPSPACE-completeness of co-emptiness and co-finiteness in the case with singleton sets of initial configurations. Section 5 presents two applications of the co-emptiness problem: we recall the structural liveness, and show the tight relation of structural deadlock-freedom to the co-emptiness problem with downward closed sets of initial configurations. We conclude by Section 6.

2. Preliminaries

By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$ we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$, we let $[i, j]$ denote the set $\{i, i{+}1, \ldots, j\}$; this set is empty when $i > j$.

For a vector $v \in \mathbb{Z}^d$, where $d \in \mathbb{N}$, by $v(i)$ we denote the i-th component of v (for $i \in [1, d]$). On $\mathbb{Z}^d$ we define the operations $+$, $-$ and the relations $\geq$, $\leq$ componentwise. For $v_1, v_2 \in \mathbb{Z}^d$, we thus have $v_1 + v_2 = w$ where $w(i) = v_1(i) + v_2(i)$ for all $i \in [1, d]$; we have $v_1 \leq v_2$ iff $v_1(i) \leq v_2(i)$ for all $i \in [1, d]$. For $k \in \mathbb{N}$ and $v \in \mathbb{Z}^d$ we put $k \cdot v = (k \cdot v(1), k \cdot v(2), \ldots, k \cdot v(d))$; we also write kv instead of $k \cdot v$.

The *norm* $\|v\|$ *of a vector* $v \in \mathbb{Z}^d$ is $\max\{|v(i)|; i \in [1, d]\}$, and the *norm* $\|V\|$ *of a finite set* $V \subseteq \mathbb{Z}^d$ is $\max\{\|v\|; v \in V\}$; here we stipulate $\max \emptyset = 0$.

When the dimension d is clear from the context, by $\mathbf{0}$ we denote the zero vector ($\mathbf{0}(i) = 0$ for all $i \in [1, d]$), and by $\mathbf{e}_i$ ($i \in [1, d]$) the vector satisfying $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for all $j \in [1, d] \smallsetminus \{i\}$.

For a set A, by A^* we denote the set of finite sequences of elements of A, and by ε we denote the empty sequence. For $w \in A^*$, $|w|$ denotes its length.

Vector addition systems with states (VASSs).

A *vector addition system with states* (a *VASS*) is a tuple $\mathcal{V} = (d, Q, \mathcal{A}, T)$ where $d \in \mathbb{N}$ is the *dimension*, Q is the finite set of *(control) states*, $\mathcal{A} \subseteq \mathbb{Z}^d$ is the finite set of *actions*, and $T \subseteq Q \times \mathcal{A} \times Q$ is the finite set of *transitions*. We often present $t \in T$ where $t = (q, \mathbf{a}, q')$ as $q \xrightarrow{\mathbf{a}} q'$ or $t : q \xrightarrow{\mathbf{a}} q'$. We also say that $\mathcal{V} = (d, Q, \mathcal{A}, T)$ has d *counters*, denoted $1, 2, \ldots, d$.

The set of *configurations* of $\mathcal{V} = (d, Q, \mathcal{A}, T)$ is the set $Q \times \mathbb{N}^d$; we prefer to denote a configuration (q, v) by $q(v)$, where $q \in Q$ and $v \in \mathbb{N}^d$. For every action $\mathbf{a} \in \mathcal{A}$, we define the relation $\xrightarrow{\mathbf{a}}_{\mathcal{V}}$ on the set $Q \times \mathbb{N}^d$ of configurations by

$$q(v) \xrightarrow{\mathbf{a}}_{\mathcal{V}} q'(v') \text{ if } q \xrightarrow{\mathbf{a}} q' \text{ is a transition in } T \text{ and } v' = v + \mathbf{a}.$$

Hence, for a transition $q \xrightarrow{\mathbf{a}} q'$ and $v \in \mathbb{N}^d$, we have $q(v) \xrightarrow{\mathbf{a}}_{\mathcal{V}} q'(v + \mathbf{a})$ if, and only if, $v + \mathbf{a} \geq \mathbf{0}$.

Relations $\xrightarrow{\mathbf{a}}_{\mathcal{V}}$ are naturally extended to relations $\xrightarrow{\alpha}_{\mathcal{V}}$ for $\alpha \in \mathcal{A}^*$; we write just $\xrightarrow{\alpha}$ instead of $\xrightarrow{\alpha}_{\mathcal{V}}$ when $\mathcal{V}$ is clear from the context. The extension is defined inductively: we put $q(v) \xrightarrow{\varepsilon} q(v)$; if $q(v) \xrightarrow{\mathbf{a}} q'(v')$ and $q'(v') \xrightarrow{\alpha} q''(v'')$, then $q(v) \xrightarrow{\mathbf{a}\alpha} q''(v'')$. We note that $q(v) \xrightarrow{\alpha} q'(v')$ where $\alpha = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m$ implies that $v' = v + \sum_{i \in [1,m]} \mathbf{a}_i$. We also note the *monotonicity*:

$$\text{if } q(v) \xrightarrow{\alpha} q'(v'), \text{ then for every } \bar{v} \geq v \text{ we have } q(\bar{v}) \xrightarrow{\alpha} q'(v' + \bar{v} - v).$$

Example 2.1. Consider the VASS depicted on the left of Figure 1. This VASS has dimension 2. It has two states A and B, five actions and five transitions. (If we replaced the transition $B \xrightarrow{(2,0)} B$ with $B \xrightarrow{(1,0)} B$, then we would have four actions and five transitions.) We can observe, e.g., that $A(1,0) \xrightarrow{(-1,0)} B(0,0)$, $B(0,0) \xrightarrow{(2,0)} B(2,0)$, $B(2,0) \xrightarrow{(1,0)} A(3,0)$, and $A(3,0) \xrightarrow{(-2,1)} A(1,1)$. Hence $A(1,0) \xrightarrow{\alpha} A(1,1)$ for $\alpha = (-1,0)(2,0)(1,0)(-2,1)$.

Reachability sets.

Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, by $q(v) \xrightarrow{*}_{\mathcal{V}} q'(v')$, or by $q(v) \xrightarrow{*} q'(v')$ when $\mathcal{V}$ is clear from the context, we denote that $q'(v')$ is *reachable from* $q(v)$, i.e., that $q(v) \xrightarrow{\alpha} q'(v')$ for some $\alpha \in \mathcal{A}^*$. The *reachability set for an* (initial) *configuration* $q(v)$ is the set

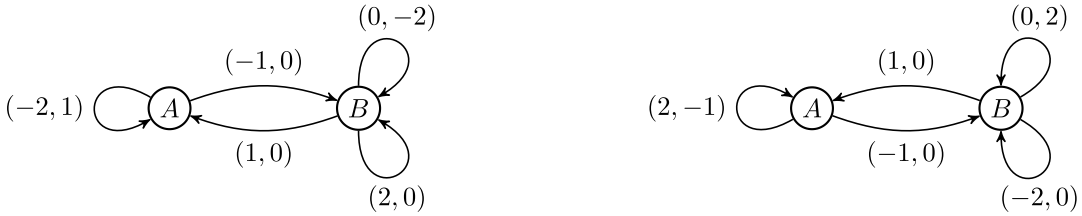

Figure 1. An example VASS of dimension 2 (left) and its reversed VASS (right).

$$\big[q(v)\big\rangle_{\mathcal{V}} = \{q'(v') \mid q(v) \xrightarrow{*}_{\mathcal{V}} q'(v')\}.$$

For a *set* $C \subseteq Q \times \mathbb{N}^d$ *of (initial) configurations* we put

$$\big[C\big\rangle_{\mathcal{V}} = \bigcup_{q(v)\in C} \big[q(v)\big\rangle_{\mathcal{V}}.$$

We also write just $\big[q(v)\big\rangle$ and $\big[C\big\rangle$ when $\mathcal{V}$ is clear from the context.

We write $q(v) \xrightarrow{*} C$ if there is $q'(v') \in C$ such that $q(v) \xrightarrow{*} q'(v')$; similarly $C \xrightarrow{*} q(v)$ if there is $q'(v') \in C$ such that $q'(v') \xrightarrow{*} q(v)$.

Semilinear sets of configurations.

A *set* $C \subseteq Q \times \mathbb{N}^d$ is *linear* if

$$C = \{q(b+n_1p_1+\cdots+n_kp_k) \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

for some $q \in Q$, $k \in \mathbb{N}$, and $b, p_1, \ldots, p_k \in \mathbb{N}^d$. A *set* $C \subseteq Q \times \mathbb{N}^d$ is *semilinear* if $C = L_1 \cup L_2 \cup \cdots \cup L_m$ for some $m \in \mathbb{N}$ and linear sets L_j, $j \in [1, m]$. We recall that semilinear sets correspond to the sets definable in Presburger arithmetic [8].

It was shown in [9] that for every 2-dimensional VASS the reachability set of a configuration is effectively semilinear. The following example illustrates this property. But this property breaks for larger dimensions, as there exists a 3-dimensional VASS with a non-semilinear reachability set [9].

Example 2.2. Continued from Example 2.1. It can be routinely checked that the reachability sets of $A(0,0)$, $A(1,0)$ and $A(2,0)$ are the following semilinear sets:

$$\begin{aligned}
\big[A(0,0)\big\rangle &= \{A(0,0)\},\\
\big[A(1,0)\big\rangle &= \{A(1+2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\} \cup \{B(2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\},\\
\big[A(2,0)\big\rangle &= \{A(2n_1, n_2) \mid n_1, n_2 \in \mathbb{N} \wedge n_1 + n_2 > 0\} \cup \{B(1+2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}.
\end{aligned}$$

Vector addition systems (VASs).

A *vector addition system* (*VAS*) is a VASS $(d, Q, \mathcal{A}, T)$ where Q is a singleton. In this case the single control state plays no role, in fact; it is thus natural to view a VAS as a pair $\mathcal{U} = (d, \mathcal{A})$ for a finite set $\mathcal{A} \subseteq \mathbb{Z}^d$. The configurations are here simply $v \in \mathbb{N}^d$, and for $\mathbf{a} \in \mathcal{A}$ we have

$$v \xrightarrow{\mathbf{a}} v' \text{ if, and only if, } v' = v + \mathbf{a}$$

for every $v, v' \in \mathbb{N}^d$. We write $[v\rangle_{\mathcal{U}}$, or just $[v\rangle$, for the reachability set of v. For a VAS the terms "action" and "transition" are identified.

Binary and unary presentations.

Instances of the problems that we will consider comprise VASSs and (presentations of semilinear sets of) configurations. We implicitly assume that the numbers in the respective vectors are presented in binary. When giving a complexity lower bound, we will explicitly refer to a unary presentation to stress the substance of the lower bound.

Multi-reachability, and semilinear reachability.

We will later make use of the following two extensions of the standard reachability problem, called the *multi-reachability problem* and the *semilinear-reachability problem*.

Multi-reachability
Instance: a positive integer k, and
a VASS $\mathcal{V}_j$ and two configurations $q_j(v_j), q'_j(v'_j)$, for each $j \in [1, k]$.
Question: is $q_j(v_j) \xrightarrow{*}_{\mathcal{V}_j} q'_j(v'_j)$ for each $j \in [1, k]$?

When k is restricted to 1, the problem is the standard *reachability problem*; as already mentioned, this problem is decidable [10], and there are recent results concerning its complexity: the long known EXPSPACE-hardness [11] has been shifted to the "least" non-elementary lower bound [5], while the best known upper bound is now given by the "least" nonprimitive recursive function [4]. The multi-reachability problem has the same complexity, as follows by the next lemma.

Lemma 2.3. The multi-reachability problem is logspace reducible to the reachability problem.

Proof:
We assume a VASS $\mathcal{V}_j = (d_j, Q_j, \mathcal{A}_j, T_j)$ and two configurations $q_j(v_j), q'_j(v'_j)$, for each $j \in [1, k]$. By renaming control states we can assume that $Q_1, \ldots, Q_k$ are pairwise disjoint, and by adding extra counters we can even assume that $d_1, \ldots, d_k$ are equal to the same dimension d and that the VASSs $\mathcal{V}_j$ are working on disjoint subsets of counters, i.e., for each $j \in [1, k]$ the non-zero components of the vectors in $\{v_j, v'_j\} \cup \mathcal{A}_j$ form a set I_j, and $I_1, \ldots, I_k$ are pairwise disjoint.

Now we consider the VASS $\mathcal{V}$ of dimension d comprising the VASSs $\mathcal{V}_1, \ldots, \mathcal{V}_k$ and the additional transitions $q'_{j-1} \xrightarrow{\mathbf{0}} q_j$, for $j \in [2, k]$. We observe that

$$q_j(v_j) \xrightarrow{*}_{\mathcal{V}_j} q'_j(v'_j) \text{ for all } j \in [1, k] \text{ if, and only if, } q_1(v_1 + \cdots + v_k) \xrightarrow{*}_{\mathcal{V}} q'_k(v'_1 + \cdots + v'_k).$$

The claimed logspace reduction is thus clear. □

Semilinear reachability

Instance: a VASS $\mathcal{V}$ and two (presentations of) semilinear sets of configurations C, D.

Question: is $q_1(x) \xrightarrow{*}_{\mathcal{V}} q_2(y)$ for some configurations $q_1(x) \in C$ and $q_2(y) \in D$?

Lemma 2.4. The semilinear-reachability problem is logspace reducible to the reachability problem.

Proof:
We assume a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, and first show a reduction in the case when C and D are linear sets, of the form

$$C = \{q_1(b+n_1p_1+\cdots+n_kp_k) \mid n_1, \ldots, n_k \in \mathbb{N}\},$$
$$D = \{q_2(b'+n_1p'_1+\cdots+n_{k'}p'_{k'}) \mid n_1, \ldots, n_{k'} \in \mathbb{N}\},$$

where $k, k' \in \mathbb{N}$ and $b, b', p_1, \ldots, p_k, p'_1, \ldots, p'_{k'} \in \mathbb{N}^d$.

From $\mathcal{V}$ we create the VASS $\mathcal{V}'$ by adding two fresh states q'_1 and q'_2, loop-transitions $q'_1 \xrightarrow{p_i} q'_1$ for all $i \in [1, k]$ and $q'_2 \xrightarrow{-p'_i} q'_2$ for all $i \in [1, k']$, and transitions $q'_1 \xrightarrow{b} q_1$, $q_2 \xrightarrow{-b'} q'_2$. It is obvious that there are $q_1(x) \in C$ and $q_2(y) \in D$ such that $q_1(x) \xrightarrow{*}_{\mathcal{V}} q_2(y)$ if, and only if, $q'_1(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} q'_2(\mathbf{0})$.

It is straightforward to extend this logspace reduction to the case where both C and D are given as finite unions of linear sets. □

3. Co-Finiteness and Co-Emptiness of Reachability Sets

Now we introduce the two main problems considered in this paper, together with their restrictions. The *co-finiteness problem* is defined as follows:

Co-finiteness

Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and
a (presentation of a) semilinear set $C \subseteq Q \times \mathbb{N}^d$.

Question: is $[C\rangle$ co-finite, i.e., is the set $(Q \times \mathbb{N}^d) \setminus [C\rangle$ finite ?

By narrowing the co-finiteness question we get the *co-emptiness problem*:

Co-emptiness

Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and
a (presentation of a) semilinear set $C \subseteq Q \times \mathbb{N}^d$.

Question: is $[C\rangle$ co-empty, i.e., is $[C\rangle = Q \times \mathbb{N}^d$?

We note that co-emptiness could also naturally be called *universality*.

We have left the semilinearity of initial sets implicit in the names of the above problems. When we restrict the problems to the instances with finite initial sets, we use the names *FMIC co-finiteness problem* and *FMIC co-emptiness problem* (where FMIC refers to "Finitely Many Initial Configurations"). Similarly, *SIC co-finiteness problem* and *SIC co-emptiness problem* refer to the restrictions with singleton initial sets (SIC refers to "Single Initial Configuration").

Example 3.1. Continued from Example 2.2. None of the three reachability sets $\big[A(0,0)\big\rangle$, $\big[A(1,0)\big\rangle$ and $\big[A(2,0)\big\rangle$ is co-finite. The reachability set $\big[\{A(1,0), A(2,0)\}\big\rangle$ is co-finite but not co-empty, as it is equal to the set $(\{A, B\} \times \mathbb{N}^2) \smallsetminus \{A(0,0)\}$. The reachability set $\big[\{A(0,0), A(1,0), A(2,0)\}\big\rangle$ is co-empty (since it is equal to $\{A, B\} \times \mathbb{N}^2$).

3.1. Decidability of the General Problems

In this section we investigate the decidability of the co-emptiness and co-finiteness problems. By combining the acceleration techniques of [3], and the decidability of Presburger arithmetic [12], it is straightforward to derive decidability of the co-emptiness problem. In fact, we show decidability of a more general problem, the *semilinear containment problem* defined as follows:

Semilinear containment
Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and
two (presentations of) semilinear sets $C, D \subseteq Q \times \mathbb{N}^d$.
Question: does $\big[C\big\rangle$ contain D, i.e., is $D \subseteq \big[C\big\rangle$?

We first recall a crucial fact.

Theorem 3.2. (reformulation of Lemma XI.1 of [3])
Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a semilinear set C of configurations, for every semilinear set $D \subseteq \big[C\big\rangle_{\mathcal{V}}$ there is a sequence $\alpha_1, \ldots, \alpha_k$ of words in $\mathcal{A}^*$ such that for every $q(v) \in D$ we have

$$C \xrightarrow{\alpha_1^{n_1} \ldots \alpha_k^{n_k}}_{\mathcal{V}} q(v)$$

for some $n_1, \ldots, n_k \in \mathbb{N}$.

We thus deduce that the semilinear containment problem can be decided by the following two procedures that are executed concurrently:

- One procedure systematically searches for some configuration $q(v) \in D$ such that $q(v) \notin \big[C\big\rangle$ which is verified by using an algorithm deciding (semilinear) reachability; this search succeeds if, and only if, $D \not\subseteq \big[C\big\rangle$.
- The other procedure systematically searches for some words $\alpha_1, \ldots, \alpha_k$ such that for every configuration $q(v) \in D$ there are $n_1, \ldots, n_k$ in $\mathbb{N}$ such that $C \xrightarrow{\alpha_1^{n_1} \ldots \alpha_k^{n_k}} q(v)$. This property (of $\alpha_1, \ldots, \alpha_k$) can be expressed in Presburger arithmetic and is thus decidable. The search succeeds if, and only if, $D \subseteq \big[C\big\rangle$.

  We can refer, e.g., to [13] for details of the above mentioned expressibility; here we just sketch a crucial fact. Given q and $\alpha = \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_m$ where $q(x) \xrightarrow{\alpha} q(x+\mathbf{a})$ for some x and $\mathbf{a} = \sum_{i \in [1,m]} \mathbf{a}_i$, we can construct the least x_0 such that $q(x_0) \xrightarrow{\alpha} q(x_0+\mathbf{a})$ and the least y_0 such that $q(y_0-\mathbf{a}) \xrightarrow{\alpha} q(y_0)$, and then the set of triples (x, n, y) in $\mathbb{N}^d \times \mathbb{N}_+ \times \mathbb{N}^d$ such that $q(x) \xrightarrow{\alpha^n} q(y)$ is captured by the Presburger formula $n \geq 1 \land x \geq x_0 \land y \geq y_0 \land y = x + n\mathbf{a}$.

However, the complexity of the semilinear containment problem is still open. Moreover, even for the restricted case of the co-emptiness problem, we do not know any better algorithm than the one presented above. In fact, we even have no reduction from or to the reachability problem.

The decidability status of the co-finiteness problem is not clear. We show how to solve the problem under a conjecture. Let us first introduce the notion of inductive set. A set of configurations D of a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ is *inductive* if for every configuration $q(v)$ in D and every transition $t : q \xrightarrow{\mathbf{a}} q'$ in T such that $v + \mathbf{a} \geq \mathbf{0}$, we have $q'(v + \mathbf{a}) \in D$. We observe that if an inductive set D contains a set of initial configurations C then $\big[C\big\rangle \subseteq D$. Moreover, we can effectively decide if a semilinear set D is inductive since in this case (given a presentation of a semilinear set D) the above condition can be easily expressed in Presburger arithmetic (which is decidable).

We note that $\big[C\big\rangle$ is inductive but it might be non-semilinear even if C is semilinear. Nevertheless, we conjecture that $\big[C\big\rangle$ can be extended to an inductive semilinear set as follows:

Conjecture 3.3. Given a VASS $\mathcal{V}$ and a semilinear set C of configurations, if $\big[C\big\rangle_{\mathcal{V}}$ is co-infinite (i.e., not co-finite), then there is an inductive semilinear set D such that $C \subseteq D$ (hence also $\big[C\big\rangle_{\mathcal{V}} \subseteq D$) and D is co-infinite.

Under this conjecture, the co-finiteness problem can be also decided by two algorithmic procedures executed concurrently:

- One procedure systematically searches for some inductive co-infinite semilinear set D that contains C; this search succeeds if, and only if, $\big[C\big\rangle$ is co-infinite (under the conjecture).
- The other procedure systematically searches for some words $\alpha_1, \ldots, \alpha_k$ and a natural number n, such that for every configuration $q(v)$ with $\|v\| \geq n$ there are $n_1, \ldots, n_k$ in $\mathbb{N}$ satisfying $C \xrightarrow{\alpha_1^{n_1} \ldots \alpha_k^{n_k}} q(v)$. This property (of $\alpha_1, \ldots, \alpha_k$ and n) can be formulated in Presburger arithmetic, and is thus decidable. The search succeeds if, and only if, $\big[C\big\rangle$ is co-finite, thanks to Theorem 3.2. Indeed, when the reachability set $\big[C\big\rangle$ is co-finite then it is semilinear, and we can apply Theorem 3.2 with $D = \big[C\big\rangle$.

Hence we have derived:

Theorem 3.4. The semilinear containment problem and the co-emptiness problem are decidable. The co-finiteness problem is decidable provided that Conjecture 3.3 holds.

Remark 3.5. We have extended the co-emptiness problem to the decidable semilinear containment problem in order to decide variants that will be introduced in the next subsection. The co-finiteness problem could be naturally extended in the same way by asking whether $D \setminus \big[C\big\rangle$ is finite for two given (presentations of) two semilinear sets C, D. This extension of the co-finiteness problem is also decidable provided that a natural extension of Conjecture 3.3 holds.

3.2. Projected Extensions

We introduce two variants of the co-finiteness and co-emptiness problems that are both reducible to the semilinear containment problem. These variants are defined as follows:

Projected co-finiteness

Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$,
a (presentation of a) semilinear set $C \subseteq Q \times \mathbb{N}^d$,
a state $q \in Q$ and a counter $i \in [1, d]$.

Question: is $\mathbb{N} \setminus \{v(i) \mid q(v) \in \lceil C \rangle\}$ finite ?

Projected co-emptiness

Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$,
a (presentation of a) semilinear set $C \subseteq Q \times \mathbb{N}^d$,
a state $q \in Q$ and a counter $i \in [1, d]$.

Question: is $\{v(i) \mid q(v) \in \lceil C \rangle\} = \mathbb{N}$?

Theorem 3.6. The projected co-emptiness and projected co-finiteness problems are decidable.

Proof:
Let us consider a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, a state $q \in Q$, a counter $i \in [1, d]$, and a semilinear set $C \subseteq Q \times \mathbb{N}^d$. From [14] we know that $N = \{v(i) \mid q(v) \in \lceil C \rangle_{\mathcal{V}}\}$ is semilinear. Since the proof of semilinearity of N given in that paper is not constructive, we cannot directly conclude the decidability of the two projected problems. To overcome this difficulty, we introduce the VASS $\mathcal{V}'$ obtained from $\mathcal{V}$ by adding the transitions $q \xrightarrow{-\mathbf{e}_j} q$ for every $j \neq i$. It is easy to verify that

$$N = \{v(i) \mid q(v) \in \lceil C \rangle_{\mathcal{V}}\} = \{n \in \mathbb{N} \mid q(n\mathbf{e}_i) \in \lceil C \rangle_{\mathcal{V}'}\}.$$

Indeed: We have $\lceil C \rangle_{\mathcal{V}} \subseteq \lceil C \rangle_{\mathcal{V}'}$, and $q(v) \xrightarrow{*}_{\mathcal{V}'} q(v(i) \cdot \mathbf{e}_i)$ due to the transitions $q \xrightarrow{-\mathbf{e}_j} q$. On the other hand, if $C \xrightarrow{*}_{\mathcal{V}'} q(n\mathbf{e}_i)$ due to a sequence of transitions (performed from some $q_0(v_0) \in C$), then we can simply omit the transitions $q \xrightarrow{-\mathbf{e}_j} q$ ($j \neq i$) in this sequence; by the monotonicity of VASSs this yields that $C \xrightarrow{*}_{\mathcal{V}} q(v)$ for some v with $v(i) = n$.

Hence $\{v(i) \mid q(v) \in \lceil C \rangle_{\mathcal{V}}\} = \mathbb{N}$ iff $\{q(n\mathbf{e}_i) \mid n \in \mathbb{N}\} \subseteq \lceil C \rangle_{\mathcal{V}'}$; the projected co-emptiness problem thus reduces to the semilinear containment problem (which is decidable by Theorem 3.4).

The projected co-finiteness can be decided by two algorithmic procedures executed concurrently:

- One procedure systematically searches for some $n_0 \in \mathbb{N}$ such that $\{q(n\mathbf{e}_i) \mid n \geq n_0\} \subseteq \lceil C \rangle_{\mathcal{V}'}$ (which is an instance of the semilinear containment problem); this search succeeds if (and only if) $\mathbb{N} \setminus N$ is finite.

- The other procedure systematically searches for some $b \in \mathbb{N}$ and $p \in \mathbb{N}_+$ such that $\lceil C \rangle_{\mathcal{V}'}$ is disjoint from $\{q(n\mathbf{e}_i) \mid n = b + mp$ for some $m\}$, which is an instance of the semilinear-reachability problem; this search succeeds if (and only if) $\mathbb{N} \setminus N$ is infinite since in that case $\mathbb{N} \setminus N$ is an infinite semilinear set, and in particular it contains an infinite linear set of the form $\{b + mp \mid m \in \mathbb{N}\}$ for some $b \in \mathbb{N}$ and $p \in \mathbb{N}_+$.

$\square$

3.3. Finitely Many Initial Configurations

As already mentioned, we have no complexity upper bound for the (decidable) co-emptiness problem in our general form (with semilinear initial sets). In this section, we focus on the *FMIC co-emptiness problem*, and the *FMIC co-finiteness problem*, where "FMIC" refers to "Finitely Many Initial Configurations". We show that both problems are decidable via reductions to the reachability problem.

Remark 3.7. In Section 4 we show that the problems are "only" EXPSPACE-complete when restricted to the SIC problems, i.e., to the singleton sets of initial configurations. Nevertheless, it is not clear how to reduce an FMIC problem to the respective SIC problem.

We fix a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, and start with some useful observations on co-emptiness and co-finiteness.

Given $n \in \mathbb{N}_+$, a set $U \subseteq Q \times \mathbb{N}^d$ is *n-upward-closed* if $q(v) \in U$ entails that $q(v+nw) \in U$ for every $w \in \mathbb{N}^d$. (Hence if in $q(v) \in U$ we change the counter values by adding nonnegative multiples of n, then the resulting configuration is in U as well.) We say that U is *$\star$-upward-closed* if U is n-upward-closed for some $n \in \mathbb{N}_+$. The following lemma gives necessary and sufficient conditions for co-emptiness and co-finiteness of the reachability set; these conditions hold for arbitrary sets of initial configurations.

Lemma 3.8. For each set $C \subseteq Q \times \mathbb{N}^d$ and each $n \in \mathbb{N}_+$ the following assertions hold:

1. $\lceil C \rangle$ is co-empty if, and only if, $\lceil C \rangle$ is 1-upward-closed and $q(\mathbf{0}) \in \lceil C \rangle$ for every $q \in Q$.
2. If $\lceil C \rangle$ is co-finite, then $\lceil C \rangle$ is $\star$-upward-closed.
3. If $\lceil C \rangle$ is n-upward-closed, then $\lceil C \rangle$ is co-finite if, and only if, for every $q \in Q$, $u \in [0, n{-}1]^d$, and $i \in [1, d]$ there exists $x \in \mathbb{N}$ such that $q(u{+}xn\mathbf{e}_i) \in \lceil C \rangle$.

Proof:
The first assertion is obvious, when we recall that the co-emptiness of $\lceil C \rangle$ means that $\lceil C \rangle = Q \times \mathbb{N}^d$.

To prove the second assertion, we assume that $\lceil C \rangle$ is co-finite. Let $m = 1 + \|(Q \times \mathbb{N}^d) \setminus \lceil C \rangle\|$. By definition of the norm, we have $v \in \lceil C \rangle$ for every $v \in \mathbb{N}^d$ such that $\|v\| \geq m$. It follows that $\lceil C \rangle$ is m-upward-closed.

To prove the third assertion, we assume that $\lceil C \rangle$ is n-upward-closed. If $\lceil C \rangle$ is co-finite then for every $q \in Q$, $u \in [0, n{-}1]^d$ and $i \in [1, d]$, the set $\lceil C \rangle$ intersects $\{q(u{+}xn\mathbf{e}_i) \mid x \in \mathbb{N}\}$ as otherwise $\lceil C \rangle$ would be co-infinite. Conversely, suppose that for every $q \in Q$, $u \in [0, n{-}1]^d$ and $i \in [1, d]$, there exists $x_{q,u,i} \in \mathbb{N}$ such that $q(u{+}x_{q,u,i}n\mathbf{e}_i) \in \lceil C \rangle$. Let x denote the maximum of the numbers $x_{q,u,i}$. We show that $q(v) \in \lceil C \rangle$ for every configuration $q(v)$ such that $\|v\| \geq xn$. Let us fix an arbitrary $q(v)$ and $i \in [1, d]$ such that $v(i) \geq xn$. We write $v = u + nw$ where $u(j) = v(j) \bmod n$ and $w(j) = v(j) \div n$, for each $j \in [1, d]$; hence $u \in [0, n{-}1]^d$, and $w(i) \geq x \geq x_{q,u,i}$. So the vector $w' = w - x_{q,u,i}\mathbf{e}_i$ is in $\mathbb{N}^d$. We get $q(v) = q(u{+}x_{q,u,i}n\mathbf{e}_i{+}nw')$, hence $q(v) \in \lceil C \rangle$ since $q(u{+}x_{q,u,i}n\mathbf{e}_i) \in \lceil C \rangle$ and $\lceil C \rangle$ is n-upward-closed. □

Now we give characterizations of $\{n, \star\}$-upward-closedness of $\left[C\right\rangle$ that will be used in the sequel. These characterizations reduce $\{n, \star\}$-upward-closedness of $\left[C\right\rangle$ to reachability conditions over a "neighbourhood" of C.

Lemma 3.9. For every set $C \subseteq Q \times \mathbb{N}^d$ and each $n \in \mathbb{N}_+$, the following assertions hold:

1. $\left[C\right\rangle$ is n-upward-closed if, and only if, $q(v{+}n\mathbf{e}_i) \in \left[C\right\rangle$ for every $q(v) \in C$ and $i \in [1, d]$.
2. $\left[C\right\rangle$ is $\star$-upward closed if, and only if, for every $i \in [1, d]$ there exists $x \in \mathbb{N}_+$ such that $q(v{+}x\mathbf{e}_i) \in \left[C\right\rangle$ for every $q(v) \in C$.

Proof:
We prove the "if" direction of the first assertion; the "only if" direction is trivial. Hence we assume that $q(v{+}n\mathbf{e}_i) \in \left[C\right\rangle$ for every $q(v) \in C$ and $i \in [1, d]$. We show that

$$q(v) \in \left[C\right\rangle \quad \text{implies} \quad q(v{+}n\mathbf{e}_i) \in \left[C\right\rangle \tag{1}$$

for all $q \in Q$, $v \in \mathbb{N}^d$, and $i \in [1, d]$. Indeed, if $q_0(v_0) \xrightarrow{*} q(v)$ for some $q_0(v_0) \in C$, then $q_0(v_0{+}n\mathbf{e}_i) \in \left[C\right\rangle$ by assumption, and $q_0(v_0{+}n\mathbf{e}_i) \xrightarrow{*} q(v{+}n\mathbf{e}_i)$ by monotonicity. Hence (1) holds, and this entails that $\left[C\right\rangle$ is n-upward-closed.

Now we prove the "if" direction of the second assertion; the "only if" direction is trivial. Hence we assume that for every $i \in [1, d]$, there is $x_i \in \mathbb{N}_+$ such that $\left[C\right\rangle$ contains $q(v{+}x_i\mathbf{e}_i)$ for every $q(v) \in C$. Let n be a positive common multiple of $x_1, \ldots, x_d$. With the same arguments as in the proof of the first assertion, we show that

$$q(v) \in \left[C\right\rangle \quad \text{implies} \quad q(v{+}x_i\mathbf{e}_i) \in \left[C\right\rangle \tag{2}$$

for all $q \in Q$, $v \in \mathbb{N}^d$, and $i \in [1, d]$. It follows that $q(v) \in \left[C\right\rangle$ entails $q(v{+}n\mathbf{e}_i) \in \left[C\right\rangle$; hence $\left[C\right\rangle$ is n-upward-closed (and thus $\star$-upward closed). □

For the rest of this section, we restrict our attention to the case where the set C of initial configurations is finite. Recall that, by Lemma 3.8, $\left[C\right\rangle$ is co-empty if, and only if, $\left[C\right\rangle$ is 1-upward-closed and $q(\mathbf{0}) \in \left[C\right\rangle$ for every $q \in Q$. It follows from Lemma 3.9 that co-emptiness of $\left[C\right\rangle$ reduces to finitely many reachability queries, and we thus obtain the following theorem.

Theorem 3.10. The FMIC co-emptiness problem is logspace reducible to the reachability problem.

Proof:
By Lemmas 3.8 and 3.9, given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a finite set $C \subseteq Q \times \mathbb{N}^d$, deciding if $\left[C\right\rangle = Q \times \mathbb{N}^d$ boils down to verifying if each configuration in the finite set

$$D \;=\; \{q(v{+}\mathbf{e}_i) \mid q(v) \in C, i \in [1, d]\} \cup \{q(\mathbf{0}) \mid q \in Q\}$$

is reachable from (a configuration in) C. For $C = \{q_1(v_1), \ldots, q_k(v_k)\}$ we can verify this condition as follows: we create $\mathcal{V}'$ from $\mathcal{V}$ by adding a fresh state q_0 and transitions $q_0 \xrightarrow{v_i} q_i$ for all $i \in [1, k]$, and verify that $q_0(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} q(v)$ for each $q(v) \in D$. This demonstrates that the FMIC co-emptiness problem is logspace reducible to the multi-reachability problem. As the latter is logspace reducible to the reachability problem by Lemma 2.3, we obtain the claimed logspace reduction. □

Now we aim to show the decidability of the FMIC co-finiteness problem, recalling the characterization from Lemma 3.8. We start with the following lemma.

Lemma 3.11. The question whether $\big[C\big\rangle$ is $\star$-upward-closed, given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a finite set $C \subseteq Q \times \mathbb{N}^d$, is logspace reducible to the reachability problem.

Proof:
We assume a given VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, and a nonempty set $C = \{q_1(v_1), \ldots, q_k(v_k)\} \subseteq Q \times \mathbb{N}^d$. We say that a vector $w \in \mathbb{N}^d$ is an *invariant* of $(\mathcal{V}, C)$ if there is $n \in \mathbb{N}_+$ such that for every $q_j(v_j) \in C$ we have $q_j(v_j{+}nw) \in \big[C\big\rangle$. We will show that, given $\mathcal{V}$, C, and w, the question if w is an invariant of $(\mathcal{V}, C)$ is logspace reducible to the reachability problem. This will be sufficient, since Lemma 3.9 shows that $\big[C\big\rangle$ is $\star$-upward-closed if, and only if, $\mathbf{e}_i$ is an invariant of $(\mathcal{V}, C)$ for every $i \in [1, d]$; the problem to decide if $\big[C\big\rangle$ is $\star$-upward-closed is thus logspace reducible to the multi-reachability problem, and thus also to the reachability problem by Lemma 2.3.

Hence now we assume $\mathcal{V}$ and C as above, and a given vector $w \in \mathbb{N}^d$. We aim to check if there is $n \in \mathbb{N}_+$ such that for each $j \in [1, k]$ we have that $q_j(v_j{+}nw)$ is reachable from (some configuration $q_\ell(v_\ell)$ in) C. We first introduce a VASS $\mathcal{V}'$ of dimension dk that comprises k disjoint copies of $\mathcal{V}$ (each copy works on its own counters); let $q|_j$ denote the jth copy of the state q (for $j \in [1, k]$), and let $v|_j$, for $v \in \mathbb{Z}^d$ and $j \in [1, k]$, denote the vector in $\mathbb{Z}^{dk}$ where $v|_j(d(j{-}1) + i) = v(i)$ for all $i \in [1, d]$ and $v|_j(m) = 0$ for all $m \in [1, dk] \setminus [d(j{-}1){+}1, dj]$.

Now we create $\mathcal{V}_0$ from $\mathcal{V}'$ by adding a fresh state denoted as $q_0|_0$, and the following transitions:

- $q_{j-1}|_{j-1} \xrightarrow{v_\ell|_j} q_\ell|_j$ for all $j, \ell \in [1, k]$ (enabling a move into C in the next copy of $\mathcal{V}$);
- $q_k|_k \xrightarrow{(-w,\ldots,-w)} q_k|_k$ (this final loop enables to remove multiples of w in all k copies of $\mathcal{V}$ synchronously).

We finish the proof by showing that $q_0|_0(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}_0} q_k|_k(v_1{+}w, \ldots, v_k{+}w)$ if, and only if, w is an invariant of $(\mathcal{V}, C)$ (which yields the announced logspace reduction to the reachability problem).

First we assume that w is an invariant of $(\mathcal{V}, C)$, i.e., there is $n \in \mathbb{N}_+$ such that $q_j(v_j{+}nw) \in \big[C\big\rangle_{\mathcal{V}}$ for all $j \in [1, k]$. Hence, for each $j \in [1, k]$ there are $\ell_j \in [1, k]$ and $\alpha_j \in \mathcal{A}^*$ such that $q_{\ell_j}(v_{\ell_j}) \xrightarrow{\alpha_j}_{\mathcal{V}} q_j(v_j{+}nw)$. By β_j we denote the word of actions obtained from α_j by working in the jth copy of $\mathcal{V}$; we thus have $q_{\ell_j}|_j(v_{\ell_j}|_j) \xrightarrow{\beta_j}_{\mathcal{V}_0} q_j|_j(v_j|_j{+}nw_j|_j)$. This entails

$$q_0|_0(\mathbf{0}) \xrightarrow{(v_{\ell_1}|_1)\beta_1\ldots(v_{\ell_k}|_k)\beta_k(-w,\ldots,-w)^{n-1}}_{\mathcal{V}_0} q_k|_k(v_1{+}w, \ldots, v_k{+}w).$$

Conversely, we assume $q_0|_0(\mathbf{0}) \xrightarrow{\alpha}_{\mathcal{V}_0} q_k|_k(v_1{+}w, \ldots, v_k{+}w)$ for a sequence α of actions of $\mathcal{V}_0$. If α contains the action $(-w, \ldots, -w)$ corresponding to the loop on $q_k|_k$, we can push the corresponding transition to the end (by monotonicity, since this loop is non-positive on all components). Hence α can be supposed to be in the form $(v_{\ell_1}|_1)\beta_1 \ldots (v_{\ell_k}|_k)\beta_k(-w, \ldots, -w)^{n-1}$ where $n \geq 1$, and where β_j is a word of actions in the jth copy of $\mathcal{V}$. It follows that $q_{\ell_j}(v_{\ell_j}) \xrightarrow{\alpha_j}_{\mathcal{V}} q_j(v_j + nw)$ where α_j is the word of actions of $\mathcal{V}$ corresponding to β_j; hence w is an invariant of $(\mathcal{V}, C)$. □

We now present a decision procedure for the FMIC co-finiteness problem. Assume that we are given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a finite set $C \subseteq Q \times \mathbb{N}^d$ of initial configurations. First, we determine, using Lemma 3.11, whether $[C\rangle$ is $\star$-upward-closed. If $[C\rangle$ is not $\star$-upward-closed, then $[C\rangle$ is not co-finite by Lemma 3.8. Otherwise, we compute a positive integer n such that $[C\rangle$ is n-upward-closed. Such an n necessarily exists as $[C\rangle$ is $\star$-upward-closed, and we can find one as n-upward-closedness of $[C\rangle$ is decidable by Lemma 3.9. Since $[C\rangle$ is n-upward-closed, we derive from Lemma 3.8 that co-finiteness of $[C\rangle$ reduces to finitely many reachability queries. We have shown the following theorem. As the enumeration of the positive integer n is not bounded, this decision procedure does not provide any complexity upper bound for the FMIC co-finiteness problem.

Theorem 3.12. The FMIC co-finiteness problem is decidable.

Remark 3.13. If C is semilinear and $[C\rangle$ is known to be n-upward-closed for a given $n > 0$, then $[C\rangle$ is computable using standard techniques [15, 16] as the limit of a growing sequence $U_0 \subseteq U_1 \subseteq \cdots$ of n-upward-closed subsets of $Q \times \mathbb{N}^d$, defined as follows:

$$\begin{aligned} U_0 &= \{q(v') \mid q(v) \in C \wedge q(v) \leq_n q(v')\} \\ U_{k+1} &= U_k \cup \{q'(v') \mid \exists q(v) \in U_k, \exists \mathbf{a} \in \mathcal{A}, q(v) \xrightarrow{\mathbf{a}} q'(v')\} \end{aligned}$$

where $\leq_n$ denotes the well-partial-order on $Q \times \mathbb{N}^d$ defined by $q_1(v_1) \leq_n q_2(v_2)$ if $q_1 = q_2$ and $v_2 = v_1 + nw$ for some $w \in \mathbb{N}^d$. Each set U_k in the sequence can be finitely represented by its minimal elements (w.r.t. $\leq_n$), and using this representation, U_{k+1} is computable from U_k. The sequence is ultimately stationary because $\leq_n$ is a well-partial-order, and $U_k = [C\rangle$ as soon as $U_k = U_{k+1}$.

We leave open the question if the FMIC co-finiteness problem is logspace reducible to the reachability problem. Another open question is if reachability can be reduced to FMIC co-finiteness or FMIC co-emptiness. In the next section, we characterize the complexity of both problems for the case of single initial configurations.

4. Single Initial Configurations

In this section we restrict our attention to the *SIC co-emptiness problem* and the *SIC co-finiteness problem* where SIC refers to “Single Initial Configuration”; the problem instances are thus restricted so that the given sets C are singletons ($C = \{q_0(v_0)\}$). In the rest of this section we prove the following theorem.

Theorem 4.1. Both the SIC co-finiteness problem and the SIC co-emptiness problem are EXPSPACE-complete.

We recall that the integers in the problem instances are presented in binary. Nevertheless the lower bound will be shown already for *unary VASs* (hence with no control states and with a unary presentation of integers).

We first recall two well-known EXPSPACE-complete problems for VASSs where the lower bound also holds for unary VASs, namely the *coverability problem* and the *boundedness problem*.

Coverability
Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, $q_0, q_1 \in Q$, $v_0, v_1 \in \mathbb{N}^d$.
Question: is $q_0(v_0) \xrightarrow{*} q_1(\bar{v}_1)$ for some $\bar{v}_1 \geq v_1$?

Boundedness
Instance: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, $q_0 \in Q$, $v_0 \in \mathbb{N}^d$.
Question: is $\big[q_0(v_0)\big\rangle$ finite ?

The EXPSPACE-hardness results follow from [11] (see also, e.g. [17]), the upper bounds follow from [6]. A generalization of [6], extending a class of problems known to be in EXPSPACE, was given in [18], which was later corrected in [19].

4.1. EXPSPACE-hardness

Showing the hardness part of Theorem 4.1 is relatively straightforward; we reduce coverability in unary VASs (which we recalled as an EXPSPACE-complete problem) to both SIC co-finiteness and SIC co-emptiness by the following lemma.

Lemma 4.2. Given a unary VAS $\mathcal{U} = (d, \mathcal{A})$ and $v_0, v_1 \in \mathbb{N}^d$, there is a logspace construction yielding a unary VAS $\mathcal{U}' = (d{+}1, \mathcal{A}')$ and $v'_0 \in \mathbb{N}^{d+1}$ such that:

a) if $v_0 \xrightarrow{*}_{\mathcal{U}} \bar{v}_1$ for some $\bar{v}_1 \geq v_1$, then $\big[v'_0\big\rangle_{\mathcal{U}'} = \mathbb{N}^{d+1}$;

b) otherwise (when $v_0 \xrightarrow{*}_{\mathcal{U}} w$ implies $w \not\geq v_1$) the set $\mathbb{N}^{d+1} \smallsetminus \big[v'_0\big\rangle_{\mathcal{U}'}$ is infinite.

Proof:
Let us assume a unary VAS $\mathcal{U} = (d, \mathcal{A})$ and vectors $v_0, v_1 \in \mathbb{N}^d$. We consider $\mathcal{U}' = (d{+}1, \mathcal{A}')$ and $v'_0 = (v_0, 0)$ where

$$\mathcal{A}' = \{(\mathbf{a}, 0) \mid \mathbf{a} \in \mathcal{A}\} \cup \{\mathbf{b}_1, \mathbf{b}_2\} \cup \{\mathbf{c}_j \mid j \in [1, d]\} \cup \{-\mathbf{e}_j \mid j \in [1, d]\}$$
$$\text{for } \mathbf{b}_1 = (-v_1, 2),\ \mathbf{b}_2 = (v_0, -1),\ \mathbf{c}_j = \mathbf{e}_j{-}\mathbf{e}_{d+1}.$$

It suffices to verify that the points a) and b) are satisfied (for $\mathcal{U}'$ and v'_0):

a) Suppose $v_0 \xrightarrow{\alpha}_{\mathcal{U}} \bar{v}_1$ for some $\bar{v}_1 \geq v_1$ and $\alpha = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m$.
For $\alpha' = (\mathbf{a}_1, 0)(\mathbf{a}_2, 0) \cdots (\mathbf{a}_m, 0)$, in $\mathcal{U}'$ we then have

$$(v_0, 0) \xrightarrow{\alpha'} (\bar{v}_1, 0) \xrightarrow{\mathbf{b}_1} (\bar{v}_1{-}v_1, 2) \xrightarrow{\mathbf{b}_2} (v_0{+}\bar{v}_1{-}v_1, 1).$$

By monotonicity, for any $k \in \mathbb{N}$ we have

$$v'_0 = (v_0, 0) \xrightarrow{(\alpha'\mathbf{b}_1\mathbf{b}_2)^k} w_k = (v_0{+}k(\bar{v}_1{-}v_1), k);$$

hence $w_k(d{+}1) = k$. For any $w \in \mathbb{N}^{d+1}$ and the sum $k = \sum_{j \in [1,d+1]} w(j)$ we have $w_k \xrightarrow{*} w$; indeed, in w_k we can first empty (i.e., set to zero) all components $j \in [1, d]$ by using actions $-\mathbf{e}_j$ ($j \in [1, d]$), and then distribute the k tokens from component $d{+}1$ by the actions $\mathbf{c}_j$ so that w is reached. Hence $\big[v'_0\big\rangle_{\mathcal{U}'} = \mathbb{N}^{d+1}$.

b) Suppose there is no $\bar{v}_1 \geq v_1$ such that $v_0 \xrightarrow{*}_{\mathcal{U}} \bar{v}_1$. Then for any $w \in \lceil v'_0 \rangle_{\mathcal{U}'}$ we have $w \not\geq (v_1, 0)$ and $w(d{+}1) = 0$, since the actions $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{c}_j$ are dead (they cannot get enabled from v'_0); indeed, by monotonicity the actions $-\mathbf{e}_j$ cannot help to cover $(v_1, 0)$ from v'_0. Hence the set $\mathbb{N}^{d+1} \setminus \lceil v'_0 \rangle_{\mathcal{U}'}$ is infinite.

This concludes the proof of the lemma. □

4.2. EXPSPACE-membership.

We now prove the EXPSPACE-membership claimed by Theorem 4.1. This is more involved; besides a closer look at the results in [6], we will also use the following result from [7], from which we derive Lemma 4.4.

Theorem 4.3. ([7])
Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and two configurations $q_0(v_0)$ and $q_1(v_1)$ reachable one from the other (i.e., $q_0(v_0) \xrightarrow{*} q_1(v_1) \xrightarrow{*} q_0(v_0)$), there is a word $\alpha \in \mathcal{A}^*$ such that

a) $q_0(v_0) \xrightarrow{\alpha} q_1(v_1)$, and

b) $|\alpha| \leq 6 \cdot (d+3)^2 \cdot x^{45(d+3)^{d+5}}$ where $x = 1 + 2|Q| + 2\|\mathcal{A}\| + 2\|v_0\| + \|v_1\|$.

Proof:
Theorem 10.1 of [7] states that for every pair (v'_0, v'_1) of configurations of a VAS $(p, \mathcal{A}')$ that are reachable one from the other there is a word $\alpha' \in (\mathcal{A}')^*$ such that:

$$v'_0 \xrightarrow{\alpha'} v'_1 \quad \text{and} \quad |\alpha'| \leq 17p^2 y^{15p^{p+2}}$$

where $y = (1 + 2\|\mathcal{A}'\|)(1 + \|v'_0\| + \|v'_1 - v'_0\|)$. We extend this result to a VASS $(d, Q, \mathcal{A}, T)$ by encoding it as a VAS $(p, \mathcal{A}')$ using [9, Lemma 2.1]. With this encoding, $p = d + 3$, $\|\mathcal{A}'\| \leq \max\{\|\mathcal{A}\|, |Q| \cdot (|Q| - 1)\}$ and the encodings of $q_0(v_0)$ and $q_1(v_1)$ provide vectors v'_0, v'_1 satisfying $\|v'_0\| \leq \|v_0\| + |Q|$ and $\|v'_1 - v'_0\| = \|v_1 - v_0\| \leq \|v_1\| + \|v_0\|$. It follows that $(1 + 2\|\mathcal{A}'\|) \leq x^2$ and $(1 + \|v'_0\| + \|v'_1 - v'_0\|) \leq x$. Thus y is bounded by x^3. Finally, since the effect of an action of the VASS is simulated by three actions of the simulating VAS, we deduce that there exists a word $\alpha \in \mathcal{A}^*$ such that $q_0(v_0) \xrightarrow{\alpha} q_1(v_1)$ and such that $|\alpha| \leq \frac{1}{3}|\alpha'|$. We derive the desired bound on $|\alpha|$ by observing that $\frac{17}{3} \leq 6$. □

Pumpability of components.

Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, we say that *component* $i \in [1, d]$ is *pumpable in* $q(v)$ if $q(v) \xrightarrow{*} q(v{+}k\mathbf{e}_i)$ for some $k \geq 1$.

Lemma 4.4. For any VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and any $q \in Q$, $v \in \mathbb{N}^d$, $i \in [1, d]$ where component i is pumpable in $q(v)$ there is $\alpha \in \mathcal{A}^*$ such that

a) $q(v) \xrightarrow{\alpha} q(v{+}k\mathbf{e}_i)$ for some $k \geq 1$, and

b) $|\alpha| \leq 6 \cdot (d+3)^2 \cdot x^{45(d+3)^{d+5}}$ where $x = 2 + 2|Q| + 2\|\mathcal{A}\| + 3\|v\|$.

The trivial fact $k \leq |\alpha| \cdot \|\mathcal{A}\|$ thus also yields a double-exponential bound on k.

Proof:
We consider a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and assume $q(v) \xrightarrow{*}_{\mathcal{V}} q(v{+}k\mathbf{e}_i)$ where $k \geq 1$. For the VASS $\mathcal{V}'$ arising from $\mathcal{V}$ by adding (action $-\mathbf{e}_i$ and) the transition $q \xrightarrow{-\mathbf{e}_i} q$ we get

$$q(v) \xrightarrow{*} q(v{+}k\mathbf{e}_i) \xrightarrow{-\mathbf{e}_i} \cdots \xrightarrow{-\mathbf{e}_i} q(v{+}\mathbf{e}_i) \xrightarrow{-\mathbf{e}_i} q(v);$$

hence $q(v)$ and $q(v{+}\mathbf{e}_i)$ are reachable one from the other (they are in the reversible-reachability relation) in $\mathcal{V}'$. Using Theorem 4.3, we derive that

$$q(v) \xrightarrow{\alpha}_{\mathcal{V}'} q(v{+}\mathbf{e}_i) \tag{3}$$

for some $\alpha \in (\mathcal{A} \cup \{-\mathbf{e}_i\})^*$ that is bounded as in the point b) of the claim.

If α in (3) is $\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m$, then there are states $q_1, q_2, \ldots, q_{m-1}$ such that

$$q(v) \xrightarrow{\mathbf{a}_1} q_1(v_1) \xrightarrow{\mathbf{a}_2} q_2(v_2) \xrightarrow{\mathbf{a}_3} \cdots q_{m-1}(v_{m-1}) \xrightarrow{\mathbf{a}_m} q(v{+}\mathbf{e}_i) \tag{4}$$

for the corresponding v_j ($j \in [1, m{-}1]$). We can view (4) as a sequence of transitions; let $\ell \geq 0$ be the number of occurrences of the transition $q \xrightarrow{-\mathbf{e}_i} q$ in (4). Due to monotonicity, we can omit these occurrences and keep performability: we get

$$q(v) \xrightarrow{\mathbf{a}_{i_1}\mathbf{a}_{i_2}\cdots\mathbf{a}_{i_{m-\ell}}}_{\mathcal{V}} q(v{+}(\ell{+}1)\mathbf{e}_i) \tag{5}$$

for the sequence $\mathbf{a}_{i_1}\mathbf{a}_{i_2} \cdots \mathbf{a}_{i_{m-\ell}}$ arising from $\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_m$ by omitting the respective ℓ occurrences of $-\mathbf{e}_i$. The proof is thus finished. ☐

We derive the following important corollary:

Corollary 4.5. There is an exponential-space algorithm that, given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $q(v)$, decides if all components $i \in [1, d]$ are pumpable in $q(v)$, and in the positive case provides an (at most double-exponential) number $n \geq 1$ such that $q(v) \xrightarrow{*} q(v{+}n\mathbf{e}_i)$ for each $i \in [1, d]$.

Proof:
It suffices to consider a nondeterministic algorithm trying to find, for each $i \in [1, d]$ separately, α_i with length bounded as in Lemma 4.4 such that $q(v) \xrightarrow{\alpha_i} q(v{+}k_i\mathbf{e}_i)$ for some $k_i \geq 1$. The algorithm just traverses along (a guessed bounded) α_i, keeping only the current configuration in memory; hence exponential space is sufficient.

By monotonicity, $q(v) \xrightarrow{*} q(v{+}k_i\mathbf{e}_i)$ implies that $q(v) \xrightarrow{*} q(v{+}xk_i\mathbf{e}_i)$ for all $x \geq 1$. Hence if $k_i \geq 1$ for all $i \in [1, d]$ are found, then the least common multiple (or even simply the product) of all k_i, $i \in [1, d]$, can be taken as the claimed number n. ☐

Before giving the algorithm deciding SIC co-emptiness we introduce some useful natural notions, namely a notion of "reversing a VASS" (letting its computations run backwards), and a notion of "transforming a VASS modulo n" (where the component-values are divided by n while the remainders are kept in the control states).

Reversed VASS.

To a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ we associate its *reversed VASS*

$$\mathcal{V}^{\leftarrow} = (d, Q, -\mathcal{A}, T^{\leftarrow})$$

where $-\mathcal{A} = \{-\mathbf{a} \mid \mathbf{a} \in \mathcal{A}\}$ and $T^{\leftarrow} = \{q' \xrightarrow{-\mathbf{a}} q \mid q \xrightarrow{\mathbf{a}} q' \text{ is in } T\}$. We note that $(\mathcal{V}^{\leftarrow})^{\leftarrow} = \mathcal{V}$. (Figure 1 shows an example of a VASS and its reversed VASS.)

The next proposition can be easily verified by induction on m.

Proposition 4.6. For any VASS $\mathcal{V}$ and $m \geq 1$, we have

$$q_0(v_0) \xrightarrow{\mathbf{a}_1} q_1(v_1) \xrightarrow{\mathbf{a}_2} q_2(v_2) \xrightarrow{\mathbf{a}_3} \cdots q_{m-1}(v_{m-1}) \xrightarrow{\mathbf{a}_m} q_m(v_m) \text{ in } \mathcal{V}$$
$$\text{if, and only if,}$$
$$q_m(v_m) \xrightarrow{-\mathbf{a}_m} q_{m-1}(v_{m-1}) \xrightarrow{-\mathbf{a}_{m-1}} \cdots q_1(v_1) \xrightarrow{-\mathbf{a}_1} q_0(v_0) \text{ in } \mathcal{V}^{\leftarrow}.$$

Modulo-n VASS.

Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $n \geq 1$, we put

$$\mathcal{V}_{(n)} = (d, Q \times \{0, 1, \ldots, n{-}1\}^d, \mathcal{A}', T_{(n)})$$

where $T_{(n)}$ arises as follows:
each transition $q \xrightarrow{\mathbf{a}} q'$ in T and each $u \in \{0, 1, \ldots, n{-}1\}^d$ determines

$$\text{the transition } (q, u) \xrightarrow{\mathbf{a}'} (q', u') \text{ in } T_{(n)}$$

where u' and $\mathbf{a}'$ are the unique vectors such that $u + \mathbf{a} = u' + n\mathbf{a}'$ and $u' \in \{0, 1, \ldots, n{-}1\}^d$. The set $\mathcal{A}'$ is simply $\{\mathbf{a}' \mid ((q, u) \xrightarrow{\mathbf{a}'} (q', u')) \in T_{(n)}\}$.

The next proposition is again easily verifiable by induction on m.

Proposition 4.7. For any VASS $\mathcal{V}$, $n \geq 1$, and $m \geq 1$, we have

$$q_0(v_0) \xrightarrow{\mathbf{a}_1} q_1(v_1) \xrightarrow{\mathbf{a}_2} q_2(v_2) \xrightarrow{\mathbf{a}_3} \cdots q_{m-1}(v_{m-1}) \xrightarrow{\mathbf{a}_m} q_m(v_m) \text{ in } \mathcal{V}$$
$$\text{if, and only if,}$$
$$(q_0, u_0)(v'_0) \xrightarrow{\mathbf{a}'_1} (q_1, u_1)(v'_1) \xrightarrow{\mathbf{a}'_2} \cdots \xrightarrow{\mathbf{a}'_m} (q_m, u_m)(v'_m) \text{ in } \mathcal{V}_{(n)}$$

where $u_j + nv'_j = v_j$ for every $j \in [0, m]$ (and $u_{j-1} + \mathbf{a}_j = u_j + n\mathbf{a}'_j$ for every $j \in [1, m]$).

Algorithm deciding SIC co-emptiness.

We define the following algorithm.

Algorithm ALG-CO-EMPT

Input: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a configuration $q_0(v_0)$.

Output: YES if $\left[q_0(v_0)\right\rangle = Q \times \mathbb{N}^d$, and NO otherwise.

1. Check if each component $i \in [1, d]$ is pumpable in $q_0(v_0)$, and in the positive case compute an (at most double-exponential) number n as described in Corollary 4.5 (hence $q_0(v_0) \xrightarrow{*} q_0(v_0{+}n\mathbf{e}_i)$ for each $i \in [1, d]$).

   In the negative case (when some component is not pumpable) **return** NO.

2. Let $\mathcal{V}'$ be the VASS $\mathcal{V}' = (\mathcal{V}^{\leftarrow})_{(n)} = (d, Q \times \{0, 1, \ldots, n{-}1\}^d, \mathcal{A}', T')$ {i.e., the reversed VASS modulo n, where n is computed in the point 1}.

   Create the configuration $(q_0, u_0)(v_0')$ of $\mathcal{V}'$ corresponding to the configuration $q_0(v_0)$ of $\mathcal{V}$ (hence $v_0 = u_0 + nv_0'$).

3. For each control state (q, u) of $\mathcal{V}'$ check if $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v_0')$ (in $\mathcal{V}'$), i.e., if $(q, u)(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(\bar{v})$ for some $\bar{v} \geq v_0'$.

   If the answer is negative for some (q, u), then **return** NO, otherwise (when all $(q, u)(\mathbf{0})$ cover $(q_0, u_0)(v_0')$) **return** YES.

Correctness and exponential-space complexity of ALG-CO-EMPT.

Lemma 4.8. Algorithm ALG-CO-EMPT satisfies its specification
(i.e., returns YES if $\left[q_0(v_0)\right\rangle = Q \times \mathbb{N}^d$, and NO otherwise).

Proof:

If ALG-CO-EMPT, when given $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $q_0(v_0)$, returns NO in the point 1, then for some $i \in [1, d]$ we have $q_0(v_0) \stackrel{*}{\not\rightarrow} q_0(v_0{+}x\mathbf{e}_i)$ for all $x \geq 1$; therefore the set $(Q \times \mathbb{N}^d) \smallsetminus \left[q_0(v_0)\right\rangle_{\mathcal{V}}$ is nonempty and even infinite.

Suppose now that the test in the point 1 has been positive, and a respective number n has been computed.

Assume first that $\left[q_0(v_0)\right\rangle_{\mathcal{V}} = Q \times \mathbb{N}^d$ and let us show that the algorithm returns YES. Let (q, u) be a control state of $\mathcal{V}' = (\mathcal{V}^{\leftarrow})_{(n)}$. Since $\left[q_0(v_0)\right\rangle_{\mathcal{V}} = Q \times \mathbb{N}^d$, we have $q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q(u)$. It follows that $(q, u)(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(v_0')$, which also entails that $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v_0')$ in $\mathcal{V}'$. We have proved that the algorithm returns YES.

Conversely, we assume that the algorithm returns YES and we prove that $\left[q_0(v_0)\right\rangle_{\mathcal{V}} = Q \times \mathbb{N}^d$. Let $q(v)$ be a configuration of $\mathcal{V}$ and let $(q, u)(v')$ be the corresponding configuration in $\mathcal{V}'$, i.e., $v = u{+}nv'$. Since $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v_0')$, there exists $\bar{v}_0' \geq v_0'$ such that $(q, u)(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(\bar{v}_0')$. It follows that $q_0(u_0 + n\bar{v}_0') \xrightarrow{*}_{\mathcal{V}} q(u)$. By monotonicity, we derive that

$$q_0(u_0 + n\bar{v}_0' + nv') \xrightarrow{*}_{\mathcal{V}} q(u + nv') = q(v).$$

By the definition of n, we get

$$q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q_0(v_0 + n(\bar{v}'_0 - v'_0) + nv') = q_0(u_0 + n\bar{v}'_0 + nv').$$

We have proved that $q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q(v)$, and thus $\left[q_0(v_0)\right\rangle_{\mathcal{V}} = Q \times \mathbb{N}^d$. □

We still need to show that ALG-CO-EMPT works in exponential space (Lemma 4.10). We first give a straightforward extension to VASSs of a result formulated in [6] for VASs.

Proposition 4.9. For any VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and any configurations $q_0(v_0)$ and $q_1(v_1)$, if $q_0(v_0) \xrightarrow{*} q_1(v_1)$ then $q_0(v_0) \xrightarrow{\alpha} q_1(\bar{v}_1)$ for some $\bar{v}_1 \geq v_1$ and $\alpha \in \mathcal{A}^*$ such that $|\alpha| < x^{(d+1)!}$, where $x = |Q| \cdot (1 + \|\mathcal{A}\| + \|v_1\|)$.

Proof:
The bounds given in [6] for VASs are easily extended to VASSs. Instead of giving a full proof, we only explain how to adapt the proof of [6] to deal with control states.

The notions of *paths*, of *i-bounded* sequences and of *i-covering* sequences from [6, pages 224–225] are extended with control states in the obvious way. For each $q \in Q$ and $v \in \mathbb{Z}^d$, define $m(i, q, v)$ to be the length of the shortest i-bounded, i-covering path in $\mathcal{V}$ starting from $q(v)$, with the convention that $m(i, q, v) = 0$ if there is none.

Now define $f(i) = \max\{m(i, q, v) \mid q \in Q, v \in \mathbb{Z}^d\}$. With the same reasoning as in [6, Lemma 3.4], we get that

$$f(0) \leq |Q| \text{ and } f(i{+}1) \leq |Q| \cdot (\max\{\|\mathcal{A}\|, \|v_1\|\} \cdot f(i))^{i+1} + f(i).$$

It follows that $f(i{+}1) \leq (xf(i))^{i+1}$. An immediate induction on i yields that $f(i) \leq x^{(i+1)!}$. In particular, we get that $m(d, q_0, v_0) \leq f(d) \leq x^{(d+1)!}$. Now, if $q_0(v_0) \xrightarrow{*} q_1(v_1)$ then $0 < m(d, q_0, v_0)$. This entails that $q_0(v_0) \xrightarrow{\alpha} q_1(\bar{v}_1)$ for some $\bar{v}_1 \geq v_1$ and some $\alpha \in \mathcal{A}^*$ whose length satisfies $|\alpha| = m(d, q_0, v_0) - 1 < x^{(d+1)!}$. □

Lemma 4.10. Algorithm ALG-CO-EMPT works (i.e., can be implemented to work) in exponential space.

Proof:
The point 1 of ALG-CO-EMPT, including the binary presentation of the computed number n, can be performed in exponential space, w.r.t. the size of the binary presentation of the input $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $q_0(v_0)$; this follows by Corollary 4.5.

The VASS $\mathcal{V}' = (\mathcal{V}^{\leftarrow})_{(n)}$ in the point 2 is not needed to be constructed explicitly. The algorithm creates the configuration $(q_0, u_0)(v'_0)$ and then stepwise generates the control states (q, u) ($q \in Q$, $u \in \{0, 1, \ldots, n{-}1\}^d$) of $\mathcal{V}'$ and checks if $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v'_0)$ in $\mathcal{V}'$.

It thus suffices to show that checking if $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v'_0)$ (i.e., if $(q, u)(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(\bar{v})$ for some $\bar{v} \geq v'_0$) can be done in exponential space (w.r.t. the binary presentation of $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $q_0(v_0)$). By Prop. 4.9, it is enough to search for witnesses of coverability $(q, u)(\mathbf{0}) \xrightarrow{\alpha} (q_0, u_0)(\bar{v})$ of length $|\alpha| < x^{(d+1)!}$, where $x = |Q|n^d \cdot (1 + \|\mathcal{A}'\| + \|v'_0\|)$. Since n is at most double-exponential, $x^{(d+1)!}$ is also at most double-exponential. As in the proof of Corollary 4.5, the algorithm just traverses along (a guessed bounded) α, keeping only the current configuration in memory; so exponential space is sufficient. □

Algorithm deciding SIC co-finiteness.

We will adjust the algorithm ALG-CO-EMPT so that, given $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and $q_0(v_0)$, it answers YES if, and only if, the set $(Q \times \mathbb{N}^d) \smallsetminus \big[q_0(v_0)\big\rangle$ is finite; this can happen even if some $(q, u)(\mathbf{0})$ does not cover $(q_0, u_0)(v'_0)$ in $\mathcal{V}'$. Informally speaking, it suffices to check if $(q, u)(\mathbf{0})$ covers $(q_0, u_0)(v'_0)$ whenever we "ignore" one-component of $\mathbf{0}$, making it "arbitrarily large".

By ω we denote an "infinite amount", satisfying $z < \omega$ and $z+\omega = \omega+z = \omega$ for all $z \in \mathbb{Z}$. Given $\mathcal{V} = (d, Q, \mathcal{A}, T)$, by the set of *extended configurations* we mean the set $Q \times (\mathbb{N} \cup \{\omega\})^d$; the relations $q(v) \xrightarrow{\mathbf{a}} q'(v')$, $q(v) \xrightarrow{\alpha} q'(v')$ ($\alpha \in \mathcal{A}^*$), and $q(v) \xrightarrow{*} q'(v')$ are then naturally extended to the relations on $Q \times (\mathbb{N} \cup \{\omega\})^d$. (Hence, e.g., if $q(v) \xrightarrow{\alpha} q'(v')$ then $v(i) = \omega$ if, and only if, $v'(i) = \omega$, for any $i \in [1, d]$.)

Let us now consider the following algorithm.

Algorithm ALG-CO-FINIT

Input: a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a configuration $q_0(v_0)$.

Output: YES if $(Q \times \mathbb{N}^d) \smallsetminus \big[q_0(v_0)\big\rangle$ is finite, and NO otherwise.

1. As in ALG-CO-EMPT.

2. As in ALG-CO-EMPT.

3. For each control state (q, u) of $\mathcal{V}'$ and each $i \in [1, d]$

   check if $(q, u)(\omega\mathbf{e}_i)$ covers $(q_0, u_0)(v'_0)$ (in $\mathcal{V}'$), i.e., if

   $$(q, u)(\omega\mathbf{e}_i) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(\bar{v}) \text{ for some } \bar{v} \geq v'_0;$$

   by $\omega\mathbf{e}_i$ we denote the d-dimensional vector where the i-th component is ω and the other components are zero.

   If the answer is negative for some (q, u) and $i \in [1, d]$, then **return** NO, otherwise (when all $(q, u)(\omega\mathbf{e}_i)$ cover $(q_0, u_0)(v'_0)$) **return** YES.

Correctness and exponential-space complexity of ALG-CO-FINIT.

Lemma 4.11. Algorithm ALG-CO-FINIT satisfies its specification (i.e., returns YES if $(Q \times \mathbb{N}^d) \smallsetminus \big[q_0(v_0)\big\rangle$ is finite, and NO otherwise).

Proof:

We reason analogously as in the proof of Lemma 4.8. We have already noted that if NO is returned in the point 1, then $(Q \times \mathbb{N}^d) \smallsetminus \big[q_0(v_0)\big\rangle$ is infinite.

Assume first that $\big[q_0(v_0)\big\rangle_{\mathcal{V}}$ is co-finite and let us show that the algorithm returns YES. Let (q, u) be a control state of $\mathcal{V}'$ and let $i \in [1, d]$. Since $\big[q_0(v_0)\big\rangle_{\mathcal{V}}$ is co-finite, there is a number $x \geq 1$ such that $q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q(u+nx\mathbf{e}_i)$. It follows that $(q, u)(x\mathbf{e}_i) \xrightarrow{*}_{\mathcal{V}'} (q_0, u_0)(v'_0)$, which also entails that $(q, u)(\omega\mathbf{e}_i)$ covers $(q_0, u_0)(v'_0)$ in $\mathcal{V}'$. We have proved that the algorithm returns YES.

Assume now that the algorithm returns YES and let us prove that $\big[q_0(v_0)\big\rangle_{\mathcal{V}}$ is co-finite. Since $(q, u)(\omega\mathbf{e}_i)$ covers $(q_0, u_0)(v'_0)$ in $\mathcal{V}'$ for every control state (q, u) of $\mathcal{V}'$ and for every $i \in [1, d]$, there is a (large enough) number x such that $(q,u)(x\mathbf{e}_i)$ covers $(q_0,u_0)(v'_0)$ for every control state (q,u) and every $i \in [1,d]$. Below we prove that every configuration $q(v)$ of $\mathcal{V}$ such that $\|v\| \geq nx$ is reachable from $q_0(v_0)$; this will entail that $\big[q_0(v_0)\big\rangle_{\mathcal{V}}$ is co-finite (i.e., $(Q \times \mathbb{N}^d) \smallsetminus \big[q_0(v_0)\big\rangle_{\mathcal{V}}$ is finite).

We thus fix an arbitrary $q(v)$ and $i \in [1,d]$ such that $v(i) \geq nx$. Let $(q,u)(v')$ be the configuration of $\mathcal{V}'$ corresponding to $q(v)$; hence $v = u + nv'$. Since $(q,u)(x\mathbf{e}_i)$ covers $(q_0,u_0)(v'_0)$ in $\mathcal{V}'$, there is $\bar{v}'_0 \geq v'_0$ such that

$$(q,u)(x\mathbf{e}_i) \xrightarrow{*}_{\mathcal{V}'} (q_0,u_0)(\bar{v}'_0)\text{; this entails } q_0(u_0{+}n\bar{v}'_0) \xrightarrow{*}_{\mathcal{V}} q(u{+}nx\mathbf{e}_i).$$

Since $v(i) \geq nx$, we have $v' - x\mathbf{e}_i \geq \mathbf{0}$. By monotonicity we derive

$$q_0(u_0{+}n\bar{v}'_0{+}n(v'{-}x\mathbf{e}_i)) \xrightarrow{*}_{\mathcal{V}} q(u{+}nx\mathbf{e}_i{+}n(v'{-}x\mathbf{e}_i)) = q(v).$$

By the definition of n, we get

$$q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q_0(v_0{+}n(\bar{v}'_0{-}v'_0){+}n(v'{-}x\mathbf{e}_i)) = q_0(u_0{+}n\bar{v}'_0{+}n(v'{-}x\mathbf{e}_i)).$$

Hence we indeed have $q_0(v_0) \xrightarrow{*}_{\mathcal{V}} q(v)$. □

Lemma 4.12. Algorithm ALG-CO-FINIT works (i.e., can be implemented to work) in exponential space.

Proof:
This is analogous to the proof of Lemma 4.10. We just note that deciding if $(q,u)(\omega\mathbf{e}_i)$ covers $(q_0,u_0)(v'_0)$ is even easier than deciding if $(q,u)(\mathbf{0})$ covers $(q_0,u_0)(v'_0)$, since the i-th component can be simply ignored. □

5. Applications of the Co-Emptiness Problem

A motivation for the study in this paper has been the decidability proof for structural liveness in [1], which is based on a particular version of the co-emptiness problem. We now give more details (in the framework of VASSs, which is equivalent to the framework of Petri nets used in [1]), and some partial complexity results. The main aim is to attract a further research effort on this topic, since the complexity of various related problems has not been answered. In particular, we have no nontrivial complexity bounds for the structural liveness problem (besides its decidability).

Assuming a VASS $\mathcal{V} = (d,Q,\mathcal{A},T)$, we are now particularly interested in the co-emptiness of $\big[D\big\rangle_{\mathcal{V}}$ for downward closed sets $D \subseteq Q \times \mathbb{N}^d$, which constitute a subclass of semilinear sets. We use the notation

$$\downarrow C = \{q(v) \mid v \leq v' \text{ for some } q(v') \in C\}$$

for the *downward closure* of a set $C \subseteq Q \times \mathbb{N}^d$ (of configurations of $\mathcal{V}$). We say that $C \subseteq Q \times \mathbb{N}^d$ is *downward closed* if $\downarrow C = C$. We write just $\downarrow q(v)$ instead of $\downarrow \{q(v)\}$.

Downward closed sets are semilinear since each such set can be presented as

$$\downarrow q_1(\bar{v}_1) \cup \downarrow q_2(\bar{v}_2) \cup \cdots \cup \downarrow q_m(\bar{v}_m)$$

for some $m \in \mathbb{N}$ and $\bar{v}_i \in (\mathbb{N} \cup \{\omega\})^d$ ($i \in [1,m]$), where we put

$$\downarrow q(\bar{v}) = \{q(v) \mid v \leq \bar{v}, v \in \mathbb{N}^d\}.$$

(Recall that $k < \omega$ for each $k \in \mathbb{N}$.)

Later we use another natural presentation of downward closed sets: for each $q \in Q$ we provide a constraint in the form of a (finite) conjunction of disjunctions of atomic constraints of the form $v(i) \leq c$ where $i \in [1,d]$ and $c \in \mathbb{N}$ (then $q(v) \in Q \times \mathbb{N}^d$ is in the set if, and only if, v satisfies the constraint associated with q).

The *DCIS co-emptiness problem* where "DCIS" stands for "Downward Closed Initial Sets of configurations" (i.e., given $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a downward closed set $D \subseteq Q \times \mathbb{N}^d$, is $\left[D\right\rangle_{\mathcal{V}} = Q \times \mathbb{N}^d$?) is decidable by Theorem 3.4 (and the fact that D is semilinear). The complexity is open, even the reductions to/from the reachability problem are unclear. Now we explain the previously mentioned motivation for such studies.

Liveness of transitions and configurations.

We recall some standard definitions and facts. Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$,

- a transition $t \in T$ is *enabled* in a configuration $q(v)$ if t is of the form $t : q \xrightarrow{\mathbf{a}} q'$ and $v + \mathbf{a} \geq \mathbf{0}$;
- a transition t is *live* in $q(v)$ if for every $\bar{q}(\bar{v}) \in \left[q(v)\right\rangle$ there is $q'(v') \in \left[\bar{q}(\bar{v})\right\rangle$ such that t is enabled in $q'(v')$;
- a transition t is *dead* in $q(v)$ if there is no $q'(v') \in \left[q(v)\right\rangle$ such that t is enabled in $q'(v')$.

We note that t is not live in $q(v)$ if, and only if, t is dead in some $q'(v') \in \left[q(v)\right\rangle$.

The next proposition (which also defines $\mathcal{D}_{t,\mathcal{V}}$ and $\mathcal{D}_{\mathcal{V}}$) is obvious, due to monotonicity.

Proposition 5.1. Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, for each $t \in T$ the set

$$\mathcal{D}_{t,\mathcal{V}} = \{q(v) \mid t \text{ is dead in } q(v)\}$$

is downward closed. Hence also the set

$$\mathcal{D}_{\mathcal{V}} = \{q(v) \mid \text{some } t \in T \text{ is dead in } q(v)\} = \textstyle\bigcup_{t \in T} \mathcal{D}_{t,\mathcal{V}}$$

is downward closed.

Example 5.2. Consider again the VASS depicted on the left of Figure 1. We have observed in Example 2.2 that $\left[A(1,0)\right\rangle = \{A(1+2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\} \cup \{B(2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}$. It follows that no transition is dead in $A(1,0)$. Similarly, no transition is dead in $B(0,0)$ since $\left[B(0,0)\right\rangle = \left[A(1,0)\right\rangle$. We derive, by monotonicity, that $\mathcal{D} \subseteq \{A(0,n) \mid n \in \mathbb{N}\}$. Conversely, for every $n \in \mathbb{N}$, the transition $A \xrightarrow{(-1,0)} B$ is dead in $A(0,n)$ since $\left[A(0,n)\right\rangle = \{A(0,n)\}$. In fact, all transitions are dead in $A(0,n)$. We have shown that $\mathcal{D} = \{A(0,n) \mid n \in \mathbb{N}\}$.

Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, a *configuration* $q(v)$ is *live* if each $t \in T$ is live in $q(v)$, i.e., if $q(v) \not\xrightarrow{*}_{\mathcal{V}} \mathcal{D}_{\mathcal{V}}$. A VASS $\mathcal{V}$ is *structurally live* if it has a live configuration, hence if the set

$$\mathcal{L}_{\mathcal{V}} = \{q(v) \mid q(v) \text{ is a live configuration of } \mathcal{V}\}$$

is nonempty. While the membership problem for ($\mathcal{D}_{t,\mathcal{V}}$ or) $\mathcal{D}_{\mathcal{V}}$ is essentially a version of (the complement of) the coverability problem, which also allows to construct a natural presentation of the (downward closed) sets $\mathcal{D}_{t,\mathcal{V}}$ and $\mathcal{D}_{\mathcal{V}}$, the membership problem for $\mathcal{L}_{\mathcal{V}}$ is close to the *reachability problem* as was already noted by Hack [20] long time ago.

The set $\mathcal{L}_{\mathcal{V}}$ is indeed more involved than $\mathcal{D}_{\mathcal{V}}$; it is obviously not downward closed but it is not upward closed either (in general), and it can be even non-semilinear; we can refer to [1] for a concrete example, as well as for the following idea of decidability.

The structural liveness can be decided as follows. We recall the reversed VASS $\mathcal{V}^{\leftarrow}$, and note that $\mathcal{V}$ is not structurally live if, and only if, $\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}}$ is co-empty:

Proposition 5.3. For any VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ we have

$$\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}} = (Q \times \mathbb{N}^d) \smallsetminus \mathcal{L}_{\mathcal{V}}.$$

Hence $\mathcal{V}$ is not structurally live if, and only if, $\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}} = Q \times \mathbb{N}^d$.

Proof:
We recall that $q(v)$ is not live if, and only if, $\big[q(v)\big\rangle_{\mathcal{V}} \cap \mathcal{D}_{\mathcal{V}} \neq \emptyset$ (i.e., iff $q(v) \xrightarrow{*}_{\mathcal{V}} q'(v')$ where some $t \in T$ is dead in $q'(v')$). Hence $q(v)$ is not live if, and only if, $q'(v') \xrightarrow{*}_{\mathcal{V}^{\leftarrow}} q(v)$ for some $q'(v') \in \mathcal{D}_{\mathcal{V}}$ (using Proposition 4.6).

Therefore $\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}} = (Q \times \mathbb{N}^d) \smallsetminus \mathcal{L}_{\mathcal{V}}$. □

Proposition 5.3 allows us to decide structural liveness of a given VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ by a reduction to the co-emptiness problem, using the above-mentioned constructability of $\mathcal{D}_{\mathcal{V}}$.

Example 5.4. Let $\mathcal{V}$ denote the VASS depicted on the left of Figure 1. We have shown in Example 5.2 that $\mathcal{D}_{\mathcal{V}} = \{A(0, n) \mid n \in \mathbb{N}\}$. The reversed VASS $\mathcal{V}^{\leftarrow}$ is depicted on the right of Figure 1. It is routinely checked that the reachability set $\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}}$ satisfies

$$\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}} \;=\; \{A(2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\} \cup \{B(1{+}2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}.$$

It follows from Proposition 5.3 that the set $\mathcal{L}_{\mathcal{V}}$ of live configurations of $\mathcal{V}$ satisfies

$$\mathcal{L}_{\mathcal{V}} \;=\; \{A(1 + 2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\} \cup \{B(2n_1, n_2) \mid n_1, n_2 \in \mathbb{N}\}.$$

In particular, $\mathcal{V}$ is structurally live since $\big[\mathcal{D}_{\mathcal{V}}\big\rangle_{\mathcal{V}^{\leftarrow}}$ is not co-empty.

Structural deadlock-freedom and DCIS co-emptiness.

We have shown that the complementary problem of the structural liveness problem (hence "non structural liveness") can be reduced to the DCIS co-emptiness problem (with downward closed sets of initial configurations). However, we have no reduction from the latter problem to the former.

We now show that a special form of structural liveness, namely structural deadlock-freedom, is closely related to the DCIS co-emptiness problem. We use the previously mentioned presentation of downward closed sets by conjunctions of disjunctions of atomic constraints of the form $v(i) \leq c$ (for each $q \in Q$).

Given a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, a configuration $q(v)$ is *deadlock-free* if every configuration in $\left[q(v)\right\rangle$ enables some transition. A VASS $\mathcal{V}$ is *structurally deadlock-free* if it has a deadlock-free configuration. The *structural deadlock-freedom problem* asks, given a VASS $\mathcal{V}$, if $\mathcal{V}$ is structurally deadlock-free. The rest of this section is devoted to the proof of the following theorem.

Theorem 5.5. The complementary problem of the structural deadlock-freedom problem is logspace interreducible with the DCIS co-emptiness problem. This entails that the structural deadlock-freedom problem is decidable.

We have already noted that the DCIS co-emptiness problem is decidable. The interreducibility claimed in Theorem 5.5 is proven in the rest of this section. We first define the set

$$\mathcal{S}_{\mathcal{V}} = \{q(v) \mid \text{no } t \in T \text{ is enabled in } q(v)\}$$

of "sink configurations" or "deadlocks" (hence $\mathcal{S}_{\mathcal{V}} = \bigcap_{t \in T} \mathcal{D}_{t,\mathcal{V}}$). It is obvious that $\mathcal{S}_{\mathcal{V}}$ is the downward closed set described so that to each $q \in Q$ we attach the constraint

$$\bigwedge_{(q \xrightarrow{\mathbf{a}} q') \in T} \quad \bigvee_{\substack{i \in [1,d] \\ \mathbf{a}(i) < 0}} \quad v(i) \leq -\mathbf{a}(i) - 1\,.$$

This presentation of $\mathcal{S}_{\mathcal{V}}$ can be clearly constructed in logarithmic space, when given a VASS $\mathcal{V}$. Hence Proposition 5.6 entails the "left-to-right" reduction in Theorem 5.5 (recall that $\mathcal{V}^{\leftarrow}$ denotes the reversed VASS of $\mathcal{V}$). The other reduction is shown by Proposition 5.7.

Proposition 5.6. A VASS $\mathcal{V}$ is not structurally deadlock-free if, and only if, $\left[\mathcal{S}_{\mathcal{V}}\right\rangle_{\mathcal{V}^{\leftarrow}}$ is co-empty.

Proof:
We consider a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$, and observe that $q(v)$ is not deadlock-free if, and only if, $\left[q(v)\right\rangle_{\mathcal{V}} \cap \mathcal{S}_{\mathcal{V}} \neq \emptyset$. Hence $q(v)$ is not deadlock-free if, and only if, $q(v) \in \left[\mathcal{S}_{\mathcal{V}}\right\rangle_{\mathcal{V}^{\leftarrow}}$ (using Proposition 4.6). It follows that $\mathcal{V}$ is not structurally deadlock-free if, and only if, $\left[\mathcal{S}_{\mathcal{V}}\right\rangle_{\mathcal{V}^{\leftarrow}} = Q \times \mathbb{N}^d$. $\square$

Proposition 5.7. Given a VASS $\mathcal{V}$ and a downward-closed set D of configurations, we can construct, in logarithmic space, a VASS $\mathcal{V}'$ such that $\left[D\right\rangle_{\mathcal{V}}$ is co-empty if, and only if, $\mathcal{V}'$ is not structurally deadlock-free.

Proof:
Let us assume a VASS $\mathcal{V} = (d, Q, \mathcal{A}, T)$ and a downward-closed set D of configurations given, for each $q \in Q$, by conjunctions of disjunctions of atomic constraints of the form $v(i) \leq c$. By negating these formulas, we derive, in logarithmic space, a collection $(B_q)_{q \in Q}$ of finite subsets of $\mathbb{N}^d$ such that

$$(Q \times \mathbb{N}^d) \setminus D = \{q(v) \mid v \geq b \text{ for some } b \in B_q\}.$$

(Hence $(B_q)_{q \in Q}$ represents the upward closed complement of D.)

We now define the VASS $\hat{\mathcal{V}} = (d, \hat{Q}, \hat{\mathcal{A}}, \hat{T})$ as follows:

a) $\hat{Q} = Q \cup \{(q, b) \mid q \in Q, b \in B_q\}$.

b) $\hat{T}$ consists of the following transitions:

   i. $q \xrightarrow{-b} (q, b)$ and $(q, b) \xrightarrow{b} q$ for all $q \in Q$, $b \in B_q$, and

   ii. $(q, b) \xrightarrow{\mathbf{a}+b} q'$ for all $(q \xrightarrow{\mathbf{a}} q') \in T$ and $b \in B_q$.

c) $\hat{\mathcal{A}} = \{\hat{\mathbf{a}} \mid q \xrightarrow{\hat{\mathbf{a}}} q' \in \hat{T} \text{ for some } q, q' \in \hat{Q}\}$.

It is obvious that for all configurations $q(v)$ and $q'(v')$ of $\mathcal{V}$ we have that

$$q(v) \xrightarrow{*}_{\hat{\mathcal{V}}} q'(v') \text{ implies } q(v) \xrightarrow{*}_{\mathcal{V}} q'(v')$$

but the converse does not hold in general. We will show that

$$\left[D\right\rangle_{\mathcal{V}^{\leftarrow}} = Q \times \mathbb{N}^d \text{ if, and only if, } \hat{\mathcal{V}} \text{ is not structurally deadlock-free.}$$

The proof will be finished, by taking $\mathcal{V}' = \widehat{\mathcal{V}^{\leftarrow}}$ (and noting that $(\mathcal{V}^{\leftarrow})^{\leftarrow} = \mathcal{V}$).

($\Rightarrow$) Assume $\left[D\right\rangle_{\mathcal{V}^{\leftarrow}} = Q \times \mathbb{N}^d$. Observe that $(q, b)(v) \xrightarrow{b} q(v{+}b)$ in $\hat{\mathcal{V}}$ for every $(q, b) \in \hat{Q}$ and $v \in \mathbb{N}^d$. We now show that no configuration $q(v)$ with $q \in Q$ is deadlock-free in $\hat{\mathcal{V}}$, which clearly entails that $\hat{\mathcal{V}}$ is not structurally deadlock-free.

We fix some $q(v) \in Q \times \mathbb{N}^d$. Since $q(v) \in \left[D\right\rangle_{\mathcal{V}^{\leftarrow}}$, in $\mathcal{V}$ we have

$$q(v) = q_0(v_0) \xrightarrow{\mathbf{a}_1} q_1(v_1) \xrightarrow{\mathbf{a}_2} \cdots q_{m-1}(v_{m-1}) \xrightarrow{\mathbf{a}_m} q_m(v_m) \in D$$

for some $\mathbf{a}_1, \ldots, \mathbf{a}_m$ in $\mathcal{A}$ and $q_0(v_0), \ldots, q_m(v_m)$ in $Q \times \mathbb{N}^d$ (recall Proposition 4.6). Moreover, we may assume w.l.o.g. that $q_i(v_i) \notin D$ for all $i \in [0, m{-}1]$. So for each $i \in [0, m{-}1]$ there is $b_i \in B_{q_i}$ such that $v_i \geq b_i$. We derive that

$$q_i(v_i) \xrightarrow{-b_i} (q_i, b_i)(v_i{-}b_i) \xrightarrow{\mathbf{a}_{i+1}+b_i} q_{i+1}(v_{i+1}) \text{ in } \hat{\mathcal{V}}$$

for all $i \in [0, m{-}1]$. It follows that $q(v) \xrightarrow{*} q_m(v_m)$ in $\hat{\mathcal{V}}$. Since $q_m(v_m) \in D$ then $v_m \not\geq b$ for all $b \in B_{q_m}$; hence no transition of $\hat{\mathcal{V}}$ is enabled in $q_m(v_m)$, and $q(v)$ is thus not deadlock-free in $\hat{\mathcal{V}}$.

($\Leftarrow$) Assume that $\hat{\mathcal{V}}$ is not structurally deadlock-free. We fix a configuration $q(v)$ of $\mathcal{V}$ and prove that $q(v) \in \left[D\right\rangle_{\mathcal{V}^{\leftarrow}}$. Since $q(v)$ is also a configuration of $\hat{\mathcal{V}}$, it is not deadlock-free in $\hat{\mathcal{V}}$. So there is a configuration $q'(v')$ of $\hat{\mathcal{V}}$ such that $q(v) \xrightarrow{*}_{\hat{\mathcal{V}}} q'(v')$ and no transition $t \in \hat{T}$ is enabled in $q'(v')$. Since $\hat{\mathcal{V}}$ contains the transition $(q, b) \xrightarrow{b} q$ for every $q \in Q$ and $b \in B_q$, we get that $q' \in Q$. No transition $q' \xrightarrow{-b} (q', b)$ of $\hat{T}$ is enabled in $q'(v')$, so $v' \ngeq b$ for every $b \in B_{q'}$. It follows that $q'(v') \in D$. Since $q(v) \xrightarrow{*}_{\hat{\mathcal{V}}} q'(v')$ implies $q(v) \xrightarrow{*}_{\mathcal{V}} q'(v')$, we get $q(v) \xrightarrow{*}_{\mathcal{V}} D$, i.e., $q(v) \in \left[D\right\rangle_{\mathcal{V}^{\leftarrow}}$. □

6. Conclusion

Motivated by the structural liveness problem for VASS, whose computational complexity is still open, in this paper we have introduced and studied the co-emptiness problem and the co-finiteness problem for VASSs. The complexity of the co-emptiness and co-finiteness problems in the case of single initial configurations has been clarified, but the complexity of general versions has been left open, even w.r.t. reductions to/from the reachability problem. This requires further work, in particular with an eye to the applications aiming to clarify structural liveness properties of VASSs, or equivalently of Petri nets.

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