# Combinatorial algorithms for binary operations on LR-tableaux with entries equal to 1 with applications to nilpotent linear operators 

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#### Abstract

In the paper we investigate an algorithmic associative binary operation $*$ on the set $\mathcal{L R}_{1}$ of Littlewood-Richardson tableaux with entries equal to one. We extend $*$ to an algorithmic nonassociative binary operation on the set $\mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$ and show that it is equivalent to the operation of taking the generic extensions of objects in the category of homomorphisms from semisimple nilpotent linear operators to nilpotent linear operators. Thus we get a combinatorial algorithm computing generic extensions in this category.


Key words: Littlewood-Richardson tableaux, partitions, invariant subspaces, nilpotent linear operators, generic extensions, pickets, degeneration partial order, combinatorial algorithms

## 1. Introduction

The main aim of the paper is to generalize results presented in [6], where an associative operation $*$ on the set $\mathcal{L} \mathcal{R}_{1}$ of Littlewood-Richardson tableaux (LR-tableaux) with entries equal to one is defined. It is proved that $*$ is associative and it is equivalent to the operation of taking the generic extension of semisimple invariant subspaces of nilpotent linear operators, see [6, Lemma 5.12].
We extend the operation $*$ to a nonassociative operation on the set $\mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$ and investigate its properties. In particular, in Theorem 4.6, we prove that this operation is equivalent to the operation of taking the generic extensions of objects in the category $\mathcal{H}_{1}$ of homomorphisms from semisimple nilpotent linear operators to nilpotent linear operators. In particular this gives a combinatorial algorithm that applying operations on LR-tableaux computes generic extensions in $\mathcal{H}_{1}$.
We are motivated by results presented in [8, 9], where there are investigated relationships between Littlewood-Richardson tableaux and geometric properties of invariant subspaces of nilpotent linear operators. It is proved there that these relationships are deep and interesting. On the other hand, in [12], the existence of generic extensions for Dynkin quivers is proved and their connections with Hall algebras are investigated. Moreover, by results presented in [2, 3, 4, 12] generic extensions of nilpotent linear operators exist and the operation of taking the generic extension provides the set of all isomorphism classes of nilpotent linear operators with a monoid structure. There are many results concerning this monoid and its properties (see [3, 4, 5, 7, 12]).
The paper is organized as follows.

- In Section 2 we define algorithmically a binary operation $*$ on the set $\mathcal{L R}{ }_{1} \times \mathbb{N}$ and illustrate it by examples. Moreover we present an example showing that $*$ is non-associative.
- Section 3 contains basic definitions and facts concerning the category of homomorphisms between nilpotent linear operators. We also recall there definitions of generic extensions and hom-order.
- In Section 4 we prove that the operation $*$ is equivalent to the operation of taking generic extensions in some subcategory of the category of homomorphisms between nilpotent linear operators (Theorem 4.6).


## 2. Operations on $\mathcal{L \mathcal { R } _ { 1 }}$ and on $\mathcal{L \mathcal { R } _ { 1 } \times \mathbb { N }}$

The aim of this section is to present algorithmic definitions of combinatorial binary operations on the sets $\mathcal{L} \mathcal{R}_{1}$ and $\mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$. Later these operations are applied to computing generic extensions in some category of homomorphisms of nilpotent linear operators.
Let $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}, \ldots\right)$ be a partition (i.e. a sequence of non-negative integers containing only finitely many non-zero terms and such that $\alpha_{1} \geq \alpha_{2} \geq \ldots \geq \alpha_{n} \geq \ldots$ ). Denote by $\bar{\alpha}=\left(\bar{\alpha}_{1}, \ldots, \bar{\alpha}_{n}, \ldots\right)$ the dual partition of $\alpha$, i.e. $\bar{\alpha}_{i}=\#\left\{j ; \alpha_{j} \geq i\right\}$, where $\# X$ denotes the cardinality of a finite set $X$. Moreover let $|\alpha|=\alpha_{1}+\alpha_{2}+\ldots$. Given partitions $\alpha, \beta$ we denote by $\alpha \cup \beta$ the union of these partitions, i.e. the multiset of parts of $\alpha \cup \beta$ is the union of multisets of parts of $\alpha$ and $\beta$.
Let $(\alpha, \beta, \gamma)$ be a partition triple such that $\alpha_{1} \leq 1$. An LR-tableau of type $(\alpha, \beta, \gamma)$ is a skew diagram of shape $\beta \backslash \gamma$ with $|\alpha|$ entries all equal to 1 and such that $\beta \backslash \gamma$ is a horizontal strip, i.e. $\gamma_{i} \leq \beta_{i} \leq \gamma_{i}+1$ for all $i$. By $\mathcal{L} \mathcal{R}_{1}$ we denote the set of all LR-tableaux with entries equal to 1 . Note that an LR-tableau of type ( $\alpha, \beta, \gamma$ ) and with entries equal to one is uniquely determined by the partitions $\beta, \gamma$. We will identify an LR-tableau $X \in \mathcal{L} \mathcal{R}_{1}$ with the corresponding pair $\left(\gamma^{X}, \beta^{X}\right)$ of partitions.
By $\emptyset$ we denote the LR tableau such that $\beta^{\emptyset}=\gamma^{\emptyset}=(0)$.
Example 1. The following LR-tableau with entries equal to 1 is uniquely determined by partitions $\beta^{X}=(7,7,5,2,2,1)$ and $\gamma^{X}=(7,6,4,1,1,1)$ :


Below we present algorithmic definition of operations

$$
*: \mathcal{L R}_{1} \times \mathcal{L R}_{1} \rightarrow \mathcal{L} \mathcal{R}_{1}
$$

and

$$
*:\left(\mathcal{L R}_{1} \times \mathbb{N}\right) \times\left(\mathcal{L R}_{1} \times \mathbb{N}\right) \rightarrow\left(\mathcal{L R}_{1} \times \mathbb{N}\right)
$$

The following algorithm is given in [6] and defines the operation $*: \mathcal{L R}_{1} \times \mathcal{L R}_{1} \rightarrow \mathcal{L} \mathcal{R}_{1}$.

Algorithm 2.1. Input. $X, Y \in \mathcal{L R}_{1}$.
Output. $Z=Y * X \in \mathcal{L} \mathcal{R}_{1}$.

1. $\gamma^{Z}=\gamma^{X}+\gamma^{Y}$
2. set $n_{0}=0$
3. for any $i=1, \ldots, \min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}$, do
(a) $\operatorname{put} \beta_{i}^{Z}=\beta_{i}^{X}+\gamma_{i}^{Y}$
(b) if $\beta_{i}^{Y} \neq \gamma_{i}^{Y}$, then put $n_{i}=n_{i-1}+1$; else put $n_{i}=n_{i-1}$
4. if $\overline{\beta_{1}^{X}}>\min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}$, then for $i=\min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}+1, \ldots, \overline{\beta_{1}^{X}}$ put

$$
\beta_{i}^{Z}=\beta_{i}^{X} \text { and } n_{i}=n_{i-1},
$$

else for $i=\min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}+1, \ldots, \overline{\beta_{1}^{Y}}$ we set
(a) if $\left(\gamma_{i}^{Y}=\beta_{i}^{Y}\right.$ and $\left.n>0\right)$ then $\left(\beta_{i}^{Z}=\beta_{i}^{Y}+1\right.$ and $\left.n_{i}=n_{i-1}-1\right)$; else ( $\beta_{i}^{Z}=\beta_{i}^{Y}$ and $\left.n_{i}=n_{i-1}\right)$
5. We set

$$
\beta^{Z}=\beta^{Z} \cup \alpha
$$

where $\alpha=(1,1, \ldots, 1)$ is a partition with $n_{s}$ copies of 1 , where $s=\max \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}$.
Example 2. Let

Then $\beta^{X}=(5,4,3,3,1), \gamma^{X}=(4,4,2,2)$, $\beta^{Y}=(4,3,2,2,1,1), \gamma^{Y}=(3,3,2,1,1,1)$. The Step 1 of Algorithm 2.1 gives

$$
\gamma^{Z}=\gamma^{X}+\gamma^{Y}=(7,7,4,3,1,1)
$$

In our case $\min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}=5$ and therefore, for $i=1,2,3,4,5$, the Step 3 of Algorithm 2.1 sets $\beta_{i}^{Z}=\beta_{i}^{X}+\gamma_{i}^{Y}$. We have

$$
\beta_{1}^{Z}=8, \beta_{2}^{Z}=7, \beta_{3}^{Z}=5, \beta_{4}^{Z}=4, \beta_{3}^{Z}=2 .
$$

Note that $n_{5}=2$.
For $i>5=\overline{\beta_{1}^{X}}=\min \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}$ the Step 4 of the algorithm gives:

$$
\beta_{6}^{Z}=\beta_{6}^{Y}+1=2 \text { and } n_{6}=n_{5}-1=1 \text {, because } \gamma_{6}^{Y}=\beta_{6}^{Y} .
$$

Coefficients of partition $\beta^{Z}$ for $i=1,2, \ldots, \max \left\{\overline{\beta_{1}^{X}}, \overline{\beta_{1}^{Y}}\right\}=s$ are constructed, but $n_{s}=$ $1 \neq 0$. Therefore the step 5 of Algorithm 2.1 sets $\beta^{Z}=\beta^{Z} \cup(1)$.

Finally, we get $\beta^{Z}=(8,7,5,4,2,2,1)$, $\gamma^{Z}=(7,7,4,3,1,1)$, and


The following algorithm is the first part of the definition of $*:\left(\mathcal{L R}{ }_{1} \times \mathbb{N}\right) \times\left(\mathcal{L R}_{1} \times \mathbb{N}\right) \rightarrow$ $\left(\mathcal{L R}_{1} \times \mathbb{N}\right)$..

Algorithm 2.2. Input. $X \in \mathcal{L} \mathcal{R}_{1} . n \in \mathbb{N}$.
Output. $(Z, m)=(\emptyset, n) *(X, 0) \in \mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$.

1. let $L=[]$ be an empty list
2. for any $i=1, \ldots, \bar{\beta}_{1}^{X}$ do
(a) if $\beta_{i}^{X}=\gamma_{i}^{X}$, then add $i$ as the last element of the list $L$
3. for any $i=\bar{\beta}_{1}^{X}, \ldots, 1$, do
(a) if $i \in L$ and $n>0$, then set

$$
\beta_{i}^{Z}=\beta_{i}^{X}, \gamma_{i}^{Z}=\gamma_{i}^{Z}-1 \text { and } n=n-1,
$$

else set

$$
\beta_{i}^{Z}=\beta_{i}^{X} \text { and } \gamma_{i}^{Z}=\gamma_{i}^{Z}
$$

4. the result is $(Z, n)$.

Example 3. Let

$$
X=\begin{array}{|l|l|l}
\hline & & \\
\hline & & \\
\hline & & 1 \\
\hline 1 & & \\
\hline
\end{array}
$$

Note that $\beta^{X}=(5,4,3,3,1)$ and $\gamma^{X}=(4,4,3,2,1)$. The list $L$ has a form $L=[2,3,5]$.

1. First we illustrate the algorithm for $n=2$.

Note that $5 \in L$ and $n>0$, so we put

$$
\beta_{5}^{Z}=\beta_{5}^{X}=1, \quad \gamma_{5}^{Z}=\gamma_{5}^{Z}-1=0
$$

and $n:=n-1=1$. Since $4 \notin L$, we put

$$
\beta_{4}^{Z}=\beta_{4}^{X}=3 \text { and } \gamma_{4}^{Z}=\gamma_{4}^{X}=2 .
$$

We have $3 \in L$ and $n>0$, then

$$
\beta_{3}^{Z}=\beta_{3}^{X}=3, \quad \gamma_{3}^{Z}=\gamma_{3}^{Z}-1=2
$$

and $n:=n-1=0$. Since $n=0$, we have:

$$
\begin{aligned}
& \beta_{2}^{Z}=\beta_{2}^{X}=4 \text { and } \gamma_{2}^{Z}=\gamma_{2}^{X}=3 \\
& \beta_{1}^{Z}=\beta_{1}^{X}=5 \text { and } \gamma_{1}^{Z}=\gamma_{1}^{X}=5
\end{aligned}
$$

Finally $\beta^{Z}=(5,4,3,3,1), \gamma^{Z}=(4,4,2,2), n=0$ and the result is $(Z, 0)$, where

2. Note that applying this procedure to $X$ and $n=5$ we get the result $(Z, 2)$, where


REmARK 2.3. By an empty column we mean a column of arbitrary length that has no entry. Let $X \in \mathcal{L} \mathcal{R}_{1}$ have $m$ empty columns and let $n \in \mathbb{N}$. Note that Algorithm 2.2 produces the pair $(Z, k)$, where $Z$ is created by putting $\min \{m, n\}$ elements 1 in empty columns of $X$ starting from the right most column. If $n$ is bigger than $m$, then $k=n-m$ and otherwise $k=0$.

Below we give an algorithmic definition of the operation $*:\left(\mathcal{L R}_{1} \times \mathbb{N}\right) \times\left(\mathcal{L R}_{1} \times \mathbb{N}\right) \rightarrow$ $\left(\mathcal{L R}_{1} \times \mathbb{N}\right)$..

Algorithm 2.4. Input. $(X, m),(Y, n) \in \mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$
Output. $(Z, k)=(Y, n) *(X, m) \in \mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$.

1. Apply Algorithm 2.2 to $X$ and $n$ and get $(T, s)=(\emptyset, n) *(X, 0)$
2. Apply Algorithm 2.1 to $T$ and $Y$ and get $Z=Y * T$.
3. The result is $(Z, s+m)$.

Example 4. Let

and $m=4, n=2$.
In the first step we apply Algorithm 2.2 to $X$ and $n$ so we obtain

and $s=0$ (see example to Algorithm (2.2).
Next by Algorithm 2.1] for $T$ and $Y$ we obtain the LR-tableau Z (see example to Algorithm (2.1).


Finally the result is $(Z, 4)$.
Example 5. Recall that the operation $*$ defined on $\mathcal{L R}_{1}$ is associative, see [6] Lemma 6.4]. Now we show that the operation $*$ defined on $\mathcal{L} \mathcal{R}_{1} \times \mathbb{N}$ is non-associative. Let $N=(\emptyset, 1), M=(\square, 0)$. Then applying Algorithm 2.4 we obtain:

$$
N * M=(\square, 0) \quad \text { and } \quad M * M=(\square, 0)
$$

and again applying Algorithm 2.4:

$$
(N * M) * M=(\square \square, 0) \quad \text { and } \quad N *(M * M)=(\square, 0)
$$

## 3. The category of homomorphisms of nilpotent linear operators

Let $k$ be an arbitrary field. For a partition $\alpha=\left(\alpha_{1} \geq \ldots \geq \alpha_{n}\right)$ we denote by $N_{\alpha}=N_{\alpha}(k)$ the nilpotent linear operator of type $\alpha$, i.e. the finite dimensional $k[T]$-module

$$
N_{\alpha}(k)=k[T] /\left(T^{\alpha_{1}}\right) \oplus \ldots \oplus k[T] /\left(T^{\alpha_{n}}\right)
$$

where $k[T]$ is the $k$-algebra of polynomials with one variable $T$ and $\oplus$ is the direct sum of $k[T]$-modules.
Denote by $\mathcal{H}=\mathcal{H}(k)$ the category of systems $\left(N_{\alpha}, N_{\beta}, f\right)$, where $\alpha, \beta$ are partitions and $f: N_{\alpha} \rightarrow N_{\beta}$ is a $k[T]$-homomorphism. Let $M=\left(N_{\alpha}, N_{\beta}, f\right), M^{\prime}=\left(N_{\alpha^{\prime}}, N_{\beta^{\prime}}, f^{\prime}\right)$ be objects of $\mathcal{H}$. A morphism $\phi: M \rightarrow M^{\prime}$ is a pair $\left(\phi_{1}, \phi_{2}\right)$, where $\phi_{1}: N_{\alpha} \rightarrow N_{\alpha^{\prime}}$, $\phi_{2}: N_{\beta} \rightarrow N_{\beta^{\prime}}$ are homomorphisms of $k[T]$-modules such that $f^{\prime} \phi_{1}=\phi_{2} f$. Denote by $\mathcal{S}$ or $\mathcal{S}(k)$ the full subcategory of $\mathcal{H}$ consisting of all systems $\left(N_{\alpha}, N_{\beta}, f\right)$, where $f$ is a monomorphism. For a natural number $n$, we write $\mathcal{H}_{n}$ or $\mathcal{H}_{n}(k)$ for the full subcategory of $\mathcal{H}$ of all systems $\left(N_{\alpha}, N_{\beta}, f\right)$ such that $\alpha_{1} \leq n$. In particular, objects of $\mathcal{H}_{1}$ are such that $N_{\alpha}$ is a semisimple $k[T]$-module. We will denote by $\mathcal{H}_{a}^{b}$ the full subcategory of $\mathcal{H}_{1}$ consisting of all systems $\left(N_{\alpha}, N_{\beta}, f\right)$ where $|\alpha|=a$ and $|\beta|=b$. For a natural number $n$, we write $\mathcal{S}_{n}$ or $\mathcal{S}_{n}(k)$ for the full subcategory of $\mathcal{S}$ consisting of objects from $\mathcal{S} \cap \mathcal{H}_{n}$.

### 3.1. Pickets

The category $\mathcal{H}_{1}(k)$ is of particular interest for us in this paper. We shortly describe its properties. An object $M=\left(N_{\alpha}, N_{\beta}, f_{M}\right) \in \mathcal{H}_{1}$ induces the following short exact sequence:

where $u$ is the natural embedding and $p$ is the canonical epimorphism. Since $\left(N_{\alpha}, N_{\beta}, f_{M}\right) \in$ $\mathcal{H}_{1}$, the module $N_{\alpha}$ is semisimple and therefore the lower row splits. So we obtain that $M \simeq M^{\prime} \oplus M^{\prime \prime}$ where $M^{\prime}=\left(\operatorname{Ker} f_{M}, 0,0\right)$ and $M^{\prime \prime}=\left(\operatorname{Coker} u, N_{\beta}, \overline{f_{M}}\right) \in \mathcal{S}_{1}$, i.e. $M^{\prime} \oplus M^{\prime \prime}=\left(\operatorname{Ker} f_{M} \oplus \operatorname{Coker} u, 0 \oplus N_{\beta}, 0 \oplus \overline{f_{M}}\right)$.
We present a description of indecomposable objects in $\mathcal{H}_{1}$. An object $\left(N_{\alpha}, N_{\beta}, f\right)$ of the category $\mathcal{H}$ such that $\beta=(m)$ we call a picket. First we note that any object of the form $M^{\prime}=\left(N_{\alpha}, 0,0\right) \in \mathcal{H}_{1}$ is isomorphic to a direct sum of indecomposable objects of the form:

$$
P_{1}^{0}=\left(N_{(1)}, 0,0\right) .
$$

It is because $\alpha=(1,1 \ldots, 1)$ and thus $N_{\alpha}$ is a semisimple $k[T]$-module.
Each indecomposable object of $\mathcal{S}_{1}$ is isomorphic to a picket that is, it has the form

$$
P_{0}^{m}=\left(0, N_{(m)}, 0\right)
$$

or

$$
P_{1}^{m}=\left(N_{(1)}, N_{(m)}, \iota\right)
$$

where $\iota(1)=T^{(m-1)}$, see [1]. Whenever we want to emphasize the dependence on the field $k$, we will write $P_{\ell}^{m}=P_{\ell}^{m}(k)$.
Thanks to this classification we can associate with any object $M$ in $\mathcal{S}_{1}(k)$ the LR-tableau $\Gamma(M) \in \mathcal{L R}_{1}$ and with any object $M \simeq M^{\prime} \oplus\left(P_{1}^{0}\right)^{m}$ in $\mathcal{H}_{1}(k)$, where $M^{\prime} \in \mathcal{S}_{1}(k)$, the pair

$$
\widehat{\Gamma}(M)=(\Gamma(M), m) \in \mathcal{L} \mathcal{R}_{1} \times \mathbb{N},
$$

where $\Gamma(M)=\Gamma\left(M^{\prime}\right)$. First we list LR-tableaux associated with pickets in the following table.

| LR-tableaux for the indecomposable objects in $\mathcal{S}_{1}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| X | $P_{0}^{m}$ | $P_{1}^{m}$ |  |
| $\Gamma(X)$ | $m\left\{\begin{array}{c}\square \\ \vdots \\ \square\end{array}\right.$ | $m\left\{\begin{array}{c}\square \\ \vdots \\ \square \\ \hline\end{array}\right.$ |  |

The LR - tableau for a direct sum $M \oplus M^{\prime}$ has a diagram given by the union $\beta \cup \beta^{\prime}$ of the partitions representing the ambient spaces, and in each row the entries are obtained by lexicographically ordering the entries in the corresponding rows in the tableaux for $M$ and $M^{\prime}$, with empty boxes coming first.

Example 6. The object $M=P_{0}^{7} \oplus P_{1}^{7} \oplus P_{1}^{5} \oplus P_{1}^{2} \oplus P_{1}^{2} \oplus P_{0}^{1} \oplus P_{1}^{0} \oplus P_{1}^{0}$ corresponds to the $\widehat{\Gamma}(M)=(\Gamma(M), 2)$, where


REMARK 3.1. The following observation is an easy consequence of the combinatorial classification of objects in $\mathcal{H}_{1}$. Let $M, N \in \mathcal{H}_{1}$ be such that $\widehat{\Gamma}(M)=(\Gamma(M), m)$ and $\widehat{\Gamma}(N)=(\Gamma(N), n)$. Assume that $\beta^{\Gamma(M)}=\beta^{\Gamma(N)}$ and $\gamma^{\Gamma(N)} \subseteq \gamma^{\Gamma(M)}$. If $P_{1}^{n}$ is a direct summand of $M$ then $P_{1}^{n}$ is a direct summand of $N$.

For each pair $(M, N)$ of indecomposable objects in $\mathcal{H}_{1}(k)$ we determine in the table below the dimension of the $k$-space $\operatorname{Hom}_{\mathcal{H}}(M, N)$ of homomorphisms $M \rightarrow N$ in the category $\mathcal{H}$, see [14, Lemma 4] and [8].

| Dimensions of Spaces $\operatorname{Hom}_{\mathcal{H}}(M, N)$ |  |  |  |
| :--- | :--- | :--- | :--- |
| $M$ | $N=P_{0}^{m}$ | $P_{1}^{m}$ | $P_{1}^{0}$ |
| $P_{0}^{\ell}$ | $\min \{\ell, m\}$ | $\min \{\ell, m\}$ | 0 |
| $P_{1}^{\ell}$ | $\min \{\ell-1, m\}$ | $\min \{\ell, m\}$ | 1 |
| $P_{1}^{0}$ | 0 | 0 | 1 |

### 3.2. Generic extensions and the hom-order

Let $M, N \in \mathcal{H}_{1}$. An object $U \in \mathcal{H}$ is an extension of $N$ by $M$ if there exists a short exact sequence of the form:

$$
0 \rightarrow M \rightarrow U \rightarrow N \rightarrow 0
$$

Note that the subcategory $\mathcal{H}_{1} \subseteq \mathcal{H}$ is not closed under extensions. In this paper we are interested only in the extensions that are objects of $\mathcal{H}_{1}$.
Since, for fixed integers $a, b$, the category $\mathcal{H}_{a}^{b} \subseteq \mathcal{H}_{1}$ has only finitely many isomorphism classes of indecomposable objects, the results of [2], [12] (see also [6, Proposition 4.2]) imply that given $M \in \mathcal{H}_{a}^{b}$ and $N \in \mathcal{H}_{c}^{d}$ there exists the unique (up to iso) extension $U \in \mathcal{H}_{a+c}^{b+d}$ of $N$ by $M$ with minimal dimension of its endomorphism ring $\operatorname{End}_{\mathcal{H}}(U)$. We call $U$ the generic extension of $N$ by $M$ and denote by $U=N * M$.
On the set of isomorphism classes of objects in $\mathcal{H}_{a}^{b}$ one can define the classical degeneration partial order $\leq_{\text {deg }}$ that has a geometric nature (see [6] for details). Algorithmic approach to degeneration order one can find for example in [8, 11]. It is known that the generic extension $U=N * M$ is the unique $\leq_{\operatorname{deg} \text {-minimal extension of } N \text { by } M \text {, see [12, 13]. }}^{\text {[ }}$. It is easy to see that the category $\mathcal{H}_{a}^{b}$ is equivalent to a subcategory of the category of modules over a finite dimensional algebra, see [6]. Therefore by [15], the degeneration order is equivalent with hom-order $\leq_{\text {hom }}$, because the category $\mathcal{H}_{a}^{b}$ has only finitely many isomorphism classes of indecomposable objects. It follows that the generic extension $U=N * M$ is the extension of $N$ by $M$ that is minimal in hom-order. In the paper instead of degeneration order we will use hom-order for testing whether an extension is the generic extension.

Definition 1. Fix natural numbers $a, b$. We say that $M, N \in \mathcal{H}_{a}^{b}$ are in the hom-order, in symbols $M \leq_{\text {hom }} N$, if

$$
[U, M] \leq[U, N]
$$

for any object $U$ in $\mathcal{H}(k)$. Here we write $[M, N]=\operatorname{dim}_{k} \operatorname{Hom}_{\mathcal{H}}(M, N)$ for $M, N \in \mathcal{H}$.
By [13, page 280] $M \leq_{\text {hom }} N$ if and only if

$$
[M, U] \leq[N, U]
$$

for any object $U$ in $\mathcal{H}(k)$.

### 3.3. Properties of pickets

We collect some elementary properties of pickets of the form $P_{1}^{0}$. We collect some elementary properties of pickets of the form $P_{1}^{0}$.

Lemma 3.2. Let $M$ be an object of $\mathcal{S}_{1}$ and $N$ be an object of $\mathcal{H}_{1}$. We have
(1) $\operatorname{Hom}_{\mathcal{H}}\left(P_{1}^{0}, M\right)=0$,
(2) $\operatorname{Ext}_{\mathcal{H}}^{1}\left(M, P_{1}^{0}\right)=0$, where $\operatorname{Ext}_{\mathcal{H}}^{1}\left(M, P_{1}^{0}\right)=0$ is the first group of extensions of $M$ by $P_{1}^{0}$.
(3) if $f: M \oplus\left(P_{1}^{0}\right)^{m} \rightarrow N$ is a monomorphism, then $\left(P_{1}^{0}\right)^{m}$ is a direct summand of $N$,
(4) $N *\left(M \oplus\left(P_{1}^{0}\right)^{m}\right)=(N * M) \oplus\left(P_{1}^{0}\right)^{m}$.

Proof. Let $M=\left(N_{\alpha}, N_{\beta}, f\right) \in \mathcal{S}_{1}$.
(1) Any homomorphism $\phi=\left(\phi_{1}, \phi_{2}\right) \in \operatorname{Hom}\left(P_{1}^{0}, M\right)$ induces the following commutative diagram:


Therefore $\phi=\left(\phi_{1}, \phi_{2}\right)=0$, because $f$ is a monomorphism .
(2) We consider an extension $U$ of $M$ by $P_{1}^{0}$ :

$$
0 \longrightarrow P_{1}^{0} \longrightarrow U \longrightarrow M \longrightarrow 0
$$

Since $\left[P_{1}^{0}, U\right] \neq 0$, by (1) the object $P_{1}^{0}$ is a direct summand of $U$. Finally $U \simeq P_{1}^{0} \oplus M$. The statement (3) is an easy consequence of (1).
(4) Suppose that there exists an extension $U \in \mathcal{H}_{1}$ of $N$ by $M \oplus\left(P_{1}^{0}\right)^{m}$ and such that

$$
\begin{equation*}
[U, U]<\left[(N * M) \oplus\left(P_{1}^{0}\right)^{m},(N * M) \oplus\left(P_{1}^{0}\right)^{m}\right] \tag{3.3}
\end{equation*}
$$

Since there is a monomorphism $M \oplus\left(P_{1}^{0}\right)^{m} \rightarrow U$, by (3) we have $U \simeq \bar{U} \oplus\left(P_{1}^{0}\right)^{m}$. It is easy to deduce that there is a short exact sequence:

$$
0 \longrightarrow M \longrightarrow \bar{U} \longrightarrow N \longrightarrow 0
$$

Since $N * M$ is the generic extension of $N$ by $M$, we have $[N * M, N * M] \leq[\bar{U}, \bar{U}]$ and $N * M \leq_{\text {hom }} \bar{U}$. It follows that $\left[N * M,\left(P_{1}^{0}\right)^{m}\right] \leq\left[\bar{U},\left(P_{1}^{0}\right)^{m}\right]$ and $\left[\left(P_{1}^{0}\right)^{m}, N * M\right] \leq$ $\left[\left(P_{1}^{0}\right)^{m}, \bar{U}\right]$. Finally we get

$$
\begin{aligned}
& {\left[(N * M) \oplus\left(P_{1}^{0}\right)^{m},(N * M) \oplus\left(P_{1}^{0}\right)^{m}\right]=} \\
& {[N * M, N * M]+\left[\left(P_{1}^{0}\right)^{m}, N * M\right]+\left[N * M,\left(P_{1}^{0}\right)^{m}\right]+\left[\left(P_{1}^{0}\right)^{m},\left(P_{1}^{0}\right)^{m}\right] \leq} \\
& {[\bar{U}, \bar{U}]+\left[\left(P_{1}^{0}\right)^{m}, \bar{U}\right]+\left[\bar{U},\left(P_{1}^{0}\right)^{m}\right]+\left[\left(P_{1}^{0}\right)^{m},\left(P_{1}^{0}\right)^{m}\right]=} \\
& \\
& \left.\qquad \bar{U} \oplus\left(P_{1}^{0}\right)^{m}, \bar{U} \oplus\left(P_{1}^{0}\right)^{m}\right]=[U, U]
\end{aligned}
$$

This contradicts (3.3). We are done.
Lemma 3.4. (1) Let $U, V \in \mathcal{H}_{1}$ be such that $U \leq_{\text {hom }} V$. If $P_{1}^{0}$ is a direct summand of $U$, then $P_{1}^{0}$ is a direct summand of $V$.
(2) For $m>k$ we have $\left(P_{0}^{m} \oplus P_{1}^{k}\right) \leq_{\text {hom }}\left(P_{1}^{m} \oplus P_{0}^{k}\right)$.
(3) For any $m>0$ we have $P_{1}^{m} \leq_{\text {hom }}\left(P_{0}^{m} \oplus P_{1}^{0}\right)$.

Proof. (1) Assume that $P_{1}^{0}$ is a direct summand of $U$. It follows that $\left[P_{1}^{0}, U\right] \neq 0$. Since $U \leq_{\text {hom }} V$, we have $\left[P_{1}^{0}, V\right] \neq 0$. Therefore $P_{1}^{0}$ is a direct summand of $V$, because $\left[P_{1}^{0}, W\right]=0$ for every $W \in \mathcal{S}_{1}$ (Lemma 3.2 (1)).
(2) Let $W \in \mathcal{H}$. Applying the functor $\operatorname{Hom}_{\mathcal{H}}(\mathrm{W},-)$ to the following short exact sequence:

$$
0 \rightarrow P_{0}^{k} \rightarrow P_{0}^{m} \oplus P_{1}^{k} \rightarrow P_{1}^{m} \rightarrow 0
$$

we obtain the exact sequence:

$$
0 \rightarrow \operatorname{Hom}_{\mathcal{H}}\left(\mathrm{W}, \mathrm{P}_{0}^{\mathrm{k}}\right) \rightarrow \operatorname{Hom}_{\mathcal{H}}\left(\mathrm{W}, \mathrm{P}_{0}^{\mathrm{m}} \oplus \mathrm{P}_{1}^{\mathrm{k}}\right) \rightarrow \operatorname{Hom}_{\mathcal{H}}\left(\mathrm{W}, \mathrm{P}_{1}^{\mathrm{m}}\right) \rightarrow \operatorname{Coker}(\mathrm{f}) \rightarrow 0
$$

Thus $\left[W, P_{0}^{m} \oplus P_{1}^{k}\right] \leq\left[W, P_{0}^{m} \oplus P_{1}^{k}\right]+\operatorname{dim}_{k} \operatorname{Coker}(f)=\left[W, P_{1}^{m}\right]+\left[W, P_{0}^{k}\right]=\left[W, P_{1}^{m} \oplus P_{0}^{k}\right]$ and $\left(P_{0}^{m} \oplus P_{1}^{k}\right) \leq_{\text {hom }}\left(P_{1}^{m} \oplus P_{0}^{k}\right)$.
(3) This follows by the same method as in the proof of (2). We have to apply the functor $\operatorname{Hom}_{\mathcal{H}}(\mathrm{W},-)$ to the following short exact sequence:

$$
0 \rightarrow P_{0}^{m} \rightarrow P_{1}^{m} \oplus P_{0}^{k} \rightarrow P_{1}^{0} \rightarrow 0
$$

## 4. The operation $*$ gives generic extensions

In this section we prove that operations defined in Section 2 are equivalent to the operation of taking the generic extensions.
The following theorem is proved in [6].
Theorem 4.1. (1) Let $M, N$ be objects of the category $\mathcal{S}_{1}$. The object $U \in \mathcal{S}_{1}$, where $\Gamma(U)=\Gamma(N) * \Gamma(M)$ is computed by Algorithm 2.1 for $\Gamma(M), \Gamma(N)$, is the generic extension of $N$ by $M$, i.e. $U=N * M$.
(2) The operation $*$ defined on $\mathcal{L R}_{1} \times \mathcal{L R}_{1}$ is associative.

We prove several facts that generalize this result.
Proposition 4.2. Let $M \in \mathcal{S}_{1}$ and let $n \in \mathbb{N}$. Let $(\Gamma(U), u)=(\emptyset, n) *(\Gamma(M), 0)$ be computed by Algorithm [2.2 for $\Gamma(M)$ and $n$. The object $U \in \mathcal{H}_{1}$, such that $\widehat{\Gamma}(U)=$ $(\Gamma(U), u)$, is the generic extension of $\left(P_{1}^{0}\right)^{n}$ by $M$, i.e. $U=\left(P_{1}^{0}\right)^{n} * M$.

Proof. Let $U$ be the object such that $\widehat{\Gamma}(U)=(\Gamma(U), u)=(\emptyset, n) *(\Gamma(M), 0)$ is constructed by Algorithm [2.2. Note that

$$
0 \rightarrow P_{0}^{c} \rightarrow P_{1}^{c} \rightarrow P_{1}^{0} \rightarrow 0
$$

is a short exact sequence in the category $\mathcal{H}_{1}$. Now, applying Remark 2.3, it is easy to see that $U$ is an extension of $\left(P_{1}^{0}\right)^{n}$ by $M$.
Let $M=P_{\varepsilon_{1}}^{k_{1}} \oplus \ldots \oplus P_{\varepsilon_{s}}^{k_{s}} \in \mathcal{S}_{1}$, where $k_{1} \geq k_{2} \geq \ldots \geq k_{s} \geq 1$ and $\varepsilon_{i} \in\{0,1\}$ for all $i$. Set $I=\{1, \ldots, s\}, I_{0}=\left\{i \in I ; \varepsilon_{i}=0\right\}$ and $I_{1}=I \backslash I_{0}$. Let $N$ be an arbitrary extension of $\left(P_{1}^{0}\right)^{n}$ by $M$ and let $\widehat{\Gamma}(N)=\left(\Gamma(N), n^{\prime}\right)$. Note that $\bigoplus_{i \in I_{1}} P_{1}^{k_{i}}$ is a direct summand of $N$. Indeed, the short exact sequence

$$
0 \rightarrow M \rightarrow N \rightarrow\left(P_{1}^{0}\right)^{n} \rightarrow 0
$$

induces the following commutative diagram with exact rows and columns

where $n^{\prime} \leq n$. It follows that $\beta^{\Gamma(M)}=\beta^{\Gamma(N)}$ and, by the Snake Lemma, we get the following exact sequence

$$
0 \rightarrow N_{(1)^{n^{\prime}}} \rightarrow N_{(1)^{n}} \rightarrow N_{\gamma^{\Gamma(M)}} \rightarrow N_{\gamma^{\Gamma(N)}} \rightarrow 0
$$

Applying [10, (3.1) page 185] we get $\gamma^{\Gamma(N)} \subseteq \gamma^{\Gamma(M)}$. Therefore Remark 3.1 implies that the object $\bigoplus_{i \in I_{1}} P_{1}^{k_{i}}$ is a direct summand of $U$. Now, applying Remark 2.3 and Lemma 3.4 (b), (c) we can deduce that extension computed by Algorithm 2.2 is minimal in hom-order among all extensions of $\left(P_{1}^{0}\right)^{n}$ by $M$. We are done.

The following result is a simple consequence of Algorithm 2.2,
Lemma 4.3. Let $X, Y \in \mathcal{L R}_{1}, m \in \mathbb{N}$ and let $(\emptyset, m) *(X, 0)=(Y, n) \in \mathcal{L R}_{1} \times \mathbb{N}$ for some $n \in \mathbb{N}$. Then $(Y, 0)=(\emptyset, m-n) *(X, 0)$.

Lemma 4.4. Let $M, N \in \mathcal{S}_{1}$ and $n \in \mathbb{N}$. Let $(T, s)=(\emptyset, n) *(\Gamma(M), 0)$ be computed by Algorithm 2.2 and $Z=\Gamma(N) * T$ be computed by Algorithm 2.1. The generic extension of $N \oplus\left(P_{1}^{0}\right)^{n}$ by $M$ is $W$, where $\widehat{\Gamma}(W)=(Z, s)$.

Proof. Let $M \in \mathcal{S}_{1}$ and $N \in \mathcal{S}_{1}$. Consider $N^{\prime \prime} \simeq N \oplus\left(P_{1}^{0}\right)^{n}$. It follows from Lemma 3.2 (2) that $N^{\prime \prime}$ is the unique extension of $N$ by $\left(P_{1}^{0}\right)^{n}$. By Proposition 4.2, the generic extension $V^{\prime \prime}$ of $\left(P_{1}^{0}\right)^{n}$ by $M$ is such that $\widehat{\Gamma}\left(V^{\prime \prime}\right)=(\emptyset, n) *(\Gamma(M), 0)$. Let $\widehat{\Gamma}\left(V^{\prime \prime}\right)=(\Gamma(V), s)$ for suitable $V \in \mathcal{S}_{1}$. By Lemma 4.3, we have $(\Gamma(V), 0)=(\emptyset, n-s) *(\Gamma(M), 0)$. Thus Proposition 4.2 implies that $V$ is the generic extension of $\left(P_{1}^{0}\right)^{n-s}$ by $M$. Fix a short exact sequence

$$
0 \longrightarrow M \xrightarrow{f} V \longrightarrow\left(P_{1}^{0}\right)^{n-s} \longrightarrow 0
$$

By Lemma 4.1, the generic extension $W$ of $N$ by $V$ is given by $\Gamma(W)=\Gamma(N) * \Gamma(V)$. Fix a short exact sequence

$$
0 \longrightarrow V \xrightarrow{g} W \longrightarrow N \longrightarrow 0
$$

We show that $W \in \mathcal{S}_{1}$ is the generic extension of $N^{\prime}$ by $M$, where $N^{\prime} \simeq N \oplus\left(P_{1}^{0}\right)^{n-s}$. The following diagram, induced by short exact sequences given above, proves that $W$ is an extension of $N^{\prime}$ by $M$ :


Note that $\bar{N} \simeq N^{\prime}$, because it follows from Lemma 3.2 (2) that $N^{\prime}$ is the unique extension of $N$ by $\left(P_{1}^{0}\right)^{n-s}$. Consequently the last column splits.
Suppose for a contrary that there exists an extension $W^{\prime} \in \mathcal{H}_{1}$ of $N^{\prime}$ by $M$ such that $W \nsucceq W^{\prime}$ and $\left[W^{\prime}, W^{\prime}\right] \leq[W, W]$. In this case $W^{\prime} \leq_{\text {hom }} W$. Since $W \in \mathcal{S}_{1}$, by Lemma 3.4
(1) we have $W^{\prime} \in \mathcal{S}_{1}$. Consider the diagram

where $V^{\prime}$ is the pullback of the morphisms $h_{1}$ and $h_{2}$. Since $W^{\prime} \in \mathcal{S}_{1}$, we have $V^{\prime} \in \mathcal{S}_{1}$, because $\left[P_{1}^{0}, \mathcal{S}_{1}\right]=0$. Since $V$ is the generic extension of $\left(P_{1}^{0}\right)^{n-s}$ by $M$ and $V^{\prime}$ is an extension of $\left(P_{1}^{0}\right)^{n-s}$ by $M$, we have $V \leq_{\operatorname{deg}} V^{\prime}$ and by [6, Lemma 5.8] also $W=N * V \leq_{\text {hom }}$ $N * V^{\prime} \leq_{\text {hom }} W^{\prime}$. We proved that $W \leq_{\text {hom }} W^{\prime}$ and $W^{\prime} \leq_{\text {hom }} W$. This implies that $W \simeq W^{\prime}$, because $\leq_{\text {hom }}$ is a partial order. This contradiction proves that $W$ is the generic extension of $N^{\prime}$ by $M$.
It remains to prove that $W \oplus\left(P_{1}^{0}\right)^{s}$ is the generic extension of $N^{\prime \prime}$ by $M$. Let $\bar{W}$ be the generic extension of $N^{\prime \prime}$ by $M$. We construct the following pullback diagram


Since $V^{\prime \prime}$ is the generic extension of $\left(P_{1}^{0}\right)^{n}$ by $M$ and $\left(P_{1}^{0}\right)^{s}$ is a direct summand of $V^{\prime \prime}$, we conclude that $\left(P_{1}^{0}\right)^{s}$ is a direct summand of $\bar{V}$. It follows that $\left(P_{1}^{0}\right)^{s}$ is a direct summand of $\bar{W}$, because $\left[P_{1}^{0}, \mathcal{S}_{1}\right]=0$. Therefore $\bar{W} \simeq \bar{W}^{\prime} \oplus\left(P_{0}^{1}\right)^{s}$. We have $\bar{W} \leq_{\text {hom }} W \oplus\left(P_{0}^{1}\right)^{s}$ (because $\bar{W}$ is the generic extension of $N^{\prime \prime}$ by $M$ ), and it follows that $\bar{W}^{\prime} \leq_{\text {hom }} W$. Since $W$ is the generic extension of $N^{\prime}$ by $M$, we get $W \simeq \bar{W}^{\prime}$ and consequently $\bar{W} \simeq W \oplus\left(P_{1}^{0}\right)^{s}$. We are done.

Lemma 4.5. Let $M \in \mathcal{S}_{1}, N \in \mathcal{H}_{1}$ and $m \in \mathbb{N}$. The generic extension of $N$ by $M \oplus\left(P_{1}^{0}\right)^{m}$ is $U \oplus\left(P_{1}^{0}\right)^{m}$, where $U$ is the generic extension of $N$ by $M$. Moreover, $\widehat{\Gamma}\left(U \oplus\left(P_{1}^{0}\right)^{m}\right)=$ $(\Gamma(U), s+m)$, where $(\Gamma(U), s)=\widehat{\Gamma}(N) *(\Gamma(M), 0)$.

Proof. It is a direct consequence of Lemma 3.2 (4) and Lemma 4.4.
TheOrem 4.6. Let $M, N \in \mathcal{H}_{1}$ be such that $\widehat{\Gamma}(M)=(\Gamma(M), m)$ and $\widehat{\Gamma}(N)=(\Gamma(N), n)$. The generic extension $U=N * M \in \mathcal{H}_{1}$ is such that $\widehat{\Gamma}(U)=(\Gamma(N), n) *(\Gamma(M), m)$ is computed by Algorithm 2.4.

Proof. Let $M=M^{\prime} \oplus\left(P_{1}^{0}\right)^{m}$ and $N=N^{\prime} \oplus\left(P_{1}^{0}\right)^{n}$, where $M^{\prime}, N^{\prime} \in \mathcal{S}_{1}$. By Lemma 4.5 we have

$$
N * M=\left(\left(N^{\prime} \oplus\left(P_{1}^{0}\right)^{n}\right) * M\right) \oplus\left(P_{1}^{0}\right)^{m} .
$$

Let $V=\left(N^{\prime} \oplus\left(P_{1}^{0}\right)^{n}\right) * M$. By Lemma 4.4, to get $\widehat{\Gamma}(V)$ we have to apply first Algorithm 2.2 to $\Gamma(M)$ and $n$ and get $(T, s)$, then we have to apply Algorithm 2.1 to $T$ and $\Gamma\left(N^{\prime}\right)$ to get $Z$. Finally, $\widehat{\Gamma}(V)=(Z, s)$ and $\widehat{\Gamma}(N * M)=(Z, s+m)=(\Gamma(N), n) *(\Gamma(M), m)$

Example 7. We rewrite the last example of Section 园 in terms of generic extensions. Let $N=P_{1}^{0}, M=P_{0}^{1}$ then we have

$$
N * M=P_{1}^{1} \quad \text { and } \quad M * M=P_{0}^{2} .
$$

Moreover, it easy to check that

$$
(N * M) * M=P_{1}^{1} * P_{0}^{1}=P_{0}^{1} \oplus P_{1}^{1} \text { and } N *(M * M)=P_{1}^{0} * P_{0}^{2}=P_{1}^{2}
$$

Therefore * is non-associative.

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